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THE SPACE $\mathcal{D}(\Omega)$ IS NOT B_r -COMPLETE

by Manuel VALDIVIA

The purpose of this paper is to study certain classes of locally convex spaces which have non-complete separated quotients. Consequently, we obtain some results about B_r -completeness. In particular, we prove that the L. Schwartz space $\mathcal{D}(\Omega)$ is not B_r -complete, where Ω is a non-empty open subset of \mathbb{R}^m .

The vector spaces which are used here are defined on the field of the real or complex numbers K . We shorten « Hausdorff locally convex space » to « space ». If $\langle G, H \rangle$ is a dual pair, we denote by $\sigma(G, H)$ and $\mu(G, H)$ the weak topology and the Mackey topology on G , respectively.

Sometimes we symbolize by $L[\mathcal{F}]$ a space L with the topology \mathcal{F} . We denote by L' the topological dual of L . If L_n are copies of L , $n = 1, 2, \dots$, we denote by L^N the product $\prod_{n=1}^{\infty} L_n$ and by $L^{(N)}$ the direct sum $\bigoplus_{n=1}^{\infty} L_n$. If A is an absolutely convex bounded and closed subset of L , L_A is the linear hull of A with the locally convex topology which has as fundamental system of neighbourhoods of the origin the family $\left\{ \frac{1}{n} A : n = 1, 2, \dots \right\}$. We say that a sequence (x_n) in L is a Cauchy (convergent) sequence in the sense of Mackey, if there exists an absolutely convex closed and bounded subset B of L , such that (x_n) is a Cauchy (convergent) sequence in L_B . Then it is natural to define a locally complete space L as a space in which every Cauchy sequence in the sense of Mackey is convergent in the sense of Mackey in L . If a subset M of L is such that for every

point x of L there exists a sequence (x_n) in M which converges to x in the sense of Mackey, M is called locally dense in L . L is called a dual locally complete space if $L'[\sigma(L', L)]$ is locally complete. If L is topologically isomorphic to P write $L \simeq P$.

Let Ω be a non-empty open subset of the m -dimensional euclidean space R^m . Then $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ are the L. Schwartz spaces with the strong topologies. By ω we denote the product of countably many one-dimensional spaces. Let s be the vector space of all the rapidly decreasing sequences, with the metric topology. The dual of s with the Mackey topology is denoted by s' .

A space T is called B_r -complete if every dense subspace M of $T'[\sigma(T', T)]$ coincides with T' when $M \cap A$ is $\sigma(T', T)$ -closed in A for every equicontinuous set A that belongs to T' .

Throughout this paper, E denotes an infinite-dimensional Fréchet space, different from ω , and F a Fréchet space which is not Banach.

In order to prove Theorem 1, we need the following result, which we gave in [8]: a) *Let G be an infinite-dimensional space such that in $G'[\sigma(G', G)]$ there is an equicontinuous total sequence. Let H be a space with a separable absolutely convex weakly compact total subset. If H is infinite-dimensional, there is a linear mapping u continuous and injective from G into H , such that $u(G)$ is separable, dense in H and $u(G) \neq H$.*

In Theorem 1, (E_n) is a sequence of infinite-dimensional spaces such that in E_n there is a weakly compact absolutely convex and separable subset which is total in E_n , $n = 1, 2, \dots$

THEOREM 1. — *Let $L = (E \widehat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$. Then there is a quotient of $L[\sigma(L', L)]$ which is topologically isomorphic to a locally dense and non-closed subspace of $\prod_{n=1}^{\infty} E_n[\sigma(E_n, E'_n)]$.*

Proof. — Since F is a Fréchet space which is not Banach, there exists a closed subspace G of F such that $F/G \simeq \omega$, (see [2], remark at the end of § 31, 4.(1), p. 432). Let f be the canonical mapping from F onto F/G and let J be the

identity mapping on E . The tensorial product $J \otimes f$ is a topological homomorphism from $E \hat{\otimes} F$ onto

$$E \hat{\otimes} (F/G) \simeq E \hat{\otimes} \omega,$$

(see [1], Chapter I p. 38). Further, if H_n is an one-dimensional space, then

$$E \hat{\otimes} \omega = E \hat{\otimes} \left(\prod_{n=1}^{\infty} H_n \right) \simeq \prod_{n=1}^{\infty} (E \hat{\otimes} H_n) \simeq E^{\mathbb{N}},$$

(see [1], Chapter I p. 46).

