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PROPERTIES OF ORLICZ-PETTIS OR NIKODYM TYPE AND BARRELLEDNESS CONDITIONS

by Philippe TURPIN

1. Introduction.

1.1. The following result is known (it is an easy consequence of [7], [21] (proposition 0.5)).

THEOREM 1.1. – *Let a vector space* F *be locally convex metrizable and complete for a linear topology* $\mathcal{T}$. *Let* $\mathcal{T}_0$ *be a Hausdorff linear topology on* E , *coarser than* $\mathcal{T}$. *Let* $\mathcal{H}$ *be a* σ*-ring of subsets of a set* T *and* $\mu : \mathcal{H} \longrightarrow$ F *an additive set function.*

Then, $\mu(\mathcal{H})$ *is bounded for* $\mathcal{T}$ *if the convex hull of* $\mu(\mathcal{H})$ *is bounded for* $\mathcal{T}_0$.

Proof. – The space $S(T, \mathcal{H})$ of $\mathcal{H}$ -simple functions is barrelled for the topology $\mathcal{S}_\infty$ of uniform convergence on T ([7] (lemma 2.4). [13] (proposition 6) and references of [13] ; see also [10] (pp. 145 and 217)). The assumption on μ implies that the mapping $x \longrightarrow \int x\, d\mu$ of $S(T, \mathcal{H})$ into F is continuous for $\mathcal{T}_0$ and therefore for $\mathcal{T}$ (closed graph theorem, [11] p. 40).

The barrelledness of $(S(T, \mathcal{H}), \mathcal{S}_\infty)$ gives also the following generalization of the Nikodym uniform boundedness principle ([5], [20], [21] (lemma 0.6)).

THEOREM 1.2. – *Let* G *be a locally convex topological vector space,* $\mathcal{H}$ *a* σ*-ring : every pointwise bounded set* M *of bounded additive set functions* $\mathcal{H} \longrightarrow$ G *is equibounded on* $\mathcal{H}$.

The following Orlicz-Pettis type theorem 1.3 ([7], [21]) is an easy consequence of theorem 1.1 ; σ-exhaustive spaces are defined in definition 1.5 below.

THEOREM 1.3. – *Let a vector space* F *be locally convex, complete, metrizable and* σ*-exhaustive for a linear topology* $\mathcal{T}$, *let* $\mathcal{H}$ *be a* σ*-ring.*

Then, if a set function $\mu : \mathcal{H} \longrightarrow$ F *is* σ*-additive for some convex Hausdorff linear topology* $\mathcal{T}_0$ *on* F *coarser than* $\mathcal{T}$, μ *is* σ*-additive for* $\mathcal{T}$.

1.2. In section 2 below, the theorem 2.3 shows (via the proposition 2.2) that theorems 1.1 and 1.3 are false if F is a suitable (non locally convex) Orlicz space of generalized sequences.

And in section 3 the theorem 3.2 shows that the uniform boundedness principle (theorem 1.2 above) is not verified by some non locally convex space G.

One may ask whether these theorems remain true under suitable hypotheses of generalized convexity ("galb" hypotheses [22]). In section 5 the problems of extending theorems 1.1 or 1.2 are reduced to the study of certain barrelledness conditions (introduced in section 4) related to the notion of galb. This shows that these problems are equivalent. A (very small) galb is evaluated for which the corresponding barrelledness condition is not fulfilled : this refines the sections 2 and 3. A positive result for the galb of p-convexity, $0 < p < 1$, would permit to generalize a theorem of Bennett and Kalton to Hardy classes H^p, $p > 0$.

In theorem 3.1 we mention a little extension of theorem 1.2, of an other kind.

1.3. Let us make precise our terminology and notations.

$\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{N}$ are the sets of real numbers, non negative real numbers, non negative integers.

An *F-seminorm* of a (real) linear space E is a function $\nu : E \longrightarrow \mathbf{R}_+$ verifying, for

$x \in E$, $y \in E$, $r \in \mathbf{R}$, $\nu(x+y) \leqslant \nu(x) + \nu(y)$, $\nu(rx) \leqslant \nu(x)$ if $|r| \leqslant 1$

and $\nu(rx) \longrightarrow 0$ when $r \longrightarrow 0$. An F-seminorm ν is an *F-norm* when

$\nu(x) = 0 \Longrightarrow x = 0$. An *F-seminormed* (resp. *F-normed*) space (E, ν) is a vector space endowed with an F-seminorm (resp F-norm) ν and with the associated linear topology.

A subset B of a topological vector space E (resp of an F-seminormed vector space (E, ν)) is said to be *bounded* (resp *metrically bounded*) when B is absorbed by every neighbourhood of 0 (resp when $\sup\{\nu(x) \mid x \in \mathrm{B}\} < \infty$).

If $\mathscr{H}$ is a ring of subsets of a set T, $\mathrm{S}(\mathrm{T}, \mathscr{H})$ is the vector subspace of $\mathbf{R}^{\mathrm{T}}$ generated by the set of the characteristic functions $\chi_{\mathrm{H}}, \mathrm{H} \in \mathscr{H}$.

If $\mathscr{H}$ is a ring and E a vector space, a fonction $\mu : \mathscr{H} \longrightarrow \mathrm{E}$ is *additive* when $\mu(\mathrm{H} \cup \mathrm{K}) = \mu(\mathrm{H}) + \mu(\mathrm{K})$ as soon as $\mathrm{H} \in \mathscr{H}$ and $\mathrm{K} \in \mathscr{H}$ are disjoint.

If E is endowed with a linear topology $\mathscr{T}$ (resp with an F-seminorm ν) $\mu : \mathscr{H} \longrightarrow \mathrm{E}$ is *bounded* (resp *metrically bounded*) when $\mu(\mathscr{H})$ is bounded (resp metrically bounded). A set M of functions $\mathscr{H} \longrightarrow \mathrm{E}$ is *equibounded* (resp *metrically equibounded*) when $\{\mu(\mathrm{H}) \mid \mu \in \mathrm{M}, \mathrm{H} \in \mathscr{H}\}$ is bounded (resp metrically bounded), *pointwise bounded* when $\{\mu(\mathrm{H}) \mid \mu \in \mathrm{M}\}$ is bounded for every $\mathrm{H} \in \mathscr{H}$.

