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FOLIATIONS WITH ALL LEAVES COMPACT

by D.B.A. EPSTEIN

Let M be a foliated manifold with each leaf compact. At first sight it seems reasonable to expect that for any compact subset K of M, the set of leaves meeting K should have bounded volumes (see § 3 for the the definition of the "volume" of a leaf). We shall talk of this belief as the *conjecture*.

The investigation of this question was started by Reeb [10]. Reeb showed that in codimension one the conjecture was true. He also gave C^∞ counterexample in every codimension greater than one, and in these counterexamples M was not compact. The author has given counter-examples in these codimensions which are real analytic, and in these examples M is also non-compact [2].

Following a question asked by Haefliger, the author proved that a compact 3-manifold foliated by circles satisfies the conjecture [2]. Edwards, Millett and Sullivan have recently shown that this proof can be modified to prove the conjecture in codimension two when M is compact [12]. By a rather different proof using ideas from measure theory, the same three authors proved the conjecture under certain homological assumptions, no matter what the codimension.

Sullivan has recently given an example of a compact five-dimensional manifold, foliated by circles, and the circles have unbounded length. Thurston has even given a version which is real analytic [15].

A.W. Wadsley has proved the conjecture under assumptions of a differential geometric nature [7]. For example he has shown that for a manifold M foliated by circles, the conjecture is equivalent to the existence of a Riemannian metric on M, in which each of the circles is geodesic [8].

The purpose of this paper is firstly to encourage work on the conjecture, secondly to provide background material which may be necessary in considering the problem, and thirdly to provide complete proofs of a result announced without proof many years ago by Ehresmann [1] p. 38. We give a number of conditions on a foliation with all leaves compact, each of which is equivalent to the holonomy of each leaf being finite. This paper should be regarded as mainly expository, rather than an original study, and it is hoped that it will be a useful source of reference. There is considerable overlap between this paper and some unpublished notes of K.C. Millett. Theorem 7.3 was proved independently by him and the author. The reader is also referred to a paper by Millett [13].

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1.

The analysis of our problem produces in an entirely natural way foliations on subspaces of M which are not manifolds. We therefore give a definitions of a foliated space which is rather more general than the usual definition for a foliated manifold. This definition is due to Ehresmann.

Let X and B be Hausdorff, locally compact spaces with countable bases. These hypotheses clearly imply metrizability and paracompactness. We say X is a B-*foliated space* (or simply a *foliated space*) if we are given in addition a family of maps $h : V \times W \to U$, which we call *charts*, satisfying the following conditions.

1.1. The charts are homeomorphisms onto open subsets U of X which form a basis for X.

1.2. The spaces W are open subsets of B. Each of the spaces denoted by V is locally connected, and, to avoid logical difficulties, we assume that V is a subspace of X. (This is no loss of generality since h maps $V \times w$ homeomorphically into U for each $w \in W$).

1.3. If $x \in U_1 \cap U_2$ and if

$$h_1 : V_1 \times W_1 \to U_1 \qquad \text{and} \qquad h_2 : V_2 \times W_2 \to U_2$$

are charts, then in some neighbourhood $h_2^{-1}x$, there are continuous maps α and β such that for (v_2, w_2) in this neighbourhood

$$h_1^{-1}h_2(v_2, w_2) = (\alpha(v_2, w_2), \beta(w_2)).$$

It follows that β is a local homeomorphism between open subsets of B.

1.4. We may, if we wish, impose the further condition that each β belong to some pseudogroup of homeomorphisms of B, and that each $h_1^{-1}h_2$ belong to some pseudogroup of homeomorphisms of X. For example, if X and B are Euclidean spaces of dimension n and q respectively, and each V is a manifold of dimension $n - q$, we could insist that each β is analytic and each α is differentiable.

