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## THE POULSEN SIMPLEX

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### 1. Introduction.

In 1961, E.T. Poulsen published a paper in which he constructed a metrizable Choquet Simplex  $S$  in which the extreme points  $\partial_e S$  form a dense set. It is evident that Poulsen's elegant construction has many degrees of freedom and thus it seemed as if there are many different simplices in which the extreme points are dense. Surprisingly it turns out that such a simplex is unique up to an affine homeomorphism. We call this unique simplex the Poulsen simplex.

We discovered the uniqueness of the Poulsen simplex after reading Lusky's paper [8] on the uniqueness of the Gurari space. Our proof of the uniqueness uses the same idea which Lusky used in [8].

Once we know that the Poulsen simplex is unique it is of course of interest to study its properties in some detail. The uniqueness proof presented in section 2 is arranged so that it shows in addition that the Poulsen simplex  $S$  has a strong homogeneity property: Any affine homeomorphism between two proper faces  $F_1$  and  $F_2$  of  $S$  can be extended to an affine automorphism of  $S$ . In section 2 we show also how a slight modification in Poulsen's construction enables us to prove that the Poulsen simplex is universal in the following sense. Any metrizable simplex can be realized as a closed face of  $S$ . It is easily verified that the homogeneity and the universality property combined characterize  $S$  among the metrizable simplices.

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In section 3 we examine the topological nature of  $\partial_e S$ . By using a deep result from infinite-dimensional topology we show that  $\partial_e S$  is homeomorphic to the Hilbert space  $l_2$ . Many topological properties of  $\partial_e S$  can be proved in a simple way directly (without using its homeomorphism with  $l_2$ ); some of these properties are exhibited in the beginning of section 3.

The notion underlying the uniqueness proof of Lusky as well as the uniqueness proof which we present in section 2 is that of a representing matrix of a simplex space and more generally of an arbitrary separable predual of  $L_1(\mu)$ , which was introduced and studied in [7]. In section 2 we do not however use this notion explicitly. This is done in section 4 where we characterize those matrices which represent the Poulsen simplex  $S$  (or more precisely the space  $A(S)$  of affine continuous functions on  $S$ ). We also prove in section 4 a general stability theorem for representing matrices of preduals of  $L_1$ .

The Poulsen simplex is evidently the opposite to the most simple and well behaved simplices, i.e. the simplices whose extreme points form a closed set (these are the probability measures on compact Hausdorff spaces  $K$ , the so called Bauer simplices). We exhibit in section 5 some other aspects in which the Poulsen simplex is the opposite to the Bauer simplices. In particular we give a necessary condition for a matrix to represent a Bauer simplex and compare this result with the result of section 4. It is hoped that by investigating representing matrices it will be possible to classify in a quantitative and meaningful way simplices which are between the two extreme cases (i.e. Bauer simplices on the one hand and the Poulsen simplex on the other hand).

The paper ends with a brief discussion of the Gurari space. We indicate how some of the results proved in sections 2-5 can be modified so as to apply to the case of the Gurari space.

## 2. Uniqueness, homogeneity and universality.

For the proof of the uniqueness and homogeneity of the Poulsen simplex we use the notions of "partition of unity" and "peaked

partition of unity" in  $A(K)^{(1)}$  where  $K$  is a simplex. We say that  $\{e_i\}_{i=1}^n \subset A(K)$  forms a partition of unity if  $e_i \geq 0$  for all  $i$  and  $\sum_{i=1}^n e_i = 1$  (i.e. the function identically equal to 1). And  $\{e_i\}_{i=1}^n$  is a peaked partition of unity if in addition  $\|e_i\| = 1$  for all  $i$ .

LEMMA 2.1. — *Let  $K$  be a metrizable simplex, let  $F$  be a closed face of  $K$ , and let  $\{e_i\}_{i=1}^n \subset A(K)$  be a peaked partition of unity such that each  $e_i$  has a peak point  $(^2)$  in  $F$ .*

*Then there exists an affine continuous projection  $Q$  from  $K$  onto  $F$  so that  $e_i = e_i \circ Q$  for all  $1 \leq i \leq n$ .*

*Proof.* — Let  $s_i \in \partial_e F$  be peak points of  $e_i$ ,  $1 \leq i \leq n$ , and define an affine continuous projection  $P$  from  $K$  into  $F$  by

$$Pk = \sum_{i=1}^n e_i(k) s_i \quad k \in K.$$

Clearly

$$e_i = e_i \circ P \quad 1 \leq i \leq n \tag{1}$$

Let  $E$  denote the completion of the linear span of  $F$ . Since  $F$  is compact it is closed in  $E$ . Define  $\psi : K \rightarrow c(E)$  (= the non-empty closed convex subsets of  $E$ ) by

$$\psi(k) = P^{-1}(Pk) \cap F \quad k \in K.$$

Then  $\psi$  is a convex map. Since  $P$  maps  $K$  onto a finite dimensional face,  $P$  is an open mapping, thus  $\psi$  is lower-semicontinuous. Now  $id/F$  is an affine continuous selection for  $\psi/F$ . From Lazar's selection theorem ([5] cor. 3.4) it follows that there exists an affine continuous selection  $Q$  for  $\psi$  which extends  $id/F$ .

Clearly  $Q$  is a projection. By the definition of  $\psi$  we get that  $P \circ Q = P$ , hence by (1) we get that  $e_i \circ Q = e_i \circ P \circ Q = e_i \circ P = e_i$  for  $1 \leq i \leq n$  and the lemma is proved.  $\square$

LEMMA 2.2. — *Let  $S$  be a metrizable simplex with  $\overline{\partial_e S} = S$ . Let  $F$  be a proper closed face of  $S$ , let  $\{e_i\}_{i=1}^n \subset A(S)$  be a peaked*

<sup>(1)</sup>  $A(K)$  is the linear space of affine continuous functions on  $K$ .

<sup>(2)</sup> i.e.  $e_i$  takes its maximum in  $K$  at a point of  $F$ .

partition of unity, let  $\{f_i\}_{i=1}^{n+1} \subset A(F)$  be a partition of unity, and let  $\{a_i\}_{i=1}^n$  be nonnegatives with  $\sum_{i=1}^n a_i = 1$  so that

$$e_i/F = f_i + a_i f_{n+1}, \quad 1 \leq i \leq n.$$

Then, for each  $\epsilon > 0$  there exists a peaked partition of unity  $\{g_i\}_{i=1}^{n+1} \subset A(S)$  so that

$$g_i/F = f_i, \quad 1 \leq i \leq n+1,$$

and

$$\|e_i - (g_i + a_i g_{n+1})\| < \epsilon, \quad 1 \leq i \leq n.$$

*Proof.* — Let  $\{s_i\}_{i=1}^n \subset \partial_e S$  be peaks of  $\{e_i\}_{i=1}^n$  respectively.

Choose  $n+1$  distinct points  $\{t_i\}_{i=1}^{n+1}$  in  $\partial_e S \setminus F$  as follows: first let  $t_{n+1} \in \partial_e S \setminus F$  be such that

$$|e_i(t_{n+1}) - e_i(\sum_{j=1}^n a_j s_j)| < \epsilon \quad \text{for } 1 \leq i \leq n. \quad (1)$$

This is possible since  $\partial_e S \setminus F$  is dense in  $S$ , and  $\sum_{j=1}^n a_j s_j \in S$ . Next for  $1 \leq j \leq n$  if  $s_j \notin F$  set  $t_j = s_j$  and if  $s_j \in F$  choose  $t_j \in \partial_e S \setminus F$  with

$$|e_i(t_j) - e_i(s_j)| < \epsilon \quad \text{for } 1 \leq i \leq n. \quad (2)$$

Set

$$F' = \text{conv}(F, t_1, \dots, t_{n+1}).$$

Observe that  $\{s_i\}_{i=1}^n \subset F'$ ; hence by Lemma 2.1 there exists a projection  $Q: S \rightarrow F'$  with  $e_i \circ Q = e_i$ ,  $1 \leq i \leq n$ .

