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HANS G. FEICHTINGER

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ON A CLASS OF CONVOLUTION ALGEBRAS OF FUNCTIONS

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Introduction.

In this note we give a general construction of convolution algebras of measurable (continuous) functions on certain locally compact groups. The spaces $\Lambda(A, B, X, G)$ constructed here will consist of those functions of a convolution algebra $A \cap B$ which can in a certain sense be "well approximated" by functions with compact support.

Although this construction seems perhaps a bit artificial there is a great number of examples of spaces of this type that have a quite natural interpretation. On the other hand the given construction demonstrates their common properties in the best way and reveals most of their structure. It is also the most direct approach to the results presented here. Several further assumptions are necessary for the theorems but if one takes concrete examples it can be shown that most of these assumptions are fulfilled in the cases of interest.

The paper is organized in the following way. In the first section the notation will be fixed and the material we need for the construction will be prepared. § 2 contains the definition of the spaces $\Lambda(A, B, X, G)$ and a demonstration of their fundamental properties. It is the main part of this paper. Examples of such spaces that are defined in a different, more natural way can be found in the last section. § 3 contains some results on inclusions between such spaces. Finally § 4 presents further results, especially for spaces defined on Abelian groups. Most of the results in this section are derived from more general theorems on Banach convolution algebras. In this connection a paper of Domar is of importance for us.

The spaces $\Lambda(A, B, X, G)$ are a generalization of the spaces $\Lambda_g(G)$ or $\Lambda_g^0(G)$ which have been considered in an earlier note [3]. Therefore most of the assertions stated in that note are special cases of the theorems presented here.

1. Preliminaries.

G shall denote a noncompact locally compact group that is σ-compact, dx shall be a fixed left invariant Haar measure on G. For a measurable set M, $|M|$ shall denote its measure. For any function f on G and $y \in G$ let $L_y f\,(R_y f)$ be defined by

$$L_y f(x) := f(y^{-1}x) \quad ; R_y f(x) := f(xy^{-1})\,\Delta(y^{-1}),$$

where Δ is the modular function on G. K(G) denotes the space of all continuous functions on G with compact support. Throughout this paper it will be more convenient to speak as usual of "measurable functions" on G identifying two functions which coïncide almost everywhere (a.e.), than to speak of equivalence classes of measurable functions.

A normed space of measurable functions will be called F-*space,* if every convergent sequence has a subsequence converging almost everywhere. If the space is complete it will be called a BF-*space.* A normed space B of measurable (continuous) functions is called *solid,* if for every function $f \in B$ and any measurable (continuous) function g satisfying $|g(x)| \leqslant |f(x)|$ a.e., $g \in B$ and $\|g\|_B \leqslant \|f\|_B$ must hold.

It is well known that any solid Banach space of measurable functions is also a BF-space. Sometimes such spaces are called Banach function spaces. Moreover the norm of a BF-space is unique up to equivalence by the closed graph theorem.

The most important solid BF-spaces are of course the spaces $L^p(G)$, $1 \leqslant p \leqslant \infty$ of absolutely p-summable or essentially bounded functions on G respectively. $C^0(G)$, the space of continuous functions on G vanishing at infinity is a solid space of continuous functions. It can be identified with the closure of K(G) in $L^\infty(G)$. The corresponding sequence spaces will be denoted by z^p and c_0.

Since we shall be concerned with spaces of functions on groups the following properties are of importance :

L1) B is *left invariant,* i.e. $L_y B \subset B$ for every $y \in G$;

L2) B satisfies L1) and $y \longrightarrow L_y f$ is a *continuous function* from G into B for all $f \in B$;

L3) B satisfies L1) and L_y is a *contraction,* i.e. $\| L_y \|_B \leqslant 1$ for all $y \in G$.

Right invariance and properties R1) – R3) are defined in a similar way, with L_y replaced by R_y. We note that the set of all BF-spaces forms a lattice, if we define for two BF-spaces B_1 and B_2 :

$$B_1 \wedge B_2 := B_1 \cap B_2 \quad , \quad \| f \|_\wedge := \| f \|_{B_1} + \| f \|_{B_2} \ ;$$

Whenever $B_1 \cap B_2$ appears it will be thought of this BF-space.

$$B_1 \vee B_2 := \{ f \mid f = f_1 + f_2 \quad , \quad f_i \in B_i \} \, .$$

$$\| f \|_\vee := \inf \{ \| f_1 \|_{B_1} + \| f_2 \|_{B_2} \quad , \quad f = f_1 + f_2 \} \, .$$

The subset of solid BF-spaces forms of course a sublattice. The same is true for all BF-spaces on G satisfying one of the conditions L1) – L3) or R1) – R3). For later reference we state here some facts concerning the above properties. For simplicity we give only the "left" versions. The "right" versions can be proved in a similar way.

LEMMA 1.1. – *Let* B *be a left invariant* BF-*space, then* L_y *is a bounded operator for every* $y \in G$. *If* $\| L_y \|_B$ *denotes the operator norm on this space the following inequality holds :*

$$\| L_{xy} \|_B \leqslant \| L_x \|_B \ \| L_y \|_B \quad \textit{for} \quad x , y \in G \, .$$

Proof. – The assertion follows from the closed graph theorem since L_y is evidently a linear operator having closed graph. The inequality is a trivial consequence therefrom.

All left invariant BF-spaces defined in a natural way satisfy L4) $y \longrightarrow \| L_y \|_B$ is a *locally bounded function.*

Remark. – 1) In many cases $y \longrightarrow \| L_y f \|_B$ is a continuous function for every f out a dense subspace of B. In this case it follows that $y \longrightarrow \| L_y \|_B$ is a measurable function on G, being semi-continuous. On the other hand B satisfies L4) if $G = \mathbf{R}$ and if $y \longrightarrow \| L_y \|_B$ is a measurable function on $\mathbf{R}$, $y \longrightarrow \log \| L_y \|_B$ being a subadditive function. For a proof see [5], Chap. VI. A similar proof applies to the torus group $G = \mathbf{T}$. It is difficult to derive therefrom a similar result for groups of the form

$$G = \mathbf{R}^m \times \mathbf{T}^n \times \mathbf{Z}^s , m , n , s \in \mathbf{N} .$$

These calculations show for example, that any space

$$L^p_w(G) = \{f | fw \in L^p(G)\} , \ 1 \leqslant p < \infty\} , \ w(x) \geqslant 1$$

which satisfies L1) also satisfies L4), K (G) being dense in it.

LEMMA 1.2. – *If* B *satisfies* L4) *then*

$$B_c \ : = \ \{f \mid f \in B, \ y \longrightarrow L_y f$$

is a continuous function from G into B} *is a closed subspace of* B.

Proof. – For a fixed compact neighbourhood U_0 of the identity $\sup \{ \| L_y \|_B , y \in U_0 \} \leqslant K_0 < \infty$ for some $K_0 \geqslant 1$ by L4). If now $\epsilon > 0$ and f in the closure of B_c are given, there is some $h \in B_c$ such that $\| f - h \|_B < \epsilon/3K_0$. Since h lies in B_c there is some $U \subseteq U_0$ such that $\| h - L_y h \|_B < \epsilon/3$ for all $y \in U$. All together we have $\| f - L_y f \|_B \leqslant \| f - h \|_B + \| h - L_y h \|_B + \| L_y \|_B \ \| h - f \|_B < \epsilon$ for all $y \in U$, showing that f lies in B_c.

Now we are going to prepare the material we need for our construction.

I) In the sequel $(B_n)_{n \geqslant 0}$ will denote a (fixed) sequence of neighbourhoods of the identity (except $n = 0$) such that we have

$$\text{S1)} \ B_0 = \phi , \ \bigcup_{n=1}^{\infty} B_n = G \ ;$$

$$\text{S2)} \ B_n B_n \subseteq B_{n+1} \quad \text{for} \quad n \geqslant 1 \ ;$$

to avoid trivialities we assume further

$$\text{S3)} \quad B_n \neq G \quad \text{for all} \quad n \geqslant 0 .$$

The characteristic function of $G \setminus B_n$ shall be denoted by χ_n.

Remarks. — 2) On every compactly generated group G one can find such a sequence, e.g. by taking $B_n = U^{2^{n-1}}$ for $n \geqslant 1$, U being an arbitrary compact neighbourhood of e, but one can take $B_n = U^{5^{n-1}}$ as well.

3) If G is connected S2) and S3) imply $B_n \neq B_{n+1}$ for $n \geqslant 1$, since the closure of B_n must be contained in the interior of B_{n+1}.

4) From S1) and S2) property S4) follows :

S4) For every compact set $K \subseteq G$ there is some $n_K \in N$ such that $K \subseteq B_n$ for $n \geqslant n_K$.

5) If $G = G_1 \times G_2$, and $(B^i_n)_{n \geqslant 0} \subset G_i$, $i = 1, 2$ are given satisfying S1) and S2), then $(B_n)_{n \geqslant 0}$, $B_n := B^1_n \times B^2_n$ also satisfies S1) and S2).