1.5. We suppose that the family of charts is maximal, subject to the above conditions.

1.6. We say X is a *foliated manifold* if each V and each W above is a manifold.

Let $h : V \times W \to U$ be a chart and let V_1 be an open subset of V and W_1 an open subset of W. Let $U_1 = h(V_1 \times W_1)$. Then h induces a homeomorphism $h_1 : V_1 \times W_1 \to U_1$ which is clearly a chart by 1.5. We therefore lose no generality in restricting ourselves to charts $h_1 : V_1 \times W_1 \to U_1$ which can be extended to charts $h : V \times W \to U$ where $\overline{U}_1$, $\overline{V}_1$ and $\overline{W}_1$ are compact subsets of U, V and W respectively.

2.

2.1. From now on the word "chart" will be restricted to charts like the h_1 just described. The image of such a chart will be called a *coordinate neighbourhood.*

We can give X another topology, called the *leaf topology*, and denoted by X_l. This has as a basis subsets, which we call *slices*, of the form $h(V \times w)$ where $w \in W$ and we use the above notation. X_l clearly has a basis of connected open sets with compact closure. The components of X_l are called *leaves*. Note that the change of coordinates function $h_1^{-1}h_2$ (as in 1.3) is continuous in the leaf topology.

The following theorem is easily proved using partitions of unity.

2.2. THEOREM. – *Let* $f : A \to B$ *be a continuous map. Let* B *be paracompact with an open covering* U_i *such that each* $f^{-1}U_i$ *is paracompact. Then* A *is paracompact.*

Applying this result to the identity map $X_l \to X$, we see that X_l is paracompact. Therefore every leaf is paracompact in the leaf topology. Since a leaf is also Hausdorff, and has a basis each member of which satisfies the second axiom of countability and has compact closure, we see by a chain argument that each leaf also satisfies the second axiom of countability in the leaf topology. It follows that a leaf meets a coordinate neighbourhood in at most a countable number of slices.

2.3. It now follows that a leaf L, which is closed in X, meets a coordinate neighbourhood in at most a finite number of slices. To see this, consider a transversal T_1 contained in a larger transversal T. (*A transversal* is a subset of the form $h(v \times W)$ where $v \in V$ and we use the above notation). If $L \cap T_1$ is infinite, then it has a limit point in $L \cap T$. The holonomy construction (which we will describe later in detail) then shows that each point of $L \cap T$ is a limit point of $L \cap T$. In other words $L \cap T$ is perfect. By the Baire category theorem, no such a space is countable. Therefore $L \cap T$ is uncountable which contradicts the consequences of 2.2.

2.4. In particular we see that the subspace topology is equal to the leaf topology on a closed leaf and therefore the same is true for a compact leaf. (An irrational linear flow on a torus gives an example of a non-closed leaf where 2.4 fails).

3.1. There is another notion which we need in order to express our results. This is the notion of a *volume function* $v : Q \to (0, \infty]$.

Here Q is the *leaf space* which is the topological space obtained from a foliated space X by identifying each leaf to a point. Usually the topology of Q is very bad and Q is seldom Hausdorff. A volume function is any (not necessarily continuous) function constructed according to the following procedure. Let $\{U_i\}_{i\in I}$ be a locally finite covering of X by coordinate neighbourhoods, with corresponding charts $h_i : V_i \times W_i \to U_i$. For each $i \in I$ let $v_i : W_i \to [0, \infty)$ be a continuous function. Suppose $W_i' = \{w \in W_i : v_i(w) > 0\}$ has compact closure in W_i. We suppose that the open sets $h_i(V_i \times W_i')$ cover X. Let L be a leaf. For each $i \in I$, let $L(i) = \pi_2 h_i^{-1} (U_i \cap L) \subset W_i$. We define

$$v(L) = \Sigma_{i\in I} \Sigma_{w\in L(i)} v_i(w).$$

3.2. Note that $v(L) > 0$.

3.3. We can "improve" the covering $\{U_i\}$ and the auxiliary functions $\{v_i\}$ in the following ways without changing the function v.

3.3.1. Without changing the indexing set I or the functions $\{v_i\}$, we can shrink $\{U_i\}$ to an open covering $\{\overline{U}_i''\}$, using the paracompactness of X, so that $U_i'' = h_i(V_i'' \times W_i''\}$ with $\overline{U}_i'' \subset U_i$ and supp $v_i \leqslant W_i''$.