Define the  $g'_i$ 's, first on  $F'$  by

$$g_i/F = f_i \quad 1 \leq i \leq n+1$$

and

$$g_i(t_j) = \delta_{i,j} \quad 1 \leq i, j \leq n+1.$$

The  $g'_i$ 's are well defined since  $\{t_j\}_{j=1}^{n+1} \cap F = \emptyset$ . Now extend the  $g'_i$ 's to the whole of  $S$  by setting  $g_i(s) = g_i(Q(s))$ ,  $s \in S$ . Clearly  $\{g_i\}_{i=1}^{n+1}$  is a peaked partition of unity, and  $g_i/F = f_i$ . Since  $e_i \circ Q = e_i$  and  $g_i \circ Q = g_i$  we get that

$$\begin{aligned} \|e_i - (g_i + a_i g_{n+1})\| &= \|e_i - (g_i + a_i g_{n+1})\|_{F'} \\ &= \max_{1 \leq j \leq n+1} |e_i(t_j) - (g_i + a_i g_{n+1})(t_j)| < \epsilon \end{aligned}$$

by (1) for  $j = n + 1$  and by (2) for  $1 \leq j \leq n$ . □

We shall prove now the uniqueness and homogeneity of the Poulsen simplex. As stated in the introduction, the proof is a modification of a technique due to Lusky [8].

**THEOREM 2.3.** — *Let  $S_1$  and  $S_2$  be metrizable simplices with  $\overline{\partial_e S_i} = S_i$  for  $i = 1, 2$ . Let  $F_i$  be a proper closed face of  $S_i$ ,  $i = 1, 2$ , and let  $\varphi$  be an affine homeomorphism which maps  $F_2$  onto  $F_1$ .*

*Then  $\varphi$  can be extended to an affine homeomorphism which maps  $S_2$  onto  $S_1$ .*

*Proof.* — Let  $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$  be sequences of unit vectors whose linear span is dense in  $A(S_1)$  and  $A(S_2)$  respectively.

We shall construct inductively peaked partitions of unity  $\{e_{i,m}^j\}_{i=1}^m \subset A(S_1)$  and  $\{f_{i,m}^j\}_{j=1}^m \subset A(S_2)$   $m = 1, 2, 3, \dots, j \geq m$  with the following properties:

There are non negatives  $\{a_{i,m}\}_{i=1}^m, m = 1, 2, \dots$  with  $\sum_{i=1}^m a_{i,m} = 1$  for each  $m$ , so that

(i)  $e_{i,m}^j = e_{i,m+1}^j + a_{i,m} e_{m+1,m+1}^j$

(i')  $f_{i,m}^j = f_{i,m+1}^j + a_{i,m} f_{m+1,m+1}^j$

for  $1 \leq i \leq m, m = 1, 2, \dots$  and  $j \geq m + 1$

(ii)  $e_{i,m}^j(\varphi(s)) = f_{i,m}^j(s) \quad s \in F_2$

$1 \leq i \leq m, m = 1, 2, \dots, j \geq m$

(iii)  $\|e_{i,m}^j - e_{i,m}^{j+1}\| < 2^{-j}$

(iii')  $\|f_{i,m}^j - f_{i,m}^{j+1}\| < 2^{-j}$

$1 \leq i \leq m, m = 1, 2, \dots, j \geq m$ .

For each  $m \geq 1$  let  $E_m$  (resp.  $F_m$ ) denote the linear span of  $\{e_{i,m}^m\}_{i=1}^m$  (resp.  $\{f_{i,m}^m\}_{i=1}^m$ ). Then

(iv) For each integer  $n > 0$  there exists some  $m > 0$  so that

$$d(x_k, E_m) < 2^{-n} \text{ for all } 1 \leq k \leq n.$$

(iv') For each integer  $n > 0$  there exists some  $m > 0$  so that

$$d(y_k, F_m) < 2^{-n} \text{ for all } 1 \leq k \leq n.$$

Let  $e_{1,1}^1 = 1$  and  $f_{1,1}^1 = 1$ . Assume that  $\{e_{i,k}^j\}_{i=1}^k$  and  $\{f_{i,k}^j\}_{i=1}^k$  have already been constructed for  $k = 1, 2, \dots, m$  and  $k \leq j \leq m$  so that (i) – (iii') holds whenever defined and (iv), (iv') holds for some integer  $n$ .

By [7 Theorem 3.1] there is a subspace  $E_{m+r}$  of  $A(S_1)$  isometric to  $l_\infty^{m+r}$  so that  $E_m \subset E_{m+r}$  and  $d(x_k, E_{m+r}) < 2^{-(n+1)}$  for all  $1 \leq k \leq n + 1$ .

Since the function 1 belongs to  $E_m$ , it follows from the Michael-Pelczynski lemma (cf [7 p. 179 & 186]) that there exist non negative  $\{a_{i,k}\}_{i=1}^k$   $k = m, m + 1, \dots, m + r - 1$  with  $\sum_{i=1}^k a_{i,k} = 1$ , and peaked partitions of unity  $\{e_{i,k}\}_{i=1}^k \subset A(S_1)$  for  $k = m, m + 1, \dots, m + r - 1$  so that

$$e_{i,m}^m = e_{i,m} + a_{i,m} e_{m+1,m+1} \quad 1 \leq i \leq m$$

$$e_{i,k} = e_{i,k+1} + a_{i,k} e_{k+1,k+1} \quad 1 \leq i \leq k, m + 1 \leq k \leq m + r - 1$$

Put  $e_{i,k}^j = e_{i,k}^m$  for  $k \leq m$ , and  $e_{i,k}^j = e_{i,k}$  for

$$k = m + 1, \dots, m + r - 1,$$

for every  $1 \leq i \leq k$  and  $k \leq j \leq m + r$ . Then (i) and (ii) are valid up to  $m + r - 1$ , and by the choice of  $E_{m+r}$  (iv) holds for  $n + 1$ .

We pass now to  $A(S_2)$ .  $\{e_{i,m+1}^{m+1} \circ \varphi\}_{i=1}^{m+1}$  is a partition of unity on  $F_2$ , (not necessarily peaked) and for  $1 \leq i \leq m$  we have

$$f_{i,m}^m / F_2 = e_{i,m+1}^{m+1} \circ \varphi + a_{i,m} e_{i,m+1,m+1}^{m+1} \circ \varphi$$

From Lemma 2.2 it follows that for each  $\epsilon > 0$  there is a peaked partition of unity  $\{f_i\}_{i=1}^{m+1} \subset A(S_2)$  so that

$$f_i(s) = e_{i,m+1}^{m+1}(\varphi(s)) \text{ for } s \in F_2 \quad 1 \leq i \leq m + 1$$

and

$$\|f_{i,m}^m - (f_i + a_{i,m} f_{n+1})\| < \epsilon \text{ for } 1 \leq i \leq m.$$

Hence, if we take  $\epsilon < 2^{-2(m+1)}$ , and define

$$\begin{aligned} f_{i,m+1}^{m+1} &= f_i & 1 \leq i \leq m+1 \\ f_{i,m}^{m+1} &= f_{i,m+1}^{m+1} + a_{i,m} f_{m+1,m+1}^{m+1} \\ &\vdots \\ f_{1,1}^{m+1} &= f_{1,2}^{m+1} + a_{1,1} f_{2,2}^{m+1} \end{aligned}$$

it follows that  $\|f_{i,k}^{m+1} - f_{i,k}^m\| < 2^{-m}$  for  $1 \leq i \leq k$  and  $k = 1, 2, \dots, m$ . In a similar manner we construct the vectors  $f_{i,k}^j$  for  $1 \leq i \leq k$ ,  $k = m+2, \dots, m+r$  and  $k \leq j \leq m+r$  so that (iii') holds for  $j \leq m+r-1$ .

