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HOMOGENEOUS SELF DUAL CONES, VERSUS JORDAN ALGEBRAS. THE THEORY REVISITED

by J. BELLISSARD* and B. IOCHUM

Introduction.

The study of ordered linear spaces has a very long history. We know that ordered structures are closely related to measure theory. In fact many Banach lattices are known to be L^p spaces for a suitable Borel measure [12, 35, 36, 40, 49].

On the other hand we know how to extend the integration theory to non commutative algebras by studying the states on C^*-algebras (for instance [20, 46]). Therefore, it is not surprising to find connections between algebras and ordered linear spaces.

Two years ago A. Connes [19] made this relation very precise in the case of von Neumann algebras, using the results of the Tomita-Takesaki theory [53]. Let $\mathfrak{M}$ be a von Neumann algebra on the Hilbert space H, ξ_0 be a cyclic and separating vector for $\mathfrak{M}$, Δ_{ξ_0} the modular operator of the triplet $(\mathfrak{M}, H, \xi_0)$. A Connes [18], H. Araki [8] and U. Haagerup [25] introduced the cone

$$\mathcal{P}^{\natural}_{\xi_0} = \overline{\{\Delta^{1/4}_{\xi_0}\,\mathfrak{M}^+\,\xi_0\}}^{\|\cdot\|} \qquad \text{(See also [58, 59])}$$

and in [19] Connes proved that $\mathcal{P}^{\natural}_{\xi_0}$ is characterized by three properties : self duality, facial homogeneity, and orientability.

A cone H^+ is *self dual* in H if $H^+ = \{\xi \in H / \langle \xi , \eta \rangle \geqslant 0 \; \forall \eta \in H^+\}$. H^+ is *orientable* when the quotient of the Lie algebra of the cone

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by its center, is a complex Lie algebra. H^+ is *facially homogeneous* if for any face F the operator $P_F - P_{F^\perp}$ belongs to the Lie algebra of H^+, P_F being the orthogonal projection on the closed linear space spanned by F.

This last property was very novel, and an interesting question was to characterize facially homogeneous self dual cones in a finite dimensional space. It was proved [1, 11, 24] that this class of cone is exactly the class of transitively homogeneous cones. A finite dimensional self dual cone is *transitively homogeneous* if its group acts transitively in its topological interior ([37, 38, 44, 45, 55, 56]).

Therefore the 15-years old papers of E.B. Vinberg (see [55, 56]) gave a classification of such objects by constructing a one-to-one correspondance between this class of cones and the class of formally real Jordan algebras.

Recall that a commutative (but not associative) real algebra $\mathfrak{M}$ is *Jordan* if the product satisfies $a(a^2b) = a^2(ab)$, $a, b \in \mathfrak{M}$. A Jordan algebra is *formally real* if $\sum_{i=1}^{n} a_i^2 = 0$ implies $a_i = 0$ for alli (see [16, 28]).

The classical representation theorem proved by P. Jordan, J. von Neumann and E. Wigner [33] says that there are five classes of irreducible such algebras: $M_n(\mathbf{R})$, $M_n(\mathbf{C})$, $M_n(\mathbf{H})$, $\mathbf{V}_n$, and M_3^8. Here, $M_n(\mathbf{K})$ is the set of self adjoints $n \times n$ matrices with elements in the field $\mathbf{K}$; $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$ are respectively the real, complex and quaternionic fields. $\mathbf{V}_n$ is the algebra of spin factors, generated by $a\mathbf{1} + b(f)$ with $f \in \mathbf{R}^n$ and $b(f)\,b(g) + b(g)\,b(f) = 2\langle f, g\rangle \mathbf{1}$ [54]. M_3^8 is the exceptional algebra of 3x3 self adjoint matrices with coefficients in the Cayley algebra (see [16, 23, 28]).

The transitively homogeneous self dual cone associated to a given class is then the set of positive elements of the Jordan algebra, with the Hilbert structure given by the natural trace.

The question arises of generalizing these results in the infinite dimensional case. In this direction the work of A. Connes is a precise guide. The paper of E.M. Alfsen, F.W. Shultz and E. Størmer [7] defines and investigates a "good" class of Jordan Banach algebras, whose norm satisfies, for arbitrary a, b:

i) $\|ab\| \leqslant \|a\|\ \|b\|$,

ii) $\|a^2\| = \|a\|^2$

iii) $\|a^2\| \leqslant \|a^2 + b^2\|$

which these authors propose to call JB algebras in analogy with B*-algebras. The analogue of a C*- (W*-) algebra was called by D. Topping [54] a JC (JW)-algebra and is a norm (weakly) closed Jordan algebra of self adjoints operator on a complex Hilbert space. As a consequence [2], M_3^8 is excluded from the class of JC algebras. This special class is in fact very well known [21, 29, 30, 31, 50, 51, 52].

In the work we present here we have restricted ourselves to the simplest case of a JB algebra $\mathfrak{M}$ with a finite faithful normal trace. A trace is defined to be a state φ on $\mathfrak{M}$ such that $\varphi((ab)c) = \varphi(a(bc)), a, b \in \mathfrak{M}$. We characterize the cone associated with positive elements of $\mathfrak{M}$ by three properties: self duality, facial homogeneity, and the existence of a trace vector (see definition 3.1). In fact we expect that the presence of a trace is useless. But for technical reasons, due to the absence of Tomita's theory for JB algebras, we prefered at first stage to assume the existence of a trace vector.

We must indicate that the connection between formally real Jordan algebra with a trace and cones in a infinite dimensional Hilbert space, was already given by G. Janssen in 1971 [29]. Therefore the ideas developed here are already known. However, since it seems to us that facial homogeneity is a very crucial property in the category of cones, we prefer to give a self consistent exposition of the results.

In the first section we recall some elementary facts about self dual cones: faces, group and Lie algebra of the cone, the ideal center introduced by W. Wils [57], the direct integral decomposition theory [14, 42].

In section 2, we give useful information about facially homogeneous self dual cones. In particular we give a detailed analysis of the set of faces. The most important difficulty comes from the fact that the closure of a face is not known to be a face, although this is known to hold for any example constructed. However we show that it is enough to restrict our attention to "completed" faces F such that $F = F^{\perp\perp}$.

Section 3 is devoted to the study of trace vectors. The main result is that a trace vector is an element of $F \oplus F^{\perp}$ for any completed face F.

Section 4 gives a spectral theorem for hermitian elements belonging to the Lie algebra of the cone. This is the main tool used in the sequel. Unfortunatly the existence of a trace is crucial for the proof, for technical reasons. However we believe this theorem to be true in any facially homogeneous self dual cone (it holds for orientable cones).

The techniques used in this theorem have been known for a long time. The spectral theorem can be found in essence in the classical book of F. Riesz and B. Sz. Nagy [43]. It can also be found in H. Freudenthal [22, 39]. The idea of the crucial theorem 4.1 is due to W. Bös [15] and the essential steps in the proof can be found in G. Janssen [29]. The consequences for the Lie algebra of the cone (theorem 4.6) and for the transitive homogeneity (corollary 4.8) are due to the authors, and generalize the techniques previously developed in [11].

Section 5 is devoted to the construction of the JB algebra of a homogeneous self dual cone with a finite trace. We adopt the formalism of [7]. The main original idea of this chapter is to use the property of the trace vector which is cyclic and separating for the hermitian part of the Lie algebra of the cone.

Section 6 proves the converse theorem. Given a monotone closed JB algebra $\mathfrak{M}$ with a faithful finite trace, we construct a self dual cone canonically associated with $\mathfrak{M}$. The main difficulty comes from the characterization of the faces, (necessary in order to get facial homogeneity).

In the last section we give additional information. We prove that any unitary operator leaving the cone invariant is given by a Jordan isomorphism of the associated Jordan algebra. We give also without proof a representation of $\mathfrak{M}$ as a direct integral of JB-factors, in analogy with the von Neumann case. In fact the most useful property comes from the fact that $\mathfrak{M}$ can be represented as a subspace (not a subalgebra) of the hermitian operators on a Hilbert space.

1. Self dual cones.

Let H be a Hilbert space. With F a subset of H, let F^* be *the dual set* of F defined by

$$F^* = \{\xi \in H \,/\, \langle \xi, \eta \rangle \geqslant 0 \quad \forall \eta \in F\} \tag{1.1}$$

F^* is a weakly closed convex cone in H. A subset H^+ is *a self dual cone* if it coincides with its dual. There is a useful characterization of self dual cones:

LEMMA 1.0. – *Let* H^+ *be a closed convex cone in the real Hilbert space* H. *The following are equivalent:*

i) H^+ *is self dual in* H.

ii) *For all* ξ *in* H *there exists a unique decomposition called the Jordan decomposition of* ξ *such that*

$$\xi = \xi^+ - \xi^-, \ \xi^\pm \in H^+, \ \langle \xi^+ | \xi^- \rangle = 0 \tag{1.2}$$

Proof. – i) $\Rightarrow$ ii) : let ξ be in H and ξ^+ be the projection of ξ on H^+. Then by a classical argument ([26], [41]), $\xi^- = \xi^+ - \xi \in (H^+)^* = H^+$ and $\langle \xi^+, \xi^- \rangle = 0$. If $\xi^+ - \xi^-$ and $\eta^+ - \eta^-$ are two decompositions of ξ then

$$\|\xi^+ - \eta^+\|^2 = \langle \xi^+ - \eta^+, \xi^- - \eta^- \rangle = - \langle \xi^+, \eta^- \rangle - \langle \eta^+, \xi^- \rangle \leqslant 0.$$

Hence $\xi^+ = \eta^+$ and $\xi^- = \eta^-$.

ii) $\Rightarrow$ i) : Suppose that $\xi = \xi^+ - \xi^- \in (H^+)^*$ and $\xi \notin H^+$. Then $0 \leqslant \langle \xi^-, \xi \rangle = \langle \xi^-, \xi^+ - \xi^- \rangle = - \|\xi^-\|^2$. Hence $\xi = \xi^+ \in H^+$, a contradiction. Suppose that $\xi \in H^+$ and $\xi \notin (H^+)^*$. If η is the projection of ξ on $(H^+)^*$ by the same argument as above $\eta - \xi \in (H^+)^{**}$ and $\langle \eta - \xi, \eta \rangle = 0$. By Hahn Banach's theorem $(H^+)^{**} = H^+$, then, $\xi = \eta - (\eta - \xi)$ is a decomposition of ξ and by hypothesis $\eta - \xi = 0$ hence a contradiction. □

From now on let H^+ be a self dual cone in the complex Hilbert space H. The following proposition is well known (see for instance [8], [19]).

PROPOSITION 1.1. – *Let* H^J *be the space* $H^+ - H^+$.

i) H^J *is a real Hilbert space and* H^+ *is self dual in* H^J.

ii) $H = H^J \oplus i H^J$ *and the map* $J : \xi_1 + i\xi_2 \longrightarrow \xi_1 - i\xi_2 \quad \xi_n \in H^J$ *is an antiunitary involution in* H.

iii) *For any face* F *of* H^+, *the set*

$$F^\perp = \{\xi \in H^+ \ ; \ \langle \xi , \eta \rangle = 0 \quad \forall \eta \in F\} \tag{1.3}$$

is a weakly closed face of H^+, *called the orthogonal face of* F.

Let $\leqslant$ be the ordering defined by H^+ in H^J. We recall that F is *a face* in the convex cone H^+ if and only if F is a cone and $0 \leqslant \eta \leqslant \xi$, $\xi \in F$ implies $\eta \in F$. Such a set satisfies $F = (F - H^+) \cap H^+$. For F a face, let P_F be the orthogonal projection on the closed subspace spanned by F. Clearly since H^J is closed, P_F commutes with J. Therefore P_F can be restricted to H^J.