3.3.2. Given a (finite) open covering $\{A_j^i\}_{j\in J(i)}$ of W_i and a subordinate partition of unity $\{\lambda_j^i\}_{j\in J(i)}$ on W_i, then we can change the covering of X to $\{h_i(V_i \times A_j^i\}$ and the auxiliary functions to $\lambda_j^i \,.\, v_i : A_j^i \to [0, \infty)$.

3.3.3. Given a (finite) covering $\{B_j^i\}_{j\in J(i)}$ of V_i by open subsets, we choose strictly positive numbers b_j^i such that for each i, $\Sigma_{j\in J(i)} b_j^i = 1$. We change the covering of X to $h_i(B_j^i \times W_i)$ and the auxiliary functions to $b_j^i \,.\, v_i : W_i \to [0, \infty)$.

3.3.4. Suppose that the shrinking described in 3.3.1 is first carried out. Then we do not need to assume that in 3.3.2 $h_i(V_i \times A_j^i)$ is a subspace of U_i'' or that in 3.3.3 $h_i(B_j^i \times W_i)$ is a subspace of U_i''. In fact $h_i(V_i \times A_j^i)$ or $h_i(B_j^i \times W_i)$ need only be contained in the larger neighbourhood U_i and not in the smaller one U_i''.

3.3.5. Given a volume function v and any covering Γ of X by open sets, we may always assume that the covering $\{U_i\}$, used in the construction of v, is subordinate to the given covering Γ. To see this start with any covering $\{U_i\}$ auxiliary to the construction of v, and shrink it. Then use the fact that any open covering of a product of two compact spaces can be refined by the product of two finite coverings.

3.4. Note that if L is compact then $v(L)$ is finite, though the converse is not necessarily true. If every leaf is compact, then we have $v : Q \to (0, \infty)$.

3.5. There are many ways of constructing a volume function.

3.5.1. R.D. Edwards pointed out to the author that the functions v_i can be chosen arbitrarily as above, giving us a volume function on every foliated space.

3.5.2. If every V_i has a measure μ_i which is positive on open sets and $\{\lambda_i\}_{i\in I}$ is a partition of unity subordinate to $\{U_i\}_{i\in I}$ then we can define

$$v_i(w) = \int_{V_i} \lambda_i(v, w)\, d\mu_i.(v).$$

This method was suggested by K.C. Millett.

3.5.3. If we have a C^1-foliation of a Riemannian manifold, then the measure in 3.5.2 can be taken to be the measure associated to the Riemannian structure on V_i. This gives the same answer as taking the usual Riemannian volume of a leaf, which was the original volume function defined for the investigation of our problem.

We can now state our main results.

4.

4.1. THEOREM. – *Let* X *be a foliated space with each leaf compact. Let* $\pi : X \to Q$ *be the quotient map which identifies each leaf to a point. The following five conditions are equivalent.*

4.1.1. π is a closed map.

4.1.2. π maps compact sets to closed sets.

4.1.3. Each leaf has arbitrarily small saturated neighbourhoods.

4.1.4. Q is Hausdorff.

4.1.5. If $K \subseteq X$ is compact, then $\hat{K} = \pi^{-1}\pi K$, the saturation of K, is also compact.

Remark. – Larry Siebenmann has pointed out to the author that this is actually a result about decompositions of a locally compact Hausdorff space X. We assume that each of the subsets in the decomposition is compact and connected. We interpret the word "leaf" as "element of the decomposition". Then the above theorem gives five equivalent conditions for upper semi-continuity of the decomposition. The author has not been able to find this result explicity in the literature, although it is very close to what is well-known. For the sake of completeness, the proofs have been included.

4.2. THEOREM. – *Let* X *be a foliated space and let* L *be a compact leaf. Then the following conditions are equivalent.*

4.2.1. For some volume function v on X and for some neighbourhood N of L, $v|N$ is bounded. (By abuse of notation we write v instead of $v \cdot \pi$ and talk as though v is defined on X instead of the quotient space Q).