Since E is different from ω , in $E'[\sigma(E', E)]$ there is an equicontinuous sequence whose closed linear hull M is infinite dimensional. Let M^\perp be the orthogonal subspace of E to M , set $T = E/M^\perp$, and let h be the canonical mapping from $E^{\mathbb{N}}$ onto $E^{\mathbb{N}}/(M^\perp)^{\mathbb{N}} \simeq (E/M^\perp)^{\mathbb{N}} = T^{\mathbb{N}}$.

Now we use the result *a*) and we obtain a linear continuous and injective mapping u_n , from T into E_n , such that $u_n(T)$ is dense in E_n and $u_n(T) \neq E_n$. We define a mapping

g from $T^{\mathbb{N}} \times \bigoplus_{n=1}^{\infty} E_n$ into $\prod_{n=1}^{\infty} E_n$ such that if

$$\begin{aligned} x &= (x_1, x_2, \dots, x_n, \dots) \in T^{\mathbb{N}} \\ y &= (y_1, y_2, \dots, y_n, \dots) \in \bigoplus_{n=1}^{\infty} E_n, \end{aligned}$$

then

$$g(x, y) = (u_1(x_1) + y_1, u_2(x_2) + y_2, \dots, u_n(x_n) + y_n, \dots).$$

The mapping g is a weak homomorphism from $T^{\mathbb{N}} \times \bigoplus_{n=1}^{\infty} E_n$ into $\prod_{n=1}^{\infty} E_n$, such that $g\left(T^{\mathbb{N}} \times \bigoplus_{n=1}^{\infty} E_n\right)$ is locally dense and non-closed in $\prod_{n=1}^{\infty} E_n$, (see the proof of Theorem 2 in [7]).

If I is the identity mapping of $\bigoplus_{n=1}^{\infty} E_n$, then it suffices to take the quotient of $L[\sigma(L, L')]$ by the kernel of the mapping $g \circ (h \circ (J \otimes f) \times I)$ in order to obtain the conclusion of the theorem. q.e.d.

COROLLARY 1.1. — *Let (E_n) be a sequence of separable infinite-dimensional Fréchet spaces. Then there is a quotient of*

$s \times \bigoplus_{n=1}^{\infty} E_n$, topologically isomorphic to a non-closed dense subspace of $\prod_{n=1}^{\infty} E_n$.

Proof. — Since E_n is a separable Fréchet space there exists an absolutely convex compact separable subset of E_n , which is total in E_n .

We put

$$T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Then $\mathcal{D}(T) \simeq s$ and $\mathcal{D}(T \times T) \simeq s$, (see [1], Chapter II, p. 54). Since $\mathcal{D}(T) \widehat{\otimes} \mathcal{D}(T) \simeq \mathcal{D}(T \times T)$, (see [1], Chapter II, p. 84-85), we have that $s \widehat{\otimes} s \simeq s$. On the other hand, s is an infinite-dimensional Fréchet space, different from ω , which is not Banach.

Using Theorem 1 we obtain a subspace P of $s \times \bigoplus_{n=1}^{\infty} E_n$ such that $\left(s \times \bigoplus_{n=1}^{\infty} E_n\right)/P$ is weakly isomorphic to a dense non-closed subspace of $\prod_{n=1}^{\infty} E_n$. Since $\left(s \times \bigoplus_{n=1}^{\infty} E_n\right)/P$ is a Mackey space and every subspace of $\prod_{n=1}^{\infty} E_n$ is also a Mackey space, we have that $\left(s \times \bigoplus_{n=1}^{\infty} E_n\right)/P$ is topologically isomorphic to a dense non-closed subspace of $\prod_{n=1}^{\infty} E_n$ q.e.d.

Note 1. — O. G. Smoljanov proves in [6] that there is a closed subspace H in $s \times \omega^{(N)}$ such that $(s \times \omega^{(N)})/H$ is a non closed dense subspace of ω . This result is a consequence of our Corollary 1.1, identifying ω with E_n , $n = 1, 2, \dots$, and noticing that $\omega^N \simeq \omega$.