We now interchange the roles of  $A(S_1)$  and  $A(S_2)$ . By [7, Theorem 3.1] we find a subspace  $F_{m+r+s}$  of  $A(S_2)$  so that  $F_{m+r} \subset F_{m+r+s}$ ,  $F_{m+r+s}$  is isometric to  $l_\infty^{m+r+s}$ , and

$$d(y_k, F_{m+r+s}) < 2^{-(n+1)} \quad \text{for } 1 \leq k \leq n+1.$$

We proceed as before, first constructing  $f_{i,k}^j$  and then  $e_{i,k}^j$ . In this way we ensure that (i) – (iii') are still valid for all relevant indices and that (iv') is also valid for  $n+1$ . This completes the inductive construction.

Put now  $e_{i,m} = \lim_{j \rightarrow \infty} e_{i,m}^j$  and  $f_{i,m} = \lim_{j \rightarrow \infty} f_{i,m}^j$ , for  $1 \leq i \leq m$ ,  $m = 1, 2, \dots$ . It is clear from (iv) and (iv') that the linear spans of  $\{e_{i,m}\}_{i=1}^m$  and  $\{f_{i,m}\}_{i=1}^m$  are dense in  $A(S_1)$  and  $A(S_2)$  respectively, and from (i) – (i') and (ii) it follows that

$$\begin{aligned} e_{i,m} &= e_{i,m+1} + a_{i,m} e_{m+1,m+1} \\ f_{i,m} &= f_{i,m+1} + a_{i,m} f_{m+1,m+1} \end{aligned}$$

for  $1 \leq i \leq m$ ,  $m = 1, 2, \dots$ , and

$$e_{i,m}(\varphi(s)) = f_{i,m}(s), \quad s \in F_2, \quad 1 \leq i \leq m, \quad m = 1, 2, \dots$$

It follows that the relations  $Te_{i,m} = f_{i,m}$  define an isometry from  $A(S_1)$  onto  $A(S_2)$ . Since  $T$  maps  $1_{S_1}$  to  $1_{S_2}$ , we get that  $T^*/S_2$  is an affine homeomorphism which maps  $S_2$  onto  $S_1$ . Since  $e(\varphi(s)) = Te(s)$  for all  $s \in F_2$  and  $e \in A(S_1)$ , it follows that  $T^*/S_2$  is an extension of  $\varphi$ .  $\square$



COROLLARY 2.4. — *Let  $F_1, F_2$  be closed faces of the Poulsen simplex  $S$  ( $\emptyset \neq F_i \neq S$   $i = 1, 2$ ) and let  $F'_i$  be the faces complementary to  $F_i$  ( $i = 1, 2$ ). Then  $F'_1$  is affinely homeomorphic to  $F'_2$ .*

*Proof.* — In the following we let  $i = 1, 2$ . Let

$$A_i = \{a \in A(S); a/F_i = 0\}.$$

Then  $A_i$  is an order ideal of  $A(S)$ , hence  $A_i$  is a simplex space. It follows that the set  $K_i = \{p \in A_i^* \mid p \geq 0, \|p\| \leq 1\}$  is a compact simplex. Let  $r_i: S \rightarrow K_i$  be the restriction map. We let  $\hat{x}_i$  be the characteristic function of  $F_i$ . Then by [1, Theorem II.3.7]  $1 - \hat{x}_i$  is an affine l.s.c. map. By Edward's theorem we have for all  $s \in S$

$$(1 - \hat{x}_i)(s) = \sup \{a(s); a \in A(S), a/F_i = 0, 0 \leq a \leq 1 - \hat{x}_i\}.$$

Since  $(1 - \hat{x}_i)^{-1}(1) = F'_i$  we get  $\|r_i(s)\| = 1$  iff  $s \in F'_i$ . Let  $K'_i = \{p \in K_i \mid \|p\| = 1\}$  then  $K'_i$  is the complementary face of  $\{0\}$ . By [1, Theorem II.6.17] it follows that  $r_i/F'_i$  is an affine homeomorphism. But  $r_i(\partial_e S) = \partial_e K_i$ , so  $\partial_e K_i$  is dense on  $K_i$ . Hence by the preceding theorem there is an affine homeomorphism  $\varphi$  which takes  $K_1$  onto  $K_2$  so that  $\varphi(0) = 0$ . Hence the map  $(r_2/F'_2)^{-1} \circ \varphi \circ (r_1/F'_1)$  is an affine homeomorphism which takes  $F'_1$  onto  $F'_2$ .  $\square$

We turn now to the universality property of the Poulsen simplex, showing that any metrizable simplex  $K$  can be realized as a face of it. We prove this by repeating the constructions of Poulsen, the only difference being that instead of starting with an interval as Poulsen did, we start with the given simplex  $K$ . The uniqueness theorem ensures to us that the final result of the construction does not depend on the simplex we started with. This result was essentially proved by Lusky [9] by different arguments.

THEOREM 2.5. — *Let  $S$  be the Poulsen simplex, and let  $K$  be a metrizable simplex. Then there is a face  $F$  of  $S$  which is affinely homeomorphic to  $K$  and a  $w^*$  continuous projection  $P$  of norm 1 on  $A(S)^*$  so that  $PS = F$  and  $(I - P)S$  is affinely homeomorphic to  $S$ .*

(We identify in an obvious way  $S$  with a subset of  $A(S)^*$ ).

*Proof.* — Let  $K$  be a compact simplex in a Banach space  $E$ . Let  $\{e_n\}_{n=1}^\infty$  denote the unit vector basis in  $l_2$ , let  $E_n = \text{span} \{e_i\}_{i=1}^n$  and let  $P_n$  be the obvious projection from  $E \oplus l_2$  onto  $E \oplus E_n$ . We identify  $K$  with the set  $\{(u, 0); u \in K\} \subset E \oplus l_2$ , and  $e_n$  with the vector  $(0, e_n) \in E \oplus l_2$ . We construct inductively a sequence  $\{z_n\}_{n=1}^\infty \subset E \oplus l_2$  and a sequence of compact simplexes  $\{S_n\}_{n=1}^\infty$  so that

$$2.6 \quad S_{n+1} = \text{conv}(\{z_{n+1}\} \cup S_n), \quad n = 0, 1, 2, \dots \quad (K = S_0)$$

$$2.7 \quad S_n \subset E \oplus E_n \quad n = 0, 1, 2, \dots$$

$$2.8 \quad \partial_e S_n \subset \partial_e S_m \quad m \geq n$$

$$2.9 \quad P_n S_m = S_n \quad m \geq n$$

2.10 For each  $\epsilon > 0$  there is an  $n$  such that every point in  $S_n$  has distance at most  $\epsilon$  from  $\partial_e S_n$ .

Assume that  $\{z_j\}_{j=1}^m$  and  $\{S_j\}_{j=1}^m$  have been constructed so that (2.6) – (2.9) hold (for  $n \leq m$ ) and that each point of  $S_m$  has distance  $\leq 2^{-k}$  from  $\partial_e S_m$ . Choose points  $\{y_i\}_{i=1}^l$  in  $S_m$  so that each point of  $S_m$  has a distance  $\leq 2^{-(k+3)}$  from the set  $\{y_i\}_{i=1}^l$ . Put

$$z_{m+i} = y_i + 2^{-(k+3)} e_{m+i} \quad i = 1, \dots, l$$

$$S_{m+i} = \text{conv}(S_m, z_1, z_2, \dots, z_i) \quad i = 1, \dots, l.$$

Then (2.6) – (2.9) hold up to  $m+l$  and (2.10) holds with  $l = 2^{-(k+1)}$ . This completes the inductive step in the construction.