4.2.2. There is a neighbourhood N of L such that every volume function is bounded on N.

4.2.3. The holonomy group of L is finite. (See §5 for a definition of the holonomy group.)

Moreover these conditions imply that some neighbourhood of L consists of a union of compact leaves and that in this neighbourhood the equivalent conditions of Theorem 4.1 are true.

4.3. THEOREM. – *Let* X *be a foliated manifold with all leaves compact. Then the conditions of Theorem 4.1 imply the conditions of Theorem 4.2 for each leaf* L. *If* X *is a* C^1*-foliation (in which case we may assume without loss of generality that the manifold – but not the foliation – is* C^∞*), then the above conditions are equivalent to the fact that we have the following local model for the foliation near a leaf* L.

Let q be the codimension. We are given a finite subgroup K of the orthogonal group $0(q)$ and a homomorphism $\Psi : \pi_1 L \to K$. Let the covering space $\widetilde{L}$ correspond to the kernel of Ψ. Then K acts on $\widetilde{L}$ by covering translations. Let $\widetilde{L} \times_K D^q$ be the quotient of $\widetilde{L} \times D^q$ obtained by identifying $(\widetilde{lk}, d)$ with $(\widetilde{l}, kd)$ for each $\widetilde{l} \in L$, $k \in K$ and $d \in D^q$. Let $p : \widetilde{L} \times D^q \to \widetilde{L} \times_K D^q$ be the quotient map. Then $\widetilde{L} \times_K D^q$ is foliated by compact leaves of the form $p(\widetilde{L} \times d)$. Our local model, whose existence is equivalent to the conditions of Theorem 4.1 or 4.2, is the assertion of the existence of K and Ψ as above and a diffeomorphism of $\widetilde{L} \times_K D^q$ onto a neighbourhood of L, preserving the leaves.

Remark. – Much of the content of Theorems 4.1, 4.2 and 4.3 is due to Reeb [10], Ehresmann [1] and Haefliger [4].

5.

5.1. At this point we expound the notion of holonomy more fully (but only for compact leaves). We also introduce notation which will be used in the rest of the paper.

Let L be a compact leaf. Given an open covering $\mathcal{A}$ of X there is open covering $\{U_i\}_{i \in I}$ of X with the following properties.

5.1.1. Each U_i is a coordinate neighbourhood equal to $h_i(V_i \times W_i)$, where V_i is connected.

5.1.2. $\{U_i\}$ is locally finite.

5.1.3. If $\overline{U}_i \cap L \neq \emptyset$ then $U_i \cap L$ consists of a single slice and $U_i \cap L = \overline{U}_i \cap L$. Moreover the indexing set I is chosen so that the sets U_i which meet L are $U_1, \ldots, U_k$.

5.1.4. For each $i \in I$ $V_i \subset L$ (see 1.2) and there is a point $w_i \in W_i$ such that $h_i(v, w_i) = v$ for each $v \in V_i$.

5.1.5. There is a covering $\{U'_j\}_{j \in J}$ with the properties 5.1.1, 5.1.2, 5.1.3.and 5.1.4, which is star refined by $\{U_i\}_{i \in I}$. By this we mean that there is a function $\alpha: I \to J$ such that if $\overline{U}_r \cap \overline{U}_s \neq \emptyset$ for some $r, s \in I$ then $\overline{U}_s \subset U'_{\alpha(r)}$. (The existence of such coverings is a well-known consequence of paracompactness). We may further assume that $(h'_{\alpha(r)})^{-1} h_r$ is the identity (see the last sentence of 3.3.5).

5.1.6. There is a covering $\{U''_k\}_{k \in K}$ which refines $\mathfrak{A}$, satisfies 5.1.1, 5.1.2, 5.1.3 and 5.1.4 and bears the same relationship to $\{U'_j\}_{j \in J}$ as $\{U'_j\}_{j \in J}$ does to $\{U_i\}_{i \in I}$.

5.1.7. If we are given a volume function v on X, then by 3.3.5 we may also assume that v can be constructed using the $\{U_i\}_{i \in I}$.