LEMMA 1.2. – *Let* F *be a face. Then*

a) $F^{\perp\perp} = (\overline{F - H^+}) \cap H^+$.

b) *The following are equivalent:*

i) $\eta \in F^\perp$

ii) $\eta \in H^+$ *and* $P_F \eta = 0$

iii) $\eta \in H^+$ *and* $P_{F^\perp} \eta = \eta$

Proof. – a) By definition $(F - H^+)^* = - F^\perp$ and if $\circ$ denotes the polar then

$$\overline{F - H^+} = (F - H^+)^{\circ\circ} = (F - H^+)^{**} = (-F^\perp)^* = \{\xi \in H \ ; \ \langle \xi , F^\perp \rangle \leqslant 0\}$$

Therefore $$(\overline{F - H^+}) \cap H^+ = F^{\perp\perp}.$$

b) i $\Rightarrow$ ii): If $\xi \in H^J$, $P_F \xi \in \overline{F - F}$. Therefore there is a sequence $(\xi_n)_n$ in F-F, converging to $P_F \xi$. Since $\eta \in F^\perp$

$$\langle P_F \xi , \eta \rangle = \lim_n \langle \xi_n , \eta \rangle = 0$$

Because ξ is arbitrary, $P_F \eta = 0$.

ii) $\Rightarrow$ i): If $\xi \in F \quad \langle \eta , \xi \rangle = \langle \eta , P_F \xi \rangle = \langle P_F \eta , \xi \rangle = 0$

i) $\Rightarrow$ iii): immediate.

iii) $\Rightarrow$ i): If $\xi \in F$ then $\xi \in F^{\perp\perp}$. Using the equivalence of

i) and ii) we have $P_{F^\perp}\xi = 0$. Therefore

$$\langle \xi, \eta \rangle = \langle \xi, (P_{F^\perp}\eta\rangle = 0 \quad \text{and} \quad \eta \in F^\perp \qquad \square$$

Corollary 1.3. – *For any face* F *in* H^+,

$$F^\perp = P_{F^\perp} H^+ \cap H^+. \tag{1.4}$$

Definition 1.4.

i) *If* A *is a subset of* H^+, *the smallest face containing* A *is denoted by* $\langle A \rangle$.

ii) $\xi \in H^+$ *is a quasi interior point if* $\langle \xi \rangle^\perp = 0$.

iii) $\xi \in H^+$ *is a weak unit order if* $\overline{\langle \xi \rangle} = H^+$.

Remarks. – The definition i) is meaningful because any intersection of faces is a face.

– It is clear that any weak order unit is a quasi interior point. It is not known if the converse is true at least in self dual cones. However we have that:

– The existence of a quasi interior point in H^+ is equivalent to H^+ is of denumerable type (see [19] def. 5.6).

– In a finite dimensional Hilbert space, a quasi interior point is a weak order unit, (and also an order unit, or an interior point).

Proposition 1.5. – *If* H *is separable, the set of weak order units is dense in* H^+.

Proof. – Since H is a separable metric space, H^+ is also a separable and metric subset. Let thus $(\xi_n)_{n\in\mathbf{N}}$ be a dense countable subset of the unit ball in H^+. Then

$$\xi = \sum_n 2^{-n}\, \xi_n \in H^+$$

and

$$0 \leqslant \xi_n \leqslant 2^n\, \xi, \forall n \in \mathbf{N}$$

Therefore $\{\xi_n\}_n \subset \langle \xi \rangle$ and $\langle \xi \rangle$ is dense.

Now let ξ be a weak order unit in H^+, and for $n \in \mathbf{N}^*$ put

$$\left[\frac{1}{n}\,\xi, n\xi\right] = \left\{\eta \in H^+ / \frac{1}{n}\,\xi \leqslant \eta \leqslant n\xi\right\} \tag{1.5}$$

Any element η in this order interval is also a weak order unit, since $\langle\eta\rangle = \langle\xi\rangle$. Therefore the set

$$Y = \bigcup_n \left[\frac{1}{n}\,\xi\,,\,n\xi\right]$$

contains only weak order units, and is dense in H^+, because $\langle\xi\rangle$ is dense in H^+ and for any $\eta \in \langle\xi\rangle$, $\eta_n = \eta + \frac{1}{n}\xi \in Y$ and $\eta_n \xrightarrow[n]{} \eta$. □

Remark. – There exist non separable self dual cones in which there is an order unit. Indeed, choose $H^J = \mathbf{R} \oplus h$ where h is a non separable real Hilbert space, and $H^+ = \{(\xi_0\,,\xi) \in H^J / \xi_0 \geqslant \|\xi\|\}$. Then, $(1,0) \in H^J$ is an order unit of H^+.

However, any maximal family of mutually orthogonal vectors of H^+ has only two elements. Therefore H^+ is of "denumerable type" ([19]).

LEMMA 1.6 ([13]).– *Let* ξ *be a quasi interior point in* H^+. *Then the set* $\phi_\xi = \{(P_F - P_{F^\perp})\,\xi\ ;\ F$ *a face of* $H^+\}$ *is total in* H^J.

Proof. – Let η be a vector in H^J orthogonal to ϕ_ξ and let $\eta = \eta^+ - \eta^-$ be its decomposition. If $F = \langle\eta^+\rangle$, then by the lemma 1.2

$$(P_F - P_{F^\perp})\,\eta = \eta^+ + \eta^- \in H^+$$

and
$$0 = \langle\eta\,,(P_F - P_{F^\perp})\,\xi\rangle = \langle\eta^+ + \eta^-\,,\xi\rangle$$

Since $\langle\xi\rangle^\perp = 0$, $\eta^+ = \eta^- = 0$ and $\eta = 0$. □

If there exist non trivial closed subcones K,L of H^+ such that $H^+ = K \oplus L$, then it is easy to see that K and L are faces satisfying $K^\perp = L$. We say that H^+ is *decomposable* (resp. *indecomposable*) if there exists (does not exist) a face $F \neq \{0\}, H^+$, such that $H^+ = F \oplus F^\perp$. If such a face exists we call it a *splitface* of H^+ ([4]).

The set of bounded operators $\mathcal{L}(H)$ leaving H^+ invariant is denoted by $\mathcal{L}(H^+)$.

For $A \in \mathcal{L}(H^+)$, $A^* \in \mathcal{L}(H^+)$ and if F is a face, $A^{-1}(F)$ is also a face.

LEMMA 1.7. – *Let* P *be an orthogonal projection in* $\mathcal{L}(H^+)$. *Then* P *commutes with* J *and* PH^+ *is a self dual cone in* PH. *Moreover*

$$PH^+ = PH \cap H^+ \tag{1.6}$$

Proof. – Immediate. □

We define $GL(H^+)$ to be the group of bounded invertible operators A on H, such that A and A^{-1} are elements of $\mathcal{L}(H^+)$ and $\mathcal{U}(H^+)$ to be the subgroup of $GL(H^+)$ whose elements are unitary operators.

PROPOSITION 1.8. – *Let* $U \in \mathcal{U}(H^+)$ *be such that* $U\langle\xi\rangle \subset \langle\xi\rangle$, $\xi \in H^+$. *Then* $U = \mathbf{1}$

Proof. – (see [19] lemme 5.4). □

Let $\mathcal{D}(H^+) = \{\delta \in \mathcal{L}(H)/e^{t\delta} \in GL(H^+), \forall t \in \mathbf{R}\}$. The elements of $\mathcal{D}(H^+)$ are called *derivations* of H^+. The following characterization of $\mathcal{D}(H^+)$ can be found in [19]; although the proof is made for facially homogeneous cone, it works in any self dual cone. (see [17] and also [47]).

PROPOSITION 1.9.

i) $\mathcal{D}(H^+)$ *is a weakly closed Lie algebra in* $\mathcal{L}(H)$.

ii) $\delta \in \mathcal{D}(H^+)$ *if and only if*

$$\langle \xi, \eta \rangle = 0 \quad \xi, \eta \in H^+, \text{ implies } \langle \delta\xi, \eta \rangle = 0 \tag{1.7}$$

The following definition is needed (see [5, 57]).

DEFINITION 1.10. – *The ideal-center* Z_{H^+} *of* (H, H^+) *is the set of bounded operators* T *such that*

$$\exists \alpha_T \geqslant 0, \ -\alpha_T \xi \leqslant T\xi \leqslant \alpha_T \xi \quad \forall \xi \in H^+ \tag{1.8}$$

We have then the following results ([13], [57]) where [,] denote the commutator.

THEOREM 1.11. – *For an orthogonal projection* P *in* $\mathcal{L}(H)$, *the following are equivalent :*

i) $P \in Z_{H^+}$.

ii) $PH^+ \subset H^+$ *and* $(1-P)\, H^+ \subset H^+$.

iii) $[P, J] = 0$ *and* $[P, P_{\langle \xi \rangle}] = 0 \quad \forall \xi \in H^+$.

iv) $P \in \mathcal{D}(H^+)$.

v) $P \in$ *Center of* $\mathcal{D}(H^+)$.

vi) $F = PH^+$ *is a split face.*

vii) P *is extremal in* $Z^+_{H^+} = \{T \in Z_{H^+} / \, 0 \leqslant T\xi \leqslant \xi \,, \quad \xi \in H^+\}$.

COROLLARY 1.12 ([13, 42]).

i) $T \in Z_{H^+}$ *implies* $T = T^*$.

ii) $T \in Z_{H^+}$ *if and only if any spectral projection of* T *is in* Z_{H^+}.

iii) $Z_{H^+} = \{P_{\langle \xi \rangle} \,, \, \xi \in H^+\}' \cap \{J\}'$ *where* ′ *denotes the commutant. In particular* $Z_{H^+} \subset \mathcal{D}(H^+)$.

iv) Z_{H^+} *is the real part of an abelian von Neumann algebra.*

v) H^+ *is indecomposable if and only if* $Z_{H^+} = \mathbf{R}1$.

vi) H^J *is a lattice (for the ordering defined by* H^+*) if and only if* Z_{H^+} *is maximal abelian.*

Associated with the abelian von Neumann algebra generated by Z_{H^+} there are direct integral decompositions of H, and also, of H^+ (see [42] for the definition).

THEOREM 1.13 ([42]). – *Let* H *be a separable Hilbert space,* H^+ *be a self dual cone in* H. *Then there exists a standard Borel space* $\mathfrak{Z}$, *a Borel positive measure* ν *on* $\mathfrak{Z}$, ν*-integrable fields* $H(\zeta)$ *of Hilbert spaces,* $H^+(\zeta)$ *of seld dual indecomposable cones,* $J(\zeta)$ *of antiunitary involutions and an isomorphism* α *of Hilbert spaces such that:*

$$\text{i)}\quad \alpha(H) = \int_{\mathfrak{Z}}^{\oplus} H(\zeta)\, d\nu(\zeta)$$

$$\text{ii)}\quad \alpha(H^+) = \int_{\mathfrak{Z}}^{\oplus} H^+(\zeta)\, d\nu(\zeta) \tag{1.9}$$

$$\text{iii)}\quad \alpha J \alpha^{-1} = \int_{\mathfrak{Z}}^{\oplus} J(\zeta)\, d\nu(\zeta)$$

iv) $\alpha Z_{H^+} \alpha^{-1}$ *is the multiplicative algebra* $L^\infty_{\text{real}}(\mathfrak{Z}, \nu)$

COROLLARY 1.14 ([40]). – *If* H^+ *defines a separable lattice ordering then* H^+ *is isomorphic to* $L^2_+(\mathfrak{Z}, \nu)$ *for a suitable standard Borel space* $\mathfrak{Z}$, *and Borel measure* ν.

In the sequel we will need only transformations of H^+ which commute with Z_{H^+}. Therefore we call *symmetry* any element of $\mathfrak{U}(H^+)$ commuting with Z_{H^+}, and the set of symmetries is denoted by $S(H^+)$. In the same way, we denote by $GL_0(H^+)$ the subgroup of elements of $GL(H^+)$ commuting with Z_{H^+}.

2. Homogeneous self dual cones.

For simplicity, H will be a separable Hilbert space in what follows. In [19], A. Connes introduced the following definition.

DEFINITION 2.1. – *Let* H *be a Hilbert space and* H^+ *be a self dual cone in* H. H^+ *is called facially homogeneous if for any face* F, *the operator*

$$N_F = P_F - P_{F^\perp} \tag{2.1}$$

is a derivation of H^+.

Note that a self dual cone H^+, such that H^J is a lattice, is facially homogeneous by corollary 1.12. So all $L^2_+(\mathfrak{Z}, \nu)$ are facially homogeneous. In the finite dimensional case, a self dual cone is facially homogeneous if and only if it is homogeneous in the ordinary sense [11] (see [55, 56] and our introduction for the definition of homogeneity). For this reason we will in the sequel write *homogeneous* for *facially homogeneous*.