Let  $\hat{S} = \overline{\bigcup_{n=1}^\infty S_n}$ . Then  $\hat{S}$  is compact and convex. Every  $x \in \partial_e S_n$  belongs to  $\partial_e \hat{S}$ . Indeed assume that  $x \pm y \in \hat{S}$ . Then  $x \pm P_m y \in S_m$  for  $m \geq n$  and hence by (2.8) and (2.9)  $P_m y = 0$ . Consequently  $y = 0$ . Hence  $\partial_e \hat{S}$  is dense in  $\hat{S}$ . We show that  $\hat{S}$  is a simplex. Let  $A_n = \{f \circ P_n, f \in A(\hat{S})\}$   $n = 1, 2, \dots$ . Since  $\{f \circ P_n\}_{n=1}^\infty$  converges pointwise to  $f$  for every  $f \in A(\hat{S})$  it follows from the Lebesgue convergence theorem that  $f \circ P_n \rightarrow f$  weakly and hence  $A(\hat{S}) = \overline{\bigcup_{n=1}^\infty A_n}$ . Since  $\bigcup_{n=1}^\infty A_n$  has the Riesz interpolation property  $A(\hat{S})$  has the weak Riesz interpolation property and hence  $\hat{S}$  is a simplex (see [1 Corollary II.3.11]). Consequently  $\hat{S} = S$  the Poulsen simplex. Clearly  $K = S_0$  is a face of  $S$ .

Let  $P$  be the projection from  $E \oplus l_2$  onto  $E$  defined by  $P(u, v) = (u, 0)$ . Then  $PS = K$ . Put

$$A = (I - P^*)A(S) = \{f \in A(S); f/K = 0\}.$$

This is an order ideal in  $A(S)$  and since  $\{x^*; x^* \in A^*, x^* \geq 0, \|x^*\| \leq 1\}$  can be identified with  $(I - P)S$  (cf [1 Corollary II.6.17]) we get that  $(I - P)S$  is a simplex. Since obviously  $\partial_e(I - P)S$  is dense in  $(I - P)S$  we get that  $(I - P)S$  is affinely equivalent to the Poulsen simplex.  $\square$

The homogeneity and universality properties stated in theorems 2.3 and 2.5 also characterize the Poulsen simplex. Actually even a weaker homogeneity property suffices.

**THEOREM 2.11.** — *Let  $S_0$  be a metrizable simplex which contains all the metrizable simplices as faces, and which has the property that for any two faces  $F_1$  and  $F_2$  of  $S_0$  with  $\dim F_1 = \dim F_2 < \infty$  there is an affine homeomorphism  $\tau$  of  $S_0$  onto itself with  $\tau(F_1) = F_2$ . Then  $S_0$  is the Poulsen simplex.*

*Proof.* — By our assumptions,  $S_0$  has a face  $F_0$  which is affinely homeomorphic to the Poulsen simplex  $S$ . Let  $F$  be any finite-dimensional face of  $S_0$ . Again by our assumptions there is an automorphism  $\tau$  of  $S_0$  so that  $\tau(F) \subset F_0$ . Hence  $\tau(F) \subset F_0 = \overline{\partial_e F_0} \subset \overline{\partial_e S_0}$  and therefore also  $F \subset \overline{\partial_e S_0}$ . Since  $S_0$ , as any metrizable simplex, can be represented as  $\bigcup_{n=1}^{\infty} F_n$  where  $\{F_n\}_{n=1}^{\infty}$  is a suitable increasing sequence of finite-dimensional faces we deduce that  $\overline{\partial_e S_0} = S_0$  i.e. that  $S_0$  is the Poulsen simplex.

### 3. Topological identification of the extreme boundary.

From the definition of the Poulsen simplex, and its properties proved in section 2, it is possible to derive some topological properties of its extreme boundary. Let us mention some of them and show how they follow. In this section we denote by  $S$  the Poulsen simplex, and by  $\partial_e S$  its extreme boundary.

1. The interior (relative to  $\partial_e S$ ) of each compact subset of  $\partial_e S$  is empty.

Let  $K \subset \partial_e S$  be compact, and assume that  $\text{int } K \neq \emptyset$ . Hence there exists some open  $\emptyset \neq U \subset S$  so that  $\text{int } K = U \cap \partial_e S$ . Since  $\overline{\partial_e S} = S$ ,  $\text{int } K = U \cap \partial_e S$  is dense in  $U$ . Let  $F = \overline{\text{conv } K}$ , then  $F$  is a face of  $S$  which contains the open subset  $U$  of  $S$ , which is impossible.

2. Every polish space (= separable complete metric space) is homeomorphic to a closed subset of  $\partial_e S$ .

This follows from the result of Haydon [4] that every polish space is the extreme boundary of some metrizable simplex  $K$  and theorem 2.5 of section 2.

3.  $\partial_e S$  is homogeneous. Moreover, for each pair of homeomorphic compact subsets  $K_1$  and  $K_2$  of  $\partial_e S$  there is an auto-homeomorphism of  $\partial_e S$  which maps  $K_1$  onto  $K_2$ .

This follows from theorem 2.3, and the observation that  $F_i = \overline{\text{conv } K_i}$   $i = 1, 2$  are affinely homeomorphic faces of  $S$ .

4.  $\partial_e S$  is arcwise connected by simple arcs.

Let  $s_1, s_2$  be any two points in  $\partial_e S$ . By 2.,  $\partial_e S$  contains some simple arc  $l$  with end points say  $t_1, t_2$ . By 3. there exists an auto-homeomorphism  $h$  of  $\partial_e S$  which carries  $\{t_i\}_{i=1}^2$  onto  $\{s_i\}_{i=1}^2$ . Hence  $h(l)$  is a simple arc in  $\partial_e S$  with endpoints  $s_1$  and  $s_2$ .

All these properties suggest that  $\partial_e S$  is homeomorphic to  $l_2$ . This is indeed the case. It is a simple consequence from a deep result in infinite-dimensional topology.

Let  $Q$  denote the Hilbert cube  $[-1, 1]^{x_0}$ , and let  $P$  denote its pseudo-interior  $P = \{(x_1, x_2, \dots) \in Q : |x_n| < 1 \quad n = 1, 2, \dots\}$ . It is a well known result of Anderson that  $P$  is homeomorphic to  $l_2$  (see e.g. [2]).

**THEOREM 3.1.** — *Let  $S$  be the Poulsen simplex. Then there exists a homeomorphism  $h$  of  $Q$  onto  $S$  which maps  $P$  onto  $\partial_e S$ .*

For the proof of theorem 3.1 we use the terminology and results of [2].

Let  $I^n$  denote the  $n$  cube  $[-1, 1]^n$ , and let  $K \subset I_2$  be an infinite dimensional compact convex set.  $C(I^n, K)$  denotes the space of continuous  $K$  valued functions on  $I^n$  with the compact-open topology.

DEFINITION 3.2. ([2] p. 161). — *A subset B of K is said to be a T set if for each  $n = 1, 2, \dots$  the set  $C_B^n = \{f \in C(I^n, K) : f[I^n] \subset B\}$  is dense in  $C(I^n, K)$ .*

By [2] cor 4.3 p. 161, if B is a  $G_\delta$  T set, and  $B \subset \partial_e K$  then there is a homeomorphism  $h$  of  $Q$  onto  $K$  with  $h[P] = B$ . Since the extreme boundary of a metrizable compact convex set is always a  $G_\delta$ , theorem 3.1 is a consequence of the following lemma:

LEMMA 3.3. — *Let S denote the Poulsen simplex. Then  $\partial_e S$  is a T set.*

*Proof.* — For an integer  $k$  set

$$3.4 \quad G_k = \left\{ x \in S : \text{if } x = \frac{1}{2}(y+z), y, z \in S \text{ then } \|y-z\| < \frac{1}{k} \right\}.$$

(where  $\|\cdot\|$  is the norm in a fixed Banach space containing S).