THEOREM 2. — *Let (E_n) be a sequence of non-Banach Fréchet spaces. Then $(E \widehat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ has a quotient which is topologically isomorphic to a non-closed dense subspace of ω .*

Proof. — Since E_n is not a Banach space there is a subspace G_n of $(E \widehat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ lying in E_n , such that $E_n/G_n \simeq \omega$.

Let $G = \bigoplus_{n=1}^{\infty} G_n$. Then

$$\left[(E \widehat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n \right] / G \simeq (E \widehat{\otimes} F) \times \bigoplus_{n=1}^{\infty} (E_n / G_n) \simeq (E \widehat{\otimes} F) \times \omega^{(N)}$$

and the conclusion follows straightforward from Theorem 1 applied to $(E \widehat{\otimes} F) \times \omega^{(N)}$. q.e.d.

COROLLARY 1.2. — *There is a quotient of $s^{(N)}$ which is topologically isomorphic to a non-closed dense subspace of ω .*

Proof. — Since $s \widehat{\otimes} s \simeq s$, it is possible to write $(s \widehat{\otimes} s) \times s^{(N)} \simeq s^{(N)}$. Now we apply Theorem 2. q.e.d.

LEMMA. — *The space $\mathcal{D}(\Omega)$ has a closed subspace topologically isomorphic to $s^{(N)}$.*

Proof. — In Ω let (K_n) be a sequence of compact sets such that $K_n \subset \overset{\circ}{K}_{n+1}$, $n = 1, 2, \dots$, being $\overset{\circ}{K}_{n+1}$ the interior

set of K_{n+1} , and $\bigcup_{n=1}^{\infty} K_n = \Omega$. Suppose that $K_1 \neq \emptyset$.

We put $K_0 = \emptyset$. Let R_n be a closed m -dimensional rectangle contained in $\overset{\circ}{K}_n \sim K_{n-1}$, $n = 1, 2, \dots$. Let E_n and F_n be the subspaces of $\mathcal{D}(\Omega)$ formed by the elements of $\mathcal{D}(\Omega)$ whose supports are contained in R_n and K_n , respectively.

We now prove that $E = \bigoplus_{n=1}^{\infty} E_n$ is a subspace of $\mathcal{D}(\Omega)$.

Let V_n be an absolutely convex neighbourhood of the origin in E_n . Since E_1 is a subspace of the Fréchet space F_1 there is an absolutely convex neighbourhood U_1 of the origin in F_1 such that $U_1 \cap E_1 = V_1$. Suppose that we have constructed in F_n an absolutely convex neighbourhood of the origin U_n such that

$$U_n \cap [F_{n-1} \oplus E_n] = \Gamma(U_{n-1} \cup V_n), \quad n \geq 1, \quad U_0 = \{0\},$$

being $\Gamma(U_{n-1} \cup V_n)$ the absolutely convex hull of $U_{n-1} \cup V_n$. Since $F_n \oplus E_{n+1}$ is closed in F_{n+1} and E_{n+1} and F_n are Fréchet spaces, we have that $F_n \oplus E_{n+1}$ is a subspace of F_{n+1} and therefore we can find an absolutely convex neigh-

neighbourhood U_{n+1} of the origin in F_{n+1} such that

$$U_{n+1} \cap (F_n \oplus E_{n+1}) = \Gamma(U_n \cup V_{n+1}).$$

Let $U = \bigcup_{n=1}^{\infty} U_n$. It is easy to see that $U \cap \left(\bigoplus_{n=1}^{\infty} E_n\right)$ coincides with the absolutely convex hull $\Gamma\left(\bigcup_{n=1}^{\infty} V_n\right)$ of $\bigcup_{n=1}^{\infty} V_n$ and $U \cap F_n$ coincides with U_n .

Since $\mathcal{D}(\Omega)$ is the inductive limit of the sequence (F_n) we have that U is a neighbourhood of the origin in $\mathcal{D}(\Omega)$ and therefore E is a subspace of $\mathcal{D}(\Omega)$. On the other hand, E is complete, hence E is closed in $\mathcal{D}(\Omega)$.