DEFINITION 1.4 ([3]). – *An additive set function $\mu : \mathscr{H} \longrightarrow \mathrm{E}$ is exhaustive when $\mu(\mathrm{H}_n) \longrightarrow 0$ for every disjoint sequence $(\mathrm{H}_n) \subset \mathscr{H}$.*

DEFINITION 1.5. – *We say that a topological vector space E is exhaustive (resp σ-exhaustive) when, for every ring (resp σ-ring) $\mathscr{H}$, every bounded additive set function $\mu : \mathscr{H} \longrightarrow \mathrm{E}$ is exhaustive.*

1.4. Every bounded subset of a topological vector space E is metrically bounded for every continuous F-seminorm of E. The converse is generally false but ([23]) it is true if E is galbed by some sequence (a_n) with $a_n > 0$ for every n (definition 4.1 infra).

Every exhaustive additive set function with values in an F-seminormed space is metrically bounded ([3]), but it may be unbounded ([24]).

A metrically bounded additive set function with values in a Musielak-Orlicz space $\mathrm{L}^{\varphi}(\Omega)$ is bounded (and even with bounded convex hull) if $\sup_{r>0} \varphi(r, \omega) = \infty$ for almost every $\omega \in \Omega$ (see [8], generalized in [16], [25], [9]).

2. Counterexample to the Orlicz-Pettis property.

2.1. For every integer $n \geqslant 1$, let us define a constant d_n in the following way. If $N = 2^{n+1}$, let $S_N = \{0,1\}^N$ be the set of points of $\mathbf{R}^N$ with coordinates equal to 0 or 1 ; let $\mathscr{L}_n$ be the (finite) set of the affine subspaces L of $\mathbf{R}^N$ generated by $n+1$ points of S_N and verifying $0 \notin L$. If $|x|_\infty = \sup|x_i|$ for $x = (x_i) \in \mathbf{R}^N$, we let

$$d_n = \inf\{|x|_\infty \mid x \in \bigcup_{L \in \mathscr{L}_n} L\} . \tag{1}$$

The d_n's will be used via the following lemma. B(T) is the space of bounded functions $T \longrightarrow \mathbf{R}$.

LEMMA 2.1. – *Let* H_i, $0 \leqslant i \leqslant n$, *be* $n+1$ *subsets of a set* T *and suppose that* χ_{H_0} *does not belong to the vector subspace* V *of* B(T) *generated by* $\{\chi_{H_i} \mid 1 \leqslant i \leqslant n\}$. *Then we have the inequality*

$$\inf_{x \in V} |\chi_{H_0} - x|_\infty \geqslant d_n$$

where $|x|_\infty = \sup_{t \in T} |x(t)|$ *for every* $x \in B(T)$.

Indeed, let $\mathscr{R}$ be the ring generated by the H_i's, $0 \leqslant i \leqslant n$, and let $\mathscr{P}$ be the partition of $\bigcup_i H_i$ associated to (H_i). Endowing the vector space $S(T, \mathscr{R})$ with the basis $\{\chi_H \mid H \in \mathscr{P}\}$, we construct an affine isometry α of $(S(T, \mathscr{R}), |.|_\infty)$ onto a subspace of $(\mathbf{R}^N, |.|_\infty)$, with $N = 2^{n+1}$, verifying $\alpha(\chi_H) \in S_N$ for every $H \in \mathscr{R}$ and $\alpha(\chi_{H_0}) = 0$. Hence $\alpha(V) \in \mathscr{L}_n$ and $|\chi_{H_0} - x|_\infty = |\alpha(x)|_\infty \geqslant d_n$ if $x \in V$.

Let us establish the following minoration of d_n (which can certainly be improved).

$$\log_2 d_n \geqslant -(n+1)\,2^n . \tag{2}$$

If $L \in \mathscr{L}_n$ and $N = 2^{n+1}$, L is contained in an affine hyperplan P of $\mathbf{R}^N$ which is generated by N linearly independent points $s_j \in S_N$, $1 \leqslant j \leqslant N$. We have $P = \{x \in \mathbf{R}^N \mid f(x) = 1\}$ for some linear form $f : x \longrightarrow \sum_{i=1}^{N} a_i x_i$ of $\mathbf{R}^N$. The a_i's are solutions of the Cramer system $\sum_{i=1}^{N} a_i s_{j,i} = 1$, $1 \leqslant j \leqslant N$, with $s_{j,i} \in \{0, 1\}$. $a_i = D_i/D$, D_i and D being determinants of matrices of order N with entries equal to 0 or 1. Hence $|D_i| \leqslant 2^{1-N} N^{N/2}$ ([2]), $|D| \geqslant 1$ and $\sum_{i=1}^{N} |a_i| \leqslant N^{N/2}$. So, for every $x \in L$, $|x|_\infty \geqslant \left(\sum_{i=1}^{N} |a_i|\right)^{-1} \geqslant N^{-N/2}$ and (2) follows.

2.2. Let φ be a subadditive Orlicz function, that is an increasing and continuous map $\varphi : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ verifying $\varphi(0) = 0$, $\varphi(r) > 0$ for $r > 0$ and $\varphi(r_1 + r_2) \leqslant \varphi(r_1) + \varphi(r_2)$ for r_1 and r_2 in $\mathbf{R}_+$.

If Ω is a set we consider the Orlicz space $1^\varphi(\Omega)$ of generalized real sequences $x = (x_\omega)_{\omega\in\Omega}$ defined by

$$1^\varphi(\Omega) = \{x \in \mathbf{R}^\Omega \mid |x|_\varphi = \sum_{\omega\in\Omega} \varphi(|x_\omega|) < \infty\} .$$

PROPOSITION 2.2. – *$1^\varphi(\Omega)$ is an exhaustive complete metrizable topological vector space for the F-norm $|.|_\varphi$.*

For the exhaustivity, see [15] (theorem 4).

The proposition 2.2 and the following theorem show that theorems 1.1 and 1.3 are false if $\mathcal{T}$ is not assumed to be locally convex.

THEOREM 2.3. – *Let $\mathcal{H}$ be an infinite ring of subsets of a set T, let Ω be a set having the cardinality of the continuum, and let φ be a subadditive Orlicz function verifying the condition*

$$\forall s < \infty , \quad \sum_{i=n}^{\infty} \varphi^{-1}(s/i) = o(d_n) \quad \text{when} \quad n \longrightarrow \infty, \tag{3}$$

where d_n is defined by (1).