Clearly  $G_k$  is open in S and

$$3.5 \quad \bigcap_{k=1}^{\infty} G_k = \partial_e S$$

It follows that for each pair  $n, k$  of integers  $C_{G_k}^n$  is open in  $C(I^n, S)$  and that  $C_{\partial_e S}^n = \bigcap_{k=1}^{\infty} C_{G_k}^n$ . Hence, lemma 3.3 will follow from the Baire category theorem and the following lemma.

LEMMA 3.6. — *Let S be the Poulsen simplex, and let k be a positive integer. Then  $G_k$  is a T set in S; i.e.  $C_{G_k}^n$  is dense in  $C(I^n, S)$  for all  $n \geq 1$ .*

To prove lemma 3.6 we shall need the following simple observation.

3.7. If K is a simplex, and X is a compact subset of  $\partial_e K$  with diameter  $X < \frac{1}{k}$  then  $F = \overline{\text{conv } X}$  is contained in  $G_k$ .

Indeed, since K is a simplex, F is a face of K. Let  $x \in F$ . Hence, if  $x = \frac{1}{2}(y+z)$ ,  $y, z \in K$  then  $y, z \in F$ .

Since diameter  $F = \text{diameter } X < \frac{1}{k}$ , it follows that  $\|y-z\| < \frac{1}{k}$  i.e.  $x \in G_k$ .

*Proof of Lemma 3.6.* — Fix  $n$  and  $k$ . Let  $f \in C(I^n, S)$  and  $\epsilon > 0$ . We shall construct a  $g \in C_{G_k}^n$  with  $\|f(t) - g(t)\| < \epsilon$  for all  $t \in I^n$ . We assume, as we clearly may, that  $\epsilon < \frac{1}{k}$ . We realize  $I^n$  as an  $n$ -simplex in  $R^n$  and denote it by  $\Delta_n$ ;  $|\cdot|$  will denote the usual norm in  $R^n$ .

Let  $\delta > 0$  be small enough so that  $\|f(t_1) - f(t_2)\| < \epsilon/3$  for  $|t_1 - t_2| < \delta$ . Let  $L = \{L_i\}_{i=1}^l$  be a simplicial partition of  $\Delta_n$  with diameter  $L_i < \delta$  for all  $1 \leq i \leq l$ .

Denote by  $\{t_j\}_{j=1}^m$  the set of vertices of  $L$ , and let  $\{s_j\}_{j=1}^m$  be distinct points in  $\partial_e S$  with  $\|f(t_j) - s_j\| < \epsilon/3$  for  $1 \leq j \leq m$ .

Set

$$F_i = \text{conv} \{s_j : t_j \in L_i\} \quad 1 \leq i \leq l.$$

and

$$F = \{F_i\}_{i=1}^l.$$

Clearly, since  $s_j \in \partial_e S$ , the complexes  $L$  and  $F$  are combinatorially and topologically equivalent.

We also have  $F_i \subset G_k$  for all  $1 \leq i \leq l$ . Indeed, let  $s_{j_1}, s_{j_2} \in F_i$ ; then

$$\begin{aligned} \|s_{j_1} - s_{j_2}\| &\leq \|s_{j_1} - f(t_{j_1})\| + \|f(t_{j_1}) - f(t_{j_2})\| + \|f(t_{j_2}) - s_{j_2}\| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 < \frac{1}{k}. \end{aligned}$$

since  $t_{j_1}, t_{j_2} \in L_i$  and diameter  $L_i < \delta$ . Hence

$$\text{diameter } \{s_j : t_j \in L_i\} < \frac{1}{k} \text{ and by 3.7 } F_i \subset G_k.$$

Let  $g$  be the canonical mapping of  $L$  onto  $F$ . Then  $g(t_j) = s_j$ , and  $g \in C_{G_k}^n$ .

Let  $t$  be any point in  $\Delta_n$ . Assume  $t \in L_i$  and  $t_j \in L_i$ . Then  $|t - t_j| < \delta$ , and  $\|g(t) - g(t_j)\| < \frac{\epsilon}{3}$ . Hence

$$\begin{aligned} \|f(t) - g(t)\| &\leq \|f(t) - f(t_j)\| + \|f(t_j) - g(t_j)\| + \|g(t_j) - g(t)\| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \leq \epsilon. \end{aligned}$$

and the lemma is proved. □

The fact that  $\partial_e S$  is homeomorphic to  $l_2$  does not characterize  $S$  among the metrizable simplices. Consider for example the tensor product  $S \otimes S$  of  $S$  with itself (in the sense of Lazar [6]). As observed in [6] there is a natural homeomorphism  $\tau$  of the cartesian product  $S \times S$  into a proper subset of  $S \otimes S$ . We have  $\partial_e(S \otimes S) = \tau(\partial_e S \times \partial_e S)$  and thus it is homeomorphic to  $l_2 \times l_2$ , i.e. to  $l_2$ . On the other hand  $\overline{\partial_e(S \otimes S)} = \tau(S \times S) \neq S \otimes S$  and thus  $S \otimes S$  is not the Poulsen simplex.

#### 4. The representing matrices of $A(S)$ .

In this section we characterize the Poulsen simplex  $S$  by a property of the representing matrices of  $A(S)$ . Besides, we prove a general fact concerning a stability property of representing matrices of preduals of  $L_1$ , which has no direct connection with the subject of this article.

We refer the reader to [7] for the definition and properties of a representing matrix of a predual of  $L_1$ .

Let  $A = \{a_{i,n}\} 1 \leq i \leq n \ n = 1, 2, \dots$  be a matrix. For each  $n \geq 1$  and  $1 \leq i \leq n$  we define inductively a sequence  $\{P_{i,n}^l\}_{l=1}^\infty$  of reals as follows:

$$4.1 \quad P_{i,n}^l = \delta_{i,l} \quad \text{for } 1 \leq l \leq n$$

and

$$P_{i,n}^l = \sum_{j=1}^{l-1} a_{j,l-1} P_{i,n}^j \quad \text{for } l = n+1, n+2, \dots$$

Observe that  $P_{i,n}^{n+1} = a_{i,n}$  and that if  $\sum_{i=1}^n a_{i,n} = 1$  for all  $n \geq 1$  then also  $\sum_{i=1}^n P_{i,n}^l = 1$  for all  $l, n \geq 1$ .

It is perhaps worthwhile to give a probabilistic interpretation of  $P_{i,n}^l$ . We associate to  $A = \{a_{i,n}\}$  a Markov process on the positive integers by letting  $a_{i,n}$  be the probability to pass from  $n+1$  to  $i$ ,  $1 \leq i \leq n$ . Then  $P_{i,n}^l$  is the probability to hit first  $i$  among the integers  $1, 2, \dots, n$  if we start from  $l$ .

For  $n, l \geq 1$  let  $P_n^l$  be the vector defined by

$$4.2 \quad P_n^l = (P_{1,n}^l, P_{2,n}^l, \dots, P_{n,n}^l, 0, 0, \dots)$$

Let  $L$  denote the subset of  $l_1$  defined by

$$4.3 \quad L = \{x = (x_1, x_2, x_3, \dots) \in l_1 : \|x\|_1 = 1, x_n \geq 0 \ n = 1, 2, \dots\}$$

THEOREM 4.4. — Let  $A = \{a_{i,n}\}$  be a matrix with  $\sum_{i=1}^n a_{i,n} = 1$  for all  $n \geq 1$ .  $A$  represents  $A(S)$ ,  $S$  the Poulsen simplex, if and only if the vectors  $\{P_n^l\}_{n,l \geq 1}$  are dense in  $L$  (in the  $l_1$  norm).

