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A WHITNEY EXTENSION THEOREM IN L_p AND BESOV SPACES

by A. JONSSON and H. WALLIN

0. Introduction.

0.1. The classical Whitney extension theorem (see [27, Ch VI] or the original paper by Whitney [30]) deals with the extension of Lipschitz continuous functions on a closed set $F \subset \mathbf{R}^n$ to Lipschitz continuous functions on $\mathbf{R}^n$. The class of Lipschitz functions on F which is involved, $\mathrm{Lip}(\alpha, F)$, $\alpha > 0$, is defined by means of the usual multi-index notation in the following way. Let k be a non-negative integer and assume that $k < \alpha \leqslant k+1$. The function f, or, to be more exact, the collection $\{f_j\}_{|j| \leqslant k}$ belongs to $\mathrm{Lip}(\alpha, F)$ if the functions f_j are defined on F, $f_0 = f$, and if f_j and the functions R_j defined by

$$f_j(x) = \sum_{|j+l| \leqslant k} \frac{f_{j+l}(y)}{l!}(x-y)^l + R_j(x, y), \tag{0.1}$$

satisfy

$$|f_j(x)| \leqslant M \text{ and } |R_j(x,y)| \leqslant M|x-y|^{\alpha-|j|}, \quad x, y \in F, \ |j| \leqslant k. \tag{0.2}$$

The norm of $f \in \mathrm{Lip}(\alpha, F)$ is the smallest constant M such that (0.2) holds. When $F = \mathbf{R}^n$ the functions f_j, $|j| \geqslant 1$, are the partial derivatives $D^j f$ of f.

The Whitney extension theorem now states that there exists a continuous mapping $\underline{E} : \mathrm{Lip}(\alpha, F) \longrightarrow \mathrm{Lip}(\alpha, \mathbf{R}^n)$ which gives an extension of $f_0 = f$ from F to $\mathbf{R}^n$.

We see from (0.2) that Whitney's theorem deals with the case when we have a supremum norm on F. We shall prove a Whitney extension theorem in L^p, $1 \leqslant p < \infty$, i.e. a theorem where we replace the supremum norm by a corresponding L^p-norm taken with respect to a fixed positive measure μ supported by the closed set F where μ is in some sense a "d-dimensional" measure, $0 < d \leqslant n$. We refer to section 1 (Definition 1.1) for the precise condition on μ and here we note only that this condition on μ also imposes a condition on F. Examples of classes of sets satisfying this condition are given in section 2. We assume that $k < \alpha < k+1$, $1 \leqslant p < \infty$ and replace the condition (0.2) by the condition that the norm

$$\|f\|_{p,\alpha,\mu} = \sum_{|j|\leqslant k} \left(\|f_j\|_{p,\mu} + \left\{ \iint_{|x-y|<1} \frac{|R_j(x,y)|^p}{|x-y|^{d+(\alpha-|j|)p}} \, d\mu(x)\, d\mu(y) \right\}^{1/p} \right), \tag{0.3}$$

is finite. Here $\| \ \|_{p,\mu}$ denotes the $L^p(\mu)$-norm and the functions f_j have to be defined only μ-a.e. on F. We now define (Definition 1.2) the *generalized Besov space* $B^p_\alpha(F)$ to consist of those functions f, or, more exactly, elements $\{f_j\}_{|j|\leqslant k}$, $f_0 = f$, such that $\|f\|_{p,\alpha,\mu} < \infty$. When $F = \mathbf{R}^n$ the functions f_j, $|j| \geqslant 1$, are the distribution derivatives $D^j f$ of $f_0 = f$ (Proposition 1.2) and $B^p_\alpha(\mathbf{R}^n)$ coincides (Proposition 1.3) with the ordinary Besov space $\Lambda^{p,p}_\alpha(\mathbf{R}^n) = \Lambda^p_\alpha(\mathbf{R}^n)$; if α is an integer we define $B^p_\alpha(\mathbf{R}^n)$ by $B^p_\alpha(\mathbf{R}^n) = \Lambda^p_\alpha(\mathbf{R}^n)$.

Our Whitney extension theorem in L^p (Section 1, Main Theorem, (A)) can now be formulated in the following way, if $k < \beta = \alpha - (n-d)/p < k+1$. There exists a continuous mapping

$$\underline{E}\colon B^p_\beta(F) \longrightarrow B^p_\alpha(\mathbf{R}^n)$$

which gives an extension of $\{f_j\}_{|j|\leqslant k}$ to a function $\underline{E}\{f_j\}$ in the sense that the restriction to F of the derivative $D^j(\underline{E}\{f_j\})$ is equal to f_j μ-a.e. on F, for $|j| \leqslant k$. Here we use the pointwise restriction of the strictly defined function (Definition 1.4).

The converse of our Whitney extension theorem in L^p, $1 \leqslant p < \infty$, also holds (Section 1, Main Theorem, (B); note that the converse in the classical Whitney case, $p = \infty$, is trivial): If $f \in B^p_\alpha(\mathbf{R}^n)$, then $\underline{R}(f)$, defined by

$$\underline{\mathrm{R}}(f) = \{\mathrm{D}^j f \mid \mathrm{F}\}_{|j| \leqslant k},$$

where $\mathrm{D}^j f|\mathrm{F}$ is the restriction to F of $\mathrm{D}^j f$, belongs to $\mathrm{B}^p_\beta(\mathrm{F})$, $k < \beta = \alpha - (n-d)/p < k+1$, and the restriction operator

$$\underline{\mathrm{R}}\colon \mathrm{B}^p_\alpha(\mathbf{R}^n) \longrightarrow \mathrm{B}^p_\beta(\mathrm{F})$$

is continuous.

0.2. A classical extension and restriction theorem by Besov and others states that if $\Lambda^{p,q}_\alpha(\mathbf{R}^n)$ is the ordinary Besov space (see Definition 1.3 for $p = q$ and [27, section V. 5] for the general case) and $\beta = \alpha - (n-d)/p > 0$, where d is a positive integer, $d < n$, then every function in $\Lambda^{p,q}_\beta(\mathbf{R}^d)$ can be extended to $\mathbf{R}^n$ so that it is a function in $\Lambda^{p,q}_\alpha(\mathbf{R}^n)$. Conversely, the restriction to $\mathbf{R}^d$ of a function in $\Lambda^{p,q}_\alpha(\mathbf{R}^n)$ belongs to $\Lambda^{p,q}_\beta(\mathbf{R}^d)$. The extension and restriction problem leading to this and to related theorems has been studied by a large number of authors: Besov [6], Stein [28], [27], Taibleson [29], Aronszajn, Mulla and Szeptycki [5], Lizorkin [19], Gagliardo [17], Nikol'skii [22], [21], Burenkov [11], and others. The case when R^d is replaced by a "smooth surface", e.g. a surface locally satisfying a Lipschitz condition has also been considered; we refer to Besov [7], [8] and [9]. Extension and restriction problems in the case when R^d is replaced by an arbitrary closed set have been investigated by Wallin [31], Sjödin [26], Jonsson [18], Adams [1] and Peetre [24].

It is easy to see from our discussion in section 0.1 that our Main Theorem in section 1 generalizes the restriction and extension theorem by Besov (in the case $p = q$, β not integral) to the case when $\mathbf{R}^d$ is replaced by closed sets F of a much more general kind than the sets which have been considered in this theorem before (see Definition 1.1 and section 2). Furthermore, we get a version of the theorem where also the derivative of order j of the extended function $\underline{\mathrm{E}}\{f_j\}$ coincides on F with the corresponding function f_j, $|j| \leqslant k$ (see the final remarks in section 1). Finally it should be noted that our method of proof gives a new proof also in the classical case of the theorem of Besov.

0.3. Let D be an open set in $\mathbf{R}^n$ with a boundary $\partial\mathrm{D}$ which has some smoothness property. If a function f belongs to a Sobolev or a "Besov" space in D, is it then possible to extend f to a function in $\mathbf{R}^n$ belonging to the analogous Sobolev or Besov space in $\mathbf{R}^n$? Extension problems of this kind have been considered by Nikol'skii [23], Calderón [13], Stein [27, Ch VI. 3], Besov [10], and others. The conditions on D are usually approximatively equal to saying that ∂D is of class Lip 1. Our extension method is applicable to this problem. From the discussion in section 0.1 we see that if the closure $\bar{\mathrm{D}}$ of D is a d-set with $d = n$, then every function in $\mathrm{B}_\alpha^p(\bar{\mathrm{D}})$, α not an integer, can be extended to a function in $\mathrm{B}_\alpha^p(\mathbf{R}^n)$, and the extension operator is continuous. Our condition on D is weaker than the conditions used in the references mentioned above (compare example 2.4).

0.4. *Summary*. The main definitions and the main results are stated in section 1 which serves as a detailed introduction of the paper. The condition imposed on F is examined in section 2. In section 3 we give the connection between our generalized Besov spaces and the classical Besov spaces. In chapter II (section 4-6) we treat the extension problem and in chapter III (sections 7-9) the restriction problem.

0.5. *Notation*. $\mathbf{R}^n$ is the n-dimensional Euclidean space with points $x = (x_1, \ldots, x_n)$. We let $\mathbf{R}^d$, $d < n$, consist of those $x \in \mathbf{R}^n$ for which $x_{d+1} = \ldots = x_n = 0$.

$\mathrm{B}(x, r)$ is the closed boll of radius r centered at x.

$d(x, \mathrm{F})$ is the distance from x to F.

$\|f\|_p$ is the L^p-norm with respect to Lebesgue measure dx; $\|f\|_{p,\mu}$ is the $\mathrm{L}^p(\mu)$ norm; $\|f\|_{p,\alpha,\mu}$ and $\|f\|_{p,\alpha,\mathrm{F}}$ are defined in Definition 1.2. Integration is over the whole space if nothing else is indicated.

$\Lambda_d(\mathrm{E})$ is the d-dimensional Hausdorff measure of E (see section 2.2).

m_d denotes the d-dimensional Lebesgue measure and $m = m_n$.

$j = (j_1, \ldots, j_n)$ is a multi-index, $j! = j_1! \ldots j_n!$, $|j| = j_1 + \ldots + j_n$, $x^j = x_1^{j_1} \ldots x_n^{j_n}$, and D^j denotes the derivative corresponding to j.

C_0^∞ is the set of C^∞-functions with compact support.

c denotes different constants at most times it appears.

CHAPTER I

THE PROBLEM

1. Definitions and main results.

1.1. In the extension problem we need a special kind of closed sets. As a preparation for the definition of these sets we define a special class of measures.

DEFINITION 1.1. – *Let* F *be a closed non-empty set. A positive measure* μ *is called a d-measure on* F $(0 \leqslant d \leqslant n)$ *if*

a) supp $\mu \subset$ F *and*

b) *there exists a number* $r_0 > 0$ *such that for some constants* $c_1, c_2 > 0$

$$\mu(\mathrm{B}(x, r)) \leqslant c_1 r^d, \ x \in \mathbf{R}^n, \ r \leqslant r_0, \ \textit{and} \tag{1.1}$$

$$\mu(\mathrm{B}(x, r)) \geqslant c_2 r^d, \ x \in \mathrm{F}, \quad r \leqslant r_0. \tag{1.2}$$

The set F is called a *d-set* if there exists a *d*-measure on F.

As an example, $\mathbf{R}^d$, d positive integer, and a closed rectangle in $\mathbf{R}^d$, are *d*-sets. See section 2 for other examples.

The *d*-sets have, of course, a close connection to the *d*-dimensional Hausdorff measure. We denote the *d*-dimensional Hausdorff measure by Λ_d and the Hausdorff dimension of a set E by dim E. These concepts are defined and the following proposition proved in section 2.

PROPOSITION 1.1. – a) *If* F *is a closed d-set, then*

$$\dim(\mathrm{F} \cap \mathrm{B}(x, r)) = d, \ \textit{for } x \in \mathrm{F}, \ r > 0,$$

and the restriction $\Lambda_d | \mathrm{F}$ *of* Λ_d *to* F *is a d-measure on* F.

b) *If* μ_1 *and* μ_2 *are d-measures on* F, *there are constants* $c_1, c_2 > 0$ *such that* $c_1 \mu_1 \leqslant \mu_2 \leqslant c_2 \mu_1$.

In other words, the closed set F is a *d*-set if and only if the restriction to F of the *d*-dimensional Hausdorff measure is a *d*-measure on F.

With a suitable normalization, the n-dimensional Hausdorff measure coincides with the n-dimensional Lebesgue measure; by Proposition 1.1, the d-dimensional Hausdorff measure serves as a "canonical measure" on a d-set in the same way as the Lebesgue measure does on $\mathbf{R}^n$.

1.2. We now define the spaces $B_\alpha^p(F)$ needed in the extension problem.

DEFINITION 1.2. (*The generalized Besov or Lipschitz space* $B_\alpha^p(F)$.) – *Let* F *be a closed d-set,* k *a non-negative integer,* $k < \alpha < k+1$, *and* $1 \leqslant p < \infty$. *We say that* $f \in B_\alpha^p(F)$, *or, for greater clarity, that* $\{f_j\}_{|j| \leqslant k} \in B_\alpha^p(F)$, *if the functions* f_j *satisfy*

a) *the functions* f_j *are defined d-a.e. on* F, *i.e. everywhere on* F *except on a subset of d-dimensional Hausdorff measure zero*

b) $f_0 = f$ *d-a.e. on* F, *and*

c) *if* R_j *are defined by*

$$f_j(x) = \sum_{|j+l| \leqslant k} \frac{f_{j+l}(y)}{l!}(x-y)^l + R_j(x, y), \quad x, y \in F,$$

and μ *is a d-measure on* F, *then the norm* $\|f\|_{p,\alpha,\mu} = \|\{f_j\}\|_{p,\alpha,\mu}$, *defined by*

$$\|f\|_{p,\alpha,\mu} = \tag{1.3}$$
$$\sum_{|j| \leqslant k} \left(\|f_j\|_{p,\mu} + \left\{ \iint_{|x-y|<1} \frac{|R_j(x,y)|^p}{|x-y|^{d+(\alpha-|j|)p}} \, d\mu(x) \, d\mu(y) \right\}^{1/p} \right),$$

is finite.

When $\mu = \Lambda_d | F$, we put

$$\|f\|_{p,\alpha,F} = \|f\|_{p,\alpha,\mu}$$

and take this as *the norm of* $\{f_j\} \in B_\alpha^p(F)$.

It follows from Proposition 1.1. that "d-a.e. on F" is equivalent to "μ-a.e. on F" so that the integration in (1.3) has a meaning. In some cases we get an equivalent norm by taking the integration in (1.3) over $\mathbf{R}^n \times \mathbf{R}^n$ instead of over the part of $\mathbf{R}^n \times \mathbf{R}^n$ determined by the condition $|x-y| < 1$ (see Proposition 3.1).

In the definition we have used the ordinary notation concerning multi-indices $j = (j_1, \ldots, j_n)$ and $l = (l_1, \ldots, l_n)$; see the introduction. It should be noted that, by Proposition 1.1 b), the norms $\|f\|_{p,\alpha,\mu_1}$ and $\|f\|_{p,\alpha,\mu_2}$ are equivalent, if μ_1 and μ_2 are d-measures on F.

The functions f_j, $0 < |j| \leqslant k$, in Definition 1.2, of course serve as derivatives of f on F. In fact, we have the following proposition when $\mathrm{F} = \mathbf{R}^n$.

PROPOSITION 1.2. – *If k is a non-negative integer, $k < \alpha < k+1$, and $\{f_j\}_{|j| \leqslant k} \in \mathrm{B}^p_\alpha(\mathbf{R}^n)$, then f_j is the distribution derivative $\mathrm{D}^j f$ of $f_0 = f$, for $|j| \leqslant k$.*

This proposition, which is proved in section 3, shows that we can talk about $f \in \mathrm{B}^p_\alpha(\mathbf{R}^n)$ without specifying f_j, $0 < |j| \leqslant k$, since these last functions are uniquely determined by f.

The next proposition, which is proved in section 3, states that, when $\mathrm{F} = \mathbf{R}^n$, the generalized Besov space $\mathrm{B}^p_\alpha(\mathrm{F})$ coincides with the ordinary Besov space which can be defined in the following way:

DEFINITION 1.3. – *If k is a non-negative integer and $k < \alpha < k+1$, the ordinary Besov space $\Lambda^p_\alpha(\mathbf{R}^n)$ consists of those $f \in \mathrm{L}^p(\mathbf{R}^n)$ for which the norm (with distribution derivatives)*

$$\|f\|_{\Lambda^p_\alpha(\mathbf{R}^n)} = \sum_{|j| \leqslant k} \|\mathrm{D}^j f\|_p + \sum_{|j|=k} \left(\iint \frac{|\mathrm{D}^j f(x) - \mathrm{D}^j f(y)|^p}{|x-y|^{n+(\alpha-k)p}}\, dx\, dy \right)^{1/p}$$

is finite. When $\alpha = k+1$ the first difference $\mathrm{D}^j f(x) - \mathrm{D}^j f(y)$ shall be replaced by the second difference $\mathrm{D}^j f(x) - 2\,\mathrm{D}^j f((x+y)/2) + \mathrm{D}^j f(y)$.

PROPOSITION 1.3. – $\mathrm{B}^p_\alpha(\mathbf{R}^n) = \Lambda^p_\alpha(\mathbf{R}^n)$ *with equivalent norms.*

The space $\mathrm{B}^p_\alpha(\mathrm{F})$ was defined (Definition 1.2) for $\alpha > 0$, α not an integer. In order to get greater unity in the notation we put, because of Proposition 1.3, $\mathrm{B}^p_\alpha(\mathbf{R}^n) = \Lambda^p_\alpha(\mathbf{R}^n)$, α positive integer.