Then, there exists an additive set function $\mu : \mathcal{H} \longrightarrow 1^\varphi(\Omega)$ which is σ-additive and bounded for some Hausdorff locally convex topology $\mathcal{T}_0$ on $1^\varphi(\Omega)$ coarser than the topology of the F-norm $|.|_\varphi$, but which verifies

$$\sup_{H\in\mathcal{H}} |\mu(H)|_\varphi = \infty$$

(and consequently [3], *which is not exhaustive for the F-norm $|.|_\varphi$) .*

Owing to (2), the condition (3) is verified if

$\log^p (|\log r|) = \mathcal{O}(\varphi(r))$ when $r \longrightarrow 0$ for some $p > -1$.

In (3) φ^{-1} is defined by $\varphi^{-1}(t) = \sup\{r \geqslant 0 \mid \varphi(r) \leqslant t\}$.

Let $\mathcal{K}$ be the σ-algebra of all subsets of $\mathbf{N}$.

We take for Ω a subset of $\mathcal{K}$ containing $\{\{h\} \mid h \in \mathbf{N}\}$ and such that $(\chi_\omega)_{\omega \in \Omega}$ is a Hamel basis of the vector space $S(\mathbf{N}, \mathcal{K})$.

If $x \in 1^\varphi(\Omega)$ the series $\sum_{\omega \in \Omega} x_\omega \chi_\omega$ converges in the Banach space $1^\infty (= 1^\infty(\mathbf{N}))$ of the bounded sequences (for the supnorm $|.|_\infty$).

Indeed, φ, being subadditive, verifies $r = \mathcal{O}(\varphi(r))$ when $r \longrightarrow 0$ and $\{\chi_\omega \mid \omega \in \Omega\}$ is bounded in 1^∞.

This allows to define a continuous linear map $u : 1^\varphi(\Omega) \longrightarrow 1^\infty$ by

$$u(x) = \sum_{\omega \in \Omega} x_\omega \chi_\omega . \tag{4}$$

LEMMA 2.4. – *The condition (3) implies that u is injective.*

Indeed, let $x \in u^{-1}(0)$. There exists an injective sequence $(\omega_i)_{i \in \mathbf{N}}$ of Ω verifying

$$u(x) = \sum_{i=0}^{\infty} r_i \chi_{\omega_i} = 0, \tag{5}$$

with $r_i = x_{\omega_i}$, $\sum_{i=0}^{\infty} \varphi(|r_i|) = s < \infty$ and $|r_i|$ decreasing, whence, for $i \geqslant 1$,

$$|r_i| \leqslant \varphi^{-1}(s/i) . \tag{6}$$

Suppose $x \neq 0$; then $r_0 \neq 0$ and, by (3), the condition

$$\sum_{n+1}^{\infty} \varphi^{-1}(s/i) < |r_0| \, d_n \tag{7}$$

holds for n large enough. From (5), (6) and (7) we deduce

$$|r_0 \chi_{\omega_0} + \sum_{i=1}^{n} r_i \chi_{\omega_i}|_\infty \leqslant \sum_{n+1}^{\infty} |r_i| < |r_0| \, d_n .$$

This contradicts the lemma 2.1 since χ_{ω_0} is not algebraïcally spanned by $\{\chi_{\omega_i} \mid i \geqslant 1\}$: the lemma 2.4 is proved.

The range of u contains $S(\mathbf{N}, \mathscr{K})$, so we can define an additive set function $m : \mathscr{K} \longrightarrow 1^{\varphi}(\Omega)$ by

$$m(\mathrm{K}) = u^{-1}(\chi_{\mathrm{K}}) \quad , \quad \mathrm{K} \in \mathscr{K}$$

There exists a sequence $(\mathrm{H}_n)_{n \in \mathbf{N}}$ of non empty pairwise disjoint elements of $\mathscr{H}$ and a map $j : \mathbf{N} \longrightarrow \mathrm{T}$ such that $j(n) \in \mathrm{H}_n$ for every $n \in \mathbf{N}$. Let $\mu = j(m) : \mathscr{H} \longrightarrow 1^{\varphi}(\Omega)$. In other words,

$$\mu(\mathrm{H}) = m(j^{-1}(\mathrm{H})) \quad , \quad \mathrm{H} \in \mathscr{H}$$

Let $\mathscr{T}_0$ be the inverse image by u of the product topology of $\mathbf{R}^{\mathbf{N}}$. $\mathscr{T}_0$ is a Hausdorff locally convex topology on $1^{\varphi}(\Omega)$, coarser than the topology defined by $|\,.\,|_{\varphi}$.

Then m, and therefore μ, are σ-additive and bounded for $\mathscr{T}_0$.

But μ is not metrically bounded for $|\,.\,|_{\varphi}$. The following lemma gives a slightly stronger result, which we shall use in theorem 3.2.

Let $\mathscr{T}_{\infty}$ be the topology on $1^{\varphi}(\Omega)$ defined by the norm $|u(.)|_{\infty}$.

LEMMA 2.5. – *If the subadditive Orlicz function φ verifies (3), then, for every $s \in \mathbf{R}_+$, $\mu(\mathscr{H})$ is not included in the $\mathscr{T}_{\infty}$-closure $\overline{\mathrm{B}(s)}$ of $\mathrm{B}(s) = \{x \in 1^{\varphi}(\Omega) \mid |x|_{\varphi} \leqslant s\}$.*

It is sufficient to prove that $m(\mathscr{K}_0) \not\subset \overline{\mathrm{B}(s)}$, if $\mathscr{K}_0$ is the ring of the finite subsets of $\mathbf{N}$.

Let $s \in \mathbf{R}_+$. By (3) there exists an integer n verifying

$$\sum_{n+1}^{\infty} \varphi^{-1}(s/i) < d_n/2 . \tag{8}$$

If $\mathrm{K} \in \mathscr{K}_0$ and $m(\mathrm{K}) \in \overline{\mathrm{B}(s)}$, we have

$$u(m(\mathrm{K})) = \chi_{\mathrm{K}} = z + \sum_{i=1}^{\infty} r_i\, \chi_{\omega_i} ,$$

where $\omega_i \in \Omega$, r_i satisfies (6) and $|z|_{\infty} < d_n/2$: from this and (8) we get

$$|\chi_K - \sum_{i=1}^{n} r_i \, \chi_{\omega_i}|_\infty < d_n$$

and, by lemma 2.1, χ_K is generated by the n elements χ_{ω_i}, $1 \leqslant i \leqslant n$, of our Hamel basis. Hence K has at most n elements since K is finite and $\chi_K = \sum_{h \in K} \chi_{\{h\}}$, with $\{h\} \in \Omega$ for every h.

The lemma 2.5 and the theorem 2.3 are proved.