*Proof.* — Assume that  $A$  represents  $A(S)$  with  $S$  the Poulsen simplex. By the definition of a representing matrix there exists for every  $n$  a peaked partition of unity  $\{e_{i,n}\}_{i=1}^n$  in  $A(S)$  so that

$$4.5 \quad e_{i,n} = e_{i,n+1} + a_{i,n} e_{n+1,n+1} \quad 1 \leq i \leq n < \infty$$

and so that the  $\{e_{i,n}\}_{i=1}^n \infty_{n=1}$  span all of  $A(S)$ .

Set  $x_i = \bigcap_{n=i}^{\infty} \{s : e_{i,n}(s) = 1\}$ . It was proved in [3] that  $x_i$  is a single point, that  $x_i \in \partial_e S$ , that the set  $\{x_l\}_{l=1}^{\infty}$  is dense in  $\partial_e S$ , and that

$$4.6 \quad P_{i,n}^l = e_{i,n}(x_l) \quad i \leq n, \quad l, n = 1, 2, \dots$$

Let  $a = (a_1, a_2, \dots, a_n, 0, 0, \dots)$  be a (finite) element of  $L$ , and let  $\epsilon > 0$ . Pick an element  $s_a \in S$  for which  $e_{i,n}(s_a) = a_i$ ,  $1 \leq i \leq n$ , (e.g. take  $s_a = \sum_{i=1}^n a_i p_i$  where  $p_i$  is a peak of  $e_{i,n}$ ,  $1 \leq i \leq n$ ).

Since  $\overline{\{x_l\}_{l=1}^{\infty}} = \partial_e S = S$  there exists an  $l \geq 1$  such that  $\sum_{i=1}^n |e_{i,n}(s_a) - e_{i,n}(x_l)| < \epsilon$  i.e.  $\|a - P_n^l\|_1 = \sum_{i=1}^n |a_i - P_{i,n}^l| < \epsilon$  and we are done.

Assume conversely that the  $\{P_n^l\}$ 's  $n, l = 1, 2, \dots$  are dense in  $L$ . The metric  $d(s, t) = \sum_{n=1}^{\infty} 2^{-n} \sum_{i=1}^n |e_{i,n}(s) - e_{i,n}(t)|$  induces the topology of  $S$ .



Let  $s \in S$  and  $\epsilon > 0$  be given, and let  $n$  be such that  $\sum_{k=n}^{\infty} k 2^{-k} < \epsilon$ . Set  $a_i = e_{i,n}(s)$ ,  $1 \leq i \leq n$  and

$$a = (a_1, a_2, \dots, a_n, 0, 0, \dots) \in L.$$

Since for every integer  $h$  the vectors  $P_h^l$ ,  $l = 1, 2, \dots$  have all coordinates beyond  $h$  equal to 0 our assumption implies that for every  $h$  the set  $\{P_m^l\} m \geq h, l = 1, 2, \dots$  is dense in  $L$ . In particular there exist an  $m \geq n$  and an  $l$  so that  $\|a - P_m^l\|_1 < \epsilon$ . Let  $s' \in S$  satisfy  $e_{i,m}(s') = a_i$  if  $1 \leq i \leq n$  and  $e_{i,m}(s') = 0$  for  $n < i \leq m$ . It follows from 4.5 that  $e_{i,n}(s') = e_{i,n}(s)$  for  $1 \leq i \leq n$  and hence also  $e_{j,k}(s') = e_{j,k}(s)$  for all  $1 \leq j \leq k \leq n$ . Consequently

$$d(s, s') = \sum_{k=n}^{\infty} 2^{-k} \sum_{i=1}^k |e_{i,k}(s) - e_{i,k}(s')| \leq \sum_{k=n}^{\infty} k \cdot 2^{-k} < \epsilon.$$

By 4.6 and our choice of  $m$  and  $l$

$$\sum_{i=1}^m |e_{i,m}(s') - e_{i,m}(x_l)| = \|a - P_m^l\|_1 < \epsilon.$$

Consequently by 4.5,  $\sum_{i=1}^k |e_{i,k}(s') - e_{i,k}(x_l)| < \epsilon$  for all  $k \leq m$  and hence

$$\begin{aligned} d(s', x_l) &= \sum_{k=1}^m 2^{-k} \sum_{i=1}^k |e_{i,k}(s') - e_{i,k}(x_l)| \\ &+ \sum_{k=1+m}^{\infty} 2^{-k} \sum_{i=1}^k |e_{i,k}(s') - e_{i,k}(x_l)| \\ &< \epsilon \sum_{k=1}^m 2^{-k} + \sum_{k=m+1}^{\infty} k 2^{-k} < 2\epsilon. \end{aligned}$$

It follows that  $d(s, x_l) < 3\epsilon$ . Since  $x_l \in \partial_e S$  we deduce that  $\partial_e S$  is dense in  $S$  i.e.  $S$  is the Poulsen simplex. □

Our next theorem shows, that if two matrices are close in a certain sense defined below, then they both represent the same Banach space.

**THEOREM 4.7.** — *Let  $A = \{a_{i,n}\}$  be a matrix representing a predual  $X$  of  $L_1$ . Let  $B = \{b_{i,n}\}$  be another matrix with  $\sum_{i=1}^n |b_{i,n}| \leq 1$  for every  $n$ , for which  $\sum_{n=1}^{\infty} \sum_{i=1}^m |a_{i,n} - b_{i,n}| < \infty$ . Then  $B$  also represents  $X$ .*

*Proof.* – Set  $\epsilon_n = \sum_{i=1}^n |a_{i,n} - b_{i,n}|$  then  $\sum_{n=1}^{\infty} \epsilon_n < \infty$ . Let  $\{e_{i,n}\}$  be the elements in  $X$  corresponding to  $A$  (i.e. which satisfy 4.5). We construct vectors

$\{e_{i,n}^j\}$   $1 \leq i \leq n, j = n, n+1, n+2, \dots, n = 1, 2, \dots$  so that

(i)  $e_{i,n}^n = e_{i,n}$  for  $1 \leq i \leq n$

(ii)  $e_{i,n}^j = e_{i,n+1}^j + b_{i,n} e_{n+1,n+1}^j$  for  $j \geq n+1$ .

It is easily seen that the  $e_{i,n}^j$ 's are uniquely defined by (i) and (ii). We claim that for all  $n$  and  $j \geq n$   $\sum_{i=1}^n \|e_{i,n}^j - e_{i,n}^{j+1}\| \leq \epsilon_j$ . Indeed, let  $j > 0$  be an integer, for  $j = n$  we get

$$\begin{aligned} \sum_{i=1}^n \|e_{i,n}^n - e_{i,n}^{n+1}\| &= \sum_{i=1}^n \|e_{i,n} - e_{i,n+1}^{n+1} - b_{i,n} e_{n+1,n+1}^{n+1}\| \\ &= \sum_{i=1}^n \|e_{i,n} - e_{i,n+1} - b_{i,n} e_{n+1,n+1}\| \\ &= \sum_{i=1}^n \|e_{i,n+1} + a_{i,n} e_{n+1,n+1} - e_{i,n+1} - b_{i,n} e_{n+1,n+1}\| \\ &= \sum_{i=1}^n |a_{i,n} - b_{i,n}| \cdot \|e_{n+1,n+1}\| = \epsilon_n = \epsilon_j. \end{aligned}$$

Let us see now that if our inequality holds for some  $n \leq j$ , then it holds for  $(n-1)$  too:

$$\begin{aligned} \sum_{i=1}^{n-1} \|e_{i,n-1}^j - e_{i,n-1}^{j+1}\| &= \sum_{i=1}^{n-1} \|e_{i,n}^j + b_{i,n-1} e_{n,n}^j - e_{i,n}^{j+1} - b_{i,n-1} e_{n,n}^{j+1}\| \\ &\leq \sum_{i=1}^{n-1} \|e_{i,n}^j - e_{i,n}^{j+1}\| + \sum_{i=1}^{n-1} b_{i,n-1} \cdot \|e_{n,n}^j - e_{n,n}^{j+1}\| \\ &\leq \sum_{i=1}^n \|e_{i,n}^j - e_{i,n}^{j+1}\| \leq \epsilon_j. \end{aligned}$$

Consequently, for each  $n$ , and  $1 \leq i \leq n$  the sequence  $\{e_{i,n}^j\}_{j=n}^{\infty}$  converges to some element  $f_{i,n}$  of  $X$ .