6) We don't suppose that the B_n's are relatively compact sets (e.g. strips in R^2).

II) X is a *solid* BK-*space* which is *right invariant,* i.e.

X1) $(X, |\ |_X)$ is a Banach space of bounded sequences ; these will be denoted by $x = (x_n) = (x_n)_{n \geqslant 0}$.

X2) X is an ideal in l^∞ (with multiplication coordinatewise) such that

X3) $|xy|_X \leqslant |x|_X\ |y|_\infty$ for all $x \in X$ and $y \in l^\infty$;

X4) X contains all "finite" sequences and $|(1, 0, \ldots)|_X = 1$;

X5) $D : (x_0, x_1, \ldots) \longrightarrow (0, x_0, x_1, \ldots)$ satisfies $DX \subseteq X$.

As an immediate consequence of the closed graph theorem X5) implies that D is a continuous operator on X. We denote its operator norm by $\|D\|_X$. Sometimes it will be necessary (this condition will be indicated seperately) to suppose.

X6) The space of all finite sequences is dense in X,

i.e. $|(0, \ldots 0, x_n, x_{n+1}, \ldots)|_X \longrightarrow 0$ as $n \longrightarrow \infty$ for every $x \in X$.

Given a solid BK-space we shall call the sequence $c = (c_n)$ of positive numbers defined by $c_n := |(1, \ldots, 1, 0, \ldots)|_X$ (n times one) the *fundamental sequence* of X.

Remark. – 7) $c = (c_n)$ is a nondecreasing sequence. If X is a proper subspace of c_0 c is unbounded (We shall use only such spaces). If X satisfies X5) we have by X4) (i.e. $c_0 = 1$) :

$$c_n \leqslant (\|D\|_X + 1) c_{n-1} \leqslant (\|D\|_X + 1)^n .$$

Note that different spaces may have the same fundamental sequences.

The most important examples of such spaces are weighted c_0 and l^p-spaces, e.g.

$$X = X_s^q = \{(x_n) \mid (x_n n^s) \in l^q\} \quad \text{for some} \quad s \geqslant -\frac{1}{q} \quad \text{or}$$

$$X = X_s^0 = \{(x_n) \mid x_n n^s \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty\} \quad \text{for some} \quad s \geqslant 0.$$

Orlicz sequence spaces should be mentioned too.

III) $(A, |\ |_A)$ shall denote a solid BF space on G which is a *Banach convolution algebra,* i.e. $|f * g|_A \leqslant K\, |f|_A\, |g|_A$ for $f, g \in A$ and a fixed constant $K < \infty$. Without loss of generality we suppose $K = 1$.

IV) $(B, |\ |_B)$ shall be a solid BF-space on G which is a *twosided Banach-A-convolution module,* i.e. for $f \in A$, $g \in B$ $f * g$ and $g * f$ are in B,

$$f * g := \int_G f(y)\, L_y g\, dy = \int_G R_y\, f g(y)\, dy$$

and $|f * g|_B \leqslant K_1\, |f|_A\, |g|_B$ and $|g * f|_B \leqslant K_1\, |f|_A\, |g|_B$ for some constant $K_1 < \infty$. We suppose again $K_1 = 1$. B is called an *essential* module if $A * B$ is dense in B.

If B is contained in A, B is called a *normed ideal* of A.

Remark. – 8) It follows from the assumptions that $A \cap B$ is a solid BF-space and furthermore a Banach convolution algebra. Thus $A \cap B$ contains K(G) if A is left invariant. This can be shown in the following way :

LEMMA 1.3. – *If* B *is a normed left (right) invariant, solid* BF-*space on* G, *containing any function* f_0, *continuous on some open set, then* K(G) *is continuously embedded into* B.

Proof. – Since B is solid, we may suppose that f_0 is a positive, continuous function with compact support, such that $f_0(x_0) > 0$ for some point $x_0 \in \mathrm{G}$. Thus for some

$$\delta > 0 \quad \mathrm{U} := \{x \in \mathrm{G} \ , \ f_0(x) \geqslant \delta > 0\}$$

is a nonvoid, open, relatively compact subset of G. If K is an arbitrary compact subset of G, then there is some

$$\Psi_{\mathrm{K}} \in \mathrm{K}_+(\mathrm{G}) \ , \ \|\Psi_{\mathrm{K}}\|_\infty \leqslant 1 \ , \ \Psi_{\mathrm{K}}(x) = 1$$

for all $x \in \mathrm{K}$. Since $\mathrm{K}_1 := \operatorname{supp} \Psi_{\mathrm{K}}$ is compact there is a finite sequence $(y_i)_{i=1}^n \subseteq \mathrm{G}$ such that $\mathrm{K}_1 \subseteq \bigcup_{i=1}^n y_i \mathrm{U}$. It follows that for an arbitrary $k \in \mathrm{K}(\mathrm{G})$, supp $k \subseteq \mathrm{K}$ the following inequality holds :

$$|k(x)| \leqslant \|k\|_\infty \Psi_{\mathrm{K}}(x) \leqslant \|k\|_\infty \delta^{-1} \sum_{i=1}^n \mathrm{L}_{y_i} f_0(x).$$ The finite sum is an element of B and thus k lies in B, B being solid. Moreover $\|k\|_{\mathrm{B}} \leqslant \|\Psi_{\mathrm{K}}\|_{\mathrm{B}} \ \|k\|_\infty$ holds, showing that the inclusion $\mathrm{K}(\mathrm{G}) \longrightarrow \mathrm{B}$ is continuous.

COROLLARY 1.1. – *Let* $\mathrm{B} \neq \{0\}$ *be a solid, left invariant space that is closed under convolution. Then* $\mathrm{K}(\mathrm{G}) \subseteq \mathrm{B}$.

Proof. – If $\mathrm{B} \neq \{0\}$ then there is a measurable set

$$\mathrm{M} \subseteq \mathrm{G} \ , \ |\mathrm{M}| > 0 \ ,$$

such that $\chi_{\mathrm{M}} \in \mathrm{B}$. Since $\chi_{\mathrm{M}} \in \mathrm{L}^1 \cap \mathrm{L}^\infty(\mathrm{G})$, $\chi_{\mathrm{M}} * \chi_{\mathrm{M}} \in \mathrm{B}$ is a positive, continuous function and $\chi_{\mathrm{M}} * \chi_{\mathrm{M}}(0) = |\mathrm{M}| > 0$. Thus lemma 1.3. is applicable.

We shall assume from now on that any solid space appearing in the context contains K(G). As we have seen this condition is rather mild.

LEMMA 1.4. – *If* B *is a left invariant* BF-*space, containing* K(G) *as a dense subspace, then* B *satisfies* L4) *and* L2).

Proof. – By remark 1) B satisfies L4). Therefore B_c is closed in B by lemma 1.2.. By lemma 1.3. B_c contains K(G) and therefore $\mathrm{B} = \mathrm{B}_c$, i.e. B satisfies L2).

The most important examples for III are of course the group algebra $L^1(G)$ itself or an arbitrary Beurling algebra

$$L^1_w(G) := \{f \mid fw \in L^1(G)\}$$

([7], Chap. 3, § 7.1.) defined by means of a weight function w satisfying

W1) $w(x) \geqslant 1$,

W2) $w(xy) \leqslant w(x)\, w(y)$ for all $x, y \in G$,

W3) w is locally bounded.

Corresponding to $L^1(G)$ one can take as A-convolution-module $B = L^p(G)$ or $L^1 \cap L^p(G)$, $1 \leqslant p \leqslant \infty$; $B = C^0(G)$ or $L^1 \cap C^0(G)$ if G is a unimodular group. All these spaces satisfy L1) – L3) and R1) – R3). If G is not unimodular R3) fails to hold and one has to replace $L^1(G)$ by $A = L^1_{w_0}(G)$ defined by $w_0(x) := \max(\Delta(x), 1)$. In this connection the following lemma is of great use.

LEMMA 1.5. – *Let* $B \subseteq L^1_{\mathrm{loc}}(G)$ *be a left invariant* BF-*space satisfying* L2), *then* B *is a left convolution module over some Beurling algebra* $L^1_w(G)$.

Proof. – If we define $w(x) := \max(1, \|L_y\|_B)$, then w satisfies W1) and W2) by lemma 1.1. and L2) implies W3). Thus $L^1_w(G)$ is a Beurling algebra. Furthermore we have for $f \in L^1_w(G)$, $g \in B$:

$$\|f * g\|_B = \left\| \int f(y)\, L_y g\, dy \right\|_B \leqslant \int |f(y)|\ \|L_y g\|_B\, dy \leqslant$$

$$\leqslant \int |f(y)|\, w(y)\ \|g\|_B\, dy \leqslant \|f\|_{w,1}\ \|g\|_B .$$

Thus B is a left $L^1_w(G)$ – convolution module.