LEMMA 2.2. – *Let* H^+ *be a homogeneous self dual cone, and* F *be a face. Then :*

i) $P_F H^+ \subset H^+$

ii) $F^\perp = P_{F^\perp} H^+$

iii) $P_F = P_{F^{\perp\perp}}$ *and* $F^{\perp\perp} = P_F H^+$

Proof. – i) [1] $e^{t(N_F - 1)} \in \mathfrak{L}(H^+)$ for all $t \in \mathbf{R}$ and

$$P_F = s - \lim_{t \to \infty} e^{t(N_F - 1)} \in \mathfrak{L}(H^+) \quad (\mathfrak{L}(H^+) \text{ is weakly closed}).$$

ii) follows from the corollary 1.3.

iii) [13] $P = P_{F^{\perp\perp}} - P_F = N_{F^{\perp\perp}} - N_F \in \mathcal{D}(H^+)$ and P is a projector. Therefore $PH^+ \subset H^+$ and if $\xi \in H^+$, $\xi = P\xi$, $P_F\xi = 0$ and $P_{F^\perp}\xi = 0$ thus $\xi = 0$ (Lemma 1.2). Since H^+ is generating, $P = 0$. □

Remarks. – As far as we are concerned with facial projections the previous result allows us to restrict ourselves to the faces F such that $F = F^{\perp\perp}$. We called them *completed faces* and we denoted by $\mathfrak{F}(H^+)$ the set of such faces. Therefore one has $F \in \mathfrak{F}(H^+)$ if and only if

$$P_F H^+ = F \tag{2.2}$$

Clearly a completed face is closed.

It is not known whether every closed face is complete in a homogeneous self dual cone but in the finite case it is known that :

A face F satisfies $F = F^{\perp\perp}$ if and only if the natural order induced on $H/_{F-F} = \{x + F - F/x \in H\}$ is archimedean. There are conter-examples in three dimensions (See [32]).

The previous result shows that if ξ is a quasi interior point in H^+, then $\langle\xi\rangle$ is a total set in H.

PROPOSITION 2.3. – *Let* H^+ *be a homogeneous self dual cone. Then either* $\dim H = 1$ *or* $\mathfrak{F}(H^+)$ *is not reduced to* $\{0\}$ *and* H^+.

Proof. – Let ξ be in H^J and $\xi = \xi^+ - \xi^-$ be its Jordan decomposition. If $\mathfrak{F}(H^+)$ is trivial then either $\langle\xi^+\rangle^{\perp\perp}$ is $\{0\}$ or it is H^+. Therefore, either $\xi^+ = 0$ or $\xi^- = 0$, and consequently the order in H^J is total.

Let now ξ_1 and ξ_2 be two linearly independent vectors in H^J. Then, without loss of generality we can choose $\xi_1 \geqslant \xi_2 \geqslant 0$ and $\|\xi_1\| = \|\xi_2\|$, since the order is total. Therefore

$$0 = \|\xi_1\|^2 - \|\xi_2\|^2 = \langle \xi_1 - \xi_2 , \xi_1 + \xi_2 \rangle$$

and $\xi_1 - \xi_2 \in \langle \xi_1 + \xi_2 \rangle^\perp$. On the other hand $0 \leqslant \xi_1 - \xi_2 \leqslant \xi_1 + \xi_2$ This implies $\xi_1 = \xi_2$ which contradict our hypothesis. Therefore $\dim_{\mathbf{C}} H = 1$. □

LEMMA 2.4 ([27]). – *Let* H^+ *be a homogeneous self dual cone. If* $F \in \mathfrak{F}(H^+)$ *then* F *is a self dual homogeneous cone in* $P_F H$.

Proof. – $F = P_F H^+ \subset H^+$ proves that F is self dual (Lemma 1.7). In order to prove the homogeneity we need the following lemma :

LEMMA 2.5. – *Let* H^+ *be as above. Let* F *and* G *be faces of* H^+, *such that* $[P_F, P_G] = 0$. *Then* $[P_F, P_{G^\perp}] = 0$.

Proof. – If $\xi \in P_F P_{G^\perp} H^+$ then $P_G \xi = 0$ by hypothesis. Since H^+ is homogeneous the lemma 2.2 shows that ξ is also in H^+. Therefore (lemma 1.2) $\xi = P_{G^\perp}\xi$. H^+ being generating:

$$P_F P_{G^\perp} = P_{G^\perp} P_F P_{G^\perp} \quad \text{and} \quad P_F P_{G^\perp} = P_{G^\perp} P_F. \qquad \square$$

Proof of the Lemma 2.4 (end). – Let G be a face in F. Then G is also a face in H^+, and $N_G \in \mathcal{D}(H^+)$; moreover P_G commutes with P_F. Therefore by lemma 2.5, N_G commutes with P_F, and

$$N_G (F - F) \subset F - F, \quad e^{tN_G} F \subset F \quad \forall t \in \mathbf{R}$$

(Note that $F - F$ is closed because F is self dual in $P_F H$).

In particular $N_G P_F = N_{G/F} \in \mathcal{D}(F)$. Now let $G^\perp/F$ be the orthogonal face of G in the cone F. We have that

$$G^\perp/F = G^\perp \cap F$$

because $\xi \in G^\perp/F$ implies $\xi \in F$ and $\langle \xi, G \rangle = 0$. Since $G^\perp$ and F are completed faces, then $G^\perp$ and F are self dual cones in the closed subspaces they span. Thus (use Proposition 1.0):

$$G^\perp \cap F - G^\perp \cap F = (G^\perp - G^\perp) \cap (F - F)$$

and consequently, $P_{G^\perp/F} = P_{G^\perp \cap F} = P_{G^\perp} \wedge P_F = P_{G^\perp} P_F$
which proves that F is homogeneous because

$$N_{G/F} = N_G P_F = P_F - P_{G^\perp/F}. \qquad \square$$

COROLLARY 2.6. – *Let* H^+ *be as above. If* H *is separable, for any face* $F \in \mathfrak{F}(H^+)$ *there exists* $\xi \in H^+$ *such that* $\langle \xi \rangle = F$.

Proof. – Apply the Proposition 1.5 to F. □

LEMMA 2.7. – *Let* H^+ *be a homogeneous self dual cone and* $\{F_\alpha\}_\alpha$ *be any family of completed faces in* H^+. *Then* $F = \bigcap_\alpha F_\alpha$ *is also a completed face.*

Proof. – Clearly F is a closed face, and $F \subset F^{\perp\perp}$. On the other hand, H^+ being homogeneous, $P_F = P_{F^{\perp\perp}}$. Therefore

$$P_{F_\alpha} P_{F^{\perp\perp}} = P_{F^{\perp\perp}} \qquad \forall \alpha$$

and $\xi \in F^{\perp\perp}$ implies $P_{F_\alpha} \xi = \xi$, hence $\xi \in F_\alpha^{\perp\perp} = F_\alpha, \forall \alpha$. Thus: $F^{\perp\perp} \subset \bigcap_\alpha F_\alpha = F$. □

DEFINITION 2.8. – *Let* H^+ *be a homogeneous cone and* $\{F_\alpha\}$ *be any family of completed faces in* H^+. *Then* $\bigwedge_\alpha F_\alpha$ *is defined to be* $\bigcap_\alpha F_\alpha$, *and* $\bigvee_\alpha F_\alpha$ *to be the smallest completed face containing all the* F_α.

LEMMA 2.9. – *Let* H^+ *be as above. For any family of completed faces* $\{F_\alpha\}_\alpha$ *in* H^+ *then*

$$\bigvee_\alpha (F_\alpha^\perp) = (\bigwedge_\alpha F_\alpha)^\perp \tag{2.3}$$

In particular $F \vee F^\perp = H^+$.

Proof. – Clearly

$$\langle \bigcup_\alpha F_\alpha^\perp \rangle = \{\eta \in H^+ / \exists \xi \in \mathrm{Conv}(\bigcup_\alpha F_\alpha^\perp),\ 0 \leqslant \eta \leqslant \xi\}$$

Therefore $$\bigvee_\alpha F_\alpha^\perp = \langle \bigcup_\alpha F_\alpha^\perp \rangle^{\perp\perp}.$$

Now $\xi \in \langle \bigcup_\alpha F_\alpha^\perp \rangle^\perp$ is equivalent to: $\xi \in H^+$ and $\langle \xi, \xi_\alpha \rangle = 0$ $\forall \alpha$, $\forall \xi_\alpha \in F_\alpha^\perp$ and to: $\xi \in F_\alpha^{\perp\perp} = F_\alpha$ $\forall \alpha$.

Therefore $$\langle \bigcup F_\alpha^\perp \rangle^\perp = \bigwedge_\alpha F_\alpha .$$ □

The following is a generalisation of [9]. Theorem 4.1. for homogeneous self dual cones.

COROLLARY 2.10. – *The set* $\mathfrak{F}(H^+)$, *ordered by inclusion, and with the operations* $\wedge, \vee, \perp$ *is an orthocomplemented lattice. This lattice is distributive if and only if* H^J *is a lattice.*

Proof. – Clearly $\mathfrak{F}(H^+)$ is an orthocomplemented lattice. Suppose that H^J is a lattice, then the algebra generated by $(P_F)_{F\in \mathfrak{F}(H^+)}$ is abelian by corollary 1.12. Let $F, G \in \mathfrak{F}(H^+)$ then $F = F\wedge G \oplus F\wedge G^\perp$ because all faces are split faces. Thus $F + G = F\wedge G \oplus F\wedge G^\perp \oplus F^\perp \wedge G$ and $P_{F+G} = P_F P_G + P_F P_{G^\perp} + P_{F^\perp} P_G = P_F + P_G - P_F P_G$. So $\mathbf{1} - P_{F+G} = (\mathbf{1} - P_F)(\mathbf{1} - P_G) = P_{F^\perp \wedge G^\perp}$ and $F \vee G = (F^\perp \wedge G^\perp)^\perp = F + G$. The application $F \longrightarrow P_F$ is an isomorphism between $\mathfrak{F}(H^+)$ and the projectors of Z_{H^+} which are distributive lattices.

Suppose H^J is not a lattice, then there exists a face F in $\mathfrak{F}(H^+)$ such that $F \oplus F^\perp \neq H^+$. Let $\xi \in H^+$ and $\xi \notin F \oplus F^\perp$, $(\mathbf{1} - N_F^2)\,\xi = \xi^+ - \xi^-$ the Jordan decomposition of $(\mathbf{1} - N_F^2)\,\xi$ (cf. 2.1) and $G = \langle \xi^+ \rangle^{\perp\perp}$. If $\eta \in F \wedge G$ then $P_F \eta = \eta$ and $\langle \eta, \xi^- \rangle = 0$. Hence $\langle \eta, \xi^+ \rangle = \langle \eta, (\mathbf{1} - N_F^2)\,\xi \rangle = 0$ and $\eta \in \langle \xi^+ \rangle^\perp \cap \langle \xi^+ \rangle^{\perp\perp} = \{0\}$. In the same way $F \wedge G^\perp = \{0\}$. Thus $(F\wedge G) \vee (F \wedge G^\perp) = \{0\}$ and $F \wedge (G \vee G^\perp) = F \wedge H^+ = F$ so $\mathfrak{F}(H^+)$ is not distributive. □

PROPOSITION 2.11. – *Let* H^+ *be a homogeneous self dual cone.* H^+ *is no lattice if and only if there exists two non trivial complemented faces* F *and* G *such that* $[P_F, P_G] \neq 0$.

Proof. – Using theorem 1.11 and corollary 1.12, if H is not a lattice, we can find $\xi \in H^+$ such that $P_{\langle \xi \rangle} \notin Z_{H^+}$. Since H^+ is homogeneous, $P_{\langle \xi \rangle^{\perp\perp}} = P_{\langle \xi \rangle} \notin Z_{H^+}$ and therefore $F = \langle \xi \rangle^{\perp\perp} \in \mathfrak{F}(H^+)$, $H^+ \neq F \oplus F^\perp$. Let η be a vector in H^+ such that $\eta \notin F \oplus F^\perp$. That means: $N_F^2 \eta = (P_F + P_{F^\perp})\,\eta \neq \eta$.
Let $\eta^+ - \eta^-$ be the Jordan decomposition of $(\mathbf{1} - N_F^2)\,\eta$, and G be the face $\langle \eta^+ \rangle^{\perp\perp}$. Then G is completed and P_G does not commute with F; for, in the other case we would have (lemma 2.5)

$$0 = N_G\, N_F^2\, (\mathbf{1} - N_F^2)\eta = N_F^2\, N_G\,(\eta^+ - \eta^-) = N_F^2(\eta^+ + \eta^-)$$

and lemma 2.2 implies $\eta^+ = \eta^- = 0$.