Problem 1. – If Ω has the continuum cardinality, for what subadditive Orlicz functions φ is it true that every $l^\varphi(\Omega)$-valued additive set function defined on a σ-ring is σ-additive for $|.|_\varphi$ if it is σ-additive for some Hausdorff locally convex linear topology $\mathscr{T}_0$ on $l^\varphi(\Omega)$ coarser than the topology defined by $|.|_\varphi$?

By theorem 1.3 and proposition 2.2 this is true if $\varphi(r) = r$.

By a theorem of Kalton ([6] or [13]), this would be always true if Ω was countable (even without assuming $\mathscr{T}_0$ locally convex).

3. Counterexample to the uniform boundedness principle.

First, let us give the announced slight generalization of theorem 1.2.

A subset B of a topological vector space E is said to be *additively bounded* when $\sup_{x \in B} \nu(x) < \infty$ for every continuous F-seminorm ν of E.

THEOREME 3.1. – *Let* $(E, \mathscr{T})$ *be a topological vector space, and suppose that there exists some* σ*-exhaustive linear topology* $\mathscr{T}'$ *on* E *such that* $(E, \mathscr{T})$ *and* $(E, \mathscr{T}')$ *have the same bounded subsets.*

Then, if $\mathscr{H}$ *is a* σ*-ring, every pointwise bounded set* M *of bounded measures* $\mathscr{H} \longrightarrow (E, \mathscr{T})$ *is "additively equibounded"* (i.e. $\{\mu(H) \mid \mu \in M, H \in \mathscr{H}\}$ *is additively bounded in* $(E, \mathscr{T})$).

If $\mathscr{T}$ is locally convex we can take for $\mathscr{T}'$ the weak topology $\sigma(E, E')$.

An imitation of the second proof of theorem 2 in [5] gives this theorem. Let $(H_n)_{n \geqslant 0}$ be a disjoint sequence of $\mathscr{H}$. If

$\{\mu(H_n) \mid \mu \in M, n \geqslant 0\}$ is not bounded, there exist scalars $\epsilon_n \longrightarrow 0$, $\epsilon_n > 0$, a subsequence (K_n) of (H_n) and a sequence (μ_n) of M such that $\epsilon_n \mu_n (K_n)$ does not tend to 0 for $\mathcal{T}'$. But, for $\mathcal{T}'$, the additive set functions $\epsilon_n \mu_n$ are exhaustive and $\epsilon_n \mu_n (H) \longrightarrow 0$ for every $H \in \mathcal{H}$. So they are equiexhaustive (theorem of Brooks and Jewett : [4], [14]), which is a contradiction. Therefore,

$$\{\mu(H_n) \mid \mu \in M, n \geqslant 0\}$$

is bounded and (lemma of [4]) M is additively equibounded for $\mathcal{T}$.

Now let us show that the conclusion of theorem 3.1 does not hold in general.

Take the space $l^{\varphi}(\Omega)$ and the operator $u : l^{\varphi}(\Omega) \longrightarrow l^{\infty}$ of section 2 (φ is a subadditive Orlicz function). Since u is continuous we define an F-norm $|.|_{\varphi,h}$ on l^{∞} topologically equivalent to the norm $|.|_{\infty}$ if we let, for $h \in \mathbf{N}$, $x \in l^{\infty}$,

$$|x|_{\varphi,h} = \inf\{|x_1|_{\varphi} + 2^h |x_2|_{\infty} \mid x = u(x_1) + x_2, x_1 \in l^{\varphi}(\Omega), x_2 \in l^{\infty}\}$$

Now let G be the F-normed space consisting of the sequences $(x_h)_{h\in\mathbf{N}}$ such that $x_h \in l^{\infty}$ for every h and $|x_h|_{\varphi,h} \longrightarrow 0$ when $h \longrightarrow \infty$, endowed with the F-norm

$$\|(x_h)\|_{\varphi} = \sup_{h\in\mathbf{N}} |x_h|_{\varphi,h} \cdot$$

THEOREM 3.2. – *If $\mathcal{H}$ is an infinite ring of subsets of a set* T *and if φ verifies the condition* (3) *of theorem 2.3, there exists a pointwise bounded set of bounded additive set functions $\mathcal{H} \longrightarrow$* (G, $\|.\|\varphi$) *which is not metrically equibounded.*

Proof. – Take the set function $\mu : \mathcal{H} \longrightarrow l^{\varphi}(\Omega)$ defined in section 2. For every $H \in \mathcal{H}$ and $h \in \mathbf{N}$, let

$$\mu_h(H) = (\epsilon_h^k\, u(\mu(H)))_{k\in\mathbf{N}}$$

where ϵ_h^k is the symbol of Kronecker. Each $\mu_h : \mathcal{H} \longrightarrow G$ is a bounded additive set function and $\{\mu_h \mid h \in \mathbf{N}\}$ is pointwise bounded : this follows from

$$\|r\mu_h(H)\|_{\varphi} = |ru(\mu(H))|_{\varphi,h} \leqslant \min\{|r\mu(H)|_{\varphi}, 2^h r\}, r \in \mathbf{R}.$$

Suppose that there exists s verifying

$$\sup\{\|\mu_h(H)\|_{\varphi} \mid h \in \mathbf{N}, H \in \mathcal{H}\} < s < \infty.$$

For every $H \in \mathscr{H}$ and every $h \in \mathbf{N}$, $|u(\mu(H))|_{\varphi,h} < s$, so $\mu(H) = x_1 + u^{-1}(x_2)$ with $|x_1|_\varphi < s$ and $|x_2|_\infty < 2^{-h}s$. By lemma 2.5 this is impossible (we should have $\mu(\mathscr{H}) \subset \overline{B(s)})$.

COROLLARY 3.3. – *There exists an* F*-normed space* G *on which no* σ*-exhaustive linear topology has the same bounded sets than the* F*-norm topology of* G.

Indeed, take the above space G, with φ verifying the condition (3), and apply the theorem 3.1.

4. Barrelledness conditions.

4.1. Let A be a bounded subset of l^1 $(= l^1(\mathbf{N}))$ and let E be a topological vector space; let $\mathscr{S}$ be its topology.

