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ON SOME ERGODIC PROPERTIES FOR CONTINUOUS AND AFFINE FUNCTIONS

by C.J.K. BATTY

1. Introduction.

Let X be a compact Hausdorff space, let $C(X)$ denote the space of continuous real-valued functions on X , and let T be a positive linear operator of $C(X)$ into itself. Choquet and Foias [1] have considered convergence properties of the iterates T^n of T and the associated arithmetic means $S_n = n^{-1} \sum_{r=0}^{n-1} T^r$. In particular, they obtained the following two results [1, Théorèmes 13, 1]:

THEOREM 1.1. — *If, for some non-negative function f in $C(X)$, $S_n f$ converges pointwise to a continuous strictly positive function, then the convergence is uniform on X .*

THEOREM 1.2. — *If, for each x in X , $\inf \{(T^n 1)(x) : n \geq 1\} < 1$, then $T^n 1$ converges to 0 uniformly on X .*

Choquet and Foias showed that the condition that the limit in theorem 1.1 is strictly positive cannot be removed [1, Exemple 11]. They then raised the following problem:

PROBLEM 1. — *Suppose that $S_n 1$ converges pointwise to a continuous limit. Is the convergence necessarily uniform?*

If $M(X)$ denotes the set of Radon measures on X , identified with $C(X)^*$, and $P(X)$ is the set of probability measures in $M(X)$, then $P(X)$ is weak*-compact and convex, its extreme boundary $\partial_e P(X)$ consists of the measures ϵ_x concentrated at one point x

of X , and there is an isometric order-isomorphism $f \mapsto \hat{f}$ of $C(X)$ onto the space $A(P(X))$ of continuous affine real-valued functions on $P(X)$, given by $\hat{f}(\mu) = \int f d\mu$. This raises a second problem.

PROBLEM 2. — *Suppose that K is a compact convex subset of a locally convex space, and T is a positive linear operator on $A(K)$ such that for each x in $\partial_e K$, $\inf \{(T^n 1)(x) : n \geq 1\} < 1$. Does it necessarily follow that $\|T^n\| \rightarrow 0$?*

In § 2 we shall show (corollary 2.5) that the answer to problem 1 is affirmative, and in § 3 we shall give an example to show that the answer to problem 2 is negative, although it becomes affirmative if $\partial_e K$ is replaced by its closure $\overline{\partial_e K}$ in K .

2. Uniform convergence of arithmetic means.

Let T be a positive linear operator on $C(X)$, and σ be a non-negative function in $C(X)$. Let $F_\sigma = \sigma^{-1}(0)$ and G_σ be the complement of F_σ in X . For x in G_σ and $n \geq 1$ there is a bounded Radon measure $\mu_{x,\sigma}^n$ on G_σ such that

$$\int g d\mu_{x,\sigma}^n = \sigma(x)^{-1} T^n(g \cdot \sigma)(x)$$

for all functions g in the space $C^b(G_\sigma)$ of continuous bounded real-valued functions on G_σ . For a Borel-measurable function f defined $\mu_{x,\sigma}^n$ -a.e. in G_σ , put $(T_\sigma^{(n)} f)(x) = \int f d\mu_{x,\sigma}^n$ if the integral exists.

LEMMA 2.1. — *For x in G_σ , $n \geq 1$ and any bounded Borel function f on G_σ . $T_\sigma^{(n)}(f \cdot \sigma^{-1})(x) = \sigma(x)^{-1} T_1^{(n)}(\chi_\sigma \cdot f)(x)$, where χ_σ is the characteristic function of G_σ , and both sides of the equality exist.*

Proof. — Suppose that f is continuous and non-negative. Let (g_λ) be an increasing net of continuous non-negative functions on X with support in G_σ and converging pointwise to χ_σ . Then $g_\lambda \cdot f \cdot \sigma^{-1} \in C^b(G_\sigma)$, and

$$\sigma(x) \int g_\lambda \cdot f \cdot \sigma^{-1} d\mu_{x,\sigma}^n = T^n(g_\lambda \cdot f)(x) = \int g_\lambda \cdot f d\mu_{x,1}^n.$$

The right-hand integral increases to the finite integral $\int \chi_\sigma \cdot f d\mu_{x,1}^n$, so the result follows immediately in this special case.