If P and Q are two compact subsets of \mathbb{R}^p and \mathbb{R}^q , respectively, then

$$\mathcal{D}_P(\mathbb{R}^p) \widehat{\otimes} \mathcal{D}_Q(\mathbb{R}^q) \simeq \mathcal{D}_{P \times Q}(\mathbb{R}^{p+q}),$$

(see [1], Chapter II, p. 84). If α_r and β_r are two real numbers such that $\alpha_r < \beta_r$, $r = 1, 2, \dots, m$, and $I_r = [\alpha_r, \beta_r]$, then $\mathcal{D}_{I_r}(\mathbb{R}^1) \simeq s$, (see [3], p. 176). We have

$$\begin{aligned} \mathcal{D}_{I_1 \times I_2}(\mathbb{R}^2) &\simeq \mathcal{D}_{I_1}(\mathbb{R}^1) \widehat{\otimes} \mathcal{D}_{I_2}(\mathbb{R}^1) \simeq s \widehat{\otimes} s \simeq s, \\ \mathcal{D}_{I_1 \times I_2 \times I_3}(\mathbb{R}^3) &\simeq \mathcal{D}_{I_1 \times I_2}(\mathbb{R}^2) \widehat{\otimes} \mathcal{D}_{I_3}(\mathbb{R}^1) \simeq s \widehat{\otimes} s \simeq s, \\ &\dots\dots\dots \\ \mathcal{D}_{I_1 \times I_2 \times \dots \times I_m}(\mathbb{R}^m) &\simeq \mathcal{D}_{I_1 \times I_2 \times \dots \times I_{m-1}}(\mathbb{R}^{m-1}) \widehat{\otimes} \mathcal{D}_{I_m}(\mathbb{R}^1) \simeq s \widehat{\otimes} s \simeq s \end{aligned}$$

and therefore $E_n \simeq s$, hence $\bigoplus_{n=1}^{\infty} E_n \simeq s^{(\mathbb{N})}$. q.e.d.

THEOREM 3. — *Given a space E , suppose that E has a subspace G topologically isomorphic to $s^{(\mathbb{N})}$. Then E has a separated quotient which is not locally complete.*

Proof. — According to Corollary 1.2, there is in G a closed subspace H so that G/H is topologically isomorphic to a non-closed dense subspace of ω . Since G/H is a closed subspace of E/H and G/H is not locally complete, we have that E/H is not locally complete. q.e.d.

COROLLARY 1.3. — *There is a quotient of $\mathcal{D}(\Omega)$ which is not locally complete.*

Proof. — It is immediate from the Lemma and Theorem 3. q.e.d.

Note 2. — O. G. Smoljanov prove in [6] that there is a quotient of $\mathcal{D}(\mathbb{R}^1)$ which is topologically isomorphic to a non-closed dense subspace of ω . Since $\mathcal{D}(\Omega)$ has a quotient topologically isomorphic to $\mathcal{D}(\mathbb{R}^1)$, [10], Corollary 1.3 can be obtained from the result of Smoljanov.

The following result, which will be used afterwards was proved in [9]: *b) Let G be a Banach space of infinite dimension. If a space L is not dual locally complete, there is a linear mapping g of L into G , with closed graph, which is not weakly continuous.*

THEOREM 4. — *In $\mathcal{D}'(\Omega)$ there is a closed subspace G such that, if H is an arbitrary infinite-dimensional Banach space, there is a linear mapping from G into H , with closed graph, which is not weakly continuous.*

Proof. — Corollary 1.3 gives a closed subspace L of $\mathcal{D}(\Omega)$ such that $\mathcal{D}(\Omega)/L$ is not locally complete. If G is the orthogonal subspace of $\mathcal{D}'(\Omega)$ to L , then G is not dual locally complete. Now it suffices to use result *b)*. q.e.d.