It follows from lemma 1.3. and lemma 1.4. that any solid BF-space which is left and right invariant and contains K(G) as a dense subspace is a twosided Banach convolution module for a suitable Beurling algebra $L^1_w(G)$. Thus we have a rather extensive assortement of examples.

2. The spaces $\Lambda(A, B, X, G)$ and their fundamental properties.

We are now able to present the main results of this paper.

DEFINITION. – Let $(B_n)_{n\geq 0}$, χ_n, X be as in I and II, A and B solid BF-spaces (containing K (G)). Then we define

$$\Lambda(A, B, X, G) := \{f \in A \cap B, (|f\chi_n|_B)_{n\geq 0} \in X\}$$

$$\|f\| := |f|_A + |(|f\chi_n|_B)_{n\geq 0}|_X .$$

If $A \cap B$ is a space of continuous functions one has to replace χ_n by a continuous function ψ_n, $\chi_{n-1} \leqslant \psi_n \leqslant \chi_n$ for $n \geqslant 1$.

$\Lambda(A, B, X, G)$ is well defined, since $f\chi_n$ or $f\psi_n$ lie in B for $n \geqslant 0$, B being solid. It is easy to see that $\| \ \|$ is a norm since it follows from S1) that it is the sum of two norms. If it is clear from the context we shall omit some of the letters, writing shortly $\Lambda(A, B, X)$ or only Λ for example. Instead of $\Lambda(A, A, X, G)$ we write $\Lambda(A, X, G)$.

We do not indicate the dependence of $\Lambda(A, B, X, G)$ on the sequence $(B_n)_{n\geq 0}$ because in the most important case it is in fact independant of the choice of $(B_n)_{n\geq 0}$. More precisely we have :

LEMMA 2.1. – *Let $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ be two sequences of relatively compact subsets satisfying* S1) *and* S2') : $B_n B_n = B_{n+1}$ *and* $C_n C_n = C_{n+1}$ *for* $n \geqslant 1$. *Then the spaces* $\Lambda(A, B, X, G)$ *defined by means of these two sequences coincide and have equivalent norms.*

Proof. – Denote the spaces derived by means of $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ by Λ_1 and Λ_2 respectively. On account of the symmetry of the assumptions it will be enough to prove one inclusion, e.g. $\Lambda_1 \subseteq \Lambda_2$. By S4) there is some $n_0 \in \mathbf{N}$ such that $B_1 \subseteq C_{n_0}$. From S2') we derive $B_2 = B_1 B_1 \subseteq C_{n_0} C_{n_0} = C_{n_0+1}$ and further $B_k \subseteq C_{n_0+k-1}$ for $k \geqslant 1$. If we denote the characteristic functions of $G \setminus B_n$ and $G \setminus C_n$ by χ_n and ψ_n respectively it follows that

$$|f\psi_{n_0+k-1}|_B \leqslant |f\chi_k|_B$$

for all $k \geqslant 1$ and $f \in B$ and $|f\psi_j|_B \leqslant |f|_B = |f\chi_0|_B$ for $j < n_0$, hence the sequence $(|f\psi_0|_B)_{n\geq 0}$ is a minorant of the sequence

$$z := (|f|_B, \dots |f|_B, |f\chi_1|_B, |f\chi_2|_B, \dots).$$

But if $f \in \Lambda_1$ it follows from X5) that $z \in X$ and

$$\|f\|_{\Lambda_2} \leqslant |f|_A + |z|_X \leqslant |f|_A + c_{n_0-1}|f|_B + (\|D\|_X + 1)^{n_0-1}\|f\|_{\Lambda_1}$$
$$\leqslant [c_{n_0-1} + (\|D\|_X + 1)^{n_0-1}]\ \|f\|_{\Lambda_1}.$$

Thus $f \in \Lambda_2$ and $\|f\|_{\Lambda_2} \leqslant K\ \|f\|_{\Lambda_1}$ for some $K < \infty$. The proof is now complete.

THEOREM 2.1. – $(\Lambda(A, B, X, G), \|\ \|)$ *is a solid* BF-*space. Moreover the inclusions* $K(G) \subseteq \Lambda(A, B, X, G) \subseteq A \cap B$ *hold and*

$$|f|_A + |f|_B \leqslant \|f\| \quad \textit{for all} \quad f \in \Lambda .$$

Proof. – It is clear, that $(\Lambda, \|\ \|)$ is a normed, solid linear space contained in $A \cap B$. Furthermore we have $|f|_B = |f\chi_0|_B$ by S1) and thus $|f|_B + |f|_A \leqslant \|f\|$ by X4). Using S4) we see that for any $f \in K(G)$ $|f\chi_n|_B = 0$ for $n \geqslant n(f)$. Thus again by X4) $K(G) \subseteq \Lambda$, $K(G)$ being contained in $A \cap B$ by our general assumption. Now we have to show the completeness of $(\Lambda, \|\ \|)$. It will be enough to show that every absolutely convergent series $\sum_{k=1}^{\infty} f^k$ of nonnegative functions $f^k \in \Lambda$ with $\sum_{k=1}^{\infty} \|f^k\| \leqslant K < \infty$ represents an element $f \in \Lambda$ with $\|f\| \leqslant K$. Since $A \cap B$ is a BF-space $f = \sum_{k=1}^{\infty} f^k$ converges in $A \cap B$ and $f(x) = \sum_{k=1}^{\infty} f^k(x)$ a.e., hence

$$|f\chi_n|_B \leqslant \sum_{n=1}^{\infty} |f^k\chi_n|_B .$$

From the completeness of X we can deduce

$$\|f\| = |f|_A + |(|f\chi_n|_B)_{n\geqslant 0}|_X \leqslant |f|_A + |\sum_{k=1}^{\infty} (|f^k\chi_n|_B)_{n\geqslant 0}|_X$$
$$\leqslant |f|_A + \sum_{k=1}^{\infty} |(|f^k\chi_n|_B)_{n\geqslant 0}|_X \leqslant K < \infty .$$

The proof is now complete.

Remark. – It is clear that $|(|f\chi_n|_B)_{n\geq 0}|_X$ is a norm equivalent to $\|f\|$ if B is contained in A.

THEOREM 2.2. – *If* A *and* B *are as in* III *and* IV *respectively, then* $\Lambda(A, B, X, G)$ *is a Banach convolution algebra.*

Proof. – Let $f, g \in \Lambda$ be given. It is known that $f * g \in A \cap B$. Since Λ is solid and since $|f * g| \leq |f| * |g|$ holds we may suppose that f and g are nonnegative functions. Now using S2) we obtain for $x \in G \setminus B_n$ a.e. :

$$f * g(x) = \int_{G\setminus B_{n-1}} f(y)\, g(y^{-1}x)\, dy + \int_{B_{n-1}} f(y)\, g(y^{-1}x)\, dy$$
$$\leq \int_G f\chi_{n-1}(y)\, g(y^{-1}x)\, dy + \int_G f(y)\, g\chi_{n-1}(y^{-1}x)\, dy.$$

Thus we have shown the fundamental inequality

$$(*) \quad (f * g)\chi_n \leq f\chi_{n-1} * g + f * g\chi_{n-1} \quad \text{for} \quad n \geq 1 .$$

It follows $|(f * g)\chi_n|_B \leq |f\chi_{n-1}|_B\, |g|_A + |f|_A\, |g\chi_{n-1}|_B$ and further

$$\begin{aligned}
\|f * g\| &= |f * g|_A + |(|f * g)\chi_n|_B)_{n\geq 0}|_X \\
&\leq |f|_A\, |g|_A + |g|_A\, |(|f|_B, |f\chi_0|_B, |f\chi_1|_B, \ldots)|_X \\
&\qquad + |f|_A\, |(|g|_B, |g\chi_0|_B, \ldots)|_X \\
&\leq |f|_A\, |g|_A + |g|_A\, (1 + \|D\|_X)\, |(|f\chi_n|_B)_{n\geq 0}|_X \\
&\qquad + |f|_A\, (1 + \|D\|_X)\, |(|g\chi_n|_B)_{n\geq 0}|_X \\
&\leq (1 + \|D\|_X)\, \|f\|\, \|g\| .
\end{aligned}$$

The proof is now complete.

Since B is also a twosided Banach convolution module over the Banach algebra $\Lambda(A, X, G)$ the space

$$\Lambda(B, A, X, G) = \Lambda(A, X, G) \cap B$$

is a Banach algebra by remark 8). Considering the last part of the proof of theorem 2.2 we see that the roles of A and B can be changed on the right side of the estimate following $(*)$. Moreover

$$\Lambda(A, B) \cap \Lambda(B, A) = \Lambda(A \cap B)$$

holds. Thus we have :

THEOREM 2.3. – $\Lambda(B, A, X, G)$ is a solid Banach convolution algebra with the norm $\|f\|' := |(|f\chi_n|_A)_{n\geq 0}|_X + |f|_B$. Moreover there is some $C < \infty$ such that for $f, g \in \Lambda(B, A, X, G)$ we have $f * g \in \Lambda(A \cap B, X, G)$ and $\|f * g\|_{\Lambda(A \cap B)} \leqslant C \|f\|' \|g\|'$; in particular $\Lambda(A \cap B, X, G)$ is a normed ideal of $\Lambda(B, A, X, G)$.