The case when f is lower semi-continuous follows by approximating f from below by continuous functions, and the general case from the fact that the bounded Borel functions form the smallest linear space containing the lower semi-continuous functions and closed under bounded monotone sequential limits.

Now suppose that $T\sigma \leq \beta\sigma$ for some real number β . Then $T_\sigma^{(n)}1 \leq \beta^n$, so $T_\sigma^{(n)}$ maps $C^b(G_\sigma)$ into itself. It follows immediately from the definitions that the following identity is valid for f in $C^b(G_\sigma)$: $T_\sigma^{(m)}(T_\sigma^{(n)}f)(x) = (T_\sigma^{(m+n)}f)(x)$. Elementary integration theory shows that this identity is valid for any Borel function f on G , in the sense that if either expression exists then so does the other and they are equal. We shall therefore write T_σ^n instead of $T_\sigma^{(n)}$. This discussion applies in particular to the case $\sigma = 1$ when it is consistent to write T instead of T_1 .

For x in F_σ , $0 \leq (T^n\sigma)(x) \leq \beta^n\sigma(x) = 0$ so $\mu_{x,1}^n(G_\sigma) = 0$. Thus $T^n(\chi_\sigma \cdot f) = 0$ on F_σ . Note that this is consistent with lemma 2.1 which gives

$$\begin{aligned} T_\sigma^m(T_\sigma^n(f \cdot \sigma^{-1})) &= \sigma^{-1}T^m(\chi_\sigma \cdot T^n(\chi_\sigma \cdot f)) \\ T_\sigma^{m+n}(f \cdot \sigma^{-1}) &= \sigma^{-1}T^{m+n}(\chi_\sigma \cdot f). \end{aligned}$$

LEMMA 2.2. — *Suppose that $T\sigma \leq \sigma$ and $(T1)(x) < 1$ for all x in F_σ . Then there is a real number α such that $(T^n\chi_\sigma)(x) \leq \alpha$ for all $n \geq 1$ and x in G_σ .*

Proof. — By continuity and compactness, there is a neighbourhood U of F_σ and real numbers $\beta_1 < 1$ and $\beta_2 \geq \beta_1(1 - \beta_1)\|\sigma\|^{-1}$ such that

$$\begin{aligned} (T1)(x) &\leq \beta_1 && (x \in U) \\ (T1)(x) &\leq \beta_2\sigma(x) && (x \in K \setminus U). \end{aligned}$$

Let $\alpha = (1 - \beta_1)^{-1}\beta_2\|\sigma\|$. Then $T1 \leq \alpha$ and $T1 \leq \beta_1 + \beta_2\sigma$. In particular, $T\chi_\sigma \leq T1 \leq \alpha$. Now suppose that $T^n\chi_\sigma \leq \alpha$ on G_σ , and take x in G_σ . Using lemma 2.1 and the fact that $T_\sigma 1 \leq 1$,

$$\begin{aligned} (T^{n+1}\chi_\sigma)(x) &= T^n(T\chi_\sigma)(x) = \sigma(x)T_\sigma^n(\sigma^{-1} \cdot T\chi_\sigma)(x) \\ &\leq \sigma(x)T_\sigma^n(\beta_1\sigma^{-1} + \beta_2)(x) \\ &\leq \beta_1(T^n\chi_\sigma)(x) + \beta_2\sigma(x) \\ &\leq \beta_1\alpha + \beta_2\sigma(x) \\ &\leq \alpha. \end{aligned}$$

LEMMA 2.3. — Let F be a Borel subset of X , χ be the characteristic function of the complement of F in X , and

$$\delta = \sup \{(T1)(x) : x \in F\}.$$

Then
$$T^n 1 \leq \delta^n + \sum_{r=1}^n \delta^{r-1} T^{n-r} (\chi \cdot T1).$$