We shall use the following result that we proved in [10]: *c) Let G be a (LF)-space. If there is in G a closed subspace H such that G/H is a non complete separable metrizable space, then $G'[\mu(G', G)]$ is not B_r -complete. We take G separable in [10], however the proof is valid also if G is non-separable.*

THEOREM 5. — *Let (E_n) be a sequence of infinite-dimensional separable Fréchet spaces. Then the dual of $(E \widehat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ with the Mackey topology is not B_r -complete.*

Proof. — It is direct consequence of result *c)* and Theorem 1. q.e.d.

THEOREM 6. — *Let (E_n) be a sequence of non-Banach Fréchet spaces. Then the dual of $(E \widehat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ with the Mackey topology is not B_r -complete.*

Proof. — It follows straightforward from result c) and Theorem 2. q.e.d.

In Theorem 7 we suppose that (E_n) is a sequence of infinite-dimensional spaces such that there is an absolutely convex weakly compact separable subset of E_n which is total in E_n , $n = 1, 2, \dots$. We can take an infinite-dimensional closed subspace M of $E'[\sigma(E', E)]$ which contains a total compact separable subset.

THEOREM 7. — *Let $L = (E \hat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$. Then there is a closed subspace G of $L'[\sigma(L', L)]$ and a topological homomorphism f from G into M^N , such that $f(G)$ is a non-closed locally dense subspace of M^N .*

Proof. — Let $T = E/M^\perp$, being M^\perp the orthogonal subspace of E to M . Then it is possible to identify $T'[\sigma(T', T)]$ with M .

If $T_n = T$, $n = 1, 2, \dots$, we can write T^N in the form $\prod_{n=1}^{\infty} T_n$. Let X be the weak dual of $E \hat{\otimes} F$. In the same way as in the proof of Theorem 1, we see that there is a topological homomorphism u from $E \hat{\otimes} F$ onto $\prod_{n=1}^{\infty} T_n$. Let ${}^t u$ be the mapping from

$$\left(\bigoplus_{n=1}^{\infty} T'_n \right) \left[\sigma \left(\bigoplus_{n=1}^{\infty} T'_n, \prod_{n=1}^{\infty} T_n \right) \right] \text{ into } X,$$

which is the transposed of u . Let $Y = {}^t u \left(\bigoplus_{n=1}^{\infty} T'_n \right)$ and $Y_n = {}^t u(T'_n)$. Obviously, ${}^t u$ is an injective topological homomorphism and therefore Y and Y_n , $n = 1, 2, \dots$, are closed subspaces of X which can be identified with the weak duals of $\prod_{n=1}^{\infty} T_n$ and T_n , respectively.

Taking into account result a) being $G = E'_n[\mu(E'_n, E_n)]$ and $H = Y_n$, we can find an injective continuous mapping ν_n from $E'_n[\sigma(E'_n, E_n)]$ into Y_n such that $\nu_n(E'_n)$ is a non-closed dense subspace of Y_n . We define a mapping f from

$$G = Y \times \prod_{n=1}^{\infty} E'_n[\sigma(E'_n, E_n)] \text{ into } \prod_{n=1}^{\infty} Y_n$$

such that if

$$y = \sum_{n=1}^{\infty} y_n \in Y, \quad y_n \in Y_n, \quad n = 1, 2, \dots,$$

with $\sum_{n=1}^{\infty} y_n$ having only a finite number of non-zero terms, and

$$x = (x_1, x_2, \dots, x_n, \dots) \in \prod_{n=1}^{\infty} E'_n,$$

then

$$f(y, x) = (y_1 + \nu_1(x_1), y_2 + \nu_2(x_2), \dots, y_n + \nu_n(x_n), \dots).$$

The mapping f is a topological homomorphism from G into $\prod_{n=1}^{\infty} Y_n \simeq M^{\mathbb{N}}$ such that its image is a non-closed locally dense subset of $\prod_{n=1}^{\infty} Y_n$ (see the proof of Theorem 1). q.e.d.

THEOREM 8. — *Let (E_n) be a sequence of infinite-dimensional separable Fréchet spaces. Let $L = (E \otimes F) \times \bigoplus_{n=1}^{\infty} E_n$. Then there is a separated quotient of $L'[\sigma(L', L)]$ which is not locally complete.*

Proof. — In E_n there is an absolutely convex total separable weakly compact subset, and therefore it is possible to apply Theorem 7.