1.3. In order to define the restriction to $\mathrm{F} \subset \mathbf{R}^n$ of a function f defined a.e. in $\mathbf{R}^n$ we need the concept of a strictly defined function

f. If f is a locally integrable function on $\mathbf{R}^n$, we define the corrected function $\bar{f}$ by

$$\bar{f}(x) = \lim_{r\to 0} \frac{1}{m(\mathrm{B}(x\,,\,r))} \int_{\mathrm{B}(x,r)} f(t)dt$$

at every point x where the limit exists. We say that f *can be strictly defined* at all points where $\bar{f}$ is defined. According to a fundamental theorem by Lebesgue, $f = \bar{f}$ a.e. By redefining, if necessary, f on a set of Lebesgue measure zero, we can consequently obtain that $f = \bar{f}$ at all points where the limit exists. If this is done we say that f is *strictly defined* and make the following definition.

DEFINITION 1.4. – *If* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$ *and* $\mathrm{F} \subset \mathbf{R}^n$, *then* $f|\mathrm{F}$ *is the pointwise restriction to* F *of the strictly defined function* f. *Of course,* $f|\mathrm{F}$ *is defined at those points only where* f *can be strictly defined.*

We now wish to formulate the main result of the paper, stating roughly speaking that the restriction of $f \in \mathrm{B}^p_\alpha(\mathbf{R}^n)$ to a d-set F is an element in $\mathrm{B}^p_\beta(\mathrm{F})$, $\beta = \alpha - (n-d)/p$, and that, conversely, every element in $\mathrm{B}^p_\beta(\mathrm{F})$ can be extended to a function in $\mathrm{B}^p_\alpha(\mathbf{R}^n)$.

MAIN THEOREM. – *Let* F *be a d-set,* $0 < d \leqslant n$, $1 \leqslant p < \infty$,

$$\beta = \alpha - \frac{n-d}{p},$$

and $k < \beta < k+1$ *where* k *is a non-negative integer.*

(A) *(Extension theorem) For every element* $\{f_j\}_{|j|\leqslant k} \in \mathrm{B}^p_\beta(\mathrm{F})$ *there exists a function* $\underline{\mathrm{E}}(\{f_j\}) \in \mathrm{B}^p_\alpha(\mathbf{R}^n)$, *which is an extension of* $\{f_j\}_{|j|\leqslant k}$ *in the sense that*

$$[\mathrm{D}^j(\underline{\mathrm{E}}\{f_j\})]|\mathrm{F} = f_j \ d\text{-a.e. on F, for } |j| \leqslant k, \tag{1.4}$$

so that the extension operator

$$\underline{\mathrm{E}}\colon \mathrm{B}^p_\beta(\mathrm{F}) \longrightarrow \mathrm{B}^p_\alpha(\mathbf{R}^n)$$

is continuous.

(B) *(Restriction theorem) If* $f \in \mathrm{B}^p_\alpha(\mathbf{R}^n)$, *then* $\underline{\mathrm{R}}(f)$, *defined by*

$$\underline{\mathrm{R}}(f) = \{(\mathrm{D}^j f)|\mathrm{F}\}_{|j|\leqslant k}\,,$$

belongs to $B^p_\beta(F)$ *and the restriction operator*

$$\underline{R}: B^p_\alpha(\mathbf{R}^n) \longrightarrow B^p_\beta(F)$$

is continuous.

Notice that $\underline{E}\{f_j\}$ denotes a function, not a collection of functions. The extension part of the theorem is proved in Chapter II and the restriction part in Chapter III. In both cases we prove more than is stated in the Main Theorem. When $F = \mathbf{R}^d$, $d < n$, the Main Theorem is reduced, by means of Proposition 1.3 and the discussion in section 3.4, to the well-known extension and restriction theorem by Besov and others (see [27, p. 193]). It should be noted, however, that, by (1.4) in our Main Theorem, not only does the (corrected) extended function $\underline{E}\{f_j\}$ coincide with $f_0 = f$ d-a.e. on F but, furthermore, that the derivatives of $\underline{E}\{f_j\}$ of orders less than or equal to k coincide d-a.e. on F with the corresponding functions f_j.

2. Examples and properties of d-sets.

2.1. In section 1 we mentioned that $\mathbf{R}^d$ and closed rectangles in $\mathbf{R}^d$ are d-sets. We give a number of further examples.

Example 2.1. – Let d be a positive integer and A a closed rectangle in $\mathbf{R}^d$, bounded or not bounded. Let $F \subset \mathbf{R}^n$ be a Lipschitz image of A in the sense that there exists a bijective mapping $f: A \longrightarrow F$ such that f satisfies a Lipschitz condition on A, $|f(x) - f(x')| \leq M|x - x'|$, $x, x' \in A$, and the inverse function f^{-1} satisfies an analogous Lipschitz condition on F. We claim that the closed set F is a d-set.

In fact, if m_d is the d-dimensional Lebesgue measure, we define a measure μ supported by F by $\mu(E) = m_d(f^{-1}(E))$, $E \subset F$.

The restriction of m_d to A is a d-measure on A and since, by the Lipschitz conditions,

$$B(f^{-1}(x), M_1 r) \subset f^{-1}(B(x, r)) \subset B(f^{-1}(x), M_2 r), \text{ for } x \in F,$$

where the constants M_1 and M_2 depend only on the Lipschitz constants, we conclude that μ is a d-measure on F.

In this example we could clearly replace A by any d-set and m_d by a d-measure on A.

Example 2.2. – Let $F \subset \mathbf{R}^1$ be the ordinary Cantor set,

$$F = \bigcap_{n=0}^{\infty} F_n ,$$

where $F_0 = [0,1]$ and F_n is the union of 2^n closed intervals, each of length 3^{-n}, obtained by removing the middle thirds of the intervals of F_{n-1}. If μ_n is the measure consisting of the unit mass uniformly distributed on F_n, μ_n converges to a measure μ, supported by F, and it is easy to see that μ is a d-measure on F for $d = \log 2/\log 3$. Consequently F is a d-set for $d = \log 2/\log 3$.

It is quite obvious that this example extends to generalized Cantor sets in $\mathbf{R}^n$.

Let μ be a d-measure on F. The conditions (1.1) and (1.2) in section 1 are obviously satisfied for any choice of the positive (finite) number r_0 (but with different constants c_1 and c_2). We can also conclude that

$$\mu(B(x, r)) \leqslant c\, r^n , \; x \in \mathbf{R}^n , \; r \geqslant r_0 . \tag{2.1}$$

This follows from the fact that $B(x, r)$ can be covered by a constant times r^n number of balls with radii 1. However, we cannot in general replace r^n in the right member of (2.1) by r^d. This follows from the next example.

Example 2.3. – Let $F = \cup\, p_\nu$ where we take the union over all integers ν and $p_\nu = \{x = (x_1, \ldots, x_n) \colon x_1 = \nu\}$. Let, for each ν, the restriction of μ to p_ν be given by the $(n-1)$-dimensional Lebesgue measure on p_ν. Then μ is clearly a d-measure on F with $d = n-1$ but for large values of r we have $\mu(B(x, r)) \geqslant c\, r^n$.

Example 2.4. – We can, of course, in different ways construct d-sets which locally are for instance of the forms described in the examples above. A general way to do this is the following. For any set $U \subset \mathbf{R}^n$ and any $\epsilon > 0$, we put $U^\epsilon = \{x \colon B(x, \epsilon) \subset U\}$. Let the closed set F be such that there exists an $\epsilon > 0$, an integer N, and a sequence $\{U_i\}$ of open sets so that:

(i) $\bigcup_i U_i^\epsilon \supset F$

(ii) no point of $\mathbf{R}^n$ is contained in more than N of the U_i'S

(iii) there exist constants $r_0, c_1, c_2 > 0$, a number d, $0 \leqslant d \leqslant n$, and positive measures μ_i, $\operatorname{supp} \mu_i \subset F$, so that $\mu_i(B(x,r)) \leqslant c_1 r^d$, $x \in R^n$, $r \leqslant r_0$ and $\mu_i(B(x,r)) \geqslant c_2 r^d$, $x \in U_i \cap F$, $r \leqslant r_0$.

We shall prove that F is a d-set.

We let ν_i be the restriction of μ_i to $O_i = U_i^{\epsilon/2}$ and put $\mu = \Sigma \nu_i$. Then $\operatorname{supp} \mu \subset F$ and we claim that μ is a d-measure on F. If $x \in F$, then, by (i), $x \in U_i^\epsilon$ for some i, and

$$\mu(B(x,r)) \geqslant \nu_i(B(x,r)) \geqslant c_2 r^d,$$

for $r \leqslant \min(\epsilon/2, r_0)$. In order to get an estimate in the other direction we put $I(x) = \{i\colon O_i \cap B(x,r) \neq \phi\}$. Then, by (ii),

$$\sum_{i \in I(x)} c_3 \epsilon^n \leqslant \sum_{i \in I(x)} m(U_i \cap B(x,r)) \leqslant Nm(B(x,r)) < c_4,$$

if $r \leqslant r_0$. Hence, the number of elements in $I(x)$ is bounded by a constant c and we get by (iii),

$$\mu(B(x,r)) = \Sigma \nu_i(B(x,r)) \leqslant c \cdot c_1 r^d, \ r \leqslant r_0, \ x \in \mathbf{R}^n,$$

proving that μ is a d-measure on F.

We notice that the sets F which are *minimally smooth* boundaries ∂D of open sets D in the terminology of Stein [27, p. 189], are d-sets with $d = n-1$. In fact, in this case the closure of the parts $U_i \cap F$ are $(n-1)$-sets of the kind considered in Example 2.1 corresponding to Lipschitz mappings with uniformly bounded Lipschitz conditions. We also notice that if D is an open set with minimally smooth boundary, then the closure $\overline{D}$ of D is a d-set with $d = n$. In fact, it is easy to see that the restriction to $\overline{D}$ of the n-dimensional Lebesgue measure is an n-measure on $\overline{D}$.

2.2. We define the d-dimensional Hausdorff measure, $0 < d$, of any set $E \subset \mathbf{R}^n$, $\Lambda_d(E)$, as follows. For a certain constant $\alpha(d)$ (see (2.2) below) and any $\epsilon > 0$, let

$$\Lambda_d^{(\epsilon)}(E) = \alpha(d) \inf \sum_i (\operatorname{diam} E_i)^d,$$

where the infimum is taken over all coverings of E by denumerably many sets $E_i \subset \mathbf{R}^n$, $\cup E_i \supset E$, with diameters $\operatorname{diam} E_i \leqslant \epsilon$. Then $\Lambda_d(E) = \lim_{\epsilon \to 0} \Lambda_d^{(\epsilon)}(E)$.

Since E_i and its convex hull have equal diameters, we get the same set function if we require all E_i to be convex. We also clearly get the same set function if all E_i are assumed to be open (closed). In case we require all E_i to be balls we get a set function which on E is not smaller than $\Lambda_d(E)$ and not larger than $2^d \Lambda_d(E)$. We define the constant $\alpha(d)$ by

$$\alpha(d) = 2^{-d}\, \Gamma\left(\frac{1}{2}\right)^d / \Gamma\left(\frac{d}{2} + 1\right) \tag{2.2}$$

which guarantees that $\Lambda_n(E)$ coincides with the n-dimensional outer Lebesgue measure of E (see for instance [14, p. 174]). The d-dimensional Hausdorff measure is an outer measure and the class of sets measurable Λ_d contains the Borel sets in $\mathbf{R}^n$. The *Hausdorff dimension* of E, dim(E), is the infimum of the set of numbers d such that $\Lambda_d(E) = 0$. It is easy to see that $\dim(E) \leqslant n$ for all $E \subset \mathbf{R}^n$. By d-a.e. we mean everywhere except on a set of d-dimensional Hausdorff measure zero. Note that $\Lambda_d(E) = 0$ implies $\mu(E) = 0$ if μ is a positive measure such that $\mu(B(x,r)) \leqslant cr^d$, $r \leqslant r_0$, $x \in \mathbf{R}^n$. In fact, $B_\nu = B(x_\nu, r_0)$, $r_\nu \leqslant r_0$, $\cup B_\nu \supset E$ implies

$$\mu(E) \leqslant \Sigma \mu(B_\nu) \leqslant c\, \Sigma r_\nu^d$$

and this sum can be made arbitrarily small if $\Lambda_d(E) = 0$.

2.3. *Proof of Proposition* 1.1, a). – Let F be a closed d-set, μ a d-measure on F, and r_0 a positive number. For $x \in F$, $0 < r \leqslant r_0$ and denumerably many closed balls B_i with radii $r_i \leqslant r_0$, $\cup B_i \supset (F \cap B(x,r))$, we obtain from Definition 1.1 (c_ν are positive constants):

$$c_1\, r^d \leqslant \mu(B(x,r)) \leqslant \sum_i \mu(B_i) \leqslant \sum_i c_2\, r_i^d .$$

However, for any $\epsilon > 0$, the last sum is, for a suitable choice of $\{B_i\}$, less than $c_3(\epsilon + \Lambda_d(F \cap B(x,r)))$, which gives

$$\Lambda_d(F \cap B(x,r)) \geqslant \frac{c_1}{c_3} r^d, \; x \in F, \; r \leqslant r_0 . \tag{2.3}$$

To get an inequality in the other direction we have to use some kind of covering argument. Let $B(x, r)$, $r < r_0$, be such that $\Lambda_d(F \cap B(x, r)) > 0$, $t < \Lambda_d(F \cap B(x, r))$, and let $0 < \epsilon \leqslant r_0 - r$. By the Heine-Borel covering lemma we can cover $F \cap B(x, r)$ by finitely many open balls $S_i \subset B(x, r+\epsilon)$ with centers in $F \cap B(x, r)$ and radii less than ϵ. By a standard argument (see for instance the proof of Lemma 8.4 in [25]), we can choose a disjoint subcollection $\{B_i\}$ of $\{S_i\}$ such that $\cup S_i \subset \cup \beta_i$ where β_i is the ball concentric with B_i whose radius is three times the radius r_i of B_i. Since $\cup \beta_i \supset \cup S_i \supset F \cap B(x, r)$, we get, by the definition of Hausdorff measure,

$$\alpha(d)\,\Sigma(6r_i)^d > t,$$

if ϵ is small enough.

But, by the properties of μ,

$$c_1\,\Sigma r_i^d \leqslant \Sigma\mu(B_i) = \mu(\cup B_i) \leqslant \mu(B(x, r+\epsilon)) \leqslant c_2(r+\epsilon)^d .$$

By letting ϵ tend to zero and t to $\Lambda_d(F \cap B(x, r))$ we conclude that

$$\Lambda_d(F \cap B(x, r)) \leqslant \alpha(d)\,6^d\,c_1^{-1}\,c_2\,r^d,\ x \in \mathbf{R}^n,\ r < r_0 . \qquad (2.4)$$

From (2.3) and (2.4) we see that $\Lambda_d | F$ is a d-measure on F. We also see that $0 < \Lambda_d(F \cap B(x, r)) < \infty$, $x \in F$, $r > 0$, and, consequently, that $\dim(F \cap B(x, r)) = d$ for $x \in F$, $r > 0$.

2.4. *Proof of Proposition* 1.1, b). – Let μ_1 and μ_2 be d-measures on F. Take an open set O such that $\mu_1(O) > 0$ and a number $t < \mu_1(O)$. Since μ_1 is a regular Borel measure (see for instance [25, Theorem 2.18]), there exists a compact set K, $K \subset O$, such that $\mu_1(K) > t$. We can cover $K \cap F$ by finitely many open balls $S_i \subset O$ with centers in $K \cap F$ and arbitrarily small radii r_i. By the same argument as in the proof of part a) of Proposition 1.1, we can choose a disjoint subcollection $\{B_i\}$ of $\{S_i\}$ such that $\cup S_i \subset \cup \beta_i$ where β_i is the ball concentric with B_i, whose radius is three times the radius r_i of B_i. We get

$$\begin{aligned} t < \mu_1(K) &\leqslant \mu_1(\cup S_i) \leqslant \mu_1(\cup\beta_i) \leqslant \Sigma\mu_1(\beta_i) \leqslant \Sigma c_1(3r_i)^d \\ &\leqslant c_1\,3^d\,\Sigma c_2\mu_2(B_i) = c_1 c_2\,3^d\mu_2(\cup B_i) \leqslant c_1\,c_2\,3^d\mu_2(O). \end{aligned}$$

By letting t tend to $\mu_1(O)$ we conclude that $\mu_1(O) \leqslant c_3\mu_2(O)$.

For an arbitrary Borel set E we have

$$\mu_1(\mathrm{E}) \leqslant \mu_1(\mathrm{O}) \leqslant c_3 \mu_2(\mathrm{O}), \quad \mathrm{O} \supset \mathrm{E}, \ \mathrm{O} \text{ open}.$$

By taking infimum over O we conclude that $\mu_1(\mathrm{E}) \leqslant c_3 \mu_2(\mathrm{E})$. Since we obtain an inequality in the other direction in the same way, we have proved what we wanted.

3. Connection to classical Besov spaces.

3.1. We first show that in some cases the domain of integration in the double integrals defining the norm of $\mathrm{B}^p_\alpha(\mathrm{F})$, may be taken to be the whole of $\mathrm{F} \times \mathrm{F}$.