5.2. Suppose $\overline{U}_r \cap \overline{U}_s \neq \emptyset$ for some r, s with $1 \leqslant r, s \leqslant k$. Then we construct the holonomy map H^r_s from a neighbourhood of w_r in W_r to a neighbourhood of w_s in W_s as follows. Let $w \in W_r$. We say $z \in W_s$ is equal to $H^r_s w$ if there exist $x \in V_r$ and $y \in V_s$ such that $h_r(x, w)$ and $h_s(y, z)$ lie on the same slice of $U'_{\alpha(r)}$. This is equivalent to them lying on the same slice of $U'_{\alpha(s)}$ since $\{U'_j\}_{j \in J}$ star refines $\{U''_k\}_{k \in K}$ and $V'_{\alpha(r)}$ is connected. H^r_s is therefore well-defined. It is continuous, since $H^r_s(w) = \pi_2 (h'_{\alpha(s)})^{-1} h_r(x, w)$.

5.3. It follows that H_s^r is a homeomorphism whose inverse is H_r^s. Moreover if $U_r \cap U_s \cap U_t \neq \emptyset$, then $H_t^s H_s^r = H_t^r$ on a small enough neighbourhood of w_r in W_r.

5.4. Now suppose that $N^1, N^2, N^3, \ldots$ is a decreasing sequence of neighbourhoods of L, such that $\cap_{j \geqslant 1} N^j = L$. For $1 \leqslant i \leqslant k$ we define by induction on $j \geqslant 1$, open neighbourhoods $C_{i,j}$ of w_i in W_i with the following properties. We choose a point $v_i \in V$ and, by abuse of notation, think of $C_{i,j}$ and W_i as embedded in X via $h_i(v_i, \;)$.

5.4.1. $\bar{C}_{i,j+1} \subset N^j$ for $j \geqslant 1$. $\bar{C}_{i,1} \subset W_i$.

5.4.2. $h_r(V_r \times (W_r \backslash C_{r,j})) \cap C_{s,j+1} = \emptyset$ for $1 \leqslant r, s \leqslant k$ and $j \geqslant 1$.

5.4.3. If $V_r \cap V_s \cap V_t \neq \emptyset$ and $1 \leqslant r, s, t \leqslant k$ then $H_t^s H_s^r$ and H_t^r are defined and equal on $C_{r,1}$.

5.4.4. The closure of $H_s^r C_{r,j+1}$ is contained in $C_{s,j}$ $(j \geqslant 1)$.

5.5. By a *chain of length r* on L, we mean a sequence $U_{i(1)}, U_{i(2)}, \ldots, U_{i(r)}$ with each $i(j)$ satisfying $1 \leqslant i(j) \leqslant k$ and such that $U_{i(j)} \cap U_{i(j+1)} \neq \emptyset$ for $1 \leqslant j < r$. A chain allows us to construct a holonomy homeomorphism from a neighbourhood of $w_{i(1)}$ in $W_{i(1)}$ into a neighbourhood of $w_{i(r)}$ in $W_{i(r)}$, namely $H_{i(r)}^{i(r-1)} H_{i(r-1)}^{i(r-2)} \ldots H_{i(2)}^{i(1)} : C_{i(1),r+1} \to C_{i(r),1}$. If we fix i, then the chains starting and ending with U_i (we call such chains *circular* chains) define a group of germs of homeomorphisms from W_i to W_i at w_i, called the *holonomy group.* The choice of i does not really matter since the holonomy group of germs of homeomorphisms of W_i at w_i and the holonomy group of germs of homeomorphisms of W_j at w_j are isomorphic via a chain joining U_i to U_j.

5.6. The holonomy group is generated by circular chains of length at most $2k + 1$, and is therefore finitely generated.

5.7. Notice that if $w \in W_i$ is mapped to $w^1 \in W_j$ by a holonomy map, then $h_i(x\ ,\, w)$ and $h_j(y\ ,\, w^1)$ are contained in the same leaf, for $x \in V_i$ and $y \in V_j$. (But the converse does not hold in general).

6.

In this section we prove Theorem 4.1. We will show that

$$4.1.1 \Rightarrow 4.1.2 \Rightarrow 4.1.3 \Rightarrow 4.1.4 \Rightarrow 4.1.5 \Rightarrow 4.1.1$$

Clearly 4.1.1 $\Rightarrow$ 4.1.2.