If H^J is a lattice, then corollary 1.12 says that any facial projection is in Z_{H^+} and any two of them do commute. □

PROPOSITION 2.12. – *Let* $\mathfrak{Z}$ *be a standard Borel space,* ν *be a Borel positive measure on* $\mathfrak{Z}$ *and* $\zeta \longrightarrow H^+(\zeta)$ *be an integrable family of self dual cones. Then* $H^+ = \int_{\mathfrak{Z}} H^+(\zeta)\, d\nu(\zeta)$ *is homogeneous if and only if* $H^+(\zeta)$ *is so for almost every* ζ *in* $\mathfrak{Z}$.

Proof. – See [14]. □

PROPOSITION 2.13. – *Let* H^+ *be a homogeneous self dual cone. Then* $\delta \in \mathcal{D}(H^+)$ *if and only if* $P_F\, \delta\, P_{F^\perp} = 0$, $\forall F \in \mathfrak{F}(H^+)$ *or equivalently if and only if* $N_F\, \delta\, N_F = N_F^2\, \delta\, N_F^2$, $\forall F \in \mathfrak{F}(H^+)$.

Proof. – See Proposition 1.9 and the properties of faces in a homogeneous self dual cone. □

3. Finite homogeneous self dual cones.

DEFINITION 3.1. – *In a self dual cone* H^+, *a trace vector is a quasi interior point* ξ_0 *such that* $U\xi_0 = \xi_0$ $\forall U \in S(H^+)$. *A homogeneous self dual cone* H^+ *is of finite type if it contains a trace vector.*

Remark. – There exist cones without trace vectors even in the class of facially homogeneous cones, for instance if $H^+ = \mathcal{P}^{\natural}_{M,\xi_0}$ where M is a type III-factor.

In this section H^+ is of finite type. The following can be partially found in [13].

PROPOSITION 3.2. – *For* ξ_0 *being a quasi interior point, the following are equivalent:*

i) ξ_0 *is a trace vector.*

ii) $\delta\xi_0 = \delta^*\xi_0$, $\quad \forall \delta \in \mathcal{D}(H^+)$.

iii) $[\delta_1, \delta_2]\,\xi_0 = 0$, $\quad \forall \delta_i = \delta_i^* \in \mathcal{D}(H^+)$.

iv) $[N_F, N_G]\,\xi_0 = 0$, $\quad \forall F, G \in \mathfrak{F}(H^+)$.

v) $N_F^2\xi_0 = \xi_0$, $\quad \forall F \in \mathfrak{F}(H^+)$.

Proof. –

i) $\Rightarrow$ ii) : because $\delta - \delta^* \in \mathcal{D}(H^+)$ and $e^{t(\delta-\delta^*)} \in S(H^+)$

ii) $\Rightarrow$ iii) $\Rightarrow$ iv) are immediate.

iv) $\Rightarrow$ v) Let F and G be completed faces. Then (see Prop. 2.13) $0 = N_F [N_F , N_G] \xi_0 = N_F^2 N_G (\mathbf{1} - N_F^2) \xi_0$.
Let $\xi = \xi^+ - \xi^-$ be the Jordan decomposition of $(\mathbf{1} - N_F^2) \xi_0$ and $G = \langle \xi^+ \rangle^{\perp\perp} \in \mathfrak{F}(H^+)$. Then iv) implies

$$0 = N_G N_F^2 (\mathbf{1} - N_F^2) \xi_0 = N_F^2 N_G (\xi^+ - \xi^-) = N_F^2 (\xi^+ + \xi^-).$$

Therefore $\xi^+ = \xi^- = 0$ and v) is proved.

v) $\Rightarrow$ i) : Let K^+ be the cone : $K^+ = \bigcap_{F \in \mathfrak{F}(H^+)} F \oplus F^\perp$
By hypothesis $\xi_0 \in K^+$; moreover, $K^+ = \pi H^+$ with $\pi = \bigwedge_{F \in \mathfrak{F}(H^+)} N_F^2$.
Therefore $\pi H^+ \subset H^+$ and K^+ is self dual in $K = \pi H$ (lemma 1.7).

The two following lemmas are needed.

LEMMMA 3.3. – *Let* G *be a face in* K^+, $\hat{G} = \langle G \rangle^{\perp\perp}$ *be the completed face generated by* G *in* H^+. *Then* $P_{\hat{G}} \in Z_{H^+}$.

Proof. – It is easy to see that $\forall F \in \mathfrak{F}(H^+)$ $N_F^2 G \subset G$; thus $\langle \xi | P_{\hat{G}^\perp} N_F^2 P_{\hat{G}} \eta \rangle = 0, \xi, \eta \in H^+$. Which implies $P_{\hat{G}^\perp} P_F P_{\hat{G}} \eta = 0$, $\eta \in H^+$; then $P_F P_{\hat{G}} \eta = P_{\hat{G}} P_F P_{\hat{G}} \eta, \eta \in H^+$ thus $[P_{\hat{G}}, P_F] = 0$.

Now returning to the corollary 1.12, the lemma is proved. □

LEMMA 3.4. – *For any face* G *in* H^+ *such that* $P_G \in Z_{H^+}$ *we have*

$$P_G K^+ = G^{\perp\perp} \cap K^+.$$

Proof. – From $P_G H^+ = G^{\perp\perp}$ and $[P_G , N_F^2] = 0$ for all $F \in \mathfrak{F}(H^+)$ we find $P_G K^+ \subset G^{\perp\perp} \cap K^+$. Conversely

$$P_G (G^{\perp\perp} \cap K^+) = G^{\perp\perp} \cap K^+ \subset P_G K^+ \quad \text{since } P_G = P_{G^{\perp\perp}}. \qquad □$$

Proof of Proposition 3.2 (end). – From the previous results, we conclude that the completed faces of K^+ are exactly the restriction to (intersection with) K^+ of central faces of H^+. Therefore, any closed face in K^+ is a split-face and K^+ is a lattice by Corollary 1.12.

If now U is a symmetry of H^+, U commutes with π because $UN_F^2 U^{-1} = N_{UF}^2$. Therefore U leaves K^+ invariant, and also any face of K^+. By the Proposition 1.8, $U/_{K^+} = \mathbf{1}$, which proves that $U\xi_0 = \xi_0$. □

COROLLARY 3.5. – *The set of trace vectors* K^+ *is a self dual cone which induces a lattice. (This result was already given in* [6]).

COROLLARY 3.6. – H^+ *is indecomposable if and only if* K^+ *is one-dimensional.*

PROPOSITION 3.7. – *Any finite dimensional homogeneous self dual cone is of finite type.*

See for instance [1, 55].

4. Spectral theorem.

The spectral theorem is one of the main tool in many algebraic constructions. It can be seen either from the algebraic point of view by mean of the functional calculus, or from the ordered space point of view by the method of Riesz and Nagy (see [43]). It is not therefore surprising to see connections between these two aspects.

The following theorem 4.1, in this form, is due to Bös who communicated his proof to us. But it can be found in very close form in [29]. However because of the importance of this construction in the sequel we have found useful to give an extensive proof.

Let $\mathcal{M} = \mathcal{D}_h(H^+)$ be the set of self adjoint derivations of H^+ and

$$\mathcal{M}_1^+ = \{\delta \in \mathcal{M} \ ; \ 0 \leqslant \delta \leqslant \mathbf{1}\}. \tag{4.1}$$

$\mathcal{M}$ is a weakly closed real linear space, and therefore $\mathcal{M}_1^+$ is a weakly compact convex set in $\mathcal{M}$.

ξ_0 being a trace vector, let $[0, \xi_0]$ be the order interval it defines. If $0 \leqslant \xi \leqslant \xi_0$, then $\|\xi\| \leqslant \|\xi_0\|$ and $[0, \xi_0]$ is also a weakly compact convex set in H^J.

THEOREM 4.1 ([15]). – *The map* $\varphi : \delta \longrightarrow \delta\xi_0$ *is an order isomorphism from* $\mathcal{M}_1^+$ *onto* $[0, \xi_0]$.

The proof requires four steps.

1) φ is injective :

LEMMA 4.2. – *Let* ξ *be a quasi interior point of* H^+. ξ *is cyclic and separating for* $\mathfrak{M}$ *in* H^J.

Proof. – The cyclicity comes from the Lemma 1.6 and the homogeneity of H^+. Now let δ be in $\mathfrak{M}$ such that $\delta\xi = 0$. Then $e^{t\delta}\xi = \xi, \forall t \in \mathbf{R}$. Since δ is a derivation, $0 \leqslant \eta \leqslant \xi$ implies $0 \leqslant e^{t\delta}\eta \leqslant \xi, t \in \mathbf{R}$. By the spectral theorem, this is possible only if $\delta\eta = 0$. ξ_0 being quasi interior $\langle \xi \rangle$ is total in H^J; therefore $\delta/_{\langle \xi \rangle} = 0$ and $\delta = 0$. □

2) $\varphi(\mathfrak{M}_1^+) \subset [0, \xi_0]$:

Let δ be a positive derivation. If $\delta\xi_0 = \xi^+ - \xi^-$ we put $F = \langle \xi^+ \rangle^{\perp\perp}$. Then

$$0 \geqslant \langle \xi_0, -\xi^- \rangle = \langle \xi_0, P_{F^\perp}\delta\xi_0 \rangle = \langle \xi_0, P_{F^\perp}\delta(P_F + P_{F^\perp})\xi_0$$
$$= \langle \xi_0, P_{F^\perp}\delta P_{F^\perp}\xi_0 \rangle \geqslant 0$$

(Recall that ξ_0 is a trace-vector and δ is a derivation). Therefore, ξ_0 being quasi interior $\xi^- = 0$. In the same way $(\mathbf{1} - \delta)\xi_0 \in H^+$ and the desired result holds.

3) Extremal points of $[0, \xi_0]$:

LEMMA 4.3 ([29]). – ξ *is an extremal point of* $[0, \xi_0]$ *if and only if there is an* $F \in \mathcal{F}(H^+)$ *such that* $\xi = P_F\xi_0$.

Proof. – Since ξ_0 is a trace-vector, $P_F\xi_0 \in [0, \xi_0]$ (Prop. 3.2).

If $P_F\xi_0 = \alpha\xi_1 + (1-\alpha)\xi_2$ with $\xi_1, \xi_2 \in [0, \xi_0]$ and $0 \leqslant \alpha \leqslant 1$ we find $P_{F^\perp}\xi_1 = P_{F^\perp}\xi_2 = 0$. Therefore $\xi_1 = P_F\xi_1 \leqslant P_F\xi_0$ and $\xi_2 = P_F\xi_2 \leqslant P_F\xi_0$ which is possible only if $P_F\xi_0 = \xi_1 = \xi_2$. Thus $P_F\xi_0$ is extremal. □

Now we need the following :

LEMMA 4.4. – *$\eta \in [-\xi_0, \xi_0]$ if and only if $\eta^+, \eta^- \in [0, \xi_0]$, where $\eta^+ - \eta^-$ is the Jordan decomposition of η.*

Proof. – Indeed if $F = \langle \eta^+ \rangle^{\perp\perp}$ then $0 \leqslant P_F \eta = \eta^+ \leqslant P_F \xi_0 \leqslant \xi_0$ and $0 \leqslant - P_{F^\perp} \eta = \eta^- \leqslant P_F \xi_0 \leqslant \xi_0$. □

Proof of Lemma 4.3 (end). – Therefore let ξ be an extremal point in $[0, \xi_0]$; we have

$$\xi = \frac{1}{2} (\xi_0 - (\xi_0 - 2\xi)) \quad \text{and} \quad \xi_0 - 2\xi \in [-\xi_0, \xi_0]$$

So
$$\xi = \frac{1}{2} (\xi_0 - (\xi_0 - 2\xi)^+) + \frac{1}{2} (\xi_0 - 2\xi)^-$$

and because ξ is extremal, $\xi = (\xi_0 - 2\xi)^- = - P_F (\xi_0 - 2\xi)$ where $F = \langle (\xi_0 - 2\xi)^+ \rangle^\perp$. Thus: $\xi = P_F \xi_0$. □

4) φ is onto :

φ is clearly a linear weakly continuous map. Therefore $\varphi(\mathfrak{M}_1^+)$ is a weakly compact convex subset of $[0, \xi_0]$. On the other hand the extremal points of $[0, \xi_0]$ are in $\varphi(\mathfrak{M}_1^+)$ because:

$$P_F \xi_0 = \left(\frac{\mathbf{1} + N_F}{2}\right) \xi_0 = \varphi(\delta_F)$$

where
$$\delta_F = \frac{\mathbf{1} + N_F}{2}$$

The Krein-Milman theorem shows that:

$$[0, \xi_0] = \overline{\text{Conv Ext}\,[0, \xi_0]} \subset \varphi(\mathfrak{M}_1^+) \subset [0, \xi_0] \qquad □$$

COROLLARY 4.5. – *δ is extremal in $\mathfrak{M}_1^+$ if and only if there exists a completed face F such that $\delta = \delta_F$.*

Such points are called *facial derivations.*

This result was already known by A. Connes for the orientable cones ([19]). The main tool of this section is then the following.