PROPOSITION 3.1. – *Let* F *be a d-set, let* μ *be a d-measure on* F, *and suppose furthermore that* μ *satisfies*

$$\mu(\mathrm{B}(x, r)) \leqslant c_1 \, r^d, \quad x \in \mathbf{R}^n \tag{3.1}$$

for all $r > 0$. *Then the norm* $\|f\|_{p,\alpha,\mu}$ *in Definition 1.2 is equivalent to the norm*

$$\|f\|^*_{p,\alpha,\mu} = \sum_{|j| \leqslant k} \left(\|f_j\|_{p,\mu} + \left\{ \iint \frac{|\mathrm{R}_j(x,y)|^p}{|x-y|^{d+(\alpha-|j|)p}} \, d\mu(x) \, d\mu(y) \right\}^{1/p} \right).$$

Proof. – We obviously have $\|f\|_{p,\alpha,\mu} \leqslant \|f\|^*_{p,\alpha,\mu}$, and from

$$\left\{ \iint_{|x-y| \geqslant 1} \frac{|\mathrm{R}_j(x,y)|^p}{|x-y|^{d+(\alpha-|j|)p}} \, d\mu(x) \, d\mu(y) \right\}^{1/p} \leqslant$$

$$\leqslant \left\{ \iint_{|x-y| \geqslant 1} \frac{|f_j(x)|^p}{|x-y|^{d+(\alpha-|j|)p}} \, d\mu(x) \, d\mu(y) \right\}^{1/p} +$$

$$+ \sum_{|j+l| \leqslant k} \left\{ \iint_{|x-y| \geqslant 1} \frac{|x-y|^{|l|p}}{(l!)^p} \frac{|f_{j+l}(y)|^p}{|x-y|^{d+(\alpha-|j|)p}} \, d\mu(x) \, d\mu(y) \right\}^{1/p} \leqslant$$

$$\leqslant \text{(see Lemma 8.1)} \leqslant \mathrm{C} \sum_{|j+l| \leqslant k} \|f_{j+l}\|_{p,\mu} \, ,$$

it easily follows that $\|f\|^*_{p,\alpha,\mu} \leqslant \mathrm{C} \, \|f\|_{p,\alpha,\mu}$.

Remark 3.1. – With a similar argument, one realizes that for any a satisfying $0 < a < \infty$ the norm $\|f\|_{p,\alpha,\mu}$ is equivalent to the norm

$$\|f\| = \sum_{|j|\leq k} \left(\|f_j\|_{p,\mu} + \left\{ \iint_{|x-y|<a} \frac{|R_j(x,y)|^p}{|x-y|^{d+(\alpha-|j|)p}} \, d\mu(x)\, d\mu(y) \right\}^{1/p} \right).$$

This holds for any d-measure μ.

Remark 3.2. – In the proof of Proposition 3.1 we never used the lower bounds of a d-measure, so an analogous statement holds for any positive measure satisfying (3.1) for all $r > 0$.

3.2. *Proof of Proposition* 1.2.

LEMMA 3.1. – *Let* $k < \alpha < k+1$, $1 \leq p < \infty$, $\{f_j\}_{|j|\leq k} \in B^p_\alpha(\mathbf{R}^n)$ *and let* $\phi \in C_0^\infty$. *Then, with* $f = f_0$,

$$D^j(f * \phi)(x) = (f_j * \phi)(x),\ x \in \mathbf{R}^n,\ |j| \leq k. \tag{3.2}$$

Proof. – Consider a fixed multiindex j with $|j| \leq k-1$, and assume that (3.2) holds for this j. It is clearly sufficient to prove that (3.2) then holds for all $j+l$ with $|l| = 1$.

Put $g = D^j(f*\phi) =$ (by our assumption) $= f_j * \phi$, and let x and h be points in $\mathbf{R}^n$. We have

$$g(x+h) - g(x) = \int (f_j(y+h) - f_j(y))\, \phi(x-y)dy =$$

$$= \int \sum_{|l|=1} h^l f_{j+l}(y)\, \phi(x-y)dy + \int \sum_{1<|l|\leq k-|j|} \frac{h^l}{l!} f_{j+l}(y)\, \phi(x-y)dy +$$

$$+ \int R_j(y+h,y)\, \phi(x-y)dy.$$

Obviously, the second term after the latter equality sign is $O(|h|^2)$, $h \longrightarrow 0$, and since we also have

$$g(x+h) - g(x) = \sum_{|l|=1} h^l D^l g(x) + O(|h|^2),\ h \longrightarrow 0,$$

it follows that

$$\int R_j(y+h,y)\, \phi(x-y)dy = \sum_{|l|=1} h^l (D^l g(x) - \int f_{j+l}(y)\, \phi(x-y)dy)$$

$$+ O(|h|^2),\ h \longrightarrow 0. \tag{3.3}$$

Now, since $\{f_j\}_{|j|\leq k} \in B^p_\alpha(\mathbf{R}^n)$ we have that

$$\int_{|h|<1} \frac{1}{|h|^{n+\epsilon}} \left| \frac{1}{|h|} \int R_j(y+h,y)\, \phi(x-y)dy \right|^p dh \leq \left(\frac{1}{p} + \frac{1}{q} = 1\right)$$

$$\leq \|\phi\|_q^p \int_{|h|<1} \frac{1}{|h|^{n+p+\epsilon}} \int |R_j(y+h,y)|^p \, dy\, dh < \infty$$

if ϵ satisfies $p+\epsilon < (\alpha-|j|)p$. From this and (3.3) it follows that for some ϵ satisfying $0 < \epsilon < 1$ we have

$$\int_{|h|<1} \frac{1}{|h|^{n+\epsilon}} \left| \sum_{|l|=1} \frac{h^l}{|h|} (\mathrm{D}^l g(x) - \int f_{j+l}(y)\, \phi(x-y)dy) \right|^p dh < \infty$$

which gives $\mathrm{D}^l g(x) - \int f_{j+l}(y)\, \phi(x-y)dy = 0$, $|l| = 1$, i.e. $\mathrm{D}^{j+l}(f * \phi)(x) = (f_{j+l} * \phi)(x)$, $|l| = 1$. With this, the lemma is proved.

Now we can easily prove Proposition 1.2. Functions $\{f_j\}_{|j|\leqslant k} \in \mathrm{B}^p_\alpha(\mathbf{R}^n)$ are given, and we shall prove that the distribution derivatives $\mathrm{D}^j f$ of $f_0 = f$ are equal to f_j. Let ϕ satisfy $\phi \geqslant 0$, $\phi \in \mathrm{C}_0^\infty$, $\int \phi\, dx = 1$, define ϕ_ϵ by $\phi_\epsilon(x) = \epsilon^{-n}\, \phi\left(\frac{x}{\epsilon}\right)$, and put $f_\epsilon = f * \phi_\epsilon$. The lemma above shows that $\mathrm{D}^j f_\epsilon = f_j * \phi_\epsilon$, and since (see e.g. [27], p. 62) $\|f_j - f_j * \phi_\epsilon\|_p \longrightarrow 0$, $\epsilon \longrightarrow 0$, we thus have

$$\|f_j - \mathrm{D}^j f_\epsilon\|_p \longrightarrow 0,\ \epsilon \longrightarrow 0.$$

This enables us to conclude from

$$\int (\mathrm{D}^j f_\epsilon)\, \psi\, dx = (-1)^{|j|} \int f_\epsilon (\mathrm{D}^j \psi) dx,\quad \psi \in \mathrm{C}_0^\infty$$

that

$$\int f_j\, \psi\, dx = (-1)^{|j|} \int f\, \mathrm{D}^j \psi\, dx,\quad \psi \in \mathrm{C}_0^\infty$$

i.e. that $\mathrm{D}^j f = f_j$ in the distribution sense.

3.3. *Proof of Proposition* 1.3. – It is immediate from Proposition 1.2, Proposition 3.1, and the fact that $\mathrm{R}_j(x,y) = f_j(x) - f_j(y)$, $|j| = k$, that

$$\|f\|_{\Lambda^p_\alpha(\mathrm{R}^n)} \leqslant \|f\|^*_{p,\alpha,\mathrm{R}^n} \leqslant \mathrm{C}\, \|f\|_{p,\alpha,\mathrm{R}^n},\ f \in \mathrm{B}^p_\alpha(\mathbf{R}^n).$$

In order to prove a converse inequality, we shall establish the inequalities

$$\left(\iint \frac{|\mathrm{R}_j(x,y)|^p}{|x-y|^{n+(\alpha-|j|)p}}\, dx\, dy\right)^{1/p} \leqslant \mathrm{C} \sum_{|j+l|=k} \left(\iint \frac{|\mathrm{D}^{j+l} f(x) - \mathrm{D}^{j+l} f(y)|^p}{|x-y|^{n+(\alpha-k)p}}\, dx\, dy\right)^{1/p},\ |j| \leqslant k-1,\ f \in \Lambda^p_\alpha(\mathbf{R}^n), \tag{3.4}$$

where the functions f_j in the definition of R_j are taken to be $\mathrm{D}^j f$.

Then it clearly follows that

$$\|f\|_{p,\alpha,\mathbf{R}^n} \leqslant C\,\|f\|_{\Lambda^p_\alpha(\mathbf{R}^n)}, \quad f \in \Lambda^p_\alpha(\mathbf{R}^n).$$

We first prove (3.4) assuming that $f \in C^\infty$. Using the exact remainder in Taylor's formula we get

$$R_j(x,y) = D^j f(x) - \sum_{|j+l| \leqslant k-1} \frac{(x-y)^l}{l!} D^{j+l} f(y) - \sum_{|j+l|=k} \frac{(x-y)^l}{l!} D^{j+l} f(y)$$

$$= (k-|j|) \int_0^1 (1-\theta)^{k-|j|-1} \Big(\sum_{|j+l|=k} \frac{(x-y)^l}{l!} D^{j+l} f(y+\theta(x-y)) d\theta \Big)$$

$$- \sum_{|j+l|=k} \frac{(x-y)^l}{l!} D^{j+l} f(y).$$

Since $(k-|j|) \int_0^1 (1-\theta)^{k-|j|-1}\, d\theta = 1$ we may put the last sum under the integral sign, and we get

$$\Big(\iint \frac{|R_j(x,y)|^p}{|x-y|^{n+(\alpha-|j|)p}}\, dx\, dy \Big)^{1/p} \leqslant$$

$$\leqslant C \Big(\iint \Big| \int_0^1 \sum_{|j+l|=k} (x-y)^l \big(D^{j+l} f(y+\theta(x-y)) - D^{j+l} f(y)\big) d\theta \Big|^p \frac{dx\, dy}{|x-y|^{n+(\alpha-|j|)p}} \Big)^{1/p}$$

$\leqslant$ (Minkowski's inequalities for integrals) $\leqslant$

$$\leqslant C \int_0^1 \Big(\iint \sum_{|j+l|=k} |D^{j+l} f(y+\theta(x-y)) - D^{j+l} f(y)|^p \frac{dx\, dy}{|x-y|^{n+(\alpha-k)p}} \Big)^{1/p} d\theta,$$

which after substituting $x' = y + \theta(x-y)$ gives (3.4) for $f \in C^\infty$.

Let now f be an arbitrary function in $\Lambda^p_\alpha(\mathbf{R}^n)$. Then there exists a sequence $\{\phi'_n\}$ of functions in C^∞ converging to f in $\Lambda^p_\alpha(\mathbf{R}^n)$ (see e.g. [29, p. 444]), and hence also a subsequence $\{\phi_n\}$ of $\{\phi'_n\}$ such that $D^j\phi_n \longrightarrow D^j f$ a.e., $|j| \leqslant k$. By Fatou's lemma and (3.4) we then have

$$\left(\iint \frac{|R_j(x,y)|^p}{|x-y|^{n+(\alpha-|j|)p}}\, dx\, dy\right)^{1/p} \leqslant$$

$$\leqslant \lim_{m\to\infty} C \sum_{|j+l|=k} \left(\iint \frac{|D^{j+l}\phi_m(x) - D^{j+l}\phi_m(y)|^p}{|x-y|^{n+(\alpha-k)p}}\, dx\, dy\right)^{1/p}$$

$$= C \sum_{|j+l|=k} \left(\iint \frac{|D^{j+l}f(x) - D^{j+l}f(y)|^p}{|x-y|^{n+(\alpha-k)p}}\, dx\, dy\right)^{1/p},$$

which is (3.4) in the general case.

3.4. If $\{f_j\}_{|j|\leqslant k} \in B^p_\alpha(F)$, $F = \mathbf{R}^n$, then by Proposition 1.2 the functions f_j are uniquely determined by $f = f_0$. This is not true in general. Let for example $F = \mathbf{R}^d \subset \mathbf{R}^n$ (see the notation in 0.5), $0 < d < n$, let $\{f_j\}_{|j|\leqslant k} \in B^p_\alpha(\mathbf{R}^d)$, and let J_1 and J_2 denote the set of n-dimensional multiindices of type $(j_1, \ldots, j_d, 0, \ldots, 0)$ and $(0, \ldots, 0, j_{d+1}, \ldots, j_n)$, respectively. Since $(x-y)^l = 0$ if $l \notin J_1$, $x, y \in \mathbf{R}^d$, the functions $R_j(x,y)$ in the definition of $B^p_\alpha(\mathbf{R}^d)$, $\mathbf{R}^d \subset \mathbf{R}^n$, are given by

$$f_j(x) = \sum_{\substack{|j+l|\leqslant k\\ l\in J_1}} \frac{(x-y)^l}{l!} f_{j+l}(y) + R_j(x,y),\ x, y \in \mathbf{R}^d.$$

This shows that for $j \in J_2$ fixed, the functions $\{f_{j+l}\}_{l\in J_1,\, |j+l|\leqslant k}$ may be considered as a collection of functions in $B^{p,(d)}_{\alpha-|j|}(\mathbf{R}^d)$, where the extra index (d) indicates that we have the Besov space in $\mathbf{R}^d$ (not the Besov space on $\mathbf{R}^d$ considered as a subset of $\mathbf{R}^n$), and it also shows that

$$\|\{f_j\}_{|j|\leqslant k}\|_{\alpha,p,\mathbf{R}^d} = \sum_{j\in J_2} \|\{f_{j+l}\}_{l\in J_1,\, |j+l|\leqslant k}\|^{(d)}_{\alpha-|j|,p,\mathbf{R}^d} \tag{3.5}$$

where the index (d) indicates that we have the Besov norm in $\mathbf{R}^d$. Together with Proposition 1.2 this shows that, if $\{f_j\}_{|j|\leqslant k} \in B^p_\alpha(\mathbf{R}^d)$, $\mathbf{R}^d \subset \mathbf{R}^n$, the functions f_j are uniquely determined by $\{f_j\}_{j\in J_2}$ by means of

$$f_j = D^{j_1} f_{j_2},\ j = j_1 + j_2,\ j_1 \in J_1,\ j_2 \in J_2,\ |j| \leqslant k. \tag{3.6}$$

Here, D^{j_1} denotes the derivative in $\mathbf{R}^d$. The norm $\|\{f_j\}\|_{p,\alpha,\mathbf{R}^d}$ is by (3.5) and Proposition 1.3 equivalent to $\sum_{j\in J_2} \|f_j\|_{\Lambda^p_{\alpha-|j|}(\mathbf{R}^d)}$.

Conversely, any set $\{f_j \in \Lambda^p_{\alpha-|j|}(\mathbf{R}^d)\}_{j\in J_2, |j|\leqslant k}$ determines through (3.6) a function in $B^p_\alpha(\mathbf{R}^d)$.

This is also of importance when one compares our extension and restriction theorem to the classical ones in the case $F = \mathbf{R}^d \subset \mathbf{R}^n$. In view of the discussion above, a restatement of the Main Theorem in Section 1.3 in terms of classical Besov spaces when $F = \mathbf{R}^d$ is as follows.

THEOREM 3.1. – *Let* $0 < d < n$, d *integer,* $1 \leqslant p < \infty$, $\beta = \alpha - \dfrac{n-d}{p}$, $k < \beta < k+1$ *where* k *is a nonnegative integer, and let* J_1 *and* J_2 *be as above.*

(A) (*Extension Theorem*) *For every collection* $\{f_j \in \Lambda^p_{\beta-|j|}(\mathbf{R}^d)\}_{j\in J_2, |j|\leqslant k}$ *there exists a function* $\underline{E}\{f_j\} \in \Lambda^p_\alpha(\mathbf{R}^n)$ *which is an extension of* $\{f_j\}_{j\in J_2}$ *in the sense that*

$$(D^j \underline{E}\{f_j\})|\mathbf{R}^d = f_j, \ |j| \leqslant k, \ j \in J_2 .$$

Also,

$$(D^j \underline{E}\{f_j\})|\mathbf{R}^d = D^{j_1} f_{j_2}, \ j = j_1 + j_2, \ j_1 \in J_1, \ j_2 \in J_2, \ |j| \leqslant k$$

and furthermore

$$\|\underline{E}\{f_j\}\|_{\Lambda^p_\alpha(\mathbf{R}^n)} \leqslant C \sum_{j\in J_2} \|f_j\|_{\Lambda^p_{\beta-|j|}(\mathbf{R}^d)} .$$

(B) (*Restriction Theorem*) *If* $f \in \Lambda^p_\alpha(\mathbf{R}^n)$, *then*

$$(D^j f)|\mathbf{R}^d \in \Lambda^p_{\beta-|j|}(\mathbf{R}^d), \ |j| \leqslant k, \ j \in J_2 \ \textit{and}$$

$$(D^j f)|\mathbf{R}^d = D^{j_1}(D^{j_2} f|\mathbf{R}^d), \ j = j_1 + j_2, \ j_1 \in J_1, \ j_2 \in J_2, \ |j| \leqslant k.$$

Furthermore,

$$\sum_{j\in J_2} \|(D^j f)|\mathbf{R}^d\|_{\Lambda^p_{\beta-|j|}(\mathbf{R}^d)} \leqslant C \|f\|_{\Lambda^p_\alpha(\mathbf{R}^n)} .$$

Obviously, this theorem can be considered as a precise form of the classical extension and restriction theorem for Besov spaces, in the case when β is not an integer and $p = q$ (see Section 0.2 in the introduction; if only one function f belonging to $\Lambda^p_\beta(\mathbf{R}^d)$ is given, and one wants to extend it to a function in $\Lambda^p_\alpha(\mathbf{R}^n)$, one may of course put $f_0 = f$ and e.g. $f_j = 0$ if $j \in J_2$, $j \neq 0$, and $|j| \leqslant k$, and then use Theorem 3.1). Compare in this connection also [27], § 4.4, p. 193.