Proof. — It is trivial that $T1 \leq \delta + \chi \cdot T1$. Suppose the lemma holds for some integer n . Then since T is positive,

$$\begin{aligned} T^{n+1} 1 &\leq \delta^n T1 + \sum_{r=1}^n \delta^{r-1} T^{n+1-r} (\chi \cdot T1) \\ &\leq \delta^{n+1} + \sum_{r=1}^{n+1} \delta^{r-1} T^{n+1-r} (\chi \cdot T1). \end{aligned}$$

THEOREM 2.4. — Let T be a positive linear operator on $C(X)$ and suppose that there is a non-negative continuous function σ on X such that $T\sigma \leq \sigma$ and $(T1)(x) < 1$ whenever $\sigma(x) = 0$. Then $\{T^n 1 : n \geq 1\}$ is uniformly bounded.

Proof. — Take α as in lemma 2.2 and

$$\delta = \sup \{(T1)(x) : x \in F_\sigma\} < 1.$$

By lemma 2.3, for x in G_σ ,

$$\begin{aligned} (T^n 1)(x) &\leq \delta^n + \alpha \|T1\| \sum_{r=1}^n \delta^{r-1} \\ &\leq \delta^n + (1 - \delta)^{-1} \alpha \|T1\|. \end{aligned}$$

Also $T^n 1 = T((1 - \chi_\sigma) T^{n-1} 1)$ on F_σ , so a simple inductive argument shows that $T^n 1 \leq 1$ on F_σ .

COROLLARY 2.5. — Suppose that $S_n 1$ converges pointwise to a continuous limit σ . Then the convergence is uniform.

Proof. — It is shown in the proof of [1, Lemme 12] that $T\sigma \leq \sigma$. Hence $\mu_{x,1}^1(G_\sigma) = 0$ for x in F_σ . so T induces a positive linear operator \tilde{T} on $C(F_\sigma)$ given by

$$(\tilde{T}f)(x) = \int_{F_\sigma} f d\mu_{x,1}^1.$$

Now $\tilde{T}^n 1$ is the restriction of $T^n 1$ to $F_\sigma = \sigma^{-1}(0)$, so $\inf \{\tilde{T}^n 1 : n \geq 1\} = 0$. By theorem 1.2 there is an integer m such that $T^m 1 < 1$ on F_σ . Applying theorem 2.4 to T^m , it follows that $\{T^{mn} : n \geq 1\}$ is uniformly bounded. Hence $\{T^n 1 : n \geq 1\}$ is uniformly bounded. The result now follows from [1, Théorème 10].

3. Affine functions.

We shall now give an example to show that the answer to problem 2 is negative in general, even if K is a simplex.

Example 3.1. — Let N be the linear span in $M[0,1]$ of $\lambda - \epsilon_0$, where λ is Lebesgue measure on $[0,1]$, let $\pi: M[0,1] \rightarrow M[0,1]/N$ be the quotient map, and let $K = \pi(P[0,1])$. Then K is a simplex with extreme boundary $\partial_e K = \{\pi(\epsilon_x): x \in (0,1)\}$, and there is an isometric isomorphism Φ between $A(K)$ and the space $C_0[0,1]$ of functions f in $C[0,1]$ satisfying $f(0) = \int_0^1 f(x) dx$, given by $\Phi^{-1}(f) \circ \pi = \hat{f}$ ($f \in C_0[0,1]$). We shall identify these spaces.

Let g be any continuously differentiable function of $[0,1]$ into itself (in the sense of one-sided derivatives at the end-points) such that

$$\begin{aligned} g(0) &= 0, & g'(0) &= 1 \\ g(x) &> x, & g'(x) &\geq 0 \quad (x \in (0,1)) \\ g(1) &= 1, & g'(1) &= 0. \end{aligned}$$

Define the operator T by $(Tf)(x) = g'(x) f(g(x))$. Then T is a positive linear operator of $C_0[0,1]$ into itself.

For any x in $(0,1)$, let $x_0 = x$, $x_r = g(x_{r-1})$. Then x_r increases to the limit 1, so $g'(x_r) \rightarrow 0$. Now

$$(T^n 1)(x) = \prod_{r=0}^{n-1} g'(x_r) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus T satisfies all the required properties. However

$$\|T^n\| \geq |(T^n 1)(0)| = 1.$$

It is noted in [1] that Mokobodzki has shown that problem 2 has an affirmative answer if $\partial_e K$ is closed. This is a special case of the following result, which deals with a general K , but assumes a strengthened condition on T . The proof is based on one of those given in [1].