In a similar way one can prove the following assertions :

THEOREM 2.4. – *If* A_2 and B_2 *are twosided Banach convolution modules over* A_1. *Then* $\Lambda(A_2, B_2, X)$ *is a twosided* $\Lambda(A_1, X)$-*convolution module.*

Proof. – A careful repetition of the proof of theorem 2.2 will convince the reader.

COROLLARY 2.1. – *Let* A_1 *and* A_2 *be solid Banach convolution algebras such that* A_2 *is a twosided* A_1-*module, then* $\Lambda(A_2, X)$ *is a twosided* $\Lambda(A_1, X)$-*module, in particular* $\Lambda(A_2, X)$ *is a normed ideal in* $\Lambda(A_1, X)$ *if* A_2 is a normed ideal of A_1.

This corollary is of special interest for the case $A_1 = L^1(G)$ and $A_2 = L^1 \cap L^p(G)$, G unimodular.

COROLLARY 2.2. – *Put*

$$\Lambda^m := \{f \in A \cap B,\ \operatorname{supp} f \subseteq B_m\} \subseteq \Lambda \subset A \cap B.$$

Then the restriction of the norm of $A \cap B$ *to* Λ^m *is equivalent to the restriction of the norm of* Λ *to* Λ^m.

Proof. – It will be sufficient to show there is some $K_m < \infty$ such that for every $f \in \Lambda^m$ $|(|f\chi_n|_B)_{n\geq 0}|_X \leqslant K_m |f|_B$ holds ; but this is a simple consequence of X4) since $|f\chi_n|_B = 0$ holds for $n > m$. Therefore we can take $K_m = c_m$.

The corollary remains true if one replaces B_m by any compact set $K \subseteq G$ (S4).

It follows from the corollary that the space $\Lambda(\mathrm{B}, \mathrm{X})$ can be considered as an approximation space ([2], Kap. 2.) with $\mathrm{P}_n = \Lambda^n$.

THEOREM 2.5. – *If* A *and* B *are left (right) invariant then the same is true for* $\Lambda(\mathrm{A}, \mathrm{B}, \mathrm{X}, \mathrm{G})$. *Moreover* Λ *satisfies* L4) *if* A *and* B *do.*

Proof. – We shall give the proof only for L_y, $y \in \mathrm{G}$. By S4) there is some n_0 such that $y \in \mathrm{B}_n$ for $n \geqslant n_0$. For $n > n_0$ the inequality $\chi_{n+1} \leqslant \mathrm{L}_y \chi_n$ holds. This follows from the fact that by S2) $y^{-1} x \in \mathrm{B}_n$ if $x \in \mathrm{B}_{n+1}$. Thus we have

$$(\mathrm{L}_y f)\chi_{n+1} \leqslant \mathrm{L}_y f. \ \mathrm{L}_y \chi_n = \mathrm{L}_y (f\chi_n) \quad \text{for} \quad n > n_0 .$$

We deduce further

$$\begin{aligned}
\|\mathrm{L}_y f\| &= \|\mathrm{L}_y f\|_{\mathrm{A}} + |(|(\mathrm{L}_y f)\chi_n|_{\mathrm{B}})_{n\geqslant 0}|_{\mathrm{X}} \\
&\leqslant \|\mathrm{L}_y\|_{\mathrm{A}} \, |f|_{\mathrm{A}} + |((\mathrm{L}_y f)\chi_0|_{\mathrm{B}}, \ldots |(\mathrm{L}_y f)\chi_{n_0}|_{\mathrm{B}}, 0 \ldots)|_{\mathrm{X}} \\
&\qquad + |(0, \ldots 0, |(\mathrm{L}_y f)\chi_{n_0+1}|_{\mathrm{B}}, \ldots)|_{\mathrm{X}} \\
&\leqslant \|\mathrm{L}_y\|_{\mathrm{A}} \, |f|_{\mathrm{A}} + \|\mathrm{L}_y\|_{\mathrm{B}} [|f|_{\mathrm{B}}\, c_{n_0} + \|\mathrm{D}\|_{\mathrm{X}} \, |(|f\chi_n|_{\mathrm{B}})_{n\geqslant 0}|_{\mathrm{X}}] \\
&\leqslant (\|\mathrm{L}_y\|_{\mathrm{A}} + \|\mathrm{L}_y\|_{\mathrm{B}} [c_{n_0} + \|\mathrm{D}\|_{\mathrm{X}}]) \, \|f\| .
\end{aligned}$$

COROLLARY 2.3. – For $y \in \mathrm{B}_{n_0}$

$$\|\mathrm{L}_y\|_{\Lambda} \leqslant \|\mathrm{L}_y\|_{\mathrm{A}} + \|\mathrm{L}_y\|_{\mathrm{B}} \, [c_{n_0} + \|\mathrm{D}\|_{\mathrm{X}}]$$

holds.

THEOREM 2.6. – *If* X6) *holds, i.e. the finite sequences form a dense subspace of* X *and if* $\mathrm{K}(\mathrm{G})$ *is a dense subspace of* $\mathrm{A} \cap \mathrm{B}$, *then* $\mathrm{K}(\mathrm{G})$ *is dense in* $\Lambda(\mathrm{A}, \mathrm{B}, \mathrm{X})$.

Proof. – Let $f \in \Lambda$, $\epsilon > 0$ be given. We have to show that there exists some $k \in \mathrm{K}(\mathrm{G})$ such that $\|f - k\| < \epsilon$. First of all we choose a function $k_1 \in \mathrm{K}(\mathrm{G})$ such that $|f - k_1|_{\mathrm{A}} < \epsilon/4$ holds. Now by S4) there is some $n_0 \in \mathbf{N}$, such that supp $k_1 \subseteq \mathrm{B}_n$ for $n \geqslant n_0$. By X6) there is some $n_1 \geqslant n_0 \in \mathbf{N}$, such that

$$|(0, \ldots 0, |f\chi_{n_1}|_{\mathrm{B}}, |f\chi_{n_1+1}|_{\mathrm{B}}, \ldots)|_{\mathrm{X}} < \epsilon/4$$

holds. Since $|f\chi_n|_A \leqslant |f - k_1|_A < \epsilon/4$ holds, we have

$$\|f\chi_{n_1}\| < \epsilon/2 .$$

Furthermore by the assumptions there is some $k \in K(G)$ such that $|f - f\chi_{n_1} - k|_{A \cap B} < \epsilon/2\, c_{n_1+1}^{-1}$ holds. We may suppose that supp k is contained in B_{n_1+1}, for otherwise we can choose a continuous function Ψ, $0 \leqslant \Psi(x) \leqslant 1$, $\Psi(x) = 1$ for $x \in B_{n_1}$ supp $\Psi \subseteq B_{n_1+1}$ and replace k by $k\Psi$ and the above inequality remains true. From supp $(f - f\chi_{n_1}) \subseteq B_{n_1}$ it follows that

$$\operatorname{supp}(f - f\chi_{n_1} - k) \subseteq B_{n_1+1}$$

and therefore $\|f - f\chi_{n_1} - k\| < \epsilon/2$ by corollary 2.2. All together we have obtained $\|f - k\| \leqslant \|f - f\chi_{n_1} - k\| + \|f\chi_{n_1}\| < \epsilon$, hence K(G) is dense in Λ.

THEOREM 2.7. – *If* X6) *holds and* $A \cap B$ *satisfies* L2) *then* $\Lambda(A, B, X)$ *satisfies* L2) *too.*

Proof. – First of all we note that $A \cap B$ satisfies L4) by remark 1) and therefore $\Lambda(A, B, X)$ satisfies L4) by corollary 2.3. On the other hand it follows from X6) that $\bigcup_{m \geqslant 1} \Lambda^m$ is a dense subset of Λ. By the assumptions and by corollary 2.2. any Λ^m is contained in Λ_c. Thus $\Lambda = \Lambda_c$ by lemma 1.2., i.e. Λ satisfies L2).

A slight modification of the proofs gives

COROLLARY 2.4. – *Let* $A \cap B$ *satisfy* L3) *and let* K(G) *be dense in* $A \cap B$. *Then* $\Lambda(A \cap B, X, G)$ *is a proper subspace of*

$$\Lambda(A, B, X, G)$$

if $A \cap B \neq B$.

COROLLARY 2.5. – *For all* $p > 1$ $\Lambda(L^1 \cap L^p, X)$ *is a proper subspace of* $\Lambda(L^1, L^p, X)$ *and each of these spaces is a proper subspace of the corresponding space with p being replaced by any q,*

$$1 \leqslant q < p < \infty .$$

3. Inclusion results.

The following relations are easily verified.