THEOREM 4.6. – *Let δ be a self-adjoint derivation. Then there exists a unique family $\{\delta_\lambda\}_{\lambda \in \mathbf{R}}$ of self-adjoint derivations such that:*

i) $\forall\lambda \in \mathbf{R}$, $\exists F_\lambda$ such that $\delta_\lambda = \delta_{F_\lambda}$.

ii) $\lambda \longrightarrow F_\lambda$ *is increasing.*

iii) *if* $a =$ Inf spectrum (δ) *and* $b =$ Sup spectrum (δ) *then* $\delta_\lambda = 0$ *for* $\lambda < a$, $\delta_\lambda = \mathbf{1}$ *for* $\lambda \geqslant b$.

iv) *if* $\mu \downarrow \lambda$, $\delta_\mu \downarrow \delta_\lambda$ weakly.

v) $\delta = \int_{a-0}^{b+0} \lambda \, d\delta_\lambda$.

Remark. – We expect the corollary 4.5 and the theorem 4.6 to be true in any homogeneous self dual cone. Unfortunatly we are not actually able to prove it if there is no trace-vector in H^+.

Proof. – By means of the theorem 4.1 it is sufficient to prove such a theorem in $[0, \xi_0]$.

Let λ be a real number, and F_λ be the completed face generated by $(\lambda\xi_0 - \xi)^+$. If $\lambda \geqslant 1$ then $(\lambda\xi_0 - \xi) \geqslant (\lambda - 1)\,\xi_0 \geqslant 0$ and if $\lambda \leqslant 0$, $(\lambda\xi_0 - \xi) \leqslant 0$. Therefore we can restrict ourself to the case $0 \leqslant \lambda \leqslant 1$, for which $(\lambda\xi_0 - \xi)^\pm \in [0, \xi_0]$.

LEMMA 4.7. – *The map* $\lambda \longrightarrow F_\lambda$ *is increasing.*

Proof (see [29]). – If $\mu \geqslant \lambda$ then, since ξ_0 is a trace vector:

$$\begin{aligned}\mu\xi_0 - \xi &= (P_{F_\lambda} + P_{F_\lambda^\perp})\,((\mu - \lambda)\,\xi_0 + \lambda\xi_0 - \xi)\\ &= P_{F_\lambda}\,[(\mu - \lambda)\,\xi_0 + (\lambda\xi_0 - \xi)^+] + P_{F_\lambda^\perp}\,[(\mu - \lambda)\,\xi_0 - (\lambda\xi_0 - \xi)^-]\\ &= \eta_1 + \eta_2 - \eta_3\end{aligned}$$

with $\eta_1 \in F_\lambda$ and $\eta_2, \eta_3 \in F_\lambda^\perp$. Therefore since $F_\lambda^\perp$ is self dual, the Jordan decomposition of $\eta_2 - \eta_3$ is $\eta^+ - \eta^-$ where $\eta^+, \eta^- \in F_\lambda^\perp$. So $\mu\xi_0 - \xi = (\eta_1 + \eta^+) - \eta^-$, $\eta_1 + \eta^+ \geqslant 0$, $\eta^- \geqslant 0$ and $\langle \eta^-, \eta_1 + \eta^+ \rangle = 0$. Thus:
$(\mu\xi_0 - \xi)^+ = \eta_1 + \eta^+ \geqslant \eta_1 = (\mu - \lambda)\,P_{F_\lambda}\xi_0 + (\lambda\xi_0 - \xi)^+ \geqslant (\lambda\xi_0 - \xi)^+$
which proves the lemma. □

Now the remainder can be proven by the Riesz and Nagy's method (see [43]). We find:

$$\xi = \int_0^1 \lambda \, dP_{F_\lambda}\xi_0 .$$

But the relation $P_{F_\lambda}\xi_0 = \delta_{F_\lambda}\xi_0$ completes the proof of the theorem 4.6. □

The following corollary completes the equivalence between homogeneity and facial homogeneity in the infinite dimensional case. (See [11]).

COROLLARY 4.8. – *Let* ξ *be in* $Y = \bigcup_{n>0}\left[\frac{1}{n}\xi_0, n\xi_0\right]$ *then there is a unique positive operator* Λ *in* $GL_0(H^+)$ *such that* $\xi = \Lambda\xi_0$ ($GL_0(H^+)$ *acts topologically transitively on* H^+).

Proof. – If $\xi \in Y$ we can find $n \in \mathbf{N}$ such that

$$\frac{1}{n}\xi_0 \leqslant \xi \leqslant n\xi_0$$

Therefore ξ can be written as follows:

$$\xi = \int_{1/n}^{n} \lambda \, dP_{F_\lambda}\xi_0$$

Now, $P_{F_\lambda}\xi_0 = (\mathbf{1} - P_{F_\lambda^\perp})\xi_0$ and the ordinary spectral theorem allows us to write:

$$\begin{aligned}\xi &= \int_{1/n}^{n} \lambda^{1/2} \, dP_{F_\lambda} \int_{1/n}^{n} \lambda^{1/2} \, d(\mathbf{1} - P_{F_\lambda^\perp})\xi_0 \\ &= \int_{1/n}^{n} e^{1/2 \operatorname{Log}\lambda} \, dP_{F_\lambda} \int_{1/n}^{n} e^{1/2 \operatorname{Log}\lambda'} \, d(\mathbf{1} - P_{F_{\lambda'}^\perp})\xi_0 \\ &= e^{\int_{1/2}^{n} \operatorname{Log}\lambda \, d\delta_{F_\lambda}} \xi_0 .\end{aligned}$$

Since the exponent is a self adjoint derivation, the existence of Λ follows. □

The uniqueness and the topological homogeneity come from the two next propositions:

PROPOSITION 4.9 ([29]). – *The stationary subgroup of* ξ_0 *into* $GL_0(H^+)$ *is* $S(H^+)$.

Proof. – Let $\Lambda \in GL_0(H^+)$ leaving ξ_0 invariant. Then Λ leaves $[0, \xi_0]$ invariant, and therefore maps extremal points onto extremal points; for all $F \in \mathfrak{F}(H^+)$ there exists $F_\Lambda \in \mathfrak{F}(H^+)$, such that $\Lambda P_F \xi_0 = P_{F_\Lambda} \xi_0$ (lemma 4.3). Thus for any face in $\mathfrak{F}(H^+)$

$$\begin{aligned}\langle \Lambda P_F \xi_0 , \Lambda P_{F^\perp} \xi_0 \rangle &= \langle \Lambda P_F \xi_0 , \Lambda(\mathbf{1} - P_F) \xi_0 \rangle \\ &= \langle P_{F_\Lambda} \xi_0 , (\mathbf{1} - P_{F_\Lambda}) \xi_0 \rangle = 0 ,\end{aligned}$$

from which we deduce, since ξ_0 is quasi interior, $P_F \Lambda^* \Lambda P_{F^\perp} = 0$. Therefore if $\xi \geqslant 0$ then $\Lambda^* \Lambda P_{F^\perp} \xi \in F^\perp$ and $\Lambda^*\Lambda$ commutes with any facial projection. It consequently belongs to Z_{H^+}.

From this, it follows that the polar decomposition of Λ is

$$\Lambda = U\,|\Lambda| = |\Lambda|\,U$$

where $|\Lambda| \in Z_{H^+} \subset \mathfrak{M}$ and $U \in S(H^+)$. Since ξ_0 is a trace vector and $\xi_0 = \Lambda \xi_0 = |\Lambda|\, \xi_0$, using the lemma 4.2, $|\Lambda| = \mathbf{1}$. □

PROPOSITION 4.10. – *Any trace vector is a weak order unit.*

Proof. – In the separable cone H^+, the set of weak order units is dense. Let ξ be a weak order unit and for any real λ, let us put

$$F_\lambda = \langle (\lambda \xi_0 - \xi)^+ \rangle^{\perp\perp}$$

As in the lemma 4.7, $\lambda \longrightarrow F_\lambda$ is increasing, therefore $\lambda \longrightarrow P_{F_\lambda}$ is also increasing, and $\lambda \longrightarrow P_{F_\lambda^\perp}$ is decreasing. We put :

$$P_\infty = \bigvee_\lambda P_{F_\lambda} , \quad P_0 = \bigwedge_\lambda P_{F_\lambda^\perp} .$$

By definition, $0 \leqslant (\xi_0 - \lambda^{-1}\xi)^- = - P_{F_\lambda^\perp}(\xi_0 - \lambda^{-1}\xi)$

thus $$0 \leqslant P_{F_\lambda^\perp} \xi_0 \leqslant \lambda^{-1} P_{F_\lambda^\perp} \xi$$

and $$P_0 \xi_0 = \lim_{\lambda \uparrow \infty} P_{F_\lambda^\perp} \xi = 0 .$$

Since ξ_0 is a trace vector,

$$P_\infty \xi_0 = \lim_{\lambda \uparrow \infty} P_{F_\lambda} \xi_0 = \lim_{\lambda \uparrow \infty} (P_{F_\lambda} + P_{F_\lambda^\perp})\, \xi_0 = \xi_0 .$$

Therefore, the family of derivations $\delta_{F_\lambda} = 2^{-1}(\mathbf{1} + P_{F_\lambda} - P_{F_\lambda^\perp})$ is increasing and converges strongly as $\lambda \longrightarrow \infty$ to

$$\delta_\infty = 2^{-1}(\mathbf{1} + P_\infty - P_0).$$

Since $\mathfrak{M}$ is weakly closed, it is strongly closed and $\delta_\infty \in \mathfrak{M}$. Moreover

$$\delta_\infty \xi_0 = \xi_0.$$

Therefore $\delta_\infty = \mathbf{1}$ (lemma 4.2) and $P_\infty = \mathbf{1}$.
Since $0 \leqslant P_{F_\lambda}(\lambda\xi_0 - \xi)$, one has $0 \leqslant P_{F_\lambda}\xi \leqslant \lambda P_{F_\lambda}\xi_0 \leqslant \lambda\xi_0$.

Therefore: $\forall\lambda \in \mathbf{R}_-$, $P_{F_\lambda}\xi \in \langle\xi_0\rangle$ and $\xi = P_\infty\xi = \lim P_{F_\lambda}\xi \in \overline{\langle\xi_0\rangle}$ which proves that $\langle\xi_0\rangle = H^+$ because ξ is an arbitrary weak order unit. □

5. Jordan algebra associated with H^+.

Now we come to the first main result of this paper, the construction of the Jordan algebra associated with a homogeneous self dual cone and a trace vector.

Before to do this we need some definitions. As good references, see [7, 16, 28].

DEFINITION 5.1. – *A* JB *algebra* $\mathfrak{M}$ *is a Jordan algebra over the reals with identity element which is a Banach space with respect to a norm satisfying the requirements*

i) $\|ab\| \leqslant \|a\|\,\|b\|$

ii) $\|a^2\| = \|a\|^2 \qquad a, b \in \mathfrak{M}$

iii) $\|a^2\| \leqslant \|a^2 + b^2\|$

A JB *algebra* $\mathfrak{M}$ *is monotone complete if any increasing bounded net has an upper bound in* $\mathfrak{M}$.

Remarks. – Note in passing that axiom i) is redundant ([6]). An equivalent requirement ([48]) is obtained by replacing i), ii) and iii) by ii) and $\|a^2 - b^2\| \leqslant \mathrm{Max}\,(\|a^2\|, \|b^2\|)$. Because $\mathfrak{M}$ is a JB algebra, $\mathfrak{M}$ is an order unit space with positive cone

$$\mathcal{M}^+ = \{a \in \mathcal{M} / \ \exists b \in \mathcal{M} \quad a = b^2\}$$

This justifies the introduction of increasing net.

A normal states ρ is a positive linear form on $\mathcal{M}$ such that $\rho(\mathbf{1}) = 1$, and such that for any decreasing net $\{a_\alpha\}_{\alpha \in \mathbf{R}}$ $a_\alpha \downarrow 0$ implies $\rho(a_\alpha) \downarrow 0$.