6.1. Suppose 4.1.2 is true. Let U be a compact neighbourhood of a leaf L and let K be the boundary of U. Then by our hypothesis $\hat{K} = \pi^{-1}\pi K$, the saturation of K, is closed. We claim that $\hat{U}\backslash\hat{K} = U\backslash\hat{K}$. For if $x \in \hat{U}$ and $x \notin \hat{K}$, then the leaf through x meets U but not K. Since the leaf is connected, it lies entirely in U. This means that $U\backslash\hat{K}$ is saturated, because it is equal to $\hat{U}\backslash\hat{K}$, and open, because it is equal to int $U\backslash\hat{K}$. This proves 4.1.3.

6.2. Clearly 4.1.3 implies 4.1.4. Now suppose 4.1.4 is true. Clearly 4.1.2 follows and it is easy to see from 6.1 that π is a closed map. It is now standard (see for example Dugundji [14]) that π is proper, so 4.1.5 follows.

6.3. Finally suppose 4.1.5 is true. Let $A \subset X$ be closed. We want to show that $\hat{A}$ is closed. Let x be in the closure of $\hat{A}$. By 4.1.5, there is compact saturated neighbourhood K of x. Then $B = K \cap A$ is compact, and so $\hat{B}$ is compact. We claim that $x \in \hat{B} \subseteq \hat{A}$, for otherwise $K\backslash\hat{B}$ would be a saturated neighbourhood of x disjoint from A and hence from $\hat{A}$.

This completes the proof of Theorem 4.1.

7.

We now prove Theorem 4.2. The main part of the proof consists of proving the following proposition. A diagram describing the situation appears at the end of this paper.

7.1. PROPOSITION. – *Let* Y *be a foliated space with a compact leaf* L, *and let* n *be an integer. Let* v *be a volume function on* Y. *Then given any neighbourhood* B *of* L *there is a finite set of coordinate neighbourhoods* $h_i(V_i \times C_i)$ $(1 \leqslant i \leqslant k)$ *and a neighbourhood* N *of* L *with the following properties.*

7.1.1. $N \subset \cup_{i=1}^{k} h_i(V_i \times C_i) \subset B$.

7.1.2. Every leaf P meeting N satisfies either

a) $vP > n \,.\, vL$,

or b) for some integer r with $1 \leqslant r \leqslant n$, $|vP - r\,.\,vL| < 1/n$ and P is a compact leaf lying in $\cup_{i=1}^{k} h_i(V_i \times C_i)$ and meeting each $h_i(V_i \times C_i)$ in exactly r leaves.

Proof of Proposition 7.1. – We use the notation of 5.1 and 5.4. We further assume that for $1 \leqslant i \leqslant k$, the neighbourhoods $C_{i,1}$ of 5.4 are chosen so that $|v_i x - v_i w_i| < 1/k(n+1)^2$ for $x \in C_{i,1}$ and $1 \leqslant i \leqslant k$. Let $B_j = \cup_{i=1}^{k} h_i(V_i \times C_{i,j})$ for $j \geqslant 1$. The sets C_i in the statement of the proposition are defined by $C_i = C_{i,1}$ for $1 \leqslant i \leqslant k$. We define the set N in the statement of the proposition by $N = B_m$ where $m = (2k+1)(n+2)$.

Let a leaf P meet N in $h_i(V_i \times C_{i,m})$. Let r be the largest integer such that $r \leqslant n+1$ and P contains at least r slices in $h_i(V_i \times C_{i,k+1})$. It follows that for each j $(1 \leqslant j \leqslant k)$, the coordinate neighbourhood $h_j(V_j \times C_{j,1})$ contains at least r slices. Therefore $vP > rvL - 1/n$ so if $r = n+1$ our proposition is proved.