PROPOSITION 3.1. –

$$\Lambda(\mathrm{A}, \mathrm{B}, \mathrm{X}) \cap \Lambda(\mathrm{A}_1, \mathrm{B}_1, \mathrm{X}) = \Lambda(\mathrm{A} \cap \mathrm{A}_1, \mathrm{B} \cap \mathrm{B}_1, \mathrm{X})\,;$$

$$\Lambda(\mathrm{A}, \mathrm{B}, \mathrm{X}) \cap \Lambda(\mathrm{A}, \mathrm{B}, \mathrm{X}_1) = \Lambda(\mathrm{A}, \mathrm{B}, \mathrm{X} \cap \mathrm{X}_1)\,;$$

if furthermore $\mathrm{A}_1 \subseteq \mathrm{A}$, $\mathrm{B}_1 \subseteq \mathrm{B}$ *and* $\mathrm{X}_1 \subseteq \mathrm{X}$ *holds, we have*

$$\Lambda(\mathrm{A}, \mathrm{B}, \mathrm{X}) \subseteq \Lambda(\mathrm{A}_1, \mathrm{B}_1, \mathrm{X}_1)\,.$$

The proof is left to the reader.

Considering the above proposition it is natural to ask whether proper inclusions lead again to proper inclusions. We don't give a full discussion of this problem but confine ourselves to the most important special cases. Thus we shall see that at least for a great number of interesting examples an affirmative answer can be given. The following technical lemma will be useful.

LEMMA 3.1. – *Let* $(\mathrm{B}_n)_{n \geqslant 0}$ *be given as in* I). *Then for any compact subset* $\mathrm{K} \subseteq \mathrm{G}$ *there exists a subsequence* $(\mathrm{B}_{n_k})_{k \geqslant 0} \subseteq \mathrm{G}$ *such that for a suitable sequence* $(y_k)_{k \geqslant 0} \subseteq \mathrm{G}$ $\;y_k \mathrm{K} \subseteq \mathrm{B}_{n_k} \setminus \mathrm{B}_{n_k - 1}$ *holds.*

Proof. – We note that it follows from S2) and S3) that for any $m \in \mathbf{N}$ there is some $n > m$ such that $\mathrm{B}_m \subset \mathrm{B}_{n-1} \neq \mathrm{B}_n$ holds. Since K is compact we have $\mathrm{K} \cup \mathrm{K}^{-1} \subseteq \mathrm{B}_{k_0}$ for some $k_0 \in \mathbf{N}$ (S4). To prove the assertion it will be enough to show that for any $n_0 \geqslant k_0$ there is some $j > n_0$ and some $y_j \in \mathrm{G}$ such that $y_j \mathrm{K} \subseteq \mathrm{B}_j \setminus \mathrm{B}_{j-1}$, i.e. $\mathrm{B}_j \mathrm{K}^{-1} \setminus \mathrm{B}_{j-1} \mathrm{K}^{-1} \neq \phi$. Suppose $\mathrm{B}_j \mathrm{K}^{-1} = \mathrm{B}_{n_0} \mathrm{K}^{-1}$ for all $j > n_0$. It follows from S2) that for all $j > n_0$

$$\mathrm{B}_j \subseteq \mathrm{B}_j \mathrm{B}_{k_0} \mathrm{K}^{-1} \subseteq \mathrm{B}_{j+1} \mathrm{K}^{-1} = \mathrm{B}_{n_0} \mathrm{K}^{-1} \subseteq \mathrm{B}_{n_0} \mathrm{B}_{k_0} \subseteq \mathrm{B}_{n_0+1}$$

holds. This is a contradiction to our first observation.

Note that $y_n \in \mathrm{B}_{n_k} \mathrm{B}_{k_0} \subseteq \mathrm{B}_{n_k+1}$ holds. For connected groups we may suppose $n_{k+1} = n_k + 1$ (confer remark 3).

THEOREM 3.1. – *Let* A *and* B *satisfy* L1) *and* L3) *and let two solid* BK-*spaces* $X_1, X_2, X_1 \subset X_2$ *be given. Denote their fundamental sequences by c and d respectively. Then any closed, left invariant subspace* $M \neq \{0\}$ *of* $\Lambda(A, B, X_2, G)$ *contains elements, which don't belong to* $\Lambda(A, B, X_1, G)$, *if* $d_n/c_n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Proof. – Suppose $M = M \cap \Lambda(X_1)$. Since M and $M \cap \Lambda(X_1)$ are BF-spaces with the norm of $\Lambda(X_2)$ and $\Lambda(X_1)$ respectively, these two norms must be equivalent when restricted to M. This will lead to a contradiction. Since $M \neq \{0\}$ holds there is some $f \in M$ and some compact subset $K \subseteq G$ such that

$$\|f\chi_K\|_{\Lambda(X_1)} \geqslant \|f\chi_K\|_{\Lambda(X_2)} \geqslant |f\chi_K|_B = \delta > 0$$

holds. If we choose $(y_k)_{k\geqslant 1}$ and $(n_k)_{k\geqslant 1}$ as in lemma 3.1. and put $f_k := d_{n_k}^{-1} L_{y_k} f$, then $\{f_k\}_{k\geqslant 1} \subseteq M$, M being left invariant. Now we have to give estimates for the norms of the f_k's. We know from lemma 3.1. that $y_K \in B_{n_k+1}$ holds. Thus by corollary 2.3. and L3) we obtain :

$$\begin{aligned} \|f_k\|_{\Lambda(X_2)} = d_{n_k}^{-1} \|L_{y_k} f\|_{\Lambda(X_2)} &\leqslant d_{n_k}^{-1} \|L_{y_k}\|_{\Lambda(X_2)} \|f\|_{\Lambda(X_2)} \\ &\leqslant d_{n_k}^{-1} (1 + \|D\|_X + d_{n_k+1}) \|f\|_{\Lambda(X_2)} \\ &\leqslant (1 + \|D\|_X)(d_{n_k}^{-1} + 1) + \|f\|_{\Lambda(X_2)}, \end{aligned}$$

showing that $\{f_k\}$ is bounded in $\Lambda(X_2)$. On the other hand we have

$$\|f_k\|_{\Lambda(X_1)} \geqslant d_{n_k}^{-1} \|L_{y_k}(f\chi_K)\|_{\Lambda(X_1)},$$

since by lemma 3.1. supp $L_{y_k}(f\chi_K) \subset B_{n_k} \setminus B_{n_k-1}$, we have by L3)

$$|L_{y_k}(f\chi_K)\chi_n|_B = \begin{cases} |f\chi_K|_B := \delta > 0 & \text{for } n \leqslant n_K \\ 0 & \text{for } n > n_K . \end{cases}$$

Thus we obtain $\|L_{y_k}(f\chi_K)\|_{\Lambda(X_1)} \geqslant \delta\, c_n$ and further

$$\|f_k\|_{\Lambda(X_1)} \geqslant \delta\, c_{n_k} d_{n_k}^{-1},$$

showing that $\{f_k\}_{k\geqslant 1}$ is unbounded in $\Lambda(X_1)$. Therefore the two norms are not equivalent on M and $M \neq M \cap \Lambda(X_1)$ must hold.

THEOREM 3.2. – *Let* A, B, X_1, X_2 *be as in theorem* 3.1. *Then any closed left ideal* I *of* $\Lambda(A, B, X_1, G)$ *is not any more a left ideal in* $\Lambda(A, B, X_2, G)$.

Proof. – Let I be any closed left ideal of $\Lambda(X_1) \subseteq \Lambda(X_2)$, then I is a Banach algebra itself. If it is also a left ideal in $\Lambda(X_2)$ we have by theorem 2.3. of [1] the following inequality :

$$\|k * f\|_{\Lambda(X_1)} \leqslant C \|k\|_{\Lambda(X_2)} \|f\|_{\Lambda(X_1)}$$

for some $C < \infty$ and all $k \in \Lambda(X_2)$, $f \in I \subseteq \Lambda(X_1)$. This implies that for every $k \in K(G) \subseteq \Lambda(X_1)$ and every $f \in I$ the set

$$\{\|L_y k * f\|_{\Lambda(X_1)} \|L_y\|^{-1}_{\Lambda(X_2)},\ y \in G\}$$

is bounded. Since $(L_y k) * f = L_y(k * f)$ for all $y \in G$ we may apply arguments as in theorem 3.1. with f replaced by $k * f$ to lead this assumption to a contradiction.

COROLLARY 3.1. – *Under the above assumptions*

$$\Lambda(A, B, X_1, G)$$

cannot be an ideal in $\Lambda(A, B, X_2, G)$.