A set S of states is *full* if it is convex and $a \geqslant 0$ in $\mathcal{M}$ if and only if $\rho(a) \geqslant 0 \ \ \forall \rho \in \mathrm{S}$. For S a full set of states ([3], Prop. II.1.7); one has

$$\|a\| = \sup_{\rho \in \mathrm{S}} |\rho(a)| .$$

If $b \in \mathcal{M}$, we define: U_b and $\mathrm{L}(b)$ by

$$\mathrm{U}_b(a) = \{bab\} = 2b(ab) - b^2 a$$

$$\mathrm{L}_b(a) = ba .$$

Then $\mathrm{U}_b(\mathcal{M}^+) \subset \mathcal{M}^+ \ \forall b \in \mathcal{M}$ and $\rho_b = \rho \circ \mathrm{U}_b$ is a positive linear normal map if ρ is a normal state. We say that a set S of states is *invariant* if $\rho \longrightarrow \rho_b$ maps S into the cone $\bigcup\limits_{\lambda \geqslant 0} \lambda \mathrm{S}$ for all $b \in \mathcal{M}$.

In [7], there is the construction of what is called the "enveloping algebra" $\widetilde{\mathcal{M}}$ of $\mathcal{M}$. It is the smallest monotone complete JB algebra containing $\mathcal{M}$ and contained in $\mathcal{M}^{**}$. In fact $\widetilde{\mathcal{M}}$ can be identified with the bidual $\mathcal{M}^{**}$ equipped with the Arens product and the usal norm (cf. F.W. Schultz : On normed Jordan algebras wich are Banach dual spaces and [6]). In particular $\widetilde{\mathcal{M}}$ has a full invariant set of normal states defining "weak" and "strong" topology. Then, monotone, weak and strong convergence coincide on monotone nets in $\widetilde{\mathcal{M}}$.

DEFINITION 5.2. – *Let $\mathcal{M}$ be a monotone complete* JB *algebra. A finite trace on $\mathcal{M}$, is a normal state φ such that $\forall a, b \in \mathcal{M}$*

$$\varphi((ab)c) = \varphi(a(bc)).$$

φ is faithful if $\varphi(a^2) = 0$ implies $a = 0$.

Remark. – If $\mathcal{M}$ is the Jordan algebra of the hermitian part of a von Neumann algebra the two definitions of trace agree.

LEMMA 5.3. – *Let* $\mathfrak{M}$ *be a monotone complete* JB *algebra with a finite faithful trace* φ. *Then the set*

$$S = \{\varphi \circ U_a / a \in \mathfrak{M}\ ,\ \varphi(a^2) = 1\}$$

is a full invariant set of normal states.

Proof. – $\forall x \in \mathfrak{M}\ ,\quad \varphi(U_a(x)) = \varphi(2a(ax) - a^2x) = \varphi(a^2x)$. Therefore $\mathfrak{M}^+$ being convex, S is convex.

If $x = x^+ - x^-$ is the polar decomposition (spectral theorem in $\mathfrak{M}$) of x in $\mathfrak{M}$ with e^+, e^- the idempotents such that $e^+ x^+ = x^+$, $e^- x^- = x^-$, $e^+ x^- = e^- x^+ = 0$ and if $\omega(x) \geqslant 0\ \forall \omega \in S$ then $0 \leqslant \varphi(U_{e^-}(x)) = -\varphi(x^-) \leqslant 0$. Since φ is faithful $x^- = 0$ and $x = x^+ \in \mathfrak{M}^+$; S is invariant because

$$\begin{aligned}\varphi(U_a\, U_b x) &= \varphi((2b(bx) - b^2x)a^2) = \varphi(2(a^2b)(bx) - (a^2b^2)x)\\ &= \varphi((2b(ba^2) - a^2b^2)x) = \varphi(U_b(a^2)x)\end{aligned}$$

which is in $\bigcup_{\lambda \geqslant 0} \lambda S$ since $U_b(a^2) \in \mathfrak{M}^+$. □

THEOREM 5.4. – *Let* H *be a Hilbert space,* H^+ *be a homogeneous self dual cone in* H, *with a trace vector* ξ_0. *Then the set* $\mathfrak{M}$ *of self adjoint derivations of* H^+ *has a canonical structure of Jordan algebra defined by:*

$$(\delta_1 \circ \delta_2)\,\xi_0 = \delta_1\,\delta_2\,\xi_0 \qquad \delta_1, \delta_2 \in \mathfrak{M}$$

Moreover with its natural Banach norm, $\mathfrak{M}$ *is a monotone complete* JB *algebra. The state*

$$\omega_{\xi_0} : \delta \longrightarrow \langle \xi_0\,, \delta\xi_0 \rangle$$

is a finite normal faithful trace, and the positive cone $\mathfrak{M}^+$ *for the Jordan structure coincides with the positive self adjoint derivations.*

Proof. – 1) From the spectral decomposition we deduce for $\delta \in \mathfrak{M}$,

$$\delta = \int_{a-0}^{b+0} \lambda\, d\delta_{F_\lambda} \quad \text{and} \quad \delta^2 \xi_0 = \int_{a-0}^{b+0} \lambda\, d\delta_{F_\lambda} \int_{a-0}^{b+0} \lambda' dP_{F_{\lambda'}} \xi_0$$

$$\delta^2 \xi_0 = \int_{a-0}^{b+0} \lambda' dP_{F_{\lambda'}} \int_{a-0}^{b+0} \lambda dP_{F_\lambda} \ \xi_0 = \int_{a-0}^{b+0} \lambda^2 dP_{F_\lambda} \ \xi_0$$
$$= \int_{a-0}^{b+0} \lambda^2 d\delta_{F_\lambda} \ \xi_0$$

Therefore $0 \leqslant \delta^2 \xi_0 \leqslant \|\delta\|^2 \xi_0$

If $\delta_1, \delta_2 \in \mathfrak{M}$, then

$$\delta_1 \delta_2 \xi_0 = \delta_2 \delta_1 \xi_0 = 2^{-1} ((\delta_1 + \delta_2)^2 \xi_0 - \delta_1^2 \xi_0 - \delta_2^2 \xi_0)$$

which implies:

$$-2^{-1} (\|\delta_1\|^2 + \|\delta_2\|^2) \xi_0 \leqslant \delta_1 \delta_2 \xi_0 \leqslant 2^{-1} \|\delta_1 + \delta_2\|^2 \xi_0$$

There exists consequently a unique element $\delta_1 \circ \delta_2$ in $\mathfrak{M}$ such that

$$(\delta_1 \circ \delta_2) \xi_0 = \delta_1 \delta_2 \xi_0 \quad \text{(theorem 4.1).}$$

2) The product $(\delta_1, \delta_2) \longrightarrow \delta_1 \circ \delta_2$ is bilinear by construction; moreover $\delta_1 \circ \delta_2 = \delta_2 \circ \delta_1$ and

$$\delta_1 \circ ((\delta_1 \circ \delta_1) \circ \delta_2) \xi_0 = \delta_1 \circ (\delta_1 \circ \delta_1) \delta_2 \xi_0 = (\delta_1 \circ \delta_1) \circ (\delta_1 \circ \delta_2) \xi_0$$

because, by 1) $\delta_1 \circ \delta_1 = \int_{a_1+0}^{b_1+0} \lambda^2 d\delta_{F_\lambda}$ commutes with δ_1 for the ordinary product.

Therefore the product $\circ$ defines a Jordan structure.

3) The spectral formula

$$\delta \circ \delta = \int_{a-0}^{b+0} \lambda^2 d\delta_{F_\lambda}$$

shows that $\|\delta \circ \delta\| = \text{Max}(|b|^2, |a|^2) = \|\delta\|^2$ and $\delta \circ \delta \geqslant 0$ for the order of operators. Conversely if $\delta \geqslant 0$ the spectral theorem shows that $\delta = \int_{0^-}^{\|\delta\|+0} \lambda d\delta_{F_\lambda}$, then $\delta = \delta' \circ \delta'$ with $\delta' = \int_{0^-}^{\|\delta\|+0} \lambda^{1/2} d\delta_{F_\lambda}$.
Therefore positivity does not depend on the two algebraic structures on $\mathfrak{M}$. Finally:

$$-\delta_2 \circ \delta_2 \leqslant \delta_1 \circ \delta_1 - \delta_2 \circ \delta_2 \leqslant \delta_1 \circ \delta_1$$

which proves that

$$\|\delta_1 \circ \delta_1 - \delta_2 \circ \delta_2\| \leqslant \text{Max}(\|\delta_1 \circ \delta_1\|, \|\delta_2 \circ \delta_2\|).$$

$\mathfrak{M}$ is thus a JB algebra for its norm operator topology.

4) Since $\mathcal{M}$ is weakly closed in the bounded operators, $\mathcal{M}$ is a monotone closed JB algebra. Now the state ω_{ξ_0} is a trace because

$$\omega_{\xi_0}((\delta_1 \circ \delta_2) \circ \delta_3) = \langle \xi_0, (\delta_1 \circ \delta_2) \circ \delta_3 \xi_0 \rangle = \langle \xi_0, \delta_1 \delta_2 \delta_3 \xi_0 \rangle$$
$$= \langle \xi_0, \delta_1 \circ (\delta_2 \circ \delta_3) \xi_0 \rangle = \omega_{\xi_0}(\delta_1 \circ (\delta_2 \circ \delta_3))$$

where we have used repeatedly the definition of $\circ$ and the fact that $\delta_i = \delta_i^*$. ω_{ξ_0} is faithful by lemma 4.2. □

Remark. – If $\theta \in \mathcal{M}$ then $\omega_{\xi_0}(U_\theta(\delta)) = \langle \xi_0, \theta\delta\theta\xi_0 \rangle = \langle \theta\xi_0, \delta\theta\xi_0 \rangle$. So the set $S = \{\omega_\xi / \omega_\xi(\delta) = \|\xi\|^{-2} \langle \xi, \delta\xi \rangle$ where $\xi \in \langle \xi_0 \rangle\}$ is a full invariant set of normal states on $\mathcal{M}$ by the theorem 4.1 and lemma 5.3.

Therefore, the weak topologies of $\mathcal{M}$ as JB algebra as well as operator algebra on H coincide.

Some other results can be useful. For instance:

PROPOSITION 5.5. – *Let δ be in $\mathcal{M}$. Then δ is an idempotent in the* JB *algebra $\mathcal{M}$ if and only if it is a facial derivation.*

Proof. – If δ_F is facial derivation then clearly:

$$\delta_F^2 \xi_0 = \delta_F P_F \xi_0 = P_F \xi_0 = \delta_F \xi_0 \quad \text{thus} \quad \delta_F \circ \delta_F = \delta_F$$

Now it is not hard to see that any idempotent δ in $\mathcal{M}$ is an extremal point in $\mathcal{M}_1^+$. Indeed if $\delta \circ \delta = \delta$ then $(\mathbf{1} - \delta) \circ (\mathbf{1} - \delta) = \mathbf{1} - \delta$ and if $\delta = \alpha\delta_1 + (1 - \alpha)\delta_2$ with $\delta_1, \delta_2 \in \mathcal{M}_1^+$ and $0 \leqslant \alpha \leqslant 1$, then

$$U_{\mathbf{1}-\delta}(\delta) = 0 = \alpha U_{\mathbf{1}-\delta}(\delta_1) + (\mathbf{1} - \alpha) U_{\mathbf{1}-\delta}(\delta_2)$$

Therefore $U_{\mathbf{1}-\delta}(\delta_i) = 0$, that is $\delta_i = U_\delta(\delta_i) \leqslant U_\delta(\mathbf{1}) = \delta \circ \delta = \delta$ (see [7] Corollary 2.10). Therefore $\delta_i = \delta$ and the corollary 4.5 proves the proposition. □

PROPOSITION 5.6. – *The center* $Z(\mathcal{M})$ *of the* JB *algebra* $\mathcal{M}$, *is equal to the ideal center of* Z_{H^+}, *and coincides with the center of* $\mathcal{D}(H^+)$.

Proof. – Recall that δ commutes with δ' if $[L(\delta), L(\delta')] = 0$. The center of $\mathcal{M}$ is then the set of element of $\mathcal{M}$ commuting with

any element of $\mathfrak{M}$. In [7] it is proved that $\delta \in Z(\mathfrak{M})$ if and only if $U_\sigma(\delta) = \delta \quad \forall \sigma \in \mathfrak{M}$ such that $\sigma^2 = \mathbf{1}$.
But $\sigma \in \mathfrak{M}$ and $\sigma^2 = \mathbf{1}$ is equivalent to $\sigma = 2\delta_F - \mathbf{1} = N_F$ where δ_F is an idempotent (use the Prop. 5.5).