CHAPTER II

THE EXTENSION THEOREM

4. The extension operators $\underline{E}_k$.

4.1. We first restate, in a slightly more precise form, the extension thoerem of this paper. See also Remark 4.1 below for a more general version of Theorem 4.1.

THEOREM 4.1. – *Let* $F \subset \mathbf{R}^n$ *be a d-set,* $0 < d \leqslant n$, $1 \leqslant p < \infty$, $\beta = \alpha - \dfrac{n-d}{p} > 0$, *and* $k < \beta < k+1$, *where* k *is a nonnegative integer. Then there exists a linear operator* $\underline{E}_k$ *on* $B^p_\beta(F)$, *such that for every* $\{f_j\}_{|j|\leqslant k} \in B^p_\beta(F)$,

(a) $\|\underline{E}_k \{f_j\}\|_{p,\alpha,\mathbf{R}^n} \leqslant c \,\|\{f_j\}\|_{p,\beta,F}$ (4.1)

where c *depends only on* F, β, p, *and* n, *i.e.* $\underline{E}_k$ *is a continuous operator into* $B^p_\alpha(\mathbf{R}^n)$,

(b) $D^j(\underline{E}_k\{f_j\})|F = f_j$ *d-a.e. on* F *for* $|j| \leqslant k$, *and*

(c) $\underline{E}_k\{f_j\} \in C^\infty(\complement F)$.

The operator $\underline{E}_k$ is defined in this section, and in sections 5 and 6 we prove that it has the stated properties. However, we shall first of all, in Section 4.2, reduce the case $d = n$ to the case $d < n$.

Remark 4.1. – In proving the theorem above, the lower bound of a d-measure is the essential one. In fact, it is obvious from the proof of Theorem 4.1, that the theorem holds if F is a closed set, μ is a fixed measure supported by F, finite on bounded sets, and satisfying (1.2), i.e.

$$\mu(B(x,r)) \geqslant cr^d,\ x \in F,\ r \leqslant r_0.$$

Then $B^p_\beta(F)$ shall be interpreted as the space of functions $\{f_j\}_{|j|\leqslant k}$ with finite norm $\|\{f_j\}\|_{p,\beta,\mu}$, where the norm is given by (1.3) with this measure μ. It is interesting to compare this to the situation in the restriction theorem (Theorem 7.1), where only the upper bound (1.1) is needed.

4.2. *The case* $d = n$. Suppose that Theorem 4.1 has been proved for $0 < d < n$. Using also the restriction part of the Main Theorem, we can then obtain the theorem for $d = n$ by the following argument (compare also the discussion in Section 3.4). Let functions $\{f_j\}_{|j|\leq k} \in B^p_\beta(F)$ be given, where $F \subset \mathbf{R}^n$ is a given d-set with $d = n$. Define for any multiindex $j' = (j, \dots, j_n, j_{n+1}) = (j, j_{n+1})$, with $|j'| \leq k$, a function $f_{j'}$ on F by $f_{j'} = f_j$ if $j_{n+1} = 0$, $f_{j'} = 0$ if $j_{n+1} > 0$. Then $\{f_{j'}\}_{|j'|\leq k} \in B^p_\beta(F)$, where F is considered as a subset of $\mathbf{R}^{n+1}$. Let $\underline{E}'$ be the operator extending $\{f_{j'}\}_{|j'|\leq k}$ continuously into $B^p_\alpha(\mathbf{R}^{n+1})$, $\alpha = \beta + \dfrac{1}{p}$, as in Theorem 4.1, and put $g_{j'} = (D^{j'}\underline{E}'\{f_{j'}\})|\mathbf{R}^n$. Then $g_{j'} = f_{j'}$ d-a.e. on F, and using also the restriction part of the Main Theorem we see that $\|\{g_{j'}\}\|_{p,\beta,\mathbf{R}^n} \leq c\,\|\{f_{j'}\}\|_{p,\beta,F} = c\,\|\{f_j\}\|_{p,\beta,F}$. Here, of course, the dimension of the multiindex indicates whether F (and $\mathbf{R}^n$) is considered as a subset of $\mathbf{R}^n$ or $\mathbf{R}^{n+1}$. Define now g_j for any n-dimensional multiindex j by $g_j = g_{(j,0)}$. Since $\|\{g_{j'}\}\|_{p,\beta,\mathbf{R}^n} = \|\{g_j\}\|_{p,\beta,\mathbf{R}^n}$, it follows that the functions $\{g_j\}_{|j|\leq k}$ give the desired extension of $\{f_j\}_{|j|\leq k}$.

4.3. As was pointed out in the introduction, our extension is of Whitney type, and in the construction of $\underline{E}_k$ we need the same type of machinery as in the Whitney extension theorem. We give here a short description of these tools and state their properties. Our presentation follows [27], p. 167-170, where details and proofs may be found.

Let F be a given closed set. Then there exists a collection of closed cubes Q_k with sides parallel to the axes with the following properties.

(a) $\complement F = \cup Q_k$.

(b) The interior of the cubes are mutually disjoint.

(c) For a cube Q_k, let diam Q_k denote its diameter and $d(Q_k, F)$ its distance to F. Then

$$\operatorname{diam} Q_k \leq d(Q_k, F) \leq 4 \operatorname{diam} Q_k. \tag{4.2}$$

(d) Suppose Q_k and Q_ν touch. Then

$$1/4 \operatorname{diam} Q_k \leqslant \operatorname{diam} Q_\nu \leqslant 4 \operatorname{diam} Q_k. \tag{4.3}$$

(e) Let ϵ be a fixed number satisfying $0 < \epsilon < \frac{1}{4}$, and let Q_k^* denote the cube which has the same center as Q_k but is expanded by the factor $1+\epsilon$. Then each point in $\complement F$ is contained in at most N_0 cubes Q_k^*, where N_0 is a fixed number. Furthermore, Q_k^* intersects a cube Q_ν only if Q_k touches Q_ν.

In connection with this decomposition, we shall use the following notation:

x_k = the center of Q_k

l_k = the diameter of Q_k

s_k = the length of the sides of Q_k (thus $l_k = \sqrt{n}\, s_k$)

Sometimes we also denote the center of Q_k by y_k.

Next we make a partition of unity. Let ψ be a C^∞-function satisfying $0 \leqslant \psi \leqslant 1$, $\psi(x) = 1$, $x \in Q$ and $\psi(x) = 0$, $x \notin (1+\epsilon)Q$, where Q denotes the cube centered at the origin with sides of length 1 parallel to the axes. Define ψ_k by $\psi_k(x) = \psi\left(\frac{x-x_k}{s_k}\right)$, and then ϕ_k by $\phi_k(x) = \psi_k(x)/\sum_k \psi_k(x)$, $x \in \complement F$. Then $\phi_k(x) = 0$ if $x \notin Q_k^*$, $\sum \phi_k(x) \equiv 1$, $x \in \complement F$, and it is easy to show that for any multiindex j we have

$$|D^j \phi_k(x)| \leqslant A_j (\operatorname{diam} Q_k)^{-|j|}. \tag{4.4}$$

4.4. Let now F be a d-set, $0 < d < n$, and let μ denote the measure $\Lambda_d | F$. Recall that μ satisfies

$$c_1 r^d \leqslant \mu(B(x,r)) \leqslant c_2 r^d, \; r \leqslant r_0, \; x \in F \tag{4.5}$$

for some constant r_0, which may be taken arbitrarily big (see Section 2.1). Let $\{f_j\}_{|j| \leqslant k}$ be a collection of functions defined on F, and summable with respect to μ on bounded sets.

Put

$$P(x,t) = \sum_{|j| \leqslant k} \frac{(x-t)^j}{j!} f_j(t), \; x \in \mathbf{R}^n, \; t \in F.$$

Define an operator $\underline{E}'_k$ by

$$(\underline{E}'_k\{f_j\})(x) = \sum_i \phi_i(x)\, c_i \int_{|t-x_i|\leqslant 6l_i} P(x,t)\, d\mu(t), \quad x \in \complement F, \tag{4.6}$$

where c_i is defined by

$$c_i^{-1} = \int_{|t-x_i|\leqslant 6l_i} d\mu(t) = \mu(B(x_i, 6l_i))$$

Note that, since F has n-dimensional Lebesgue measure zero, the function $\underline{E}'_k\{f_j\}$ becomes defined a.e. in $\mathbf{R}^n$ by (4.6).

Next fix a function Φ such that $\Phi \in C^\infty$, $\Phi(x) = 1$ if $d(x,F) \leqslant 3$, $\Phi(x) = 0$ if $d(x,F) \geqslant 4$, and such that $D^j\Phi$ is bounded for every j, with a bound which may depend on j. The extension operator $\underline{E}_k$ is now defined by

$$(\underline{E}_k\{f_j\})(x) = \Phi(x)\,(\underline{E}'_k\{f_j\})(x).$$

4.5. From (4.2) we see that there exists a point $p_i \in F$ with $|p_i - x_i| \leqslant 5l_i$. This gives $\mu(B(x_i, 6l_i)) \geqslant \mu(B(p_i, l_i)) \geqslant c_1 l_i^d$ if $l_i \leqslant r_0$ or

$$c_i \leqslant \frac{1}{c_1} l_i^{-d} \ \text{ if } l_i \leqslant r_0, \tag{4.7}$$

where $c_i^{-1} = \mu(B(x_i, 6l_i))$, an estimate which is important in what follows.

5. Lemmas.

5.1. It will be convenient to make some more agreements on notation. Let k and m be nonnegative integers, and let $\{f_j\}_{|j|\leqslant k}$ be a collection of functions on the d-set F, locally summable with respect to the d-measure on F. Below the function $\underline{E}'_k\{f_j\}$ will be denoted by f, so $\underline{E}_k\{f_j\} = \Phi f$. The remainders corresponding to $\{f_j\}_{|j|\leqslant k}$, $\{D^j f\}_{|j|\leqslant m}$, and $\{D^j(\Phi f)\}_{|j|\leqslant m}$ will be denoted by $r_j(t,s)$, $R_j(x,y)$, and $R_j^\Phi(x,y)$, respectively, i.e.

$$r_j(t,s) = f_j(t) - \sum_{|j+l|\leqslant k} \frac{(t-s)^l}{l!} f_{j+l}(s), \quad s, t \in F, \tag{5.1}$$

$$R_j(x,y) = D^j f(x) - \sum_{|j+l| \leqslant m} \frac{(x-y)^l}{l!} D^{j+l} f(y), \ x,y \in \complement F, \quad (5.2)$$

and $R_j^{\Phi}(x,y)$ is the same expression with f replaced by Φf.

We also put

$$P_j(x,t) = \sum_{|j+l| \leqslant k} \frac{(x-t)^l}{l!} f_{j+l}(t), \ x \in \mathbf{R}^n, \ t \in F, \ |j| \leqslant k.$$

Note that $P_0(x,t) = P(x,t)$ as defined in 4.3, and that $\frac{\partial^j}{\partial x_j} P(x,t) = P_j(x,t)$, $|j| \leqslant k$.

The following identities will be useful below.

LEMMA 5.1. – *Suppose* $x,y \in \mathbf{R}^n$ *and* $s,t \in F$. *Then*

$$P_j(x,t) - P_j(x,s) = \sum_{|j+l| \leqslant k} r_{j+l}(t,s) \frac{(x-t)^l}{l!} \quad (5.3)$$

and

$$P_j(x,s) = \sum_{|j+l| \leqslant k} P_{j+l}(y,s) \frac{(x-y)^l}{l!}. \quad (5.4)$$

For a proof of (5.3), see e.g. [27], p. 177.

The identity (5.4) is just the Taylor expansion of the polynomial in x, $P_j(x,s)$, around the point y.

5.2. In the following lemma, the fundamental estimates on the extended function in terms of the given functions $\{f_j\}$ on F are given. Recall that $\complement F = \bigcup_{i=1}^{\infty} Q_i$, where Q_i are cubes with centers x_i (or y_i) and diameters l_i.

LEMMA 5.2. – *Let* F *be a d-set,* $0 < d < n$, *let* $\{f_j\}_{|j| \leqslant k} \in B_\beta^p(F)$, $k < \beta < k+1$, $1 \leqslant p < \infty$, *let* m *be a nonnegative integer,* $m \geqslant k$, *and let* $f = \underline{E}'_k\{f_j\}$ *be given by (4.6). Let also* $x \in Q_i$ *and* $y \in Q_\nu$ *be points with distance from* F *not greater than 4, and put*

$$J_u(x_i) = \iint_{\substack{|t-x_i| \leqslant 30\, l_i \\ |s-x_i| \leqslant 30\, l_i}} |r_u(t,s)|^p \, d\mu(t)\, d\mu(s).$$

(a) *Then for any multiindex* j

$$|D^j f(x)|^p \leqslant c \sum_{|u|\leqslant k} l_i^{(|u|-|j|)p-2d}\, J_u(x_i) \tag{5.5}$$
$$+ c \sum_{|j+l|\leqslant k} l_i^{-d+|l|p} \int_{|t-x_i|\leqslant 30\, l_i} |f_{j+l}(t)|^p \, d\mu(t)$$

(b) *For* j *with* $|j| \leqslant m$ *and* $R_j(x,y)$ *given by (5.2) we have*

$$|R_j(x,y)|^p \leqslant c \sum_{|j+l|\leqslant k} l_i^{|l|p}\, l_i^{-d}\, l_\nu^{-d} \iint_{\substack{|t-x_i|\leqslant 30\, l_i \\ |s-y_\nu|\leqslant 30\, l_\nu}} |r_{j+l}(t,s)|^p \, d\mu(t)\, d\mu(s)$$
$$+ c \sum_{|u|\leqslant k} \sum_{|j+l|\leqslant m} |x-y|^{|l|p}\, l_\nu^{(|u|-|j+l|)p-2d}\, J_u(y_\nu) \tag{5.6}$$
$$+ c \sum_{|u|\leqslant k} l_i^{(|u|-|j|)p-2d}\, J_u(x_i).$$

Here the constants c *depend only on* j, m, F, β, p, *and* n.

Note that the second sum in (5.5) and the first in (5.6) vanish if $|j| > k$. The number 4 in the assumption $d(x,\mathrm{F})$, $d(y,\mathrm{F}) \leqslant 4$, may be replaced by any positive number.

Proof. – For convenience, we first make the following *change of notation: We assume that* $x \in Q_I$ *and* $y \in Q_N$, and we shall consequently prove that the lemma holds with i and ν replaced by I and N, respectively.

From the definition of f,

$$f(x) = \sum_i \phi_i(x)\, c_i \int_{|t-x_i|\leqslant 6\, l_i} P(x,t)\, d\mu(t), \quad x \in \complement\, \mathrm{F},$$

it is easy to see that $D^j f(x)$ equals

$$A_j(x) = \sum_i \phi_i(x)\, c_i \int_{|t-x_i|\leqslant 6\, l_i} P_j(x,t)\, d\mu(t)$$

plus terms of type

$$B_{j'}(x) = \sum_i D^{j'} \phi_i(x)\, c_i \int_{|t-x_i|\leqslant 6\, l_i} P_{j''}(x,t)\, d\mu(t),$$
$$j'+j'' = j,\ j' \neq 0 = (0,0,\ldots,0).$$

Here, $P_j(x,t)$ shall be interpreted as zero if $|j| > k$. Similarly, we have that $R_j(x,y) = D^j f(x) - \sum_{|j+l|\leqslant m} \dfrac{(x-y)^l}{l!} D^{j+l} f(y)$ equals

$$H(x,y) = A_j(x) - \sum_{|j+l|\leqslant m} \frac{(x-y)^l}{l!} A_{j+l}(y)$$

plus terms of type $B_{j'}(x)$, $j' \neq 0$, plus terms of type

$$\sum_{|j+l|\leqslant m} \frac{(x-y)^l}{l!} B_{(j+l)'}(y), \quad (j+l)' \neq 0.$$

The proof consists of estimating $A_j(x)$, $B_{j'}(x)$ and $H(x,y)$. The lemma follows from the estimates (5.9), (5.10) and (5.11) below.

Let Q_i be a cube touching Q_I. Then, by (4.3),

$$|t-x_I| \leqslant |t-x_i| + |x_i - x_I| \leqslant 6l_i + l_i + l_I \leqslant 30 l_I \text{ if } |t-x_i| \leqslant 6l_i \tag{5.7}$$

and by (4.7) and (4.3)

$$c_i \leqslant \frac{1}{c_1} l_i^{-d} \leqslant c\, l_I^{-d}. \tag{5.8}$$

Since $\phi_i(x) \neq 0$ only if $x \in Q_i^*$, and $x \in Q_i^*$ iff Q_i and Q_I touch, it follows that (5.7) and (5.8) hold for the at most N_0 numbers i such that $\phi_i(x) \neq 0$.

Recalling that $c_i = \{\mu(B(x_i, 6l_i))\}^{-1}$, we see from Hölder's inequality that

$$|A_j(x)| \leqslant \sum_i \phi_i(x)\, c_i \int_{|t-x_i|\leqslant 6l_i} |P_j(x,t)|\, d\mu(t)$$

$$\leqslant \sum_i \phi_i(x)\, c_i^{1/p} \left\{ \int_{|t-x_i|\leqslant 6l_i} |P_j(x,t)|^p\, d\mu(t) \right\}^{1/p}.$$

Since the sum has at most N_0 terms not equal to zero and $\phi_i(x) \leqslant 1$, we get using (5.7) and (5.8)

$$|A_j(x)|^p \leqslant N_0^p\, c^p\, l_I^{-d} \int_{|t-x_I|\leqslant 30 l_I} |P_j(x,t)|^p\, d\mu(t)$$

which gives

$$|A_j(x)|^p \leqslant c\, l_I^{-d} \sum_{|j+l|\leqslant k} l_I^{|l|p} \int_{|t-x_I|\leqslant 30 l_I} |f_{j+l}(t)|^p\, d\mu(t). \tag{5.9}$$

Since $\sum_i \phi_i(x) \equiv 1$, $x \in \complement F$ we have $\sum_i D^j \phi_i(x) = 0$, $j \neq 0$.