It is clear that  $f_{i,n} = f_{i,n+1} + b_{i,n} f_{n+1,n+1}$ , and that  $\{f_{i,n}\}_{i=1}^n$  is an admissible basis of  $l_n^{\infty}$ . Hence  $F_n = \text{span} \{f_{i,n}\}_{i=1}^n$  is isometric to  $l_n^{\infty}$ ,  $F_n \subset F_{n+1}$ , and  $\bigcup_{n=1}^{\infty} F_n$  is a predual of  $L_1$  represented by the matrix  $B$ . Therefore, to conclude the proof we have to show that

$X = \overline{\bigcup_{n=1}^{\infty} F_n}$ . Let  $f = \sum_{i=1}^n \lambda_i e_{i,n} \in X$ . Recall that  $\|f\| = \max_{1 \leq i \leq n} |\lambda_i|$ , and that  $e_{i,n} - f_{i,n} = \sum_{j=n}^{\infty} (e_{i,n}^j - e_{i,n}^{j+1})$ . Hence

$$\begin{aligned} \|f - \sum_{i=1}^n \lambda_i f_{i,n}\| &= \left\| \sum_{i=1}^n \lambda_i (e_{i,n} - f_{i,n}) \right\| \leq \|f\| \sum_{i=1}^n \|e_{i,n} - f_{i,n}\| \\ &= \|f\| \sum_{i=1}^n \sum_{j=n}^{\infty} \|e_{i,n}^j - e_{i,n}^{j+1}\| \leq \|f\| \sum_{j=n}^{\infty} \epsilon_j \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

□

**5. Comparison with Bauer simplices.**

In this section we compare the Poulsen simplex with the Bauer simplices i.e. simplices whose set of extreme points is closed. It is evident from the definitions that the Poulsen simplex is in a sense the direct opposite to the Bauer simplices. Naturally this is also reflected by other properties, of the simplices themselves, and of the corresponding spaces of continuous affine functions.

As usual we denote the Poulsen simplex by  $S$  and  $B$  will denote any Bauer simplex.

**DEFINITION 5.1.** ([1] p. 164). – *A simplex  $K$  is said to be prime, if for any two closed faces  $F_1$  and  $F_2$  with  $K = \text{conv}(F_1 \cup F_2)$ , either  $K = F_1$  or  $K = F_2$ .*

Clearly, a Bauer simplex is never prime. But for  $S$  we have:

**PROPOSITION 5.2.** – *The Poulsen simplex is prime.*

*Proof.* – Assume  $S = \text{conv}(F_1 \cup F_2)$  where  $F_i$  are closed faces of  $S$ . Then  $\partial_e S \subset F_1 \cup F_2$ , and hence

$$S = \overline{\partial_e S} \subset F_1 \cup F_2. \text{ Thus } S = F_1 \text{ or } S = F_2. \quad \square$$

**DEFINITION 5.3.** – *An ordered space  $A$  is called an antilattice if  $\max(a, b)$   $a, b \in A$  exists if and only if either  $a \leq b$  or  $b \leq a$ .*

In ([1] Th 7.15) it is proved that a simplex  $K$  is prime if and only if  $A(K)$  is an antilattice. Hence we have:

PROPOSITION 5.4. —  $A(S)$  is an antilattice, while  $A(B)$  is a lattice.

The difference between  $S$  and  $B$  is reflected in the nature of their representing matrices too. In section 4 we have shown that a matrix  $A = \{a_{i,n}\}$  represents  $A(S)$  if and only if the  $P_n^l$ 's are dense in  $L$ . For Bauer simplices we have:

PROPOSITION 5.5. — Let  $A = \{a_{i,n}\}$  be a matrix with  $\sum_{i=1}^n a_{i,n} = 1$  for all  $n$ . If  $A$  represents a  $C(K)$  space then for each integer  $k$

$$\lim_{n \rightarrow \infty} \left( \sum_{i=1}^k P_{i,n}^l - \max_{1 \leq i \leq k} P_{i,n}^l \right) = 0$$

uniformly in  $l$ . (\*)

*Proof.* — Let  $\{e_{i,n}\}_{i=1}^n$  be the elements of  $C(K)$  corresponding to  $A = \{a_{i,n}\}$ , (i.e. satisfying 4.5). Let  $x_i = \bigcap_{n=i}^{\infty} \{x \in K : e_{i,n}(x) = 1\}$   $i = 1, 2, \dots$  and set

$$U_{i,n}^\epsilon = \{x \in K : e_{i,n}(x) > \epsilon\}.$$

It has been proved in ([3] lemma 12) that

$$\lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} \text{diameter } U_{i,n}^\epsilon = 0 \text{ for all } \epsilon > 0.$$

It follows that the sequence  $f_n = \sum_{i=1}^k e_{i,n}$  ( $n \geq k$ ) converges pointwise to  $1_{\{x_i\}_{i=1}^k}$ . Indeed, if  $x \notin \{x_i\}_{i=1}^k$  let  $\delta > 0$  be such that  $d(x, x_i) > \delta$  for  $1 \leq i \leq k$ . Let  $\epsilon > 0$ , and let  $n_0$  be big enough so that  $\max_{1 \leq i \leq k} \text{diameter } U_{i,n}^{\epsilon/k} < \delta$  for  $n \geq n_0$ . Then  $x \notin U_{i,n}^{\epsilon/k}$  for  $n \geq n_0$  i.e.  $e_{i,n}(x) \leq \frac{\epsilon}{k}$  and hence  $f_n(x) \leq \epsilon$ . Clearly  $0 \leq \max_{1 \leq i \leq k} e_{i,n} \leq f_n \leq 1$ , and  $\max_{1 \leq i \leq k} e_{i,n}(x_j) = f_n(x_j) = 1$  for  $1 \leq j \leq k$ . It follows that  $h_n = \max_{1 \leq i \leq k} e_{i,n}$  converges pointwise to  $1_{\{x_i\}_{i=1}^k}$  too. Hence  $g_n = f_n - h_n$  tends to 0 pointwise on  $K$ .

We claim that  $g_n \rightarrow 0$  uniformly on  $K$ . To prove this it is sufficient to show that  $g_n \geq g_{n+1}$ , and then to use Dini's theorem.

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(\*) The class of simplices for which the conclusion of 5.5 holds contains other simplices besides the Bauer simplices. It consists exactly of those simplices  $\sigma$  for which each boundary measure representing a point of  $\overline{\partial_\epsilon \sigma}$  has at most one atom. See [11].