DEFINITION 4.1. – *We say that* $\{A\}$ *galbs* E (*or its topology* $\mathscr{S}$), *or that* E (*or* $\mathscr{S}$) *is* $\{A\}$*-galbed, when, for every zero-neighbourhood* V *in* E, *there exists a zero-neighbourhood* U *in* E *verifying*

$$\forall (a_n)_{n \geqslant 0} \in A\,,\ \forall N \geqslant 0\,,\ \forall (x_n)_{0 \leqslant n \leqslant N} \in U^{1+N}\,,\ \sum_{n=0}^{N} a_n x_n \in V\,.$$

We say that a point $a \in l^1$ *galbs* E (*or* $\mathscr{S}$), *or that* E (*or* $\mathscr{S}$) *is* a*-galbed, when* $\{\{a\}\}$ *galbs* E.

In the words of [22], $\{A\}$ galbs E iff A is bounded in the galb $\mathscr{G}(E)$ of E ([22], n° 2.3.2.1).

For example, if $0 < p \leqslant 1$, E is locally p-convex ([18], [26]) if and only if E is galbed by $\{B^p\}$, with

$$B^p = \left\{ a \in l^1 \mid \sum_{n=0}^{\infty} |a_n|^p \leqslant 1 \right\}. \tag{9}$$

PROPOSITION 4.2. – *Every* $\{A\}$*-galbed linear topology* $\mathscr{S}$ *is the lower upper bound of a set of semimetrizable* (*i.e.* F*-seminormable*) $\{A\}$*-galbed linear topologies.*

Indeed, for every zero-neighbourhood V for $\mathscr{S}$, there exists a sequence $(V_n)_{n \geqslant 0}$ of balanced zero-neighbourhoods such that

$V_0 \subset V$, $V_{n+1} + V_{n+1} \subset V_n$ and (V_n, V_{n+1}) verifies the condition of the definition 4.1. (V_n) is a basis of zero-neighbourhoods for an {A}-galbed semimetrizable linear topology. $\mathcal{S}$ is the lower upper bound of these topologies.

PROPOSITION 4.3. – *If* A *contains some sequence* (a_n) *verifying* $a_n > 0$ *for every* n *(or, more generally, if* A *is not bounded in the space* 1_b^0 *of* [22], n° 0.1.7.1), *a subset* B *of an* {A}*-galbed (resp* {A}*-galbed and metrizable) topological vector space* E *is bounded if* B *is metrically bounded for every continous* F*-seminorm of* E *(resp for every* F*-norm defining the topology of* E).

Proof. – Apply [23] (propositions 3 and 4). When E is metrizable, observe that if an F-norm p defines the topology of E and if a continuous F-seminorm q of E is unbounded on B, the F-norm $\sup\{p, q\}$ enjoys both these properties.

DEFINITION 4.4. – *We say that* E *(or its topology* $\mathcal{S}$) *is* {A}*-barrelled when* $\mathcal{S}$ *is finer than any* {A}*-galbed linear topology* $\mathcal{T}$ *on* E *which admits a basis of* $\mathcal{S}$*-closed zero-neighbourhoods.*

If $a \in 1^1$, *we shall write "a-barrelled" instead of "*{{a}}*-barrelled".*

Remark 4.5. – In view of the proposition 4.2, the above definition 4.3 is unaltered if $\mathcal{T}$ is assumed to be semimetrizable.

For example, the ultrabarrelled spaces of [11], [12], [17], [26] are the 0-barrelled spaces, where 0 is the null element of 1^1.

DEFINITION 4.6. – *If* $0 < p \leqslant 1$, *a* p*-barrelled space is a* $\{B^p\}$*-barrelled space, where* B^p *is defined by (9).*

So, the usual barrelled spaces are the 1-barrelled locally convex spaces.

Remark 4.7. – a) Every {A}-barrelled space is {B}-barrelled if $A \subset B$.

More precisely, every {A}-barrelled space is {B}-barrelled if and only if A is bounded in the strict galb $G_{\{B\}}^{\text{strict}}$ generated by {B} : cf. [22], n° 5.5.8.

b) If $(E, \mathscr{S})$ is $\{A\}$-barrelled, the $\{A\}$-galbed hull of $\mathscr{S}$ (i.e. the finest $\{A\}$-galbed linear topology on E which be coarser than $\mathscr{S}$) is $\{A\}$-barrelled.

Only the "only if" part of a) need a proof. If A is not bounded in $G^{\text{strict}}_{\{B\}}$ there exists a complete metrizable linear space $(X, \mathscr{S})$ which is galbed by $\{B\}$ but not by $\{A\}$ ([22], théorème 5.7.2). Now we use an argument of [17] : $(X, \mathscr{S})$ is 0-barrelled, so the $\{A\}$-galbed convex hull $\mathscr{S}_{\{A\}}$ of $\mathscr{S}$ is $\{A\}$-barrelled (by b) above) ; if $\mathscr{S}_{\{A\}}$ was $\{B\}$-barrelled, we should have $\mathscr{S} = \mathscr{S}_{\{A\}}$ (theorem 4.8 below), a contradiction.

THEOREM 4.8. – E *is* $\{A\}$-*barrelled if and only if, for every* $\{A\}$-*galbed complete metrizable topological vector space* F, *every linear operator* $u : E \longrightarrow F$ *with closed graph is continuous.*

Proof. – If $\{A\}$ galbs F and if $u : E \longrightarrow F$ is linear, it is easily seen that $\{A\}$ galbs the linear topology on E which admits as a basis of zero-neighbourhoods the closures in E of the sets $u^{-1}(V)$, V zero-neighbourhood in F. Therefore, u is almost continuous if E is $\{A\}$-barrelled, hence continuous if, moreover, F is complete and metrizable and if the graph of u is closed ([11] or [26]).

Conversely, if $\mathscr{T}$ is a semimetrizable (remark 4.5) $\{A\}$-galbed linear topology endowed with a basis of zero-neighbourhoods closed in $(E, \mathscr{S})$, the complete Hausdorff space F associated to $(E, \mathscr{T})$ is metrizable, complete and $\{A\}$-galbed and it is known that the graph of the canonical map $(E, \mathscr{S}) \longrightarrow F$ is closed.

PROPOSITION 4.9. – *If* $\{A\}$ *galbs* E, *for* E *to be* $\{A\}$-*barrelled, it is sufficient that, for every* $\{A\}$-*galbed complete and metrizable linear space* $(F, \mathscr{T})$ *and for every* $\{A\}$-*galbed Hausdorff linear topology* $\mathscr{T}_0$ *on* F *coarser than* $\mathscr{T}$, *every continuous linear operator* $u : E \longrightarrow (F, \mathscr{T}_0)$ *is still continuous from* E *to* $(F, \mathscr{T})$.