Using this and the definition of c_i we get

$$B_{j'}(x) = \sum_i D^{j'} \phi_i(x) \left(c_i \int_{|t-x_i|\leqslant 6 l_i} P_{j''}(x,t)\, d\mu(t) - P_{j''}(x,s)\right)$$

$$= \sum_i D^{j'} \phi_i(x)\, c_i \int_{|t-x_i|\leqslant 6 l_i} (P_{j''}(x,t) - P_{j''}(x,s))\, d\mu(t)$$

so, by Hölder's inequality,

$$|B_{j'}(x)| \leqslant \sum_i |D^{j'} \phi_i(x)|\, c_i^{1/p} \left\{ \int_{|t-x_i|\leqslant 6 l_i} |P_{j''}(x,t) - P_{j''}(x,s)|^p\, d\mu(t)\right\}^{1/p}$$

and thus, using (4.4), (4.3), (5.7), and (5.8),

$$|B_{j'}(x)|^p \leqslant c\, l_I^{-|j'|p}\, l_I^{-d} \int_{|t-x_I|\leqslant 30 l_I} |P_{j''}(x,t) - P_{j''}(x,s)|^p\, d\mu(t).$$

Integrating this inequality with respect to s over $B(x_I, 30 l_I)$ we obtain, since clearly $\mu(B(x_I, 30 l_I)) \geqslant c\, l_I^d$,

$$|B_{j'}(x)|^p \leqslant c\, l_I^{-|j'|p}\, l_I^{-2d} \iint_{\substack{|t-x_I|\leqslant 30 l_I \\ |s-x_I|\leqslant 30 l_I}} |P_{j''}(x,t) - P_{j''}(x,s)|^p\, d\mu(t)\, d\mu(s).$$

Since by Lemma 5.1 we have

$$P_{j''}(x,t) - P_{j''}(x,s) = \sum_{|j''+l|\leqslant k} r_{j''+l}(t,s) \frac{(x-t)^l}{l!}$$

and since $|x-t| \leqslant 31 l_I$ in the domain of integration, we obtain

$$|B_{j'}(x)|^p \leqslant c \sum_{|j''+l|\leqslant k} l_I^{(|l|-|j'|)p-2d} \iint_{\substack{|t-x_I|\leqslant 30 l_I \\ |s-x_I|\leqslant 30 l_I}} |r_{j''+l}(t,s)|^p\, d\mu(t)\, d\mu(s)$$

so

$$|B_{j'}(x)|^p \leqslant c \sum_{|u|\leqslant k} l_I^{(|u|-|j|)p-2d}\, J_u(x_I). \tag{5.10}$$

In order to estimate $H(x,y)$, we first rewrite it using $\Sigma\, \phi_i(x) = 1$ and the definition of c_i, in the form

$$\mathrm{H}(x,y) = \sum_i \phi_i(x)\, c_i \int_{|t-x_i|\leqslant 6 l_i} \{\mathrm{P}_j(x,t) -$$

$$- \sum_{|j+l|\leqslant m} \frac{(x-y)^l}{l!} \sum_\nu \phi_\nu(y)\, c_\nu \int_{|s-y_\nu|\leqslant 6 l_\nu} \mathrm{P}_{j+l}(y,s)\, d\mu(s)\}\, d\mu(t) =$$

$$= \sum_i c_i\phi_i(x) \sum_\nu c_\nu\phi_\nu(y) \iint_{\substack{|t-x_i|\leqslant 6 l_i\\ |s-y_\nu|\leqslant 6 l_\nu}} (\mathrm{P}_j(x,t) - \mathrm{P}_j(x,s) + \mathrm{P}_j(x,s) -$$

$$- \sum_{|j+l|\leqslant m} \frac{(x-y)^l}{l!} \mathrm{P}_{j+l}(y,s))\, d\mu(s)\, d\mu(t).$$

Since $m \geqslant k$ and $\mathrm{P}_{j+l}(y,s) = 0$ if $|j+l| > k$, the identity (5.4) shows that the two last terms inside the brackets are zero, and hence we get

$$|\mathrm{H}(x,y)| \leqslant \sum_i c_i^{1/p}\, \phi_i(x) \sum_\nu c_\nu^{1/p}\, \phi_\nu(y)$$

$$\left\{ \iint_{\substack{|t-x_i|\leqslant 6 l_i\\ |s-y_\nu|\leqslant 6 l_\nu}} |\mathrm{P}_j(x,t) - \mathrm{P}_j(x,s)|^p\, d\mu(t)\, d\mu(s) \right\}^{1/p}.$$

Using among other things also (5.3) we get

$$|\mathrm{H}(x,y)|^p$$

$$\leqslant c\, l_{\mathrm{I}}^{-d}\, l_{\mathrm{N}}^{-d} \iint_{\substack{|t-x_{\mathrm{I}}|\leqslant 30 l_{\mathrm{I}}\\ |s-y_{\mathrm{N}}|\leqslant 30 l_{\mathrm{N}}}} \left| \sum_{|j+l|\leqslant k} r_{j+l}(t,s) \frac{(x-t)^l}{l!} \right|^p d\mu(t)\, d\mu(s)$$

$$\leqslant c \sum_{|j+l|\leqslant k} l_{\mathrm{I}}^{-d}\, l_{\mathrm{N}}^{-d}\, l_{\mathrm{I}}^{|l|p} \iint_{\substack{|t-x_{\mathrm{I}}|\leqslant 30 l_{\mathrm{I}}\\ |s-y_{\mathrm{N}}|\leqslant 30 l_{\mathrm{N}}}} |r_{j+l}(t,s)|^p\, d\mu(t)\, d\mu(s). \tag{5.11}$$

5.3. The following simple observation will be used in Section 6.

LEMMA 5.3. – *Let* $\gamma > 0$, $a > 0$, $h \geqslant 0$, *let* μ_1 *and* μ_2 *be positive measures and put* $h_m = 2^{-m}$, m *integer. Then there exist non-negative constants* a_1 *and* a_2, *depending only on* γ *and* a, *such that*

$$a_1 \iint_{|t-s|\leqslant ah_{m_0}} \frac{h(t,s)}{|t-s|^\gamma}\, d\mu_1(t)\, d\mu_2(s)$$
$$\leqslant \sum_{m=m_0}^{\infty} h_m^{-\gamma} \iint_{|t-s|\leqslant ah_m} h(t,s)\, d\mu_1(t)\, d\mu_2(s)$$
$$\leqslant a_2 \iint_{|t-s|\leqslant ah_{m_0}} \frac{h(t,s)}{|t-s|^\gamma}\, d\mu_1(t)\, d\mu_2(s).$$

Proof. – From

$$\sum_{m=m_0}^{\infty} h_m^{-\gamma} \iint_{ah_{m+1}<|t-s|\leqslant ah_m} h(t,s)\, d\mu_1(t)\, d\mu_2(s)$$
$$\leqslant \sum_{m=m_0}^{\infty} h_m^{-\gamma} \iint_{0<|t-s|\leqslant ah_m} h(t,s)\, d\mu_1(t)\, d\mu_2(s)$$
$$= \sum_{m=m_0}^{\infty} h_m^{-\gamma} \sum_{\nu=m}^{\infty} \iint_{ah_{\nu+1}<|t-s|\leqslant ah_\nu} h(t,s)\, d\mu_1(t)\, d\mu_2(s),$$

the first inequality is obvious, and the second follows after a change of order in the summation.

5.4. Lemma 5.2 gives estimates on $|R_j(x,y)|$ and $|D^j f(x)|$, which are independent of x and y, as long as $x \in Q_i$ and $y \in Q_\nu$. The next lemma, and some consequences of it given after its proof, is our main tool when we shall put these local estimates together, to get an estimate of the norm $\|f\|_{\alpha,p,\mathbf{R}^n}$.

LEMMA 5.4. – *Let* $a > 0$, *let* h *be a non-negative function defined on a closed set* $F \subset \mathbf{R}^n$, *and let* μ *be a measure supported by* F. *Put* $h_I = 2^{-I}$ *and*

$$\Delta_I = \{x \mid h_{I+1} < d(x,F) \leqslant h_I\},\ I \text{ integer.}$$

Let the function g *be given by*

$$g(x) = \int_{|t-x_i|\leqslant al_i} h(t)\, d\mu(t),\quad x \in (\operatorname{int} Q_i) \cap \Delta_I.$$

Then for $x_0 \in \mathbf{R}^n$, $0 < r \leqslant \infty$

$$\int_{\substack{x\in\Delta_I \\ |x-x_0|\leqslant r}} g(x)dx \leqslant c\, h_I^n \int_{|t-x_0|\leqslant r+(a+1)h_I} h(t)\, d\mu(t), \tag{5.12}$$

especially for $r = +\infty$,

$$\int_{x \in \Delta_I} g(x)dx \leqslant c\, h_I^n \int h(t)\, d\mu(t). \tag{5.13}$$

Here the constant c *depends only on* a *and* n.

Proof. – If i and I are such that Q_i intersects Δ_I, we obtain from (4.2) that $(h_{I+1} - l_i)/4 \leqslant l_i \leqslant h_I$ and hence

$$h_I/10 \leqslant l_i \leqslant h_I \quad \text{if} \quad Q_i \cap \Delta_I \neq \emptyset. \tag{5.14}$$

Put $M = \{i \mid Q_i \cap \Delta_I \cap B(x_0, r) \neq \emptyset\}$. Then

$$\int_{\substack{x \in \Delta_I \\ |x - x_0| \leqslant r}} g(x)dx \leqslant \sum_{i \in M} \int_{Q_i \cap \Delta_I} g(x)dx \leqslant \sum_{i \in M} h_I^n \int_{|t - x_i| \leqslant a h_I} h(t)\, d\mu(t). \tag{5.15}$$

Now, since by (5.14) $|x_i - x_j| \geqslant h_I/(10\sqrt{n})$ if $i, j \in M$, $i \neq j$, it is easy to realize that there exists a constant c, only depending on a and n, such that a fix point in $\mathbf{R}^n$ is covered by the balls $B(x_i, a h_I)$, $i \in M$, at most c times. Furthermore, none of these balls covers a point x with $d(x, x_0) > r + (a+1)h_I$. This gives

$$\sum_{i \in M} \int_{|t - x_i| \leqslant a h_I} h(t)\, d\mu(t) \leqslant c \int_{|t - x_0| \leqslant r + (a+1)h_I} h(t)\, d\mu(t),$$

which together with (5.15) proves the lemma.

For further reference, we point out some consequences of this lemma. If g is given by

$$g(x) = \iint_{\substack{|t - x_i| \leqslant a l_i \\ |s - x_i| \leqslant a l_i}} |r_j(t,s)|^p\, d\mu(t)\, d\mu(s), \; x \in \operatorname{int} Q_i,$$

then using (5.14) we see that

$$g(x) \leqslant \int_{|t - x_i| \leqslant a l_i} \int_{|s - t| \leqslant 2 a h_I} |r_j(t,s)|^p\, d\mu(s)\, d\mu(t), \; x \in (\operatorname{int} Q_i) \cap \Delta_I,$$

so

$$\int_{\substack{x \in \Delta_I \\ |x - x_0| \leqslant r}} g(x)dx \leqslant c\, h_I^n \int_{|t - x_0| \leqslant r + (a+1)h_I} \int_{|s - t| \leqslant 2 a h_I} |r_j(t,s)|^p\, d\mu(s)\, d\mu(t). \tag{5.16}$$

If $g(x,y)$ is given by

$$g(x,y) = \iint\limits_{\substack{|t-x_i|\leqslant 30\, l_i \\ |s-y_\nu|\leqslant 30\, l_\nu}} |r_j(t,s)|^p \, d\mu(t)\, d\mu(s), \; x \in \operatorname{int} Q_i, \; y \in \operatorname{int} Q_\nu$$

then for $h_I, h_N \leqslant c_0 \, h_K$, $x \in (\operatorname{int} Q_i) \cap \Delta_I$ we have by (5.12) and (5.14)

$$\int\limits_{\substack{y\in\Delta_N \\ |y-x|\leqslant h_K}} g(x,y)dy$$

$$\leqslant c \, h_N^n \int\limits_{|s-x|\leqslant h_K+31h_N} \int\limits_{|t-x_i|\leqslant 30\, l_i} |r_j(t,s)|^p \, d\mu(t)\, d\mu(s)$$

$$\leqslant c \, h_N^n \int\limits_{|t-x_i|\leqslant 30\, l_i} \int\limits_{|s-t|\leqslant (1+62\, c_0)h_K} |r_j(t,s)|^p \, d\mu(t)\, d\mu(s).$$

Using (5.13) we thus obtain

$$\iint\limits_{\substack{x\in\Delta_I, y\in\Delta_N \\ |x-y|\leqslant h_K}} g(x,y)\, dy\, dx \leqslant c \, h_I^n \, h_N^n \iint\limits_{|s-t|\leqslant (1+62c_0)h_K} |r_j(t,s)|^p \, d\mu(t)\, d\mu(s). \tag{5.17}$$

6. Proof of the extension theorem, $d < n$.

6.1. Throughout this section the assumptions are as in Theorem 4.1 with the exception that we assume that $d < n$ (see 4.2), i.e. F is a d-set, $0 < d < n$, $1 \leqslant p < \infty$, $\beta > 0$, β non-integer, the integer k satisfies $k < \beta < k+1$, and α is given by $\beta = \alpha - \dfrac{n-d}{p}\cdot$ We also define the integer m by $m < \alpha \leqslant m+1$. Let now functions $\{f_j\}_{|j|\leqslant k} \in B_\beta^p(F)$ be given, and consider the function $f = \underline{E}_k \{f_j\}$. Our task in this section is to prove that f fulfills the requirements (a) – (c) in Theorem 4.1.

It is obvious from the definition of $\underline{E}_k$ that f satisfies (c). The proof of (b) is relatively short, and will be carried out in 6.5. The main problem is to prove that (a) holds. We assume until later that $m < \alpha < m+1$. Statement (a) is then equivalent to

$$\|D^j(\Phi f)\|_p \leqslant c \, \|\{f_j\}\|_{p,\beta,F}, \quad |j| \leqslant m, \tag{6.1}$$

and

$$\Big(\iint_{|x-y|<1} |R_j^{\Phi}(x,y)|^p \frac{dx\,dy}{|x-y|^{n+(\alpha-|j|)p}}\Big)^{1/p} \leqslant c\,\|\{f_j\}\|_{p,\beta,F}, \quad |j| \leqslant m. \tag{6.2}$$

We shall obtain these inequalities by showing

$$\Big(\int_{d(x,F)<4} |D^j f(x)|^p\,dx\Big)^{1/p} \leqslant c\,\|\{f_j\}\|_{p,\beta,F}, \quad |j| \leqslant m, \tag{6.3}$$

$$\Big(\iint_{\substack{|x-y|<1\\ d(x,F)<2}} |R_j(x,y)|^p \frac{dx\,dy}{|x-y|^{n+(\alpha-|j|)p}}\Big)^{1/p} \leqslant c\,\|\{f_j\}\|_{p,\beta,F}, \quad |j| \leqslant m, \tag{6.4}$$

and

$$\Big(\iint_{\substack{|x-y|<1\\ 2\leqslant d(x,F)<5}} |R_j^{\Phi}(x,y)|^p \frac{dx\,dy}{|x-y|^{n+(\alpha-|j|)p}}\Big)^{1/p} \leqslant c \sum_{|j|\leqslant k} \|f_j\|_{p,\mu}, \quad |j| \leqslant m. \tag{6.5}$$

Clearly (6.4) and (6.5) give (6.2), and since all derivatives $D^j\Phi$ are bounded, (6.3) implies (6.1).

6.2. We first prove (6.3). Let Δ_I and h_I be as in Lemma 5.4, let $I \geqslant -2$ and $|j| \leqslant m$. Integrating (5.5) over Δ_I, using (5.14) and (5.16) with $r = +\infty$ on the first sum of (5.5), and (5.13) on the second sum, we get

$$\int_{\Delta_I} |D^j f(x)|^p\,dx \leqslant c \sum_{|u|\leqslant k} h_I^{(|u|-|j|)p-2d}\, h_I^n \iint_{|t-s|\leqslant 60h_I} |r_u(t,s)|^p\,d\mu(t)\,d\mu(s) + c \sum_{|j+l|\leqslant k} h_I^{n-d+|l|p} \int |f_{j+l}(t)|^p\,d\mu(t).$$

Note that

$$(|u|-|j|)p - 2d + n > (|u|-\alpha)p - 2d + n = (|u|-\beta)p - d.$$

Replace as we then may the factor $h_I^{n+(|u|-|j|)p-2d}$ by $h_I^{(|u|-\beta)p-d}$ in the formula above, and sum over all I with $I \geqslant -2$. Using Lemma 5.3 on the first sum and summation of $\Sigma\, h_I^{n-d}$ on the second, we get

$$\int_{d(x,F)\leqslant 4} |D^j f(x)|^p\,dx \leqslant c \sum_{|u|\leqslant k} \iint_{|t-s|\leqslant 240} |r_u(t,s)|^p \frac{d\mu(t)\,d\mu(s)}{|s-t|^{d+(\beta-|u|)p}} + c \sum_{|j+l|\leqslant k} \int |f_{j+l}(t)|^p\,d\mu(t).$$

In view of Remark 3.1, this proves (6.3).

6.3. In order to prove (6.4), we shall prove that

$$\sum_{K=0}^{\infty} h_K^{-n-(\alpha-|j|)p} \iint_{\substack{|x-y|\leqslant h_K \\ d(x,F)<2}} |R_j(x,y)|^p \, dx\, dy \leqslant c \, \|\{f_j\}\|^p_{\beta,p,F}, \quad |j| \leqslant m \tag{6.6}$$

which by Lemma 5.3 is equivalent to (6.4).