On the other hand:

$$U_{N_F}(\delta)\,\xi_0 = (2N_F \circ (N_F \circ \delta) - (N_F \circ N_F) \circ \delta)\,\xi_0 = (2N_F^2 - \mathbf{1})\,\delta\xi_0$$
$$= U_F\,\delta\,U_F\,\xi_0 . \qquad \text{with} \qquad U_F = 2N_F^2 - \mathbf{1}$$

where we have used the properties of the traces. Therefore $U_{N_F}(\delta) = U_F\,\delta\,U_F$ and, using the tehorem 1.11:

$\delta \in Z(\mathfrak{M})$ is equivalent to:
- $U_F\,\delta = \delta\,U_F \quad \forall F \in \mathfrak{F}(H^+)$
- $[\delta, P_F] = 0 \quad \forall F \in \mathfrak{F}(H^+)$
- $\delta \in Z_{H^+}$
- $\delta \in$ Center of $\mathcal{D}(H^+)$. □

6. The homogeneous self dual cone of a JB algebra.

Let us now come to the converse. In this section $\mathfrak{M}$ is supposed to be a JB algebra monotonically closed with a finite faithful trace φ.

The support of a normal state ω on $\mathfrak{M}$ is defined as in the von Neumann case. Let $\mathfrak{M}_\omega = \{a \in \mathfrak{M} \,/\, \omega(|a|) = 0\}$. $\mathfrak{M}_\omega$ is a JB ideal by the Schwarz inequality, and it is monotone closed because ω is normal. Therefore, there is in $\mathfrak{M}_\omega$ a greatest idempotent denoted by $\mathbf{1} - e_\omega$. Clearly, by the spectral decomposition in $\mathfrak{M}$, $\mathfrak{M}_\omega = U_{\mathbf{1}-e_\omega}\mathfrak{M}$. Now, ω is a faithful normal state on $\mathfrak{M}_\omega$, because if $a \in U_{e_\omega}(\mathfrak{M})$ and $\omega(|a|) = 0$, then $0 < |a| \leqslant \|a\|\,(\mathbf{1} - e_\omega)$, which implies $a = 0$, since $|a| = U_{e_\omega}(|a|)$.

e_ω is called *the support* of ω.

THEOREM 6.1. — *There exists a Hilbert space* H, *a self dual homogeneous cone* H^+ *in* H *with a trace vector* ξ_0, *such that* $\mathfrak{M}$ *coincides with the* JB *algebra of self adjoint derivations of* H^+.

Proof. – *1) Construction of* H^+:

Since φ is a faithful trace, there is a separable prehilbertian structure on $\mathfrak{M}$ defined by:

$$\langle a, b \rangle = \varphi(ab).$$

Let H^J be the real completion of $\mathfrak{M}$, H a complexication of H^J, and H^+ be the closure of $\mathfrak{M}^+$ in H^J.

LEMMA 6.2. – H^+ *is self dual in* H.

Proof. – (See [29]).

If $a \in \mathfrak{M}^+$ and $b \in \mathfrak{M}^+$ then

$$\varphi(ab) = \varphi((a^{1/2})^2 b) = \varphi(a^{1/2}(ba^{1/2})) = \varphi(U_{a^{1/2}}(b)) \geqslant 0.$$

Therefore by completion H^+ is included in its dual.

If $a \in \mathfrak{M}$ and $|a| = (a^2)^{1/2}$, $a^{\pm} = 2^{-1}(|a| \pm a)$, then by spectral theorem $|a|, a^+, a^-$ are in $\mathfrak{M}^+$ and $\langle a^+, a^- \rangle = 0$. Since $\varphi(ab) = \varphi((a^+ - a^-)(b^+ - b^-)) \leqslant \varphi((a^+ + a^-)(b^+ + b^-)) = \varphi(|a|\,|b|)$
$0 \leqslant \varphi((|a| - |b|)^2) \leqslant \varphi((a - b)^2)$

Therefore $a \longrightarrow |a|$, $a^{\pm}$ are continuous maps with respect to the Hilbert topology. By extension, if $\xi \in H^J$ is in the dual of H^+, then $0 \leqslant \langle \xi, \xi^- \rangle = -\|\xi^-\|^2 \leqslant 0$ and $\xi^- = 0$. □

2) The structure of the faces of H^+:

THEOREM 6.3. – *The map* $e \longrightarrow \overline{\langle e \rangle}$ *is an order isomorphism of the set of idempotents in* $\mathfrak{M}$ *onto* $\mathfrak{F}(H^+)$. *For every idempotent* e *in* $\mathfrak{M}$, $P_{\langle e \rangle} = U_e$ *and* $P_{\langle e \rangle^\perp} = U_{1-e}$.

LEMMA 6.4. – *The operators* $L(a): b \longrightarrow ab$ *and* $U_a: b \longrightarrow \{aba\}$ *defined on* $\mathfrak{M}$ *can be continued as bounded self adjoint operators on* H. *Moreover* $a \longrightarrow L(a)$ *is an isometric linear map from* $\mathfrak{M}$ *into* $\mathcal{L}(H)$; *and* $a \longrightarrow U_a$ *is a continuous map.*

Proof. – $L(a)$ is densely defined on H^J and is symmetric because

$$\forall a, b, c \in \mathfrak{M} \quad \langle b, L(a)c \rangle = \varphi(b(ac)) = \varphi((ab)c) = \langle L(a)b, c \rangle$$

L(a) is bounded on H because the set $S = \{\varphi \circ U_b , b \in \mathfrak{M}\}$ is full, $\mathfrak{M} = H^J$ and:

$$\|a\| = \sup_{b \in \mathfrak{M}} \frac{|\varphi(U_b(a))|}{|\varphi(b^2)|} = \sup_{b \in \mathfrak{M}} \frac{|\langle b \mid L(a) b\rangle|}{\|b\|^2} = \|L(a)\|_{\mathcal{L}(H)} .$$

Therefore, $a \longrightarrow L(a)$ is linear and isometric from $\mathfrak{M}$ to $\mathcal{L}(H)$. Finally $U_a = 2L(a^2) - L(a)^2$ which completes the proof. □

COROLLARY 6.5. – *If* e *is an idempotent in* $\mathfrak{M}$, $U_e(H^+)$ *is the closure of the face* $\langle e \rangle$ *generated by* e *in* H^+.

Proof. – If $\mathfrak{M}$ is special (i.e. $a \circ b = 2^{-1}(ab + ba)$ for some product in $\mathfrak{M}$), then: $U_b^2(a) = \{b\{bab\}b\} = \{b^2 ab^2\} = U_{b^2}(a)$. Since this is a polynomial identity in less than 3 variables which is linear in one of them, by the Mac Donald's theorem [28, p. 41] this identity holds in any Jordan algebra. If $e^2 = e$ then $U_2 = U_e$. Finally: $\mathfrak{M}^+$ is the face generated by $\mathbf{1}$, and $U_e(\mathfrak{M}^+) = \langle e \rangle$. Since U_e is a projector $U_e(H^+) = \overline{\langle e \rangle}$. □

LEMMA 6.6. – *Let* ξ *be in* H^+ *and* e *be an idempotent of* $\mathfrak{M}$. *The following proposition are equivalent:*

i) $U_e \xi = \xi$

ii) $U_{\mathbf{1}-e} \xi = 0$

iii) $\langle \xi , (\mathbf{1} - e)\rangle = 0$

iv) $L(e)\, \xi = \xi$

Proof. – i) $\Longrightarrow$ ii) $\Longrightarrow$ iii) are immediate.

iii) $\Longrightarrow$ ii): For any $a \in \mathfrak{M}^+$,

$$\langle \xi , U_{\mathbf{1}-e}(a)\rangle \leqslant \|a\| \langle \xi , U_{\mathbf{1}-e}(\mathbf{1})\rangle = \|a\| \langle \xi , \mathbf{1} - e\rangle = 0$$

Since $\overline{\mathfrak{M}^+ - \mathfrak{M}^+} = H^J$, we have $U_{\mathbf{1}-e}(\xi) = 0$.

ii) $\Rightarrow$ i): Since $U_b \in \mathfrak{M}^+$ for all b in $\mathfrak{M}$, choosing $b = (\mathbf{1} - e) + \lambda e$ with $\lambda \in \mathbf{R}$

$$\begin{aligned} U_b \xi &= \lambda^2 U_e \xi + \lambda(\mathbf{1} - U_e - U_{\mathbf{1}-e}) \xi + U_{\mathbf{1}-e} \xi \\ &= \lambda^2 U_e \xi + \lambda(\mathbf{1} - U_e - U_{\mathbf{1}-e}) \xi \geqslant 0 \qquad \forall \lambda \in \mathbf{R}. \end{aligned}$$

This is possible only if $\xi = U_e \xi$.

i) $\Longleftrightarrow$ iv): Because of the formula $L(e) = 2^{-1}(\mathbf{1} + U_e - U_{\mathbf{1}-e})$, $L(e)\xi = \xi$ implies

$$\|\xi\|^2 = \langle \xi , L(e)\xi \rangle = 2^{-1}(\|\xi\|^2 + \|U_e \xi\|^2 - \|U_{\mathbf{1}-e}\xi\|^2)$$

and $U_e \xi = \xi$ (U_e is an orthogonal projection).

Conversely $U_e \xi = \xi$ implies $U_{\mathbf{1}-e} \xi = 0$ and $L(e)\xi = \xi$. □

Proof of Theorem 6.3. – If e is an idempotent in $\mathfrak{M}$, the previous lemmas imply that $\overline{\langle e \rangle} = \langle \mathbf{1} - e \rangle^\perp$ is a completed face. Conversely:

(i) Let $\xi \in H^+$ and e_ξ the support of the state

$$\omega_\xi : a \longrightarrow \|\xi\|^{-2} \langle \xi , L(a)\xi \rangle$$

Then $\langle \xi , L(e_\xi)\xi \rangle = \|\xi\|^2 = 2^{-1}(\|\xi\|^2 + \|U_{e_\xi}\xi\|^2 - \|U_{\mathbf{1}-e_\xi}\xi\|^2)$,

which proves that $\xi = U_{e_\xi}(\xi)$ and $\xi \in \langle \mathbf{1} - e_\xi \rangle^\perp = \overline{\langle e_\xi \rangle}$; thus $\overline{\langle \xi \rangle} \subset \overline{\langle e_\xi \rangle}$.

(ii) Let λ be a positive number, we have:

$$(\lambda\mathbf{1} - \xi) = (\lambda\mathbf{1} - \xi)^+ - (\lambda\mathbf{1} - \xi)^- \quad \text{in } H^J.$$

Therefore by i) we can define the idempotent $e_\lambda = e_{(\lambda\mathbf{1}-\xi)^+}$. Then for all $\lambda > 0$, $\lambda e_\lambda = \lambda U_{e_\lambda}(\mathbf{1}) \geqslant U_{e_\lambda}(\lambda\mathbf{1} - \xi) = (\lambda\mathbf{1} - \xi)^+ \geqslant (\lambda\mathbf{1} - \xi)$ and $\xi \geqslant \lambda(\mathbf{1} - e_\lambda)$, $\lambda > 0$. Therefore $\mathbf{1} - e_\lambda \in \langle \xi \rangle$ $\forall \lambda > 0$. On the other hand, as in the lemma 4.7, $\lambda \leqslant \mu$ implies $e_\lambda \leqslant e_\mu$. Therefore, if $\lambda \downarrow 0$ then $e_\lambda \downarrow e_0$ since monotone convergence coincides in $\mathfrak{M}$ and in H^+, on monotone nets. Then:

$$\mathbf{1} - e_0 \in \overline{\langle \xi \rangle}.$$

(iii) Now, using the lemma 6.4

$$0 \leqslant U_{e_0}\xi = \lim_{\lambda \downarrow 0} U_{e_\lambda}\xi = \lim_{\lambda \downarrow 0} U_{e_\lambda}(\xi - \lambda\mathbf{1}) = \lim_{\lambda \downarrow 0} -(\lambda\mathbf{1} - \xi)^+ \leqslant 0.$$

Thus $U_{e_0}\xi = 0$. Therefore $U_{\mathbf{1}-e_0}\xi = \xi$, $e_\xi \leqslant \mathbf{1} - e_0$ and $e_\xi \in \overline{\langle \mathbf{1} - e_0 \rangle}$ by corollary 6.5. So for any $\xi \in H^+$, there exists a unique idempotent e_ξ in $\mathfrak{M}$ such that $\overline{\langle e_\xi \rangle} \subset \overline{\langle \mathbf{1} - e_0 \rangle} \subset \overline{\langle \xi \rangle} \subset \overline{\langle e_\xi \rangle}$.