Let  $x \in K$ .

$$g_n(x) - g_{n+1}(x) = \left( \sum_{i=1}^k e_{i,n}(x) - \sum_{i=1}^k e_{i,n+1}(x) \right) \\ - \left( \max_{1 \leq i \leq k} e_{i,n}(x) - \max_{1 \leq i \leq k} e_{i,n+1}(x) \right)$$

Now

$$\sum_{i=1}^k e_{i,n}(x) - \sum_{i=1}^k e_{i,n+1}(x) = \left( \sum_{i=1}^k a_{i,n} \right) e_{n+1,n+1}(x)$$

And if  $\max_{1 \leq i \leq k} e_{i,n}(x) = e_{i_0,n}(x)$  then

$$\max_{1 \leq i \leq k} e_{i,n}(x) - \max_{1 \leq i \leq k} e_{i,n+1}(x) = \max_{1 \leq i \leq k} (e_{i,n+1}(x) + a_{i,n} e_{n+1,n+1}(x)) \\ - \max_{1 \leq i \leq k} e_{i,n+1}(x) = e_{i_0,n+1} + a_{i_0,n} e_{n+1,n+1}(x) - \max_{1 \leq i \leq k} e_{i,n+1}(x) \\ \leq e_{i_0,n+1}(x) + a_{i_0,n} e_{n+1,n+1}(x) - e_{i_0,n+1}(x) = a_{i_0,n} e_{n+1,n+1}(x)$$

Hence

$$g_n(x) - g_{n+1}(x) \geq \left( \sum_{i=1}^k a_{i,n} \right) e_{n+1,n+1}(x) - a_{i_0,n} e_{n+1,n+1}(x) \geq 0.$$

By 4.6  $P'_{i,n} = e_{i,n}(x_l)$ , hence if we put  $x = x_l$  in  $g_n(x)$  we get

$$0 = \lim_{n \rightarrow \infty} g_n(x_l) = \lim_{n \rightarrow \infty} \left( \sum_{i=1}^k e_{i,n}(x_l) - \max_{1 \leq i \leq k} e_{i,n}(x_l) \right) \\ = \left( \sum_{i=1}^k P'_{i,n} - \max_{1 \leq i \leq k} P'_{i,n} \right)$$

uniformly on  $l$ .

Given a triangular matrix  $A = \{a_{i,n}\}$ , let  $P(A)$  denote the subset of  $l_1$  defined by  $P(A) = \overline{\bigcap_{k=1}^{\infty} (\{P'_{n'l=1, n=k}\})}$  (closure in  $l_1$ ).

The following proposition follows from theorem 4.4 and Proposition 5.5.

**PROPOSITION 5.6.** — *Let  $A$  be a non-negative triangular matrix with  $\sum_{i=1}^n a_{i,n} = 1$  for all  $n$ . Then  $A$  represents  $A(S)$  if and only if  $P(A) = L$  while if  $A$  represents  $A(B)$ , then  $P(A)$  is the set of unit vectors of  $l_1$ .*

## 6. Some remarks on the Gurari space.

Most of the results of sections 2-5 have analogues in the setting of the Gurari space.

The Gurari space  $G$  is a separable Banach space with the following property :

For each two finite dimensional Banach spaces  $E \subset F$ , every isometry  $T: E \rightarrow G$ , and every  $\epsilon > 0$  there exist an extension  $\hat{T}$  of  $T$  to  $F$  so that

$$(1 - \epsilon) \|x\| \leq \|\hat{T}x\| \leq (1 + \epsilon) \|x\|, x \in F.$$

Lusky proved recently that the Gurari space is *unique*. The following theorem follows easily from results stated in the works [8], [9] of Lusky, and clarifies that the role played by  $G$  among separable preduals of  $L_1$  is similar to the role of  $A(S)$  where  $S$  is the Poulsen simplex among separable ordered preduals of  $L_1$ .

**THEOREM 6.1.** — *A separable predual of  $L_1$  is the Gurari space if and only if the extreme points are  $w^*$ -dense in the unit ball of its dual.*

The existence of the Gurari space, as well as its universality among separable preduals of  $L_1$  first proved by Wojtaszczyk [12] (cf. also [9]) can be proved in a manner which is very similar to the proof of theorem 2.5.

We state this result on the Gurari space formally and then show the minor changes which have to be made in the proof of 2.5.

**THEOREM 6.2.** — *Every separable Banach space  $X$  whose dual is an  $L_1(\mu)$  space is isometric to a subspace of the Gurari space  $G$  on which there is a projection of norm 1.*

*Proof.* — Let  $E = X^*$  and  $K = B(E)$  the unit ball of  $X^*$ . Proceed with  $E$  and  $L$  as in the proof of 2.5, with “ $L$ -ball” instead of “simplex” in each place, and with  $S_{n+1} = \text{conv}(\pm \{z_{n+1}\} \cup S_n)$  instead of (2.6) there. In the inductive, step one should also put  $S_{m+i} = \text{conv}(S_m, \pm z_1, \pm z_2, \dots, \pm z_i)$  instead of

$$S_{m+i} = \text{conv}(S_m, z_1, \dots, z_i) \text{ there.}$$

Acting in this manner, we end up with an  $L$ -ball  $S \subset E \oplus l_2$ , with  $\partial_e S$  dense in  $S$ . Hence, by theorem 6.1,  $S$  is the unit ball  $B(G^*)$  of the dual  $G^*$  of  $G$ , and  $G$  thus can be realized as the space of continuous affine symmetric functions on  $S$  (and  $X$  as the space of such functions on  $K$ ).

Let  $Q$  be the natural projection of  $E \oplus l_2$  onto  $E$ . Then  $QS = K$ , and  $Tf = f \circ Q$  is a simultaneous extension operator from  $X$  to  $G$ . Thus the restriction operator  $g \rightarrow g/K$  defines a (norm one)-projection of  $G$  onto  $X$ .  $\square$

Concerning the representing matrices we have the following characterization of the Gurari space with proof and notation both as in theorem 4.4.

**THEOREM 6.3.** — *A triangular matrix  $A$  represents the Gurari space if and only if the vectors  $\{\pm P_{n^1 n, l}^1\}_{l \geq 1}$  are norm dense in the unit ball of  $l_1$ .*

As for the topology  $\partial_e B(G^*)$ , the results of section 3 holds:

**THEOREM 6.4.** — *Let  $G$  be the Gurari space. Then there exists a  $w^*$ -homeomorphism  $h$  of the Hilbert cube  $Q$  onto  $B(G^*)$  which maps  $P$  onto  $\partial_e B(G^*)$ .*

The proof is the same as that of theorem 3.1. The only difference is that observation 3.7 has to be stated now in the following form: If  $B$  is the unit ball in a dual  $L_1$  space and  $X$  is a compact subset of  $\partial_e B$  with  $X \cap (-X) = \emptyset$  and  $\text{diam } X < 1/k$  then  $F = \overline{\text{conv}}(X \cup -X)$  is in  $G_k$ . This is true since  $F$  is a face of  $B$ .

In one point however there is a difference between  $G$  and  $A(S)$ ,  $S$  the Poulsen simplex. Recall that a (norm one)-point  $x$  in a Banach space  $X$  is called smooth if there is a unique  $x^* \in X^*$  with  $\|x^*\| = 1$  and  $x^*(x) = 1$ . The set of smooth points is a dense  $G_\delta$  in the unit sphere of any separable space  $X$ .

Lusky [8] proved the following transitivity property of  $G$ : if  $x$  and  $y$  are smooth points in  $G$  then there exists an isometry of  $G$  onto itself which carries  $x$  to  $y$ .

This is false in  $A(S)$ : Let  $f, g \in A(S)$  be such that  $0 \leq f, g \leq 1$ ,  $f^{-1}(1)$  and  $g^{-1}(1)$  are single points, with  $f^{-1}(0)$  a 2 dimensional

face and  $g^{-1}(0)$  a 3 dimensional face. Then  $f$  and  $g$  are smooth points in  $A(S)$  and there is no isometry of  $A(S)$  which maps  $f$  to  $g$ . Indeed suppose  $T$  is such an isometry. Since  $\partial_e S$  is connected, either  $T$  or  $-T$  is positive. Hence  $Tf = g$  would imply that  $T^*/S$  or  $(-T)^*/S$  maps a face of  $S$  onto another face of different dimension, which is impossible.

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