The strategy of the proof of (6.6) is as follows. If y is close to x compared to the distance from x to F, we use (5.5) as an estimate for $R_j(x,y)$. This is possible, since then f is infinitely differentiable in a neighbourhood of the line segment between x and y, and we can, via the remainder in Taylor's formula, give an estimate of $R_j(x,y)$ in terms of derivatives of f. If y is not close to x, we instead use (5.6) as an estimate for $R_j(x,y)$.

Let K be fixed, and assume first that $I \leqslant K-2$. Let $x \in Q_i \cap \Delta_I$, let y satisfy $|x-y| \leqslant h_K$ and let L denote the line segment between x and y. Then

$$|R_j(x,y)| \leqslant c\, |x-y|^{m-|j|+1} \sum_{|j+l|=m+1} \sup_{\xi \in L} |D^{j+l} f(\xi)|. \tag{6.7}$$

Now, if $\xi \in L$ and, say, $\xi \in Q_\nu$, then

$$h_{I+1} - h_K - l_\nu \leqslant d(Q_\nu, F) \leqslant h_I + h_K,$$

so by (4.2) $\dfrac{h_I}{20} \leqslant l_\nu \leqslant 5h_I/4$. Also $|t-x_\nu| \leqslant 30 l_\nu$ implies $|t-x_i| \leqslant$ $\leqslant |t-x_\nu| + |x_\nu-\xi| + |\xi-x| + |x-x_i| \leqslant 30 l_\nu + l_\nu + h_K + l_i \leqslant 39 h_I +$ $+ l_i \leqslant$ (by 5.14) $\leqslant 400 l_i$. In view of this, (6.7) and (a) of Lemma 5.2 give

$$|R_j(x,y)|^p$$
$$\leqslant c\, h_K^{(m-|j|+1)p} \sum_{|u|\leqslant k} h_I^{(|u|-m-1)p-2d} \iint_{\substack{|t-x_i|\leqslant 400 l_i \\ |s-x_i|\leqslant 400 l_i}} |r_u(t,s)|^p \, d\mu(t)\, d\mu(s).$$

Using (5.16) with $r = +\infty$ we obtain

$$\iint_{\substack{|x-y|\leqslant h_K \\ x\in\Delta_I}} |R_j(x,y)|^p \, dx\, dy$$
$$\leqslant c\, h_K^{n+(m-|j|+1)p} \sum_{|u|\leqslant k} h_I^{(|u|-m-1)p-2d+n} \iint_{|t-s|\leqslant 800 h_I} |r_u(t,s)|^p \, d\mu(t)\, d\mu(s). \tag{6.8}$$

We note here that it is easy to see that a similar formula holds for $\displaystyle\iint_{\substack{|x-y|<1\\ x\in\Delta_{\mathrm{I}}}} |\mathrm{R}_j^{\Phi}(x,y)|^p\,dx\,dy$, $\mathrm{I}=-2,-3$, and that this formula gives (6.5).

Assume next that $\mathrm{I}>\mathrm{K}-2$. Integrating formula (5.6), using (5.14), (5.17), and (5.16) with $r=+\infty$, gives

$$\iint_{\substack{|x-y|\leqslant h_{\mathrm{K}}\\ d(x,\mathrm{F})\leqslant h_{\mathrm{K}-1}}} |\mathrm{R}_j(x,y)|^p\,dx\,dy \leqslant$$

$$\leqslant \sum_{\mathrm{I}=\mathrm{K}-1}^{\infty}\ \sum_{\mathrm{N}=\mathrm{K}-2}^{\infty}\ \iint_{\substack{|x-y|\leqslant h_{\mathrm{K}}\\ x\in\Delta_{\mathrm{I}},y\in\Delta_{\mathrm{N}}}} |\mathrm{R}_j(x,y)|^p\,dx\,dy$$

$$\leqslant c\sum_{\mathrm{I}=\mathrm{K}-1}^{\infty}\ \sum_{\mathrm{N}=\mathrm{K}-2}^{\infty}\ \sum_{|j+l|\leqslant k} h_{\mathrm{I}}^{n-d+|l|p}\, h_{\mathrm{N}}^{n-d} \iint_{|s-t|\leqslant 300h_{\mathrm{K}}} |r_{j+l}(t,s)|^p\,d\mu(t)\,d\mu(s)$$

$$+\,c\sum_{\mathrm{N}=\mathrm{K}-2}^{\infty}\ \sum_{|u|\leqslant k}\ \sum_{|j+l|\leqslant m} h_{\mathrm{K}}^{n+|l|p}\, h_{\mathrm{N}}^{(|u|-|j+l|)p-2d+n}$$

$$\iint_{|s-t|\leqslant 60h_{\mathrm{N}}} |r_u(t,s)|^p\,d\mu(t)\,d\mu(s)$$

$$+\,c\sum_{\mathrm{I}=\mathrm{K}-1}^{\infty} h_{\mathrm{K}}^{n} \sum_{|u|\leqslant k} h_{\mathrm{I}}^{(|u|-|j|)p-2d+n} \iint_{|s-t|\leqslant 60h_{\mathrm{I}}} |r_u(t,s)|^p\,d\mu(t)\,d\mu(s).$$

(To obtain the last term above, and similarly the term in the middle, use (5.16) with $r=+\infty$ after arguing as follows with $g(x)=\mathrm{J}_u(x_i)$, $x\in\mathrm{Q}_i$:

$$\sum_{\mathrm{N}=\mathrm{K}-2}^{\infty}\ \iint_{\substack{|x-y|\leqslant h_{\mathrm{K}}\\ x\in\Delta_{\mathrm{I}},y\in\Delta_{\mathrm{N}}}} g(x)\,dx\,dy \leqslant \iint_{\substack{|x-y|\leqslant h_{\mathrm{K}}\\ x\in\Delta_{\mathrm{I}}}} g(x)\,dx\,dy$$

$$= c\,h_{\mathrm{K}}^{n}\int_{x\in\Delta_{\mathrm{I}}} g(x)dx.)$$

Together with (6.8) this gives, if we take the two last terms above together,

$$\iint_{\substack{|x-y|\leqslant h_{\mathrm{K}}\\ d(x,\mathrm{F})\leqslant 2}} |\mathrm{R}_j(x,y)|^p\,dx\,dy \leqslant c\sum_{|j+l|\leqslant k} h_{\mathrm{K}}^{2n-2d+|l|p}$$

$$\iint_{|s-t|\leqslant bh_{\mathrm{K}}} |r_{j+l}(t,s)|^p\,d\mu(t)\,d\mu(s)\,+$$

$$+ c \sum_{|u| \leqslant k} \sum_{|j+l| \leqslant m} h_{K}^{n+|l|p} \sum_{N=K}^{\infty} h_{N}^{(|u|-|j+l|)p-2d+n} \iint_{|s-t| \leqslant bh_N} |r_u(t,s)|^p \, d\mu(t) \, d\mu(s) +$$

$$+ c \sum_{|u| \leqslant k} h_{K}^{n+(m-|j|+1)p} \sum_{I=0}^{K} h_{I}^{(|u|-m-1)p-2d+n} \iint_{|s-t| \leqslant bh_I} |r_u(t,s)|^p \, d\mu(t) \, d\mu(s). \tag{6.9}$$

Here we may take $b = 3200$. A straightforward summation gives

$$\sum_{K=0}^{\infty} h_{K}^{-n-(\alpha-|j|)p} \iint_{\substack{|x-y| \leqslant h_K \\ d(x,F)<2}} |R_j(x,y)|^p \, dx \, dy$$

$$\leqslant c \sum_{|u| \leqslant k} \sum_{M=0}^{\infty} h_{M}^{-d-(\beta-|u|)p} \iint_{|s-t| \leqslant bh_M} |r_u(t,s)|^p \, d\mu(t) \, d\mu(s)$$

which by Lemma 5.3 and Remark 3.1 proves (6.6).

For example,

$$\sum_{K=0}^{\infty} h_{K}^{-n-(\alpha-|j|)p+n+(m-|j|+1)p} \sum_{I=0}^{K} h_{I}^{(|u|-m-1)p-2d+n} \iint_{|s-t| \leqslant bh_I} |r_u(t,s)|^p \, d\mu(t) \, d\mu(s) =$$

$$= \sum_{I=0}^{\infty} \sum_{K=I}^{\infty} h_{K}^{(m+1-\alpha)p} h_{I}^{(|u|-m-1)p-2d+n} \iint_{|s-t| \leqslant bh_I} |r_u(t,s)|^p \, d\mu(t) \, d\mu(s) \leqslant$$

$$\leqslant (\text{since } \alpha < m+1) \leqslant c \sum_{I=0}^{\infty} h_{I}^{-d-(\alpha p-n+d)+|u|p} \iint_{|s-t| \leqslant bh_I} |r_u(t,s)|^p \, d\mu(t) \, d\mu(s).$$

It is only in this performed summation the condition $\alpha < m+1$ is needed.

6.4. *The case* $\alpha = m+1$. Recall that if α is an integer, we use the classical definition of $B_{\alpha}^{p}(\mathbf{R}^n)$, that is $f \in B_{\alpha}^{p}(\mathbf{R}^n)$, $\alpha = m+1$, iff the norm

$$\|f\|_{p,\alpha,\mathbf{R}^n} = \sum_{|j| \leqslant m} \|D^j f\|_p + \sum_{|j|=m} \left(\iint \frac{|D^j f(x) - 2D^j f\left(\frac{x+y}{2}\right) + D^j f(y)|^p}{|x-y|^{n+(\alpha-m)p}} \, dx \, dy \right)^{1/p} \tag{6.10}$$

is finite. Here the double integrals may be taken over $|x-y| < 1$ (compare Proposition 3.1).

Let $\{f_j\}_{|j|\leqslant k} \in B^p_\beta(F)$. From the preceding calculations, it follows that our extension f of $\{f_j\}_{|j|\leqslant k}$ belongs to $B^p_{\alpha'}(\mathbf{R}^n)$ for $\alpha' < \alpha$. Hence by Proposition 1.2 the distributional derivatives of orders j, $|j| \leqslant m$, are functions in L^p, and we have

$$\sum_{|j|\leqslant m} \|D^j f\|_p \leqslant c \, \|\{f_j\}\|_{p,\beta,F}\,.$$

It remains to estimate the double integrals in (6.10). These are estimated as the integrals $\iint \frac{|R_j(x,y)|^p}{|x-y|^{n+(\alpha-m)p}}\, dx\, dy$, $|j| = m$, were estimated in Section 6.3, the only difference of significance being that by using the mean value theorem twice, the estimate (6.7) is replaced by

$$|D^j f(x) - 2D^j f\left(\frac{x+y}{2}\right) + D^j f(y)| \leqslant c\, |x-y|^2 \sum_{|j+l|=m+2} \sup_{\xi = L} |D^{j+l} f(\xi)|,$$

which gives convergence in the last summation of 6.3 also in the case $\alpha = m+1$.

6.5. *Proof of statement (b) of Theorem 4.1.*

It is easy to see that the following variant of (5.5) holds

$$|D^j f(x) - P_j(x,t_0)|^p \leqslant c \sum_{|u|\leqslant k} l_i^{(|u|-|j|)p-2d} \iint_{\substack{|t-x_i|\leqslant 30\, l_i \\ |s-x_i|\leqslant 30\, l_i}} |r_u(t,s)|^p \, d\mu(t)\, d\mu(s) +$$
$$+ c \sum_{|j+l|\leqslant k} l_i^{|l|p-d} \int_{|t-x_i|\leqslant 30\, l_i} |r_{j+l}(t,t_0)|^p \, d\mu(t), \; x \in Q_i\,.$$

Let I_r be the smallest integer such that $h_{I_r+1} \leqslant r$. Then by (5.14), (5.16) and (5.12)

$$\int_{|x-t_0|\leqslant r} |D^j f(x) - P_j(x,t_0)|^p \, dx$$
$$\leqslant c \sum_{|u|\leqslant k} \sum_{I_r}^{\infty} h_I^{n+(|u|-|j|)p-2d} \int_{|t-t_0|\leqslant cr} \int_{|t-s|\leqslant 60 h_I} |r_u(t,s)|^p \, d\mu(t)\, d\mu(s) +$$
$$+ c \sum_{|j+l|\leqslant k} \sum_{I_r}^{\infty} h_I^{n+|l|p-d} \int_{|t-t_0|\leqslant cr} |r_{j+l}(t,t_0)|^p \, d\mu(\mathrm{t}).$$

Since $h_I^{n+|u|p-|j|p-2d+\beta p-\beta p} \leqslant r^{n-d+(\beta-|j|)p}\, h_I^{-d-(\beta-|u|)p}$ we get, using among other things Lemma 5.3,

$$\frac{1}{r^n}\int_{|x-t_0|\leqslant r} |D^j f(x) - P_j(x,t_0)|^p\, dx \leqslant$$

$$\leqslant c \sum_{|u|\leqslant k} r^{(\beta-|j|)p-d} \iint_{\substack{|s-t|\leqslant 120r\\ |t-t_0|\leqslant cr}} \frac{|r_u(t,s)|^p}{|t-s|^{d+(\beta-|u|)p}}\, d\mu(s)\, d\mu(t)$$

$$+ c \sum_{|j+l|\leqslant k} r^{|l|p-d+(\beta-|j+l|)p+d} \int_{|t-t_0|\leqslant cr} \frac{|r_{j+l}(t,t_0)|^p}{|t-t_0|^{d+(\beta-|j+l|)p}}\, d\mu(t).$$

Since $\dfrac{1}{\mu(B(t_0,r))}\displaystyle\int_{B(t_0,r)} g(t)\, d\mu(t) \longrightarrow g(t_0)$ μ-a.e., $r \longrightarrow 0$, if $g \in L^1(\mu)$, see e.g. [14, p. 156] and [25], Theorem 2.18, for the required regularity of μ, and since $r^{-d} \leqslant c(\mu(B(t_0,r)))^{-1}$, this shows that

$$\frac{1}{r^n}\int_{|x-t_0|\leqslant r} |D^j f(x) - P_j(x,t_0)|^p\, dx = O(r^{(\beta-|j|)p}),\ r \longrightarrow 0,\ |j| \leqslant k,$$

for μ-almost all t_0.

Since obviously $\dfrac{1}{r^n}\displaystyle\int_{|x-t_0|\leqslant r} |P_j(x,t_0) - f_j(t_0)|^p\, dx = O(r^p)$, $r \longrightarrow 0$, for μ-almost all t_0, it follows, since $|j| < \beta$, that $\dfrac{1}{r^n}\displaystyle\int_{|x-t_0|\leqslant r} |D^j f(x) - f_j(t_0)|^p\, dx \longrightarrow 0$, $r \longrightarrow 0$, μ-almost all t_0, which gives (b) of Theorem 4.1.

CHAPTER III

THE RESTRICTION THEOREM

7. Main Theorem.

7.1. The purpose in this chapter is to prove the following theorem.

THEOREM 7.1. – *Let* $0 < d \leqslant n,\ 1 \leqslant p < \infty$ *and*

$$\beta = \alpha - \frac{n-d}{p}, \quad k < \beta < k+1,$$

where k *is a nonnegative integer. Let* μ *be a positive measure such that, for some constants* c_1 *and* $r_0 > 0$,

$$\mu(\mathrm{B}(x,r)) \leqslant c_1\, r^d, \quad x \in \mathbf{R}^n, \quad r \leqslant r_0. \tag{7.1}$$

For $u \in \mathrm{B}_\alpha^p(\mathbf{R}^n)$, *let* R_j *be defined by*

$$\mathrm{D}^j u(x) = \sum_{|j+l| \leqslant k} \frac{\mathrm{D}^{j+l} u(y)}{l!} (x-y)^l + \mathrm{R}_j(x,y), \quad \text{for } |j| \leqslant k, \tag{7.2}$$

and put

$$\|u\|_{p,\beta,\mu} = \sum_{|j| \leqslant k} \|\mathrm{D}^j u\|_{p,\mu} + \sum_{|j| \leqslant k} \left\{ \iint_{|x-y|<1} \frac{|\mathrm{R}_j(x,y)|^p}{|x-y|^{d+(\beta-|j|)p}}\, d\mu(x)\, d\mu(y) \right\}^{1/p}. \tag{7.3}$$

Then for all $u \in \mathrm{B}_\alpha^p(\mathbf{R}^n)$,

$$\|u\|_{p,\beta,\mu} \leqslant c\, \|u\|_{p,\alpha,\mathrm{R}^n}, \tag{7.4}$$

where c *is a constant depending only on* α, β, p, d *and* μ.

Here $\mathrm{R}_j(x,y)$ is defined at all points where the other terms in (7.2) are strictly defined. It follows from the assumptions (see 7.4) that the derivatives $\mathrm{D}^j u$, $|j| \leqslant k$, can be strictly defined d-a.e. and hence μ-a.e. (see Section 2.2). If F is a d-set and μ a d-measure on F, the theorem gives, in the notation of the Main Theorem of Section 1 that the restriction $\underline{\mathrm{R}}(u) \in \mathrm{B}_\beta^p(\mathrm{F})$ and that the restriction operator $\underline{R}$: $B^p_\alpha(\mathbf{R}^n) \longrightarrow B^p_\beta(F)$ is continuous, which is part B of the Main Theorem in Section 1.

Remark 7.1. – In Theorem 7.1 we put conditions on the derivatives of u. It is possible to prove analogous theorems where we instead put conditions on the differences of u.