Indeed, if $u : E \longrightarrow F$ is linear, let $\mathscr{T}_0$ be the linear topology on F which admits as a basis of zero-neighbourhoods the set of subsets $u(U) + V$, where U (resp V) runs over the filter of zero-neighbourhoods in E (resp $(F, \mathscr{T})$). $\mathscr{T}_0$ is coarser than $\mathscr{T}$, $\{A\}$-galbed if E and $\mathscr{T}$ are $\{A\}$-galbed, and Hausdorff if the graph of

$$u : E \longrightarrow (F, \mathscr{T})$$

is closed.

THEOREM 4.10. – *Let us consider the following conditions.*

(i) E *is* {A}-*barrelled.*

(ii) *For every* {A}-*galbed topological vector space* F, *every pointwise bounded family of continuous linear operators*

$$u_i : \mathrm{E} \longrightarrow \mathrm{F},\ i \in \mathrm{I},$$

is equicontinuous.

Then, (i) *implies* (ii) *in any case, and* (ii) *implies* (i) *if* {A} *galbs* E.

Remark. – We can say also that (ii) holds if and only if the {A}-galbed hull of the topology $\mathcal{S}$ of E is {A}-barrelled.

L. Waelbroeck established the implication (ii) $\Longrightarrow$ (i) for $\mathrm{A} = \{0\}$ in [26]. And we use essentially the method of [26].

Proof. – It is seen as usual that (i) implies (ii), observing that {A} galbs the coarsest topology on E for which $\{u_i \mid i \in \mathrm{I}\}$ is equicontinuous.

Conversely, let us assume that {A} galbs the topology $\mathcal{S}$ of E and let $\mathcal{T}$ be an {A}-galbed semimetrizable linear topology on E with a basis of $\mathcal{S}$-closed zero-neighbourhoods. By proposition 4.2 and [22] (n° 0.1.4.1) there exists on E an F-seminorm ρ and a family of F-seminorms $(\nu_i)_{i\in \mathrm{I}}$ defining respectively the topologies $\mathcal{T}$ and $\mathcal{S}$ and verifying the condition

$(\mathrm{R}_{h,k})$ $\forall \mathrm{N} \geqslant 0$, $\forall (x_n) \in \mathrm{E}^{\mathrm{N}+1}$,

$$\sup \nu(x_n) < 2^{-k} \Longrightarrow \sup_{a\in \mathrm{A}} \nu\Big(\sum_{n=0}^{\mathrm{N}} a_n x_n\Big) \leqslant 2^{-h}$$

with $k = h + 1$, for $\nu = \rho$ and also for $\nu \in \{\nu_i \mid i \in \mathrm{I}\}$, and for every $h \in \mathbf{N}$.

Let $\mathcal{N} = \{2^h \nu_i \mid h \in \mathbf{N}, i \in \mathrm{I}\}$. For every $\nu \in \mathcal{N}$, let us define an F-seminorm ρ_ν on E by

$$\rho_\nu(x) = \inf \{\rho(x_1) + \nu(x_2) \mid x = x_1 + x_2,\ x_1 \in \mathrm{E},\ x_2 \in \mathrm{E}\}.$$

Let G be the subspace of $\mathrm{E}^{\mathcal{N}}$ consisting of the points $x = (x_\nu)_{\nu\in\mathcal{N}}$ for which $\{\nu \in \mathcal{N} \mid x_\nu \neq 0\}$ is finite. Equip G with the F-seminorm

$$r(x) = \sup_{\nu\in\mathcal{N}} \rho_\nu(x).$$

{A} galbs G. Indeed, every $\nu \in \mathcal{N}$ verifies $(R_{h,h+1})$, hence ρ_ν and r verify $(R_{h,h+2})$.

For every $\nu \in \mathcal{N}$, define $u_\nu : E \longrightarrow G$ by $(u_\nu(x))_{\nu'} = \epsilon_\nu^{\nu'} x$, where $\epsilon_\nu^{\nu'}$ is the symbol of Kronecker. From $(r(u_\nu(x)) = \rho_\nu(x) \leqslant \inf\{\rho(x), \nu(x)\}$ we deduce that $\{u_\nu \mid \nu \in \mathcal{N}\}$ is pointwise bounded and that the $u_\nu' s$ are continuous for $\mathcal{S}$.

That (ii) implies (i) now follows from the following observation : for every $\epsilon > 0$, $\{x \in E \mid \sup_{\nu \in \mathcal{N}} r(u_\nu(x)) < \epsilon\}$ is contained in the $\mathcal{S}$-closure of $\{x \in E \mid \rho(x) < \epsilon\}$.

Indeed, since $r(u_\nu(x)) = \rho_\nu(x)$, if x is in the first set, for every $i \in I$ and $h \in \mathbf{N}$, there exists $x_1 \in E$ and $x_2 \in E$ verifying $x = x_1 + x_2$, $\rho(x_1) < \epsilon$, $\nu_i(x_2) < 2^{-h}\epsilon$.

5. Application to vector valued set functions.

Let $\mathcal{H}$ be a ring of subsets of a set T and A a bounded subset of 1^1. Let $\mathcal{S}_{\{A\}}$ be the finest {A}-galbed linear topology on $S(T, \mathcal{H})$ for which $\{\chi_H \mid H \in \mathcal{H}\}$ is bounded. If $a \in 1^1$, we write $\mathcal{S}_a$ instead of $\mathcal{S}_{\{\{a\}\}}$.

If F is an {A}-galbed topological vector space, an additive set function $\mu : \mathcal{H} \longrightarrow F$ is bounded if and only if the map $f \longrightarrow \int f\, d\mu$ of $S(T, \mathcal{H})$ into F is continuous for the topology $\mathcal{S}_{\{A\}}$.

If, for some $s \in]0, \infty[$, $(2^{-n^s})_{n \geqslant 0} \in A$, $\mathcal{S}_{\{A\}}$ is the topology $\mathcal{S}_\infty$ of uniform convergence on T (for $s \leqslant 1$ this is essentially a theorem of Rolewicz and Ryll-Nardzewski : [19]).

Indeed $\mathcal{S}_{\{A\}}$ is obviously finer than $\mathcal{S}_\infty$. On the other hand, $\mathcal{S}_{\{A\}}$ is galbed by the sequence $(2^{-n})_{n \geqslant 0}$ ([22], theorem 5.6.2), the additive set function $\chi : H \longrightarrow \chi_H$ ($\mathcal{H} \longrightarrow S(T, \mathcal{H})$) is bounded for the topology $\mathcal{S}_{\{A\}}$; therefore, by [25] (theorem 3.5), or from the argument of [19] or [22] (n° 7.2.7.2), the identical mapping $f \longrightarrow \int f d\chi$ of $(S(T, \mathcal{H}), \mathcal{S}_\infty)$ into $(S(T, \mathcal{H}), \mathcal{S}_{\{A\}})$ is continuous.