COROLLARY 3.2. – *Let* A, B, X_1, X_2 *as in theorem* 3.1. *Let* G *be an Abelian group. Define for a compact subset* $K \subseteq \hat{G}$ *with nonvoid interior* $I_i(K) = \{f \in \Lambda(X_i),\ \text{supp}\ \hat{f} \subseteq K\}$, $i = 1, 2$. *Then for any* K *the inclusion* $I_1(K) \subseteq I_2(K)$ *is proper.*

Proof. – I_2 is a closed ideal of $\Lambda(X_2)$, thus $I_1 = I_2$ implies that I_1, being a closed ideal of $\Lambda(X_1)$, must be an ideal in $\Lambda(X_2)$, in contradiction to theorem 3.2.

A similar result for Beurling algebras has been proved by R. Spector ([8], Theorem III.1.6.).

PROPOSITION 3.2. – *Let* A, B, B_1 *satisfy* L1) *and* L3),

$$A \cap B \subseteq A \cap B_1$$

and K(G) *be dense in both spaces. Then for any* X *the inclusion* $\Lambda(A, B, X, G) \subseteq \Lambda(A, B_1, X, G)$ *is proper if the inclusion*

$$A \cap B \subseteq A \cap B_1$$

is proper.

Proof. – Since K(G) is dense in both spaces the inclusion is proper if and only if the two norms are not equivalent, when restricted to K(G). Thus for any $n \in \mathbf{N}$ there is some $k \in K(G)$ satisfying

$$|k|_A + |k|_B = 1, \ |k|_A + |k|_{B_1} \geqslant n + 1$$

and thus $|k|_{B_1} \geqslant n$. By lemma 3.1. there exist $y \in G$ and $m \in \mathbf{N}$ such that $y\,(\mathrm{supp}\, k) \subseteq B_m \setminus B_{m-1}$. Therefore we can calculate the norm of $L_y k$, using L1) and L3)

$$\|L_y k\|_{\Lambda(A,B)} = |k|_A + c_n \, |k|_B \leqslant 1 + c_n \leqslant 2c_n ,$$

$$\|L_y k\|_{\Lambda(A,B_1)} = |k|_A + c_n \, |k|_{B_1} \geqslant nc_n ,$$

showing that the norms of $\Lambda(A, B)$ and $\Lambda(A, B_1)$ are not equivalent. Thus the inclusion must be proper.

Finally we want to consider the following problem. Let us denote the closure of K(G) in Λ by Λ^0. What can we say about the inclusions $\Lambda^0 \subseteq \Lambda_c \subseteq \Lambda$? Suppose X6) holds. Then $\Lambda_c = \Lambda$ if $A \cap B$ satisfies L2 (theorem 2.7.) and $\Lambda^0 = \Lambda$ if K(G) is dense in $A \cap B$ (theorem 2.6.). Now we shall give an example showing that both of the inclusions may be proper in case X6) is not satisfied, even when K(G) is dense in $A \cap B$. Since X6) is not fulfilled essentially if l^∞ is involved in the construction, the counterexample concerns the most important case.

PROPOSITION 3.3. – *For* $G = \mathbf{R}^m$ *and* $(B_n)_{n \geqslant 0}$ *as usual we consider* $\Lambda(L^1, L^p, X_s, \mathbf{R}^m)$, $1 < p < \infty$ *with*

$$X_s := \{(x_n),\ (x_n n^s) \in l^\infty\}$$

for some $s > 1 - 1/p > 0$. *Then* Λ^0 *is a proper subspace of* Λ_c *which is in turn a proper subspace of* Λ. *Moreover* Λ^0 *is not an ideal in* Λ_c (resp. Λ).

Proof. – 1) By S4) it will be sufficient to show that there is some $f \in \Lambda$ such that $y \to L_y f$ is a continuous function from G into B, but $\|f\chi_k\| \geqslant \epsilon_0 > 0$ for all $k \in \mathbf{N}$. To this aim we take some cube Q in $\mathbf{R}^m$ such that for some neighbourhood U_0 of zero and a suitable sequence $(y_n)_{n\geqslant 1} \subseteq \mathbf{R}^m$ $y_n + U_0 + Q \subseteq B_{n+1} \setminus B_n$ for $n \geqslant 1$ holds If we put now for a given s $r := s + 1/p$ then $f := \sum_{n=1}^{\infty} n^{-r} L_{y_n} \chi_Q$ will be such a function. $r > 1$ implies $f \in L^1(\mathbf{R}^m)$. Moreover we have

$$|f\chi_k|_p = \left(\sum_{n=k}^{\infty} |Q| n^{-rp}\right)^{1/p}$$

$$\leqslant |Q|^{1/p}(k-1)^{-r+1/p} = |Q|^{1/p}(k-1)^{-s},$$

showing that f lies in Λ. On the other hand $|f\chi_k| \geqslant |Q|^{1/p} k^{-s}$ implies $\|f\chi_k\|_\Lambda \geqslant \sup_{n\geqslant k} k^s |f\chi_n|_p \geqslant |Q|^{1/p} > 0$ for all $k \in \mathbf{N}$, thus $f \notin \Lambda^0$. On the other we have for $y \in U_0$ and

$$|y| < |Q|^{1/m} \quad |Q\Delta y + Q| \leqslant K|y|$$

for some $K < \infty$. Thus we have

$$|(f - L_y f)\chi_k| \leqslant \left|\sum_{n=k}^{\infty} n^{-r}(L_{y_n}\chi_Q - L_{y_n+y}\chi_Q)\right|_p$$

$$\leqslant |Q\Delta y + Q|^{1/p}(k-1)^{-r+1/p} \leqslant K'|y|k^{-s} \text{ for all } k \in \mathbf{N}.$$

Since $L^1(\mathbf{R}^m)$ satisfies L2) this implies that $y \longrightarrow L_y f$ is continuous at $y = 0$. Lemma 1.1. gives the assertion.

2) First of all we can choose some $a > 0$ such that

$$s > 1 - 1/p + \frac{am}{p}$$

holds. Furthermore we put $r := s + 1/p - \dfrac{am}{p}$. Let Q_n be a cube with the length of his edges n^{-a}, $Q_{n+1} \subseteq Q_n$ for $n \geqslant 2$ and $(y_n)_{n\geqslant 2}$ as above. Then we put $f := \sum_{n=2}^{\infty} n^{-r} L_{y_n}\chi_{Q_n}$. $r > 1$ and $a > 0$ imply $f \in L^1(\mathbf{R}^m)$. Furthermore

$$|f\chi_k|_p \leqslant \left(\sum_{n\geqslant k} n^{-rp} n^{-am}\right)^{1/p} \leqslant (k-1)^{-r-\frac{am}{p}+1/p} = (k-1)^{-s}$$

implies $f \in \Lambda$. But since for any $y \in U_0 \subseteq \mathbf{R}^m$ $Q_n \cap y + Q_n = \phi$ for

$n \geqslant n(y)$ we have for $k \geqslant n(y) : (f - \mathrm{L}_y f) \chi_k |_p \geqslant |f \chi_k|_p \geqslant k^{-s}$ for all $y \in \mathrm{U}_0$. Thus $\| f - \mathrm{L}_y f \|_\Lambda \geqslant \epsilon_0 > 0$ for all $y \in \mathrm{U}_0$, showing that f must lie in $\Lambda \setminus \Lambda_c$.

3) The last assertion follows directly from 1). Take f as in 1) and $k \subseteq \mathrm{K}_+(\mathrm{G}) \subseteq \Lambda_c$, supp $k \subseteq \mathrm{U}_0$; then it is easily shown that $k * f \notin \Lambda^0$.

4. Further properties ; spaces on abelian groups.

To be able to derive further results we state without proof the following results concerning general solid Banach convolution algebras. Thereby we shall use the following notation : A Banach algebra has (*multiple*) *left approximate units* if for $f_1 \in \mathrm{A}$ ($f_1 \ldots f_n \in \mathrm{A}$), $\epsilon > 0$ there is some $g \in \mathrm{A}$ such that $\| g * f_i - f_i \|_\mathrm{A} < \epsilon$ for $i = 1$ ($i = 1, \ldots n$).

THEOREM A. – Let A be a solid, left invariant Banach convolution algebra, then the following properties are equivalent :

i) A satisfies L2)
ii) A has multiple left approximate units in K (G).
iii) K (G) $*$ A is dense in A.

COROLLARY 4.1. – If A is a solid, left invariant Banach convolution algebra satisfying L4), A_c is just the closure of K (G) $*$ A (cf. theorem 2.5.).

PROPOSITION B. – Let A satisfy one of the properties of theorem A. Then any essential, closed ideal M of A is left invariant.

THEOREM C. – Let A be a solid, left (right) invariant Banach convolution algebra. If K (G) is dense in A, then the closed left (right) invariant subspaces and the closed left (right) ideals of A coincide.