7.2. In the proof of Theorem 7.1 we need the Bessel potentials. A function u is the *Bessel potential of order* α, $0 < \alpha$, of the function $f \in L^p(\mathbf{R}^n)$ if

$$u = G_\alpha * f,$$

where the Bessel kernel G_α has Fourier transform

$$\hat{G}_\alpha(x) = (1 + 4\pi^2 \,|x|^2)^{-\alpha/2}.$$

The norm of the potential u is denoted by $\|u\|_{p,\alpha}$ and defined by

$$\|u\|_{p,\alpha} = \|f\|_p .$$

The Bessel kernel is a positive, decreasing function of $|x|$, analytic on $\mathbf{R}^n\backslash\{0\}$, satisfying, for a number c_1 not depending on x (see e.g. [5, § 2])

$$|D^j\, G_\alpha(x)| \leqslant c_1\, |x|^{\alpha-|j|-n}, \quad \text{for} \quad \alpha < n+|j|, \tag{7.5}$$

$$|D^j\, G_\alpha(x)| \leqslant c_1\, \log\frac{1}{|x|}, \quad 0 < |x| < 1, \quad \text{for} \quad \alpha = n+|j|, \tag{7.5'}$$

$$D^j\, G_\alpha(x) \text{ is finite, continuous at } x = 0, \text{ for } \alpha > n+|j|, \tag{7.5''}$$

and, for all derivatives,

$$|D^j\, G_\alpha(x)| \leqslant c_1\, e^{-c|x|}, \quad 1 \leqslant |x| < \infty, \quad \text{for some} \quad c > 0. \tag{7.6}$$

If $f \in L^p(\mathbf{R}^n)$ we claim that

$$D^j (G_\alpha * f) = (D^j\, G_\alpha) * f \quad \text{for} \quad |j| \leqslant k < \alpha, \tag{7.7}$$

in the distribution sense, where the convolutions in the right member of (7.7) can be written as integrals since $D^j G_\alpha \in L^1(\mathbf{R}^n)$ for $|j| \leqslant k < \alpha$. If the support of f is compact, formula (7.7) is obvious. However, the formula is true – and, of course, well-known – even if f does not have compact support. In fact, by writing $f = f_1 + f_2$ where $f_1 = f$ for $|x| \leqslant r_1$ and $f_2 = f$ for $|x| > r_1$, we conclude that $D^j(G_\alpha * f_i) = (D^j G_\alpha) * f_i$ for $i = 1, 2$, $|x| < r_1$, where, for $i = 2$, we can differentiate under the integral sign and get a continuous function for $|x| < r_1$.

7.3. It is an important fact that the strictly defined function $u = G_\alpha * f$ coincides with the integral at all points where the integral defining the Bessel potential is absolutely convergent. We need a version of this result also for derivatives of potentials.

PROPOSITION 7.1. – *Let* $u = G_\alpha * f$, $f \in L^p(\mathbf{R}^n)$, $\alpha-(n-d)/p > k$, $0 < d \leqslant n$, $1 \leqslant p < \infty$, *and* k *a nonnegative integer. Then* $D^j u$, $|j| \leqslant k$, *can be strictly defined* d-a.e. *and the integral* $(D^j G_\alpha) * f$ *is absolutely convergent and coincides* d-a.e. *with the strictly defined function* $D^j u$, $|j| \leqslant k$.

Proof. – (Compare [3, p. 13].) By putting $D^j u = (D^j G_\alpha) * f$ (see (7.7)) and changing the order of integration we obtain, for a point x where $(D^j G_\alpha) * f$ is absolutely convergent,

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |D^j u(y) - ((D^j G_\alpha) * f)(x)| dy \leqslant$$

$$\leqslant \int \left[\frac{1}{m(B(x,r))} \int_{B(x,r)} |D^j G_\alpha(y-z) - D^j G_\alpha(x-z)| dy \right] |f(z)| dz.$$

For $z \neq x$, the function in square brackets converges pointwise to zero, as $r \longrightarrow 0$. Hence, if we have dominated convergence, the right member tends to zero, as $r \longrightarrow 0$. We consider first the case when $n-\alpha+|j| > 0$. Take a point x where

$$\int_{|z-x| \leqslant 1} \frac{|f(z)|}{|z-x|^{n-\alpha+|j|}} dz < \infty. \tag{7.8}$$

By a well-known property of Riesz (and Bessel) potentials (see [16, pp. 287 and 294] or [2, § 4]) this integral is convergent d-a.e., since $d > n-p(\alpha-|j|)$ for $|j| \leqslant k$. It follows from (7.5), (7.6) and (7.8) that $(D^j G_\alpha) * f$ is absolutely convergent at x. Furthermore, by (7.5),

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |D^j G_\alpha(y-z)| dy \leqslant \frac{c}{m(B(x,r))} \int_{B(x,r)} \frac{dy}{|y-z|^{n-\alpha+|j|}} . \tag{7.9}$$

But this is less than a constant times $|x-z|^{-(n-\alpha+|j|)}$ by the Frostman mean value theorem [15, p. 27]. We use this estimate when $|x-z| \leqslant 1$. When $|x-z| > 1$ we estimate the left member of (7.9) by means of (7.6) which works when r is small. Altogether this gives the desired dominated convergence and completes the proof of the proposition when $n-\alpha+|j| > 0$. The case $n-\alpha+|j| = 0$ can be treated similarly and the case $n-\alpha+|j| < 0$ is trivial because of (7.5″).

7.4. From Proposition 7.1 we can among other things conclude that $D^j u$, $|j| \leqslant k$, can be strictly defined d-a.e. if $u \in B^p_\alpha(\mathbf{R}^n)$ and $\alpha-(n-d)/p > k$. In fact, $u \in B^p_\alpha(\mathbf{R}^n)$ implies that, for $\epsilon > 0$, $u = G_{\alpha-\epsilon} * f_\epsilon$ where $f_\epsilon \in L^p(\mathbf{R}^n)$ (see [4, p. 46]) and therefore Proposition 7.1 gives the desired result if ϵ is chosen small enough.

8. Lemmas on potentials.

In this section we collect a number of lemmas on potentials for the proof of Theorem 7.1. The main lemma is Lemma 8.4 which should be compared to [1], [18, Theorem 2] and [24]. Lemma 8.4 is a weaker form of Theorem 7.1 for potentials.

8.1. We start by the following very simple lemma.

LEMMA 8.1. – *Let $0 < d \leqslant n$ and let ν be a positive measure such that, for some constants c_1 and $r_0 > 0$,*

$$\nu(B(x,r)) \leqslant c_1\, r^d, \quad x \in \mathbf{R}^n, \quad r < r_0. \tag{8.1}$$

Then

a) $\displaystyle\int_{|x-t|\leqslant a} \frac{d\nu(t)}{|x-t|^\gamma} = O(a^{d-\gamma})$ *if* $d > \gamma$, $a \leqslant r_0$, *and*

b) $\displaystyle\int_{|x-t|>a} \frac{d\nu(t)}{|x-t|^\gamma} = O(a^{d-\gamma})$ *if* $d < \gamma$, $r_0 = \infty$.

Here O *stands for a constant depending on c_1, γ and d.*

Proof. – If we write

$$\int_{|x-t|\leqslant a} \frac{d\nu(t)}{|x-t|^{\gamma}} = \int_0^a \frac{d\nu(B(x,r))}{r^{\gamma}}$$

and make a partial integration we get a). In a similar way b) is proved.

8.2. We next need two lemmas on the Bessel kernel G_α.

LEMMA 8.2. – *For a fixed* α, $0 < \alpha < n+k+1$, k *a nonnegative integer,* $\alpha - n$ *not a nonnegative integer, we define* H_j *by*

$$D^j G_\alpha(x) = \sum_{|j+l|\leqslant k} \frac{D^{j+l}G_\alpha(y)}{l!}(x-y)^l + H_j(x,y). \tag{8.2}$$

Then

$$\left\{ \int_{\mathbf{R}^n} |H_j(x-t, y-t)|^s \, dt \right\}^{1/s} \leqslant c\,|x-y|^{\gamma-|j|} \quad \text{for } |j| \leqslant k, \tag{8.3}$$

$$s > 0, \quad \gamma = \frac{n}{s} - (n-\alpha), \quad k < \gamma < k+1,$$

if c *is a certain constant.*

Proof. – The proof essentially proceeds by a straightforward use of the estimates of G_α given by (7.5) and (7.6). A complication is that, due to convergence problems, the calculations have to be organized in different manners for different α.

By changing $y-t$ to t and putting $x-y=h$, where we assume $h \neq 0$, we write the left member of (8.3) raised to the power s in the form

$$\int_{\mathbf{R}^n} |D^j G_\alpha(t+h) - \sum_{|j+l|\leqslant k} \frac{D^{j+l}G_\alpha(t)}{l!} h^l|^s \, dt$$
$$= \int_{|t|\leqslant 2|h|} + \int_{|t|>2|h|} = \mathrm{I} + \mathrm{II}.$$

Estimate of II. We use Taylor's formula on the integrand and get a remainder with derivatives of G_α of order $k+1$ at a point $t+\theta h$, $0<\theta<1$, such that $|t+\theta h| \geqslant |t| - |h| > |t|/2$ since $|t| \geqslant 2|h|$. By means of this and (7.5) we get (c denotes different constants)

$$\mathrm{II} \leqslant c \int_{|t|>2|h|} \left(\frac{|h|^{k+1-|j|}}{|t|^{n-\alpha+k+1}} \right)^s dt, \quad \text{since } \alpha < n+k+1$$

and thus, e.g. by means of Lemma 8.1,

$$\mathrm{II} \leqslant c\,|h|^{n-s(n-\alpha+|j|)} \quad \text{if} \quad n < s(n-\alpha+k+1).$$

Thus $$\mathrm{II} \leqslant c\,|h|^{(\gamma-|j|)s} \quad \text{since} \quad \gamma < k+1.$$

Estimate of I *when* $0 < \alpha < n$. We estimate each term in the integrand separately. The estimation of a typical term proceeds in the following way by means of (7.5)

$$\int_{|t|\leqslant 2|h|} \left| \frac{D^{j+l}G_\alpha(t)}{l!}\, h^l \right|^s dt$$

$$\leqslant c\,|h|^{|l|s} \int_{|t|\leqslant 2|h|} \frac{dt}{|t|^{s(n-\alpha+|j+l|)}}, \quad \text{if} \quad \alpha < n+|j+l|.$$

The last member is, e.g. by Lemma 8.1, less than

$$c\,|h|^{|l|s}\,|h|^{n-s(n-\alpha+|j+l|)} = c\,|h|^{n-s(n-\alpha+|j|)},$$

if $|j+l| \leqslant k$ and $n > s(n-\alpha+k)$. We get in a similar way

$$\int_{|t|\leqslant 2|h|} |D^j G_\alpha(t+h)|^s\, dt \leqslant c\,|h|^{n-s(n-\alpha+|j|)}$$

if $\alpha < n+|j|$, $|j| \leqslant k$ and $n > s(n-\alpha+k)$.

Consequently,

$$\mathrm{I} \leqslant c\,|h|^{(\gamma-|j|)s} \quad \text{since} \quad \gamma > k \quad \text{and} \quad \alpha < n.$$

Estimate of I *when* $n < \alpha < n+k+1$, $\alpha - n$ *not an integer.*

For a fixed α, let ν be an integer, $1 \leqslant \nu \leqslant k+1$, such that $n+\nu-1 < \alpha < n+\nu$. We put $\delta = n-\alpha+\nu$ and observe that $0 < \delta < 1$. Since $\alpha < n+\nu$, we can proceed exactly as we did in the case $\alpha < n$, when we want to estimate the terms in the integrand containing a derivative of G_α of order $j+l$ with $|j+l| \geqslant \nu$ or order j with $|j| \geqslant \nu$. In fact, for these terms we have $\alpha < n+|j+l|$ and $\alpha < n+|j|$, respectively. These terms consequently give, exactly as in the case $\alpha < n$, a contribution $c\,|h|^{(\gamma-|j|)s}$ in the estimation of I.

If $|j| < \nu$ we get, by Taylor's formula with exact remainder, for the other terms in the integrand,

$$\mathrm{D}^j \mathrm{G}_\alpha(t+h) - \sum_{|j+l|<\nu} \frac{\mathrm{D}^{j+l}\mathrm{G}_\alpha(t)}{l!} h^l \qquad (8.4)$$
$$= (\nu-|j|) \int_0^1 (1-\rho)^{\nu-1-|j|} \sum_{|j+l|=\nu} \frac{\mathrm{D}^{j+l}\mathrm{G}_\alpha(t+\rho h)}{l!} h^l \, d\rho.$$

It should be noted that this formula is true also when the closed line segment between t and $t+h$ contains the origin. This follows since the singularity at the origin of $\mathrm{D}^{j+l}\mathrm{G}_\alpha(\mathrm{t})$, $|j+l| = \nu$, is of type $|t|^{-\delta}$, $\delta < 1$, and consequently integrable along a line segment through the origin which means that the derivative of order $\nu-1-|j|$ of the function $\rho \longrightarrow \mathrm{F}(\rho) = \mathrm{D}^j \mathrm{G}_\alpha(t+\rho h)$ is absolutely continuous in $[0,1]$.

We obtain by (7.5) that the right member of (8.4) is dominated by

$$c\,|h|^{\nu-|j|} \int_0^1 \frac{d\rho}{|t+\rho h|^{n-\alpha+\nu}} \quad \text{since} \quad \alpha < n+\nu.$$

We denote this last integral by A and use the estimate

$$|t+\rho h| \geqslant |t_i+\rho h_i|, \quad t = (t_1, \ldots, t_n), \quad h = (h_1, \ldots, h_n),$$

where for a fixed h, i is chosen so that $|h_i| \geqslant |h|/\sqrt{n}$. This gives, since $0 < \delta < 1$,

$$\mathrm{A} \leqslant \int_0^1 \frac{d\rho}{|t_i+\rho h_i|^\delta} \leqslant c\,|h|^{-\delta}, \quad \text{for} \quad |t| \leqslant 2\,|h|, \quad \delta = n-\alpha+\nu.$$

The terms in the left member of (8.4) consequently give a contribution to I which is bounded by

$$c\,|h|^n \;\; |h|^{s(\nu-|j|)}\; |h|^{-\delta s} = c\,|h|^{n-s(n-\alpha+|j|)} = c\,|h|^{(\gamma-|j|)s}.$$

Hence, we get the same estimate of I and by combining this with the estimate of II we finally obtain the desired estimate (8.3).

Remark 8.1. – The latter method of estimation of I gives, with some extra effort, a proof of the lemma in the case when $\alpha-n$ is a nonnegative integer, $n > 1$, also. We omit the proof of this since we do not need the lemma in this case – in fact, we need the lemma for a dense set of α-values only.

8.3. The next lemma is similar to Lemma 8.2 but technically a little more complicated.

LEMMA 8.3. – *Let* α, k *and* H_j *be as in Lemma 8.2 and* $0 < d \leqslant n$. *Let, for some constants* c_1 *and* $r_0 > 0$,

$$\mu(B(x,r)) \leqslant c_1\, r^d, \quad x \in \mathbf{R}^n, \quad r \leqslant r_0 .$$

Then, for $i \leqslant 0$,

$$\Big\{ \iint_{2^i \leqslant |x-y| < 2^{i+1}} |H_j(x-t, y-t)|^s \, d\mu(x)\, d\mu(y)\Big\}^{1/s} \leqslant c\, 2^{i(\frac{d}{s}+\gamma-|j|)} \tag{8.5}$$

for $t \in \mathbf{R}^n$, $|j| \leqslant k$, $s > 0$, $\gamma = \dfrac{d}{s} - (n-\alpha)$, $k < \gamma < k+1$, *if* c *is a certain constant.*

Proof. – For a fixed t we put

$$E_1 = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : |y-t| \leqslant 4\cdot 2^i,\ 2^i \leqslant |x-y| < 2^{i+1}\} \text{ and}$$

$$E_2 = \{(x,y) : |y-t| > 4\cdot 2^i,\ 2^i \leqslant |x-y| < 2^{i+1}\},$$

and estimate the left member of (8.5) raised to the power s by

$$\iint_{2^i \leqslant |x-y| < 2^{i+1}} \Big| D^j G_\alpha(x-t) - \sum_{|j+l| \leqslant k} \frac{D^{j+l} G_\alpha(y-t)}{l} (x-y)^l \Big|^s d\mu(x)\, d\mu(y)$$

$$= \iint_{E_1} + \iint_{E_2} = \mathrm{I} + \mathrm{II}.$$

Estimate of II. We estimate H_j by means of Taylor's formula at $y-t$ and get a point $y-t+\theta(x-y)$, $0 < \theta < 1$, so that

$$\mathrm{II} \leqslant c \iint_{E_2} \{2^{i(k+1-|j|)} \sum_{|j+l|=k+1} |D^{j+l} G_\alpha(y-t+\theta(x-y))|\}^s \, d\mu(x)\, d\mu(y).$$

But $|y-t+\theta(x-y)| \geqslant |y-t| - |x-y| \geqslant |y-t|/2$ since $|y-t| > 2|x-y|$. Since $\alpha < n+k+1$ we use this combined with (7.5) for $|y-t| \leqslant 1$ and with (7.6) for $|y-t| > 1$. After that we perform the x-integration, from which we get a factor bounded by $c\, 2^{id}$, and obtain

$$\mathrm{II} \leqslant c\, 2^{is(k+1-|j|)}\, 2^{id} \Big\{ \int_{4\cdot 2^i < |y-t| \leqslant 1} \frac{d\mu(y)}{|y-t|^{s(n-\alpha+k+1)}} + \int_{|y-t|>1} e^{-cs|y-t|}\, d\mu(y) \Big\}. \tag{8.6}$$

The first of the last two integrals is estimated by means of Lemma 8.1, b) to be

$$O(2^{i(d-s(n-\alpha+k+1))}) \text{ if } d < s(n-\alpha+k+1).$$

The second integral is estimated by means of the same type of calculation as in the proof of Lemma 8.1 which gives – remembering that, by (2.1), $\mu(B(x,r)) \leqslant c\, r^n$, $r \geqslant r_0$ – that the integral is bounded. Together we get

$$\mathrm{II} \leqslant c\, 2^{i(2d-s(n-\alpha+|j|))} = c\, 2^{is(\frac{d}{s}+\gamma-|j|)} \quad \text{since} \quad \gamma < k+1.$$

Estimate of I *when* $0 < \alpha < n$. In this estimate we have $|x-t| \leqslant |x-y| + |y-t| \leqslant 2^{i+1} + 4\cdot 2^i = 6\cdot 2^i$. We proceed in the same way as in the corresponding case in the proof of Lemma 8.2. For a typical term we get by means of (7.5), since $\alpha < n+|j+l|$,

$$\iint_{E_1} \left| \frac{D^{j+l}G_\alpha(y-t)}{l!}(x-y)^l \right|^s d\mu(x)\, d\mu(y)$$

$$\leqslant c\, 2^{is|l|} \iint_{E_1} \frac{d\mu(x)\, d\mu(y)}{|y-t|^{s(n-\alpha+|j+l|)}}\,.$$

By Lemma 8.1 and the assumption on μ this is dominated by

$$c\, 2^{is|l|}\, 2^{i(d-s(n-\alpha+|j+l|))} \int_{|x-t|\leqslant 6\cdot 2^i} d\mu(x)$$

$$= c\, 2^{i(2d-s(n-\alpha+|j|))} \quad \text{if} \quad d > s(n-\alpha+|j+l|).$$

Thus $\mathrm{I} \leqslant c\, 2^{i(\frac{d}{s}+\gamma-|j|)s}$ since $\gamma > k$.