But $\mathcal{S}_a$ is strictly finer than $\mathcal{S}_\infty$ if $\mathcal{H}$ is an infinite ring and if, for every $s \in \mathbf{R}$, $a = (a_n)$ verifies $a_n = \mathcal{O}(2^{-n^s})$.

Indeed, $\mathcal{S}_a = \mathcal{S}_\infty$ would imply that, for every a-galbed space E, every bounded additive set function $\mu : \mathcal{H} \longrightarrow E$ would be

"L^∞-bounded" in the sense of [22]. But this would contradict the theorem 7.4, c) of [22] (which remains true when $\mathscr{H}$ is a ring), where $\lambda = \sum_0^\infty 2^{-n} \delta_n$ for Dirac measures δ_n carried by disjoint elements of $\mathscr{H}$.

The following theorem is an immediate consequence of the proposition 4.9 and the theorem 4.10.

THEOREM 5.1. – *For every ring* $\mathscr{H}$ *of subsets of a set* T, *the following conditions are equivalent.*

(α) S(T , $\mathscr{H}$) *is* {A}-*barrelled for the topology* $\mathscr{S}_{\{A\}}$.

(β) *For every* {A}-*galbed complete metrizable topological vector space* (F , $\mathscr{T}$), *an additive set function* $\mu : \mathscr{H} \longrightarrow F$ *is bounded for if it is bounded for some Hausdorff* {A}-*galbed linear topology* $\mathscr{T}_0$ *coarser than* $\mathscr{T}$.

(γ) *For every* {A}-*galbed topological vector space* G, *every pointwise bounded set of bounded additive set functions* $\mathscr{H} \longrightarrow G$ *is equibounded on* $\mathscr{H}$.

Remark 5.2. – If $\mathscr{H}$ is an infinite σ-ring and if $\mathscr{K}$ is the σ-algebra of all subsets of $\mathbf{N}$, S(T , $\mathscr{H}$) and S($\mathbf{N}$, $\mathscr{K}$) are simultaneously {A}-barrelled or not for their respective topologies $\mathscr{S}_{\{A\}}$.

S($\mathbf{N}$, $\mathscr{K}$) is {A}-barrelled if S(T , $\mathscr{H}$) is because the first space is a quotient of the second (transpose the map $j : \mathbf{N} \longrightarrow T$ of section 2).

Conversely let us suppose that S($\mathbf{N}$, $\mathscr{K}$) is {A}-barrelled for its topology $\mathscr{S}_{\{A\}}$. The lemma of [5] shows that the set function $\mu : \mathscr{H} \longrightarrow F$ of the above condition (β) is metrically bounded for every continuous F-seminorm on (F , $\mathscr{T}$), and therefore bounded for $\mathscr{T}$ (proposition 4.3). Indeed, $\mathscr{T}$ is {A}-galbed and A is not bounded in l_b^0 because the barrelledness hypothesis implies that {A} does not galb the space $l^\varphi(\Omega)$ of theorem 2.3.

Problem 2. – For what bounded subsets A of l^1 is the topology $\mathscr{S}_{\{A\}}$ on S($\mathbf{N}$, $\mathscr{K}$) {A}-barrelled ?

Let us observe that if $\mathscr{S}_{\{A\}}$ is {A}-barrelled, then $\mathscr{S}_{\{B\}}$ is {B}-barrelled if B $\supset$ A (or, more generally, if A is bounded in the strict galb generated by $\{B\}$) since this condition implies that $\mathcal{S}_{\{B\}}$ is the $\{B\}$-galbed hull of $\mathcal{S}_{\{A\}}$ (remark 4.7).

It is known ([10]. [13] and references of [13]) that $S(\mathbf{N}, \mathcal{K})$ is barrelled for the topology $\mathcal{S}_\infty$ of uniform convergence on $\mathbf{N}$.

Is $\mathcal{S}_\infty$ p-barrelled for $0 < p < 1$?

This would give a generalization to the case $p > 0$ of a theorem of Bennett and Kalton ([1]) on Hardy classes H^p, $p \geqslant 1$.

May be, $\mathcal{S}_\infty (= \mathcal{S}_a)$ would be a-barrelled when $a_n = 2^{-n^s}$ for some $s > 0$ and $\mathcal{S}_a$ would not be a-barrelled when $a_n = \mathcal{O}(2^{-n^s})$ for every real s.

The second point would be given by a suitable improvement of the minoration (2). We are only able to prove that $\mathcal{S}_a$ is not a-barrelled under a stronger decreasciency condition on a (theorem 5.3 below).

Note that $\mathcal{S}_\infty$ is not a-barrelled if $a_n = \quad (2^{-n^s})$ for every s. Indeed the proof of theorem 7.4 of [22] gives an a-galbed complete metrizable topological vector space E and a measure $\mu : \mathcal{K} \longrightarrow E$ which is not "L^∞-bounded" for the given topology of E, but is L^∞-bounded for some Hausdorff coarser linear topology.

THEOREM 5.3. – *If $\mathcal{H}$ is an infinite ring of subsets of a set* T *and if a given sequence $a \in l^1$ verifies $n^s = \mathcal{O}(\log(\log(|a_n^{-1}|)))$ when $n \longrightarrow \infty$ for every real s, then* $S(T, \mathcal{H})$ *is not a-barrelled for the topology $\mathcal{S}_a$.*

Observe that this theorem contains the most important part of theorems 2.3 and 3.2 ; namely, the existence of a non metrically bounded additive set function with values in a complete F-normed space F, which is bounded for some Hausdorff linear topology on F coarser than the F-norm topology of F, and the existence of a pointwise bounded family of bounded additive set functions which is not metrically equibounded ; moreover, these set functions take their values in spaces verifying some galb condition.

Indeed, apply the theorem 5.1 and the proposition 4.3.