As consequences of these results we have :

THEOREM 4.1. – *Let* X6) *hold and* K (G) *be dense in* A. *If* A $\cap$ B *has left approximate units then* Λ(A , B , X , G) *has multiple left approximate units.*

Proof. – First of all we show that it follows from the assumptions that $A \cap B$ has approximate units in $K(G)$. Let $f \in A \cap B$, $\epsilon > 0$ be given. Then there is some $g \in A \cap B$ such that

$$\|g * f - f\|_{A \cap B} < \epsilon/2 .$$

If we choose $k \in K(G) \subseteq A \cap B$, $\|k - g\|_A < \epsilon/2 \quad \|f\|_{A \cap B}$ we obtain

$$\|k * f - f\|_{A \cap B} \leq \|k - g\|_A \quad \|f\|_{A \cap B} + \|g * f - f\|_{A \cap B} < \epsilon .$$

Therefore by theorem A $A \cap B$ satisfies L2). Now by theorem 2.7. Λ satisfies L2) and again by theorem A Λ has multiple left approximate units in $K(G)$.

THEOREM 4.2. – *Let* X6) *hold and* $K(G)$ *be dense in* $A \cap B$. *Then the closed left (right) ideals and the closed left (right) invariant subspaces of* $\Lambda(A, B, X, G)$ *coincide.*

Proof. – This theorem follows from theorem C and theorem 2.6.

We shall now give a number of further results concerning the case of an abelian group G. To obtain them we apply results of Y. Domar who has given an analysis of certain commutative Banach algebras in his fundamental paper [3].

DEFINITION (cf. [3], p. 5) – Let A be a solid Banach convolution algebra on a locally compact abelian group, $A \subseteq L^1(G)$. We say that $\hat{A} = \{\hat{f},\ f \in A\}$ *is of type* F if A satisfies

F1) for any $a \in \hat{G}$ and any neighbourhood U of a there is some $\hat{f} \in$ A such that $\hat{f}(a) \neq 0$ and supp $\hat{f} \subseteq$ U holds.

F2) $K(G)$ is dense in A.

Remarks. – 1) This a simplified version of the definition given in [3], adapted to our situation.

2) Since A is a commutative Banach algebra condition F1) is equivalent to the assumption that $\hat{A}$ is a standard function algebra in the sense of [7] (cf. [7], Chap. 2 § 1.1.). In any such standard function algebra $\hat{A}$ there is for any compact set $K \subseteq \hat{G}$ and any neighbourhood U of K some $\hat{f} \in \hat{A}$ such that $\hat{f}(x) = 1$ for $x \in K$, and supp $\hat{f} \subseteq U$.

THEOREM D ([3] Th. 2.11, [7], Chap. 6, § 3.1). – $L^1_w(G)\hat{}$ is of type F if and only if the condition BD) holds :

$$\text{(BD)} \qquad \sum_{n=1}^{\infty} n^{-2} \log |w(x^n)| < \infty \quad \text{for all} \quad x \in G .$$

LEMMA 4.1. – *Let* A *be a left invariant solid subalgebra of* $L^1(G)$ *satisfying* L4). *Then* A *satisfies* F1) *if* BD′) *holds* :

$$\text{(BD}'\text{)} \qquad \sum_{n=1}^{\infty} n^{-2} \log |w(x^n)| < \infty \quad \textit{for all} \quad x \in G ,$$

with $w(x) = \max(1, \|L_x\|_A)$.

Proof. – By lemma 1.2. A_c is a closed subspace of A satisfying L2). Moreover $K(G) \subseteq A_c$ by corollary 1.1. It will be sufficient to show that $\hat{A}_c$ satisfies F1). By lemma 1.5. A_c is a left convolution module over some Beurling algebra $L^1_w(G)$ (with w as above according to lemma 1.3.). It follows from BD′) that $L^1_w(G)\hat{}$ is of type F and therefore satisfies F1), i.e. given $a \in \hat{G}$ and U there is some $f \in L^1_w(G)$, $\hat{f}(a) \neq 0$, supp $\hat{f} \subseteq U$. But there is certainly some $\hat{h} \in \hat{A}_c$, $\hat{h}(a) \neq 0$, $\hat{A}_c$ being left invariant since A_c is of course character invariant (i.e. $|\chi h|_A = |h|_A$ for any $h \in A$ and any character χ on G). But now $\widehat{f * h} = \hat{f}\hat{h} \in \hat{A}_c$, $\hat{f}(a)\hat{h}(a) \neq 0$ and supp $\hat{h}\hat{f} \subseteq$ supp $\hat{f} \subseteq U$. Thus $\hat{A}_c$ satisfies F1).

THEOREM 4.3. – *If both* A *and* B *satisfy* BD′) *then* $\Lambda(A, B, X, G)$ *satisfies* F1). *Moreover* $\Lambda(A, B, X, G)$ *is of type* F, *if furthermore* K(G) *is dense in* $\Lambda(A, B, X, G)$.

Proof. – By lemma 1.4. $\Lambda(A, B, X, G)$ satisfies L2).

Put $w_1(x) := \max(1, \|L_y\|_A)$, $w_2(x) := \max(1, \|L_y\|_B)$ and $w_3(x) := \max(1, \|L_y\|_\Lambda)$. By lemma 4.1. it will be enough to show that w_3 satisfies BD). To this aim we note that by corollary 2.3. we have

$$\begin{aligned} w_3(x) &\leqslant w_1(x) + w_2(x)\,[c_{n_0} + \|D\|_X] \\ &\leqslant K\,w_1(x)\,w_2(x)\,c_{n_0} \qquad \text{for} \qquad x \in B_{n_0}, K < \infty . \end{aligned}$$

Now given any $x \in G$ there is some $n_0 \in N$ such that $x \in B_{n_0}$. If follows from S2) that $x^n \in B_{n_0+m}$ for $2^m < n \leqslant 2^{m+1}$ holds. Using this fact, the monotony of c_n and the inequality

$$c_{n+m} \leqslant (1 + \|D\|_X)^m c_n \qquad n, m \in N$$

we obtain

$$\sum_{n=1}^{\infty} n^{-2} \log w_3(x^n) = \sum_{n=1}^{\infty} \sum_{n=2^{m-1}+1}^{2^m} n^{-2} \log w_3(x^n)$$

$$\leqslant \sum_{m=1}^{\infty} \sum_{n=2^{m-1}+1}^{2^m} n^{-2} [\log w_1(x^n) + \log w_2(x^n) + \log (c_{n_0+m})] + c$$

$$\leqslant c_1(x) + c_2(x) + \sum_{m=1}^{\infty} 2^{m-1} (2^{m-1})^{-2} \log (c_{n_0+m})$$

$$\leqslant c_1(x) + c_2(x) + \sum_{m=1}^{\infty} 2^{-(m-1)} \log c_{n_0} + \sum_{m=1}^{\infty} 2^{-(m-1)} m \log (1 + \|D\|_X) < \infty .$$

The proof is now complete.

COROLLARY 4.2. – $\Lambda(A, B, X, G)$ *satisfies* F1) *if* A *and* B *satisfy* L3).

By [3] we have a number of results concerning solid Banach convolution algebras contained in $L^1(G)$, if $\hat{A}$ is of type F. We state some of them.

The only multiplicative linear functionals on A are of the form $f \longmapsto \hat{f}(x_0)$, $x_0 \in \hat{G}$. The toplogy on $\hat{G}$ is the weakest among all topologies for which all $\hat{f} \in \hat{A}$ are continuous functions on $\hat{G}$. Therefore the space of regular maximal ideals of A coincides with $\hat{G}$ and the Fourier transform coincides with the Gelfand transform. Moreover we have $\lim_{n \to \infty} \|f * f * \cdots * f\|_A^{1/n} = \|\hat{f}\|_\infty$ (n-fold convolution of f).

The functions $f \in A$ with $\hat{f}$ having compact support are dense in A and moreover A has multiple approximate units if it is translation invariant. Thus $\hat{A}$ is a Wiener algebra in the sense of [7], Chap. 2. § 2.4. It follows that any closed ideal I of A contains all functions $f \in A$ such that $\operatorname{supp} \hat{f} \subseteq \hat{G} \setminus \operatorname{cosp} I$ holds. In particular the only closed ideal with empty cospectrum is A itself.

At the end of this section we shall be concerned with the factorization problem.

DEFINITION ([9], def. 2.1.). – Let be a commutative Banach algebra. We say that A has the *(weak) factorization property* if for every $x \in A$ there exist elements $y, z \in A$ $(y_1 \dots y_n, z_1 \dots z_n \in A)$ such that $yz = x$ $(y_1 z_1 + \cdots + y_n z_n = x)$.

THEOREM 4.4. – *Let* G *be a nondiscrete abelian group. Then* $\Lambda(A, B, X, G)$ *doesn't have the weak factorization property, if* $\hat{\Lambda} \subseteq L^{p_0}(\hat{G})$ *for some* $p_0 < \infty$ *and if* A *and* B *satisfy* BD′), *e.g. if they satisfy* L3).