Estimate of I *when* $n < \alpha < n+k+1$, $\alpha - n$ *not an integer.* Again we proceed as in the proof of Lemma 8.2 with $\delta = n-\alpha+\nu$ where ν is an integer, $1 \leqslant \nu \leqslant k+1$, such that $0 < \delta < 1$. The terms on which we use Taylor's formula give a contribution to I which is dominated by

$$c\, 2^{i(\nu-|j|)s} \iint_{E_1} \left| \int_0^1 \frac{d\rho}{|y-t+\rho(x-y)|^\delta} \right|^s d\mu(x)\, d\mu(y).$$

The inner integral is estimated in the same manner as in the proof of Lemma 8.2 to be $O(2^{-i\delta})$. Consequently, the whole expression is dominated by

$$c\, 2^{i(\nu-|j|)s}\, 2^{-is\delta} 2^{2id} = c\, 2^{i(d+s(\gamma-|j|))}.$$

By combining this with the contribution to I which we get from the other terms, and with the estimate of II, we obtain (8.5) and the lemma is proved.

8.4. We now come to the main lemma.

LEMMA 8.4. – *Let* $0 < \alpha < n+k+1$, k *a nonnegative integer,* $\alpha - n$ *not a nonnegative integer,* $0 < d \leqslant n$, $1 \leqslant p < \infty$, *and suppose that*

$$k < \alpha - \frac{n-d}{p} < k+1.$$

Let $u = G_\alpha * f$, $f \in L^p(\mathbf{R}^n)$, *and define* R_j *by (7.2). Let* μ *be a positive measure satisfying (7.1). Then, for a certain constant* c,

$$\left\{ \iint_{2^i \leqslant |x-y| < 2^{i+1}} |R_j(x,y)|^p \, d\mu(x)\, d\mu(y) \right\}^{1/p} \leqslant c\, 2^{i(\gamma - |j|)} \|u\|_{p,\alpha}, \tag{8.7}$$

for $i \leqslant 0$, $|j| \leqslant k$, $\gamma = \dfrac{d}{p} + \alpha - \dfrac{n-d}{p}$,

$$\text{and } \|D^j u\|_{p,\mu} \leqslant c\, \|u\|_{p,\alpha}, \quad |j| \leqslant k. \tag{8.8}$$

Here $\|u\|_{p,\alpha} = \|f\|_p$ *is the potential space norm and* $\|D^j u\|_{p,\mu}$ *the* $L^p(\mu)$ *norm.*

Proof. – We consider the case $p > 1$ only; the case $p = 1$ is formally slightly different and, in fact, somewhat simpler. We first prove (8.7). We observe that, by (7.2), $R_j(x,y)$ is defined at all points where $D^j u(x)$ and $D^{j+l} u(y)$, $|j+l| \leqslant k$, are strictly defined. Also, by Proposition 7.1, we can in (8.7) put

$$R_j(x,y) = \int H_j(x-t,\, y-t)\, f(t)\, dt$$

where H_j is defined by (8.2). Now take a function ϕ defined on $\mathbf{R}^n \times \mathbf{R}^n$ such that

$$\iint_{2^i \leqslant |x-y| < 2^{i+1}} |\phi(x,y)|^{p'} \, d\mu(x)\, d\mu(y) = 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Let $0 < a < 1$. By means of Hölder's inequality we obtain

$$\left| \iint_{2^i \leqslant |x-y| < 2^{i+1}} R_j(x,y)\, \phi(x,y)\, d\mu(x)\, d\mu(y) \right|$$
$$= \left| \iiint H_j(x-t,\, y-t)\, f(t)\, \phi(x,y)\, dt\, d\mu(x)\, d\mu(y) \right|$$
$$\leqslant \left\{ \iiint |H_j(x-t,\, y-t)|^{ap} \, |f(t)|^p \, dt\, d\mu(x)\, d\mu(y) \right\}^{1/p}$$
$$\cdot \left\{ \iiint |H_j(x-t,\, y-t)|^{(1-a)p'} \, |\phi(x,y)|^{p'} \, dt\, d\mu(x)\, d\mu(y) \right\}^{1/p'}.$$

By using Lemma 8.3 on the first and Lemma 8.2 on the second of the integrals in the last member we get, remembering the normalization on ϕ , that the left member is less than

$$c\ \|f\|_p\ \{2^{i(\frac{d}{ap}+\gamma_1-|j|)ap}\}^{1/p}.\ \{2^{i(\gamma_2-|j|)(1-a)p'}\}^{1/p'}, \tag{8.9}$$

if
$$\gamma_1 = \frac{d}{ap} - (n-\alpha),\ \ k < \gamma_1 < k+1, \tag{8.10}$$

and
$$\gamma_2 = \frac{n}{(1-a)p'} - (n-\alpha),\ \ k < \gamma_2 < k+1. \tag{8.11}$$

By simplifying (8.9) and using that $\|f\|_p = \|u\|_{p,\alpha}$, we obtain (8.7) by the converse of Hölder's inequality if we verify that it is possible to choose a, $0 < a < 1$, so that (8.10) and (8.11) hold. Solving for a in the conditions (8.10) and (8.11) we get, when $\alpha < n+k$ (the case $\alpha > n+k$ is simpler),

$$0 < \frac{d}{p(n-\alpha+k+1)} < a < \frac{d}{p(n-\alpha+k)}$$

and
$$1 - \frac{n}{p'(n-\alpha+k)} < a < 1 - \frac{n}{p'(n-\alpha+k+1)} < 1,$$

respectively. It is, consequently, possible to choose a if

$$\frac{d}{p(n-\alpha+k+1)} < 1 - \frac{n}{p'(n-\alpha+k+1)}$$

and
$$1 - \frac{n}{p'(n-\alpha+k)} < \frac{d}{p(n-\alpha+k)}.$$

These last two conditions can be simplified to

$$k < \alpha - \frac{n-d}{p} < k+1$$

which is true by our assumption. Hence (8.7) is proved.

Proof of (8.8). – The proof of (8.8) is, of course, simpler and does not depend on the lemmas 8.2 and 8.3. By taking a function ϕ such that $\|\phi\|_{p',\mu} = 1$ and a number a, $0 < a < 1$, we obtain, for $|j| \leqslant k$, by means of Hölder's inequality,

$$|\int (D^j u)\, \phi\, d\mu(x)| \leqslant \iint |D^j G_\alpha(x-y)\, f(y)\, \phi(x)|\, dy\, d\mu(x)$$
$$\leqslant \left\{ \iint |D^j G_\alpha(x-y)|^{ap}\, |f(y)|^p\, dy\, d\mu(x) \right\}^{1/p} \cdot$$
$$\left\{ \iint |D^j G_\alpha(x-y)|^{(1-a)p'}\, |\phi(x)|^{p'}\, dy\, d\mu(x) \right\}^{1/p'}.$$

We need to estimate the integrals containing the Bessel kernel. We get

$$\int |D^j G_\alpha(x-y)|^{ap}\, d\mu(x) = \int_{|x-y|\leqslant 1/2} + \int_{|x-y|>1/2} = \mathrm{I} + \mathrm{II}.$$

By (7.5) and Lemma 8.1

$$\mathrm{I} \leqslant \int_{|x-y|\leqslant 1/2} \frac{d\mu(x)}{|x-y|^{(n-\alpha+|j|)ap}} \leqslant c \left(\frac{1}{2}\right)^{d-(n-\alpha+|j|)ap} < c,$$

for $\alpha < n+|j|$, if

$$d > (n-\alpha+|j|)ap. \tag{8.13}$$

For $\alpha = n+|j|$ we use (7.5′) and a calculation analogous to the proof of Lemma 8.1 and for $\alpha > n+|j|$ we use (7.5″) to conclude that I is finite. By (7.6) we get

$$\mathrm{II} \leqslant c \int_{|x-y|>1/2} e^{-c|x-y|}\, d\mu(x)$$

which is bounded by a calculation of the kind used to estimate (8.6).

Similarly we obtain that

$$\int |D^j G_\alpha(x-y)|^{(1-a)p'}\, dy \leqslant c,$$

if

$$n > (n-\alpha+|j|)\,(1-a)p'. \tag{8.14}$$

By simplifying (8.12) with these estimates we conclude by the converse of Hölder's inequality that

$$\|D^j u\|_{p,\mu} \leqslant c\, \|f\|_p = c\, \|u\|_{p,\alpha}\,, \quad |j| \leqslant k,$$

if it is possible to choose a, $0 < a < 1$, so that (8.13) and (8.14) hold. Since $|j| \leqslant k$, (8.13) and (8.14) are satisfied if

$$a < \frac{d}{(n-\alpha+k)p} \quad \text{and} \quad a > 1 - \frac{n}{(n-\alpha+k)p'}.$$

Since these conditions are the same as some of the conditions on a which we had in the proof of (8.7), it is possible to choose a, and the proof of Lemma 8.4 is complete.

9. Proof of Theorem 7.1.

Theorem 7.1 is now proved by means of Lemma 8.4 and the theory of interpolation spaces. In this section we follow Peetre [24] who showed how a special case of Theorem 7.1 ($k = 0$, $\alpha < n$, $\|D^j u\|_{p,\mu}$ not included, μ a little more special) can be obtained by means of interpolation theory and an estimate of the type (8.7).

9.1. Let A_0 and A_1 be a couple of Banach spaces continuously embedded in a topological vector space, and B_0 and B_1 another such couple. One introduces certain intermediate spaces

$$A_{\theta p} = (A_0, A_1)_{\theta p}, \quad 0 < \theta < 1, \quad 1 \leqslant p \leqslant \infty,$$

and $B_{\theta p}$ by means of the so called K-method. We refer to [24] and, for a complete treatment, to [12] or [20] for the basic facts on interpolation spaces. As an example we mention, that if we denote by $L^p_\alpha(\mathbf{R}^n)$ the space of Bessel potentials $u = G_\alpha * f$, $f \in L^p(\mathbf{R}^n)$, with norm $\|u\|_{p,\alpha} = \|f\|_p$, then

$$(L^p_{\alpha_0}(\mathbf{R}^n),\ L^p_{\alpha_1}(\mathbf{R}^n))_{\theta q} = B^{p,q}_\alpha(\mathbf{R}^n), \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \quad (9.1)$$

where $B^{p,q}_\alpha(\mathbf{R}^n)$ is the usual Besov space with three indexes [20, Ch. 6]. A basic fact which is used below is that if T is a bounded linear mapping from A_ν to B_ν, for $\nu = 0, 1$, then T is a bounded linear mapping also from $A_{\theta p}$ to $B_{\theta p}$. We also need the following lemma.

LEMMA 9.1 (Peetre [24, Theorem 1.3]). – *Let* $T = \sum_{-\infty}^{\infty} T_i$ *where* $T_i : A_\nu \longrightarrow B_\nu$ *is a bounded linear operator with norm* $M_{i,\nu}$ *such that* $M_{i,\nu} \leqslant c\, \omega^{i(\theta-\nu)}$, $\nu = 0,1$, *i integer, where* ω *is a fixed number,* $\neq 1$, *and* $0 < \theta < 1$. *Then* $T : A_{\theta 1} \longrightarrow B_{\theta\infty}$ *is a bounded linear operator.*

We now turn to the proof of Theorem 7.1 throughout using the notation and assumptions of Theorem 7.1. The more difficult part in the proof of (7.4) is to take care of the terms in (7.3) involving R_j (this is done in 9.2); in fact the following straightforward interpolation will take care of the terms involving $D^j u$. We interpolate by using (8.8) with a fixed p but with α changed to α_ν, $\nu = 0, 1$, where $0 < \alpha_0 < \alpha < \alpha_1 < n+k+1$, $\alpha = (1-\theta)\ \alpha_0 + \theta\alpha_1$, and

$k < \alpha_\nu - (n-d)/p < k+1$. (9.2) We get an inequality analogous to (8.8) for the corresponding intermediate spaces. In the left member the intermediate space is still $L^p(\mu)$ and in the right member we get, by (9.1),

$$(L^p_{\alpha_0}(\mathbf{R}^n),\ L^p_{\alpha_1}(\mathbf{R}^n))_{\theta,p} = B^{p,p}_\alpha(\mathbf{R}^n) = B^p_\alpha(\mathbf{R}^n).$$

This gives

$$\|D^j u\|_{p,\mu} \leqslant c\ \|u\|_{p,\alpha,\mathrm{R}^n} \quad \text{for} \quad |j| \leqslant k. \tag{9.3}$$

9.2. Following Peetre we shall use interpolation in two steps to prove the remaining part of (7.4).

Step 1. – We use Lemma 9.1 with $A_\nu = L^p_{\alpha_\nu}(\mathbf{R}^n)$ where α_ν satisfies (9.2), $\alpha_\nu - n$ not a non-negative integer,

$$B_\nu = L^p\left(\mathbf{R}^n \times \mathbf{R}^n, \frac{d\mu(x)\ d\mu(y)}{|x-y|^d}\right), \quad \nu = 0,\ 1, \text{ and}$$

$$T_i = T_{i,j} \quad \text{where} \quad T_{i,j} = 0 \ \text{ if } \ i \geqslant 0 \ \text{ and, for } \ i < 0,$$

$$(T_{i,j}u)\,(x,y) = \frac{R_j(x,y)}{|x-y|^{\beta-|j|}} \quad \text{if} \quad 2^i \leqslant |x-y| < 2^{i+1}$$

and $(T_{i,j}u)\,(x,y) = 0$ otherwise. According to (8.7)

$$\|T_{i,j}u\|_{B_\nu} \leqslant c\ \|u\|_{A_\nu}\ 2^{i(\alpha_\nu - \frac{n-d}{p} - \beta)} = c\ \|u\|_{A_\nu}\ 2^{i(\alpha_\nu - \alpha)},$$

$\nu = 0, 1$, $|j| \leqslant k$, since $\beta = \alpha - (n-d)/p$. Since $\theta = (\alpha - \alpha_0)/(\alpha_1 - \alpha_0)$, the norm $M_{i,\nu}$ in Lemma 9.1 satisfies

$$M_{i,\nu} \leqslant c\ 2^{i(\alpha_\nu - \alpha)} = c\ \omega^{i(\theta-\nu)}, \quad \nu = 0, 1, \quad \text{if} \quad \omega = 2^{\alpha_0 - \alpha_1}.$$

We can thus use Lemma 9.1 to conclude that

$$\|\sum_i T_{i,j}u\|_{B_{\theta\infty}} \leqslant c\ \|u\|_{A_{\theta 1}}, \quad |j| \leqslant k.$$

But $B_{\theta\infty} = B_0 = B_1$ and by (9.1), $A_{\theta 1} = B^{p,1}_\alpha(\mathbf{R}^n)$. Hence

$$\sum_{|j| \leqslant k} \left\{ \iint_{|x-y|<1} \frac{|R_j(x,y)|^p}{|x-y|^{(\beta-|j|)p+d}}\ d\mu(x)\ d\mu(y) \right\}^{1/p} \leqslant c\ \|u\|_{B^{p,1}_\alpha(\mathbf{R}^n)}. \tag{9.4}$$

Step 2. – We now interpolate by using (9.4) with α changed to α_ν and β to $\beta_\nu = \alpha_\nu - (n-d)/p$, $\nu = 0, 1$, where β_ν shall satisfy the same condition as β in Theorem 7.1, $\alpha_0 < \alpha < \alpha_1$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. We get an inequality analogous to (9.4) for the corresponding intermediate spaces. In the right member we get the intermediate space [20, Ch 6]

$$(\mathrm{B}_{\alpha_0}^{p,1}(\mathbf{R}^n),\ \mathrm{B}_{\alpha_1}^{p,1}(\mathbf{R}^n))_{\theta p} = \mathrm{B}_\alpha^{p,p}(\mathbf{R}^n) = \mathrm{B}_\alpha^p(\mathbf{R}^n).$$

The intermediate space in the left member is obtained by means of the Stein-Weiss interpolation theorem which gives intermediate spaces between L^p-spaces with different weights [20, Ch. 5]. This gives, since $\beta = \beta_0(1-\theta) + \beta_1\theta$, that (9.4) is true with $\mathrm{B}_\alpha^{p,1}(\mathbf{R}^n)$ in the right member changed to $\mathrm{B}_\alpha^p(\mathbf{R}^n)$. If we combine this with (9.3) we see that (7.4) and by that Theorem 7.1 is proved.

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