With the same notation as in Theorem 7, let B be the kernel of f . Then G/B is a closed subspace of $L'[\sigma(L', L)]/B$. Since G/B is not locally complete, we can conclude that $L'[\sigma(L', L)]/B$ is not locally complete. q.e.d.

COROLLARY 1.8. — *Let H be a space which contains a subspace M topologically isomorphic to $s^{(\mathbb{N})}$. Then there is a separated quotient of $H'[\mu(H', H)]$ which is not locally complete.*

Proof. — Let M^{\perp} be the subspace of $H'[\mu(H', H)]$ orthogonal to M . Let φ be the canonical mapping from $H'[\mu(H', H)]$ onto $H'[\mu(H', H)]/M^{\perp}$. Since

$$s^{(\mathbb{N})} \simeq s \times s^{(\mathbb{N})} \simeq (s \widehat{\otimes} s) \times s^{(\mathbb{N})},$$

we apply Theorem 8 and obtain a separated quotient P of $H'[\mu(H', H)]/M^\perp$ which is not locally complete. Let ψ be the canonical mapping from $H'[\mu(H', H)]/M^\perp$ onto P . If Q is the kernel of $\psi \circ \varphi$ then $H'[\mu(H', H)]/Q$ is topologically isomorphic to P and therefore it is not locally complete.
q.e.d.

Note 3. — In [5] D. A. Raikov proves that $\mathcal{D}'(\mathbb{R}^1)$ has a separated non-complete quotient. Considering that $\mathcal{D}(\Omega)$ contains a subspace topologically isomorphic to $s^{(\mathbb{N})}$, according to the Lemma, Raikov's result is a particular case of our Corollary 1.8.

THEOREM 9. — *There is a closed subspace G of $\mathcal{D}(\Omega)$ such that, if L is an arbitrary infinite dimensional Banach space, it is possible to find a linear mapping from G into L , with closed graph, which is not weakly continuous.*

Proof. — Because of Corollary 1.8 and the Lemma, there is a closed subspace H of $\mathcal{D}'(\Omega)$ so that $\mathcal{D}'(\Omega)/H$ is not locally complete. If G is the orthogonal subspace of $\mathcal{D}(\Omega)$ to H , then G is not dual locally complete. It suffices to apply result b).
q.e.d.

In order to prove Theorem 10 and Theorem 12 we shall need the following result that we have proved in [10]: *d) Let E be a separable space. Let (E_m) be an increasing sequence of subspaces of E so that $E = \bigcup_{m=1}^{\infty} E_m$. If there is a bounded set A in E such that $A \not\subset E_m$, $m = 1, 2, \dots$, there is a dense subspace F of E , $F \neq E$, so that $F \cap E_m$ is finite-dimensional for every m positive integer.*

THEOREM 10. — *Let (E_n) be a sequence of infinite-dimensional Fréchet spaces. Then $(E \otimes F) \times \bigoplus_{n=1}^{\infty} E_n$ is not B_r -complete.*

Proof. — Since every closed subspace of a B_r -complete spaces is B_r -complete, [4], we can suppose that E, F, E_n , $n = 1, 2, \dots$, are separable spaces.

With the same notation as in the proof of Theorem 7, let B

be the kernel of f . Then B is a closed subspace of

$$P = X \times \prod_{n=1}^{\infty} E'_n[\sigma(E'_n, E_n)]$$

which lies in G .

Let (U_n) be a fundamental system of neighbourhoods of the origin in $E \hat{\otimes} F$ such that $U_n \supset U_{n+1}$, $n = 1, 2, \dots$. Being A_n the polar set of U_n in X , let H_n be the linear hull of A_n with the induced topology of $\sigma(X, E \hat{\otimes} F)$ and let

$$G_n = H_n \times \prod_{n=1}^{\infty} E'_n[\sigma(E'_n, E_n)].$$

Let z_n be an element in Y_n such that $z_n \notin \nu_n(E'_n)$. Then the n th coordinate of $f(z_n, 0)$ coincides with z_n and the remaining are equal to zero, hence $(f(z_n, 0))$ is a bounded sequence in $\prod_{n=1}^{\infty} Y_n$.