Using the following lemma, the theorem is proved. □

LEMMA 6.7. – *For any face* F *in* H^+, *there exists an idempotent* e_F *such that* $\overline{F} = \overline{\langle e_F \rangle}$, $P_F = \overline{U_{e_F}}$, $\delta_F = L(e_F)$ *is a derivation of* H^+ *and* H^+ *is facially homogeneous.*

Proof. – Since $F = \bigcup_{\eta \in F} \langle \eta \rangle$ the family $\eta \longrightarrow \langle \eta \rangle$ is an increasing net. Moreover, $0 \leqslant \eta_1 \leqslant \eta_2 \in F$ implies (lemma 6.6)

$$U_{1-e_{\eta_2}}(\eta_1) = 0 \quad \text{and} \quad U_{e_{\eta_2}}(\eta_1) = \eta_1 .$$

Therefore, by the definition of e_η, $e_{\eta_1} \leqslant e_{\eta_2}$ and $\eta \longrightarrow e_\eta$ is an increasing net of idempotents. Let

$$e_F = \bigvee_{\eta \in F} e_\eta = s - \lim_{\eta \in F} e_\eta$$

(clearly e_F belong to $\mathfrak{M}$ since $\mathfrak{M}$ is monotone closed). Now if $\xi \in U_{e_F}(H^+) = \overline{\langle e_F \rangle}$ then (lemma 6.4) $\xi = \lim_{\eta \in F} U_{e_\eta}(\xi)$ and therefore $\xi \in \overline{\bigcup_{\eta \in F} \langle \eta \rangle} = \overline{F}$.

Conversely, $\xi \in F$ implies $e_\xi \leqslant e_F$ and $U_{e_F} \xi = \xi$. So $F \subset U_{e_F} H^+ \subset \overline{F}$, which proves that $\overline{F} = \overline{\langle e_F \rangle}$ and $P_F = U_{e_F}$.

It remains to prove that δ_F is a derivation. Using the proposition 2.13 this is a consequence of the identity $U_{e'} L(e) U_{1-e'} = 0$ for all idempotent e' in $\mathfrak{M}$ (Use Mac Donald's theorem). □

Remarks. – The lemma 6.7 shows that in H^+, the closure of a face is a face and $\mathcal{F}(H^+) = \{\overline{F}/F \text{ face of } H^+\}$.

– $U_F = 2N_F^2 - 1$ is a unitary such that

$$U_F H^+ = \overline{\langle 2e - 1 \rangle} \subset H^+ \quad \text{if} \quad P_F = U_e$$

3) Self adjoint derivations:

From now, H^+ is self dual and homogeneous.

LEMMA 6.8. – *For any* a *in* $\mathfrak{M}$, $L(a)$ *is a self adjoint derivation of* H^+.

Proof. – Comes from the identity $U_e L(a) U_{1-e} = 0$, valid for all idempotent e in $\mathfrak{M}$, Proposition 2.13 and Lemma 6.4. □

LEMMA 6.9. – *Any vector in* H^+ *belonging to* $[0, 1]$ *is in fact in* $\mathfrak{M}_1^+$ *and* 1 *is a trace vector in* H^+.

Proof. – $[0, \mathbf{1}]$ is a "weakly" compact convex set as well for the topology of H^+ as for the topology of $\mathfrak{M}^+$. Using lemma 4.3, and the fact that extremal points of $\mathfrak{M}_1^+$ are idempotents, we see that if we assume $\mathbf{1}$ to be a trace vector, then $[0, \mathbf{1}]$ and $\mathfrak{M}_1^+$ have same extremal points. Monotone convergence being the same in $\mathfrak{M}$ and H^+ the result follows.

It remains only to prove that $\mathbf{1}$ is a trace vector:
Since $\langle \mathbf{1} \rangle = \mathfrak{M}^+$, $\mathbf{1}$ is a weak order unit and therefore it is a quasi interior point in H^+. $\forall F_e \in \mathcal{F}(H^+)$, $\overline{F} = \overline{\langle e \rangle}$ and using theorem 6.3

$$N_F^2 \mathbf{1} = (P_F + P_{F^\perp})\mathbf{1} = (U_e + U_{\mathbf{1}-e})\mathbf{1} = e + \mathbf{1} - e = \mathbf{1}$$

The proposition 3.2 asserts the result. □

LEMMA 6.10. – *Let δ be a self adjoint derivation of H^+. Then there exists a unique $a \in \mathfrak{M}$ such that $\delta = L(a)$.*

Proof. – We can restrict ourselves to the case for which $0 \leqslant \delta \leqslant \mathbf{1}$ Then, $a = \delta\mathbf{1} \in [0, \mathbf{1}]_{H^+} = [0, \mathbf{1}]_{\mathfrak{M}^+}$ (use theorem 4.1); and therefore $L(a)\mathbf{1} = \delta\mathbf{1}$. Because $\mathbf{1}$ is separating (Lemma 4.2), $\delta = L(a)$. □

LEMMA 6.11. – *The map $a \longrightarrow L(a)$ is an order and JB-isomorphism between $\mathfrak{M}$ and $\mathcal{D}_h(H^+)$.*

Proof. – Clearly $a \longrightarrow L(a)$ is linear and isometric (lemma 6.4). Now if $a \geqslant 0$, $\langle b, L(a)b \rangle = \varphi(b(ab)) = \varphi(U_b(a)) \geqslant 0$ for any b in $\mathfrak{M}$. Therefore $L(a) \geqslant 0$.
Finally by the definition of the Jordan product in $\mathcal{D}_h(H^+)$ (theorem 5.4) : $L(a) \circ L(b)\mathbf{1} = L(a)\,L(b)\mathbf{1} = L(a)b = ab = L(ab)\mathbf{1}$. □

This completes the proof of theorem 6.1.

7. Some other results.

The following statements can be useful, in order to complete our knowledge of homogeneous cones.

LEMMA 7.1. – *Let $\mathfrak{M}$ be a monotone closed JB algebra and $Z(\mathfrak{M})$ its center. If φ_1 and φ_2 are two finite normal faithful traces on $\mathfrak{M}$, then there exists an $h \in \mathfrak{M}^+ \cap Z(\mathfrak{M})$ such that for all $a \in \mathfrak{M}$, $\varphi_1(a) = \varphi_2(ha)$.*

Proof. – Similar to the von Neumann algebra case ([20, 46]): Suppose that φ_1, φ_2 are not normalized traces ($\varphi_i' = \varphi_i(\mathbf{1})^{-1}\varphi_i$ are normalized) and that $\varphi_1 \leqslant \varphi_2$ (otherwise, compare φ_1 and $\varphi_1 + \varphi_2$). These traces induce two prehilbertian structures on $\mathfrak{M}$

$$\langle a, b\rangle_i = \varphi_i(ab) \,, \quad \|a\|_1^2 \leqslant \|a\|_2^2$$

If $H_i = \overline{\mathfrak{M}}^{\|\cdot\|_{\varphi_i}}$ then $H_2 \subset H_1$ and $H_2^+ \subset H_1^+$. There exists $A \in \mathcal{L}(H_2)$ such that $0 \leqslant A \leqslant \mathbf{1}$ and $\langle a, b\rangle_1 = \langle Aa, b\rangle_2$ $a, b \in H_2$. Since $\forall a, b \in \mathfrak{M}^+$, $\langle a, b\rangle_1 \leqslant \langle a, b\rangle_2$, the same holds for a, b in H_2^+. Self duality of H_2^+ implies $0 \leqslant A\xi \leqslant \xi$, for all $\xi \in H_2^+$. Thus $A \in Z_{H_2^+}$. If $h = A(\mathbf{1})$, then by Prop. 5.6 and lemma 6.9, $h \in \mathfrak{M}^+ \cap Z(\mathfrak{M})$; thus for all $a \in \mathfrak{M}$,

$$\varphi_1(a) = \langle \mathbf{1}, a\rangle_1 = \langle h, a\rangle_2 = \varphi_2(ha) \qquad \square$$

The following proposition is to be conferred with theorems 3.2 and 3.3 of [19]. (See also [34]).

PROPOSITION 7.2. – *Let* $\mathfrak{M}$ *be a* JB *monotone closed algebra with a finite faithful trace, and* H^+ *the homogeneous self dual cone of* $\mathfrak{M}$. *Then* $U \in S(H^+)$ *if and only if* U *can be restricted on* $\mathfrak{M}$ *as a Jordan isomorphism leaving* $Z(\mathfrak{M})$ *invariant.*

Proof. – Because $[0, \mathbf{1}]_{H^+} = [0, \mathbf{1}]_{\mathfrak{M}}$ (lemma 6.9), and because $U \in S(H^+)$, $[0, \mathbf{1}]_{\mathfrak{M}}$ is invariant by U. U being invertible, e is an idempotent if and only the same holds for $U(e)$, since idempotents are extremal points of $[0, \mathbf{1}]$. Moreover $e_1 e_2 = 0$ is equivalent to $e_1 + e_2$ is an idempotent; thus $U(e_1)\,U(e_2) = 0$.
Using the spectral theorem in $\mathfrak{M}$ we find:

$$U(a^2) = U(a)^2 \qquad \forall a \in \mathfrak{M}$$

and $U/\mathfrak{M}$ is a Jordan isomorphism, which leaves the center of $\mathfrak{M}$ invariant because U commutes with U_e if $e \in Z(\mathfrak{M})$ (using the fact that $U \in S(H^+)$, Proposition 5.6, and lemma 2.11 of [7]).

Conversely, let α be a Jordan isomorphism of $\mathfrak{M}$ leaving $Z(\mathfrak{M})$ invariant. Then $\varphi \circ \alpha$ is a trace if φ is. Using the lemma 7.1, we can find $h \in Z(\mathfrak{M})^+$ such that $\varphi(\alpha(a)) = \varphi(ha)$ $\forall a \in \mathfrak{M}$. But if $a \in Z(\mathfrak{M})$, $\alpha(a) = a$. Therefore $\varphi(a) = \varphi(ha)$ for all $a \in \mathfrak{M}$. $Z(\mathfrak{M})$ is now the real part of an abelian von Neumann algebra, and $\varphi/Z(\mathfrak{M})$ is a probability measure on the spectrum of $Z(\mathfrak{M})$. That means that $\varphi(a) = \varphi(ha)$ $\forall a \in Z(\mathfrak{M})$ and $h = \mathbf{1}$. Therefore

φ is invariant under α. Clearly, α can be continued on H^J. Invariance of φ under α says that α is a unitary operator on H^J. Since $\alpha(a^2) = \alpha(a)^2$, $\alpha(H^+) \subset H^+$ and $\alpha(\mathbf{1}) = \mathbf{1}$. Therefore $\alpha \in S(H^+)$. □

PROPOSITION 7.3. – *Let* $\mathfrak{M}$ *be a monotone closed* JB *algebra with a finite faithful trace* φ. *Then, there exists a standard Borel space* $\mathfrak{Z}$, *a positive Borel measure* ν *on* $\mathfrak{Z}$, *a fields* $\zeta \longrightarrow \mathfrak{M}(\zeta)$ *of Jordan algebras which is* ν*-integrable such that:*

i) $\mathfrak{M}$ *is Jordan isomorphic to* $\int^{\oplus} \mathfrak{M}(\zeta)\, d\nu(\zeta)$

ii) $Z(\mathfrak{M})$ *is Jordan isomorphic to* $L^{\infty}_{\text{real}}(\mathfrak{Z}, \nu)$

iii) *for* ν*-almost every* ζ, $\mathfrak{M}(\zeta)$ *is a* JB *factor.*

Sketch of the proof. – Using the representation theorem given in Section **6**, $\mathfrak{M}$ is the hermitian part of $\mathcal{D}(H^+)$. Because H, H^+ are decomposable with respect to Z_{H^+} (= Center of $\mathcal{D}(H^+) = Z(\mathfrak{M})$), the same is true for $\mathfrak{M}$.
Now, if δ is a derivation so is $\delta(\zeta)$ for almost every ζ ([14, 42]) and $\mathfrak{M}(\zeta)$ is well defined. Since for a.e. ζ, $H^+(\zeta)$ is indecomposable, therefore $Z(\mathfrak{M}(\zeta)) = \mathbf{R1}_{H(\zeta)}$ almost everywhere and $\mathfrak{M}(\zeta)$ is a JB factor. □

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