To prove the theorem 5.3 we show the condition (β) of the theorem 5.1 is not satisfied. In the situation of theorem 2.3, let

$$F = \bigcap_{p<0} l^{\varphi_p}(\Omega)$$

where $\varphi_p(r) = \log^p(|\log r|)$ for $r > 0$ small enough. F is endowed with the lower upper bound of the topologies induced by the spaces $l^{\varphi_p}(\Omega)$. F is complete, metrizable and galbed by a. Let us verify the last point. If $-1 < p < q < 0$, $h_{p,q}(\lambda) = \sup_{0<r\leqslant 1} \varphi_p(\lambda r)/\varphi_q(r)$ is of the same order than $\varphi_{q-p}(\lambda)$ for $\lambda \longrightarrow 0$ ([22], proposition 3.1.13), whence $\sum_{n=0}^{\infty} h_{p,q}(a_n) < \infty$ and a galbs F by [22] (theorem 3.2.1). φ_p verifies (3) for $p > -1$. So the additive set function $\mu : \mathscr{H} \longrightarrow$ F constructed as in section 2 is not bounded in $l^{\varphi_p}(\Omega)$ if $p > -1$ (theorem 2.3), whence not bounded in F. But if $u : \mathrm{F} \longrightarrow l^\infty$ is defined by (5), μ is bounded for the inverse image by u of the topology of l^∞, which is separated (lemma 2.4), locally convex and coarser than the topology of F.

We could deduce directly from theorem 2.3 the weaker result that, if $\mathscr{H}$ is an infinite ring, $\mathscr{S}_0$ (i.e. the finest linear topology on $\mathrm{S}(\mathrm{T}, \mathscr{H})$ for which $\{\chi_{\mathrm{H}} \mid \mathrm{H} \in \mathscr{H}\}$ is bounded) is not 0-barrelled (i.e. ultrabarrelled). This would be less satisfactory. Indeed, without galb condition, the (topological) unboundedness of an additive set function is not very significant, since it does not forbid the σ-additivity ([24]).

Problem 3. – $\mathscr{H}$ being an infinite σ-ring, is $\mathrm{S}(\mathrm{T}, \mathscr{H})$ 1-barrelled for the above topology $\mathscr{S}_0$?

In other words, if $(\mathrm{F}, \mathscr{T})$ is a Banach space and if an additive set function $\mu : \mathscr{H} \longrightarrow$ F is bounded for some Hausdorff (non locally convex) linear topology on F coarser than $\mathscr{T}$, is μ bounded for $\mathscr{T}$?

The answer is probably no.

BIBLIOGRAPHY

[1] G. BENNETT, N.J. KALTON, Inclusion theorems for K-spaces, *Canad. J. Math.*, 25 (1973), 511-524.

[2] J.H.E. COHN, On the value of determinants, *Proc. Amer. Math. Soc.*, 14 (1963), 581-588.

[3] L. DREWNOWSKI, Topological rings of sets, continuous set functions, integration, I, II and III, *Bull. Acad. Polon. Sci.*, 20 (1972), 269-276, 277-286, 439-445.

[4] L. DREWNOWSKI, Equivalence of Brooks-Jewett, Vitali-Hahn-Saks and Nikodym theorems, *Bull. Acad. Polon. Sci.*, 20 (1972), 725-731.

[5] L. DREWNOWSKI, Uniform boundedness principle for finitely additive vector measures, *Bull. Acad. Polon. Sci.*, 21 (1973), 115-118.

[6] L. DREWNOWSKI, On the Orlicz-Pettis type theorems of Kalton, *Bull. Acad. Polon. Sci.*, 21 (1973), 515-518.

[7] L. DREWNOWSKI, I. LABUDA, Sur quelques théorèmes du type d'Orlicz-Pettis II, *Bull. Acad. Polon. Sci.*, 21 (1973), 119-126.

[8] W. FISCHER, U. SCHOLER, The range of vector measures into Orlicz spaces, *Studia Math.*, 59 (1976), 53-61.

[9] W. FISCHER, U. SCHOLER, Sur la bornitude d'une mesure vectorielle, *C.R. Acad. Sci,.*, Paris, 282 A (1976), 519-522.

[10] A. GROTHENDIECK, Espaces vectoriels topologiques, Instituto de Matematica Pura e Aplicada, Universidade de Sao Paulo, 3d ed., Sao Paulo, 1964.

[11] T. HUSAIN, The open mapping and closed graph theorems in topological vector spaces, Clarendon Press, Oxford, 1965.

[12] S.O. IYAHEN, On certain classes of linear topological spaces, *Proc. London Math. Soc.*, 18(1968), 285-307.

[13] N.J. KALTON, Toplogies on Riesz groups and applications to measure theory, *Proc. London Math. Soc.*, (3) 28 (1974), 253-273.

[14] I. LABUDA, Sur quelques généralisations des théorèmes de Nikodym et de Vitali-Hahn-Saks, *Bull. Acad. Polon. Sci.*, 20 (1972), 447-456.

[15] I. LABUDA, Sur quelques théorèmes du type d'Orlicz-Pettis I, *Bull. Acad. Polon. Sci.*, 21 (1973), 127-132.

[16] I. LABUDA, Ensembles convexes dans les espaces d'Orlicz, *C.R. Acad. Sci.*, Paris, 281 (1975), 443-445.

[17] W. ROBERTSON, Completions of topological vector spaces, *Proc. London Math. Soc.*, (3) 8 (1958), 242-257.

[18] S. ROLEWICZ, Metric linear spaces, *Monografie Matematyczne* 56, PWN, Warsaw 1972.

[19] S. ROLEWICZ C. RYLL-NARDZEWSKI, On unconditional convergence in linear metric spaces, *Coll. Math.*, 17 (1967), 327-331.

[20] G.L. SEEVER, Measures on F-spaces, *Trans. Amer. Math. Soc.*, 133 (1968), 267-280.

[21] E. THOMAS, The Lebesgue-Nikodym theorem for vector valued Radon measures, *Mem. Amer. Math. Soc.*, 139 (1974).

[22] Ph. TURPIN, Convexités dans les espaces vectoriels topologiques généraux, *Dissertationes Math.*, 131, PWN, Warsaw, 1975.

[23] Ph. TURPIN, Conditions de bornitude et espaces de fonctions mesurables, *Studia Math.*, 56 (1975), 69-91.

[24] Ph. TURPIN, Une mesure vectorielle non bornée, *C.R. Acad. Sci.*, Paris, 280 A (1975), 509-511.

[25] Ph. TURPIN, Intégration par rapport à une mesure à valeurs dans un espace vectoriel topologique non supposé localement convexe, *Colloque sur l'Intégration vectorielle et multivoque*, Caen 22 et 23 mai 1975.

[26] L. WAELBROECK, Topological vector spaces and algebras, *Lecture Notes in Mathematics*, 230, Springer, Berlin, 1971.

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