Proof. – All essential calculations for this proof can be found in [9]. On account of [9], theorem 4.1. we have : Let A be a Banach algebra contained in $L^1(G)$ having properties F. and P. ([9], def. 2.9.), then A has not the weak factorization property. Property F. is exactly the assumption $\hat{\Lambda} \subseteq L^{p_0}(\hat{G})$ for some $p_0 < \infty$. Using the proof of [9], theorem 2.10. we see that A has property P, if it satisfies F1) since it is solid. This is the case here by theorem 4.3.

COROLLARY 4.3. – *If the assumption of theorem* 4.4. *are fulfilled* $\Lambda(A, B, X, G)$ *cannot have bounded approximate units.*

COROLLARY 4.4. – *Let* G *be as above. Then none of the algebras* $\Lambda(L^1, L^p, X, G)$, $1 < p \leqslant \infty$ *has the weak factorization property.*

5. Examples.

There is a number of examples of spaces $\Lambda(A, B, X, G)$ which can be defined in a natural way, different from the definition given in section 2. The most natural examples are the spaces defined on $G = \mathbf{R}^m$ or $G = \mathbf{Z}^m$, defined by means of

$$B_n := \{x \in G, |x| \leqslant 2^{n-1}\}$$

for $n \geqslant 1$ and $A = L^1(G)$. It is not very difficult to verify that the space Λ^0 defined by $B = L^\infty(G)$ and

$$X = X_\alpha := \{(x_n), (2^{\alpha n} x_n) \in c_0\}$$

for some $\alpha > 0$ is the same as

$$\Lambda_\alpha(G) = \{f \mid f \in L^1(G), f(x)(1 + |x|)^\alpha \in C^0(G)\}$$

with the norm $\|f\|_{\Lambda_\alpha} := \|f\|_1 + \|f w_\alpha\|_\infty$, $w_\alpha(x) := (1 + |x|)^\alpha$. These spaces stood at the beginning of our work.

More general any space $\Lambda(L^1, L^\infty, X, G)$ on a locally compact group defined by some space

$$X = X_a = \{(x_n), (a_n^{-1} x_n) \in l^\infty\}, \; a = (a_n)$$

being a fixed, nonincreasing sequence in c_0 can be identified with a space $\Lambda_g(G)$ as defined in [4] (def. 3). To prove this assertion it will be enough to show that there exists a so called "gage function" g ([4], def. 2) such that $\Lambda(L^1, L^\infty, X_a, G) = \Lambda_g(G)$. This can easily be shown if one defines g by $g(x) := a_n$ for $x \in B_{n+1} \setminus B_n$ and $n \geqslant 0$. We have to show that G1) – G4) ([4], def. 2) are satisfied. First of all we observe that $a_{n+1} \geqslant \delta_0 a_n$ must hold for some δ_0, $1 \geqslant \delta_0 > 0$ and all $n \geqslant 0$, since X_a must satisfy X5). We put now $U_x := B_{n-1}$ for $x \in B_{n+1} \setminus B_n$ if $n \geqslant 2$ and $U_x \subseteq B_{n-1}$ such that $x \in U_x U_x$ if $x \in B_2$ and $A := \delta_0^{-1} \geqslant 1$. Then G1) holds, G2) follows from S2) and G3) follows from X5), G4) is a consequence of X4). It is not difficult to see that g can be replaced by a continuous function defining the same space $\Lambda_g(G)$.

On the other hand any space $\Lambda_g(\mathbf{R}^m)$ which has been defined by means of any "special gage function" g ([4], def. 1) on $\mathbf{R}^m$ can be identified with some space $\Lambda(L^1, L^\infty, X_a, \mathbf{R}^m)$. A number of such functions has been given in [4]. The same is true for all known spaces $\Lambda_g(G)$ defined by means of a general gage function. One also readily verifies that in this case $\Lambda_g^0(G)$ can be identified with

$$\Lambda(L^1, C^0, X_a^0, G)$$

with $X_a^0 = \{(x_n, (a_n^{-1} x_n) \in c_0\}$. Since $K(G)$ is dense in $\Lambda_g^0(G)$ all theorems derived in this paper are applicable to the spaces $\Lambda_g^0(G)$. Thus the theory developed here represents in many points a generalization of the results obtained in [4].

As in the case of $B = L^\infty(G)$ the spaces $\Lambda(L^1, L^p, X^0_\alpha, \mathbf{R}^m)$ also have a natural representation as

$$\{f \mid f \in L^1(\mathbf{R}^m),\ h_f \in C^0(0, \infty) \quad \text{with}$$
$$h^p_f(x) = (1 + |x|)^{\alpha p} \int_{|y| \geqslant x} |f(y)|^p \, dy\}$$

with the norm of Λ equivalent to the norm defined by $\|f\|_1 + \|h_f\|_\infty$. A similar method is applicable if X^0_α is replaced by some space X defined by means of spaces l^q, $1 \leqslant q < \infty$, e.g. X^q_s (use $L^q[0, \infty)$). For the case that l^∞ is involved in the construction confer proposition 3.3.

It is worth mentioning that the spaces $\Lambda(B, X, G)$ can be considering as approximation spaces ([2], Chap. 2). For example it follows from [2], Satz 2.1.1. that the space $\Lambda(B, X^q_\alpha, \mathbf{R}^m)$ defined by

$$X^q_\alpha = \{(x_n),\ (2^{\alpha n} x_n) \in l^q\},\ \alpha > 0$$

is the same as $B^K_{\theta, q}$ with $\theta = \alpha > 0$ and $a = 2$.

$$P_n = \{f \in B,\ \operatorname{supp} f \subseteq [-n, n]^m\}.$$

Since B is solid we have $E_n(f) = |f\psi_n|_B$, ψ_n being the characteristic function of $\mathbf{R}^m \setminus [-n, n]^m$ and therefore $E_{2^{n-1}}(f) = |f\chi_n|_B$. It also follows from [2], Satz 2.1.1. that

$$\Lambda(B, X^q_\alpha, \mathbf{R}^m) = \{f \mid f \in B,\ |f\psi_n|_B \in X^q_s\}$$

with $s = \theta - 1/q$, $X^q_s := \{(x_n),\ (n^s x_n) \in l^q\}$.

Finally we observe that the fact that

$$\Lambda(L^1, X^0_r),\ X^0_r = \{(x_n),\ (x_n n^s) \in c_0\}$$

is a convolution algebra has been used implicitely in the definition of rapidly decreasing functions in [6]. Furthermore any space

$$\Lambda(L^1_w(G),\ X^1_\alpha, G)$$

can be identified with a suitable Beurling algebra $L^1_{w_1}(G)$. Therefore lemma 1.5. is in many cases a consequence of theorem 2.4., e.g. for $B = (L^1_w, L^p, X)$, since $X \supseteq X^1_\alpha$ for a suitable $\alpha > 0$ (e.g. $2^\alpha \geqslant 1 + \|D\|_X$).

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Added in proof :

After the submission of this paper the author became acquainted with a very recent paper [10] and with [11]. We note that a slight modification of the proof of theorem A 7 of [10] shows that this theorem remains true for general locally compact groups. In particular we have (confer remark 1)) : If $y \longrightarrow \|L_y\|_B$ is measurable, B satisfies L4), without any restriction on the group G.

Furthermore we observe that the functions g constructed in [4] are in fact WSA functions in the sense of [10]. Therefore theorem 3 of [10] provides an elegant and elementary proof for the fact that the spaces $\Lambda_g(G)$ or $\Lambda_g^0(G)$ are Banach convolution algebras. On the other hand the spaces L_ϕ or L_ϕ^* and ϵ_ϕ or ϵ_ϕ^* are essentially special cases of the algebras $\Lambda(A, B, X, G)$, e.g. in case $G = \mathbf{R}^\nu$ or $\mathbf{Z}^\nu$ and the WSA function ϕ is increasing for $|z| \uparrow \infty$ we have

$$L_\phi = (L^1, L^\infty, l_w^\infty)$$

and $L_\phi^* = (L^1, C^0, c_{0,w})$ where $l_w^\infty(c_{0,w}) := \{x \mid (x_n w_n) \in l^\infty(c_0)\}$ for a suitable sequence w, $w_n \geqslant 1$. Under the same conditions we obtain $\epsilon_\phi = (L^1, L^2, l_w^\infty)$ and $\epsilon_\phi^* = (L^1, L^2, c_{0,w})$. For example we have (using the notation of [11]) :

$$\epsilon(\alpha) = \{ f \in L^1(\Pi), \hat{f} \in \Lambda(L^1, L^2, X_\alpha^\infty, \mathbf{Z})\}$$

with equivalence of norms. As a consequence lemma 8a and theorem 8b of [11] are special cases of theorem 2.2.. Moreover in most cases the spaces $\Lambda(A, B, X, G)$ satisfy condition b) of theorem 2 of [10] (replace S by Λ and B by A). Thus theorem 1 of [10] gives in many cases an alternative approach to the consequences of theorem 4.3. above.

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Proposé par J. Dieudonné.

Hans G. FEICHTINGER,
Mathematisches Institut
der Universität Wien
Strudlhofgasse 4
1090 Wien (Autriche).