We therefore assume $r \leqslant n$. Let S be the set of holonomy maps induced by chains of length at most $2k+1$, starting and ending with U_i. S generates the holonomy group of L. Let $x \in C_{i,m}$ be a point such that $h_i(V_i \times x) \subseteq P$. Then

$$\{x\} \subseteq S\{x\} \subseteq S^2\{x\} \ldots\ldots \subseteq S^{n+1}\{x\} \subseteq C_{i,2k+1}$$

and $S^{n+1}\{x\}$ consists of at most r points with $r \leqslant n$. Therefore for some t with $0 \leqslant t \leqslant n$, $S(S^t\{x\}) = S^t\{x\}$.

We claim that P is equal to the set P_1 of all points of the form $h_j(V_j \times TS^t\{x\})$ where T is a holonomy map corresponding to a chain of length at most k, starting at U_i. P_1 is obviously open in the leaf topology, so we have only to prove it is closed in the leaf topology. Because $t \leqslant n$, any point z in the closure of P_1 lies in the closure of $h_s(V_s \times y)$ for some $y \in C_{s,3k+2}$ and some s, $1 \leqslant s \leqslant k$, where $y \in TS^t$ for some T as above. Now $z \in U_q$ for some q, $1 \leqslant q \leqslant k$ and $z = h_q(v, c)$ where $v \in V_q$ and $c \in C_{q,3k+1}$. Then $c = H^s_q(y)$. There is a chain of length at most k from U_q to U_i. The corresponding holonomy map T_1 takes c to a point $c_1 \in C_{i,2k+1}$. Therefore

$$c_1 \in T_1 H^s_q TS^t\{x\} \subset S^{t+1}\{x\} = S^t\{x\}.$$

Now $c = T_1^{-1} c_1$ and so $z \in P_1$ as required.

Therefore P is contained in a finite union of compact sets and is therefore compact. Moreover, we have shown that $P \subset B_{3k+2}$. It follows that $P \cap h_j(V_j \times C_j)$ consists of exactly r slices for each j $(1 \leqslant j \leqslant k)$. The inequality

$$vP < rvL + 1/n \text{ follows.}$$

This completes the proof of our proposition.

7.2. We are now in a position to prove Theorem 4.2. We prove the implications 4.2.3 ⇒ 4.1.3, 4.2.3 ⇒ 4.2.2 ⇒ 4.2.1 ⇒ 4.2.3.

So suppose the holonomy group G of a compact leaf L is finite. Each element of the group is induced by a circular chain. Let K be an integer such that $K > k$ and K is greater than the length of any of these circular chains. Without loss of generality we assume the circular chains are based at U_1. Each element $g \in G$ corresponds to a homeomorphism into $\bar{g} : C_{1,K} \to C_{1,1}$. By taking K large enough, we can ensure that for every pair of elements $g_1, g_2 \in G$, $\overline{g_1 g_2} = \bar{g}_1 \bar{g}_2$ on $C_{1,2K}$. Let $C = \cap_{g \in G} \bar{g} C_{1,3K}$. Then for each $g \in G$, $\bar{g}C = C$ and so $\bar{g}$ is a homeomorphism of C onto C, and G acts as a group of homeomorphisms of C.

Let $x \in C$ be fixed. Then the leaf containing $h_1(V_1 \times x)$ is equal to the set of points of the form $h_j(V_j \times y)$ where $y \in TGx$ and T is a holonomy map corresponding to a chain of length at most k. The proof of this is the same as the proof that $P_1 = P$ in Proposition 7.1. Therefore the saturation of $h_1(v \times c)$, where v is some fixed point of V_1, consists of compact leaves and each leaf lies inside $\cup_{i=1}^k h_i(V_i \times C_{i,1})$. This proves that 4.2.3 implies 4.1.3.

If $r = |G|$ we see that the volume of any leaf meeting C is less than $rvL + 1/n$ if $n > r$ and K is sufficiently large. So 4.2.3 implies 4.2.2. Obviously 4.2.2 implies 4.2.1.

In order to show that 4.2.1 implies 4.2.3 we choose an integer n and a compact neighbourhood N^1 on which the volume function is bounded by $n.vL$. Then by Proposition 7.1 there is a smaller saturated neighbourhood N^2 with $N^2 \subset \cup_{i=1}^k h_i(V_i \times C_{i,4k+4})$ and such that each leaf P in N^2 is compact and contains n or fewer slices in each $