If h is the canonical mapping from P onto P/B , h coincides with f on G and consequently $(h(z_n, 0))$ is a bounded sequence in P/B .

X_{A_n} is a Banach space with $Y \cap X_{A_n}$ as a closed subspace. Since

$$\left(\sum_{m=1}^p Y_m \right) \cap X_{A_n}$$

is a closed subset of X_{A_n} and

$$\cup \left\{ \sum_{m=1}^p Y_m : p = 1, 2, \dots \right\} = Y,$$

we can use the Baire theorem and we obtain a positive integer r such that

$$\left(\sum_{m=1}^r Y_m \right) \cap X_{A_n} = Y \cap X_{A_n}.$$

Therefore, if q is an integer larger than r

$$Y_q \cap X_{A_n} = \{0\},$$

and thus $z_q \notin H_n$.

Now fix $q > r$ and assume that for some n there exists an element t_n in P such that

$$h(t_n) = h(z_q, 0)$$

with

$$t_n = (y_n, (x_1, x_2, \dots, x_m, \dots)),$$

$$y_n \in H_n, (x_1, x_2, \dots, x_m, \dots) = x \in \prod_{m=1}^{\infty} E'_m.$$

Therefore $h(t_n - z_q, 0) = 0$ and, since $(z_q, 0) \in G$ and B lies in G , it results that $t_n \in G$. Thus $y_n \in Y$. Hence

$$y_n \in Y \cap H_n = Y \cap X_{A_n} = \left(\sum_{m=1}^r Y_m \right) \cap X_{A_n}.$$

We can write

$$y_n = \sum_{p=1}^r y_{np}, \quad y_{np} \in Y_p$$

and therefore

$$h(t_n) = f(y_n, x) = (y_{n_1} + \nu_1(x_1), y_{n_2} + \nu_2(x_2), \dots, \\ y_{n_r} + \nu_r(x_r), \nu_{r+1}(x_{r+1}), \nu_{r+2}(x_{r+2}), \dots)$$

and thus the q th component in $h(t_n)$ coincides with $\nu_q(x_q)$. On the other hand, $h(z_q, 0) = f(z_q, 0)$ has its q th component equal to z_q and since $z_q \notin \nu_q(E'_q)$ this a contradiction. Consequently $h(z_q, 0) \notin h(G_n)$.

From the last considerations we can assert that the subset of P/B

$$\{h(z_1, 0), h(z_2, 0), \dots, h(z_n, 0), \dots\}$$

is bounded and is not included in $h(G_n)$ for any positive integer n . Moreover,

$$P/B = \cup \{h(G_n) : n = 1, 2, \dots\}.$$

We can apply result *d*) in order to obtain a dense non closed subspace V in P/B such that $V \cap h(G_n)$ is finite-dimensional, $n = 1, 2, \dots$. If x_0 is a point lying in the closure of $G_n \cap h^{-1}(V)$ in G_n , then $h(x_0)$ is a point of the closure of $h(G_n) \cap V$ in $h(G_n)$. The space $h(G_n) \cap V$ is finite-dimensional and thus $h(x_0) \in h(G_n) \cap V$. Hence $x_0 \in G_n \cap h^{-1}(V)$ and therefore this space is closed in G_n .

If A is a compact subset of P , let $A^{(1)}$ and $A^{(2)}$ be the projection of A onto X and $\prod_{n=1}^{\infty} E'_n$, respectively. Since (A_n) is a fundamental system of compact subsets in X , there is a

positive integer n_0 such that $A^{(1)} \subset A_{n_0}$. Therefore, $A \in G_{n_0}$. Moreover $G_{n_0} \cap h^{-1}(V)$ is closed in G_{n_0} , thus $A \cap h^{-1}(V)$ is compact. Since $h^{-1}(V)$ is a non-closed dense subspace of P it results that $(E \widehat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$ is not B_r -complete. q.e.d.

THEOREM 11. — *Let G be a space which contains a subspace L topologically isomorphic to $s^{(N)}$. Then G is not B_r -complete.*

Proof. — Since every closed subspace of a B_r -complete space is B_r -complete, [4], and

$$L \simeq s^{(N)} \simeq s \times s^{(N)} \simeq (s \widehat{\otimes} s) \times s^{(N)},$$

it suffices to apply Theorem 10.

q.e.d.