THEOREM 3.2. — *Let K be a compact convex set, let $\overline{\partial_e K}$ be the closure of its extreme boundary, and let T be a positive linear operator on $A(K)$. If, for each x in $\overline{\partial_e K}$, $\inf \{(T^n 1)(x): n \geq 1\} < 1$, then $\|T^n 1\| \rightarrow 0$.*

Proof. — For a bounded real-valued function g on K , and x in K , put $(\tilde{T}g)(x) = \inf \{(Ta)(x) : a \in A(K), a \geq g \text{ on } \partial_e K\}$. Then $\tilde{T}(\lambda g) = \lambda \tilde{T}g$, $\tilde{T}g_1 \leq \tilde{T}g_2$ if $g_1 \leq g_2$ on $\partial_e K$, and $\tilde{T}a = Ta$ for a in $A(K)$.

By compactness of $\overline{\partial_e K}$, there is an integer r and constant $\alpha > 0$ such that if $g_0(x) = \min \{(T + \alpha)^n 1(x) : 1 \leq n \leq r\}$, then $g_0 \leq 1$ on $\partial_e K$. Then $(\tilde{T} + \alpha)g_0 \leq (T + \alpha)1$ on $\partial_e K$. Also $g_0 \leq (T + \alpha)^n 1$, so $(\tilde{T} + \alpha)g_0 \leq (T + \alpha)^{n+1} 1$ ($1 \leq n \leq r$). Hence, on $\partial_e K$, $(\tilde{T} + \alpha)g_0 \leq g_0$, so $\tilde{T}g_0 \leq (1 - \alpha)g_0$.

Now $g_0 \geq \alpha^r$, so $T^n 1 \leq \alpha^{-r} \tilde{T}^n g_0 \leq \alpha^{-r} (1 - \alpha)^n g_0$ on $\partial_e K$. The result now follows.

Similarly one may modify the proof of Théorème 2 of [1] to show that if, under the conditions of theorem 3.2,

$$\sup \{(T^n 1)(x) : n \geq 1\} > 1$$

for each x in $\overline{\partial_e K}$, then $\|T^n\| \rightarrow \infty$.

Example 3.3. — Let \mathcal{H} be a complex Hilbert space, and x be an operator on \mathcal{H} such that $x - \alpha$ is compact for some scalar α with $|\alpha| < 1$. Suppose that for each unit vector ξ in \mathcal{H} , $\|x^n \xi\| < 1$ for some n (possibly dependent on ξ). If x is self-adjoint, the spectral theorem may be used to deduce that $\|x\| < 1$. However it is easily verified for example that any non-self-adjoint operator x of rank 1 and norm 1 also satisfies $\|x^2\| < 1$.

Let A be the C^* -algebra spanned by the identity and the compact operators on \mathcal{H} , and let K be its state space. It is well-known that the evaluation map is an isometric order-isomorphism of the self-adjoint part A^s of A onto $A(K)$, and that $\partial_e K$ consists of the vector states ω_ξ ($\xi \in \mathcal{H}$, $\|\xi\| = 1$) given by $\omega_\xi(a) = \langle a\xi, \xi \rangle$ together with the unique state ϕ_0 annihilating the compacts [2, Corollaire 4.1.4]. Using the weak compactness of the unit ball of \mathcal{H} it is easy to see that $\overline{\partial_e K}$ consists of states of the form $\beta \omega_\xi + (1 - \beta) \phi_0$ ($\beta \in [0, 1]$).

If x satisfies the above conditions, and T is defined by $Ta = x^*ax$ then T is a positive linear operator on A^s , and

$$(\beta \omega_\xi + (1 - \beta) \phi_0)(T^n 1) = \beta \|x^n \xi\|^2 + (1 - \beta) |\alpha|^{2n} < 1$$

for some n . Theorem 3.2 now shows that $\|T^n 1\| \rightarrow 0$, so $\|x^n\| \rightarrow 0$.

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