COROLLARY 1.11. — *The space $\mathcal{D}(\Omega)$ is not B_r -complete.*

Proof. — It suffices to apply the Lemma and Theorem 11. q.e.d.

THEOREM 12. — *Let (E_n) be a sequence of infinite-dimensional separable Fréchet spaces. Let G be a separable countable inductive limit of Fréchet spaces. If there is a subspace L of G which is topologically isomorphic to $(E \widehat{\otimes} F) \times \bigoplus_{n=1}^{\infty} E_n$, then $G'[\mu(G', G)]$ is not B_r -complete.*

Proof. — Let us suppose that G is the inductive limit of the increasing sequence (G_n) of Fréchet spaces. From Theorem 1 it follows that there is a closed subspace Q of L such that L/Q is topologically isomorphic to a non-closed dense subspace of $\prod_{m=1}^{\infty} E_m$.

If φ is the canonical mapping from G onto G/Q , it leads up to $\varphi(G_n) \not\supset \varphi(L)$, $n = 1, 2, \dots$, for if there is a positive integer n_0 such that $\varphi(G_{n_0}) \supset \varphi(L)$, the restriction ψ of φ to $L \cap G_{n_0}$, considering this space as a subspace of G_{n_0} , is continuous and $\psi(L \cap G_{n_0}) = \varphi(L)$. Since the subspace $\varphi(L)$ of G/Q is topologically isomorphic to L/Q , which is barrelled, we use the open-mapping theorem, [4],

and we obtain that ψ is open and so $\varphi(L)$ is a Fréchet space. But we arrive to a contradiction since $\varphi(L)$ is topologically isomorphic to a non-closed dense subspace of $\prod_{n=1}^{\infty} E_n$.

Consequently, let (n_p) be an increasing sequence of positive integers such that

$$\varphi(G_{n_p}) \cap \varphi(L) \neq \varphi(G_{n_{p+1}}) \cap \varphi(L), \quad p = 1, 2, \dots$$

Let us pick an element $x_{p+1} \in \varphi(G_{n_{p+1}}) \cap \varphi(L)$ which is not in $\varphi(G_{n_p})$. Since $\varphi(L)$ is metrizable, there exists a sequence of strictly positive real numbers (α_p) such that the set

$$A = \{\alpha_1 x_1, \alpha_2 x_2, \dots, \alpha_p x_p, \dots\}$$

is bounded. Hence A is a bounded set in G/Q which is not contained in $\varphi(G_{n_p})$, $p = 1, 2, \dots$, and therefore, using result *d*), there is a dense subspace F of G/Q , $F \neq G/Q$, such that $F \cap \varphi(G_n)$ is finite-dimensional, $n = 1, 2, \dots$.

Let Q^\perp be the orthogonal subspace of Q in $G'[\mu(G', G)]$. If B is an absolutely convex closed subset of G/Q equicontinuous on Q^\perp then $(G/Q)_B$ is a Banach space. Since G/Q is a countable inductive limit of Fréchet spaces, we can use Grothendieck's closed graph theorem, [1], applied to the canonical injection from $(G/Q)_B$ into G/Q . Then there is a positive integer n_i such that B is contained in $\varphi(G_{n_i})$, hence

$$B \cap F = B \cap \varphi(G_{n_i}) \cap F$$

and since $\varphi(G_{n_i}) \cap F$ is a finite dimensional space, $B \cap F$ is closed in B . Hence Q^\perp is not B_r -complete.

Since each closed subspace of a B_r -complete space is itself B_r -complete, [4], it follows that $G'[\mu(G', G)]$ is not B_r -complete. q.e.d.

Note. — In [10] we proved that the space $\mathcal{D}'(\Omega)$ is not B_r -complete. This result follows also from our Theorem 12, considering that $\mathcal{D}(\Omega)$ is a countable strict inductive limit of separable Fréchet spaces and that it has a subspace L such that $L \simeq s^{(N)} \simeq (s \widehat{\otimes} s) \times s^{(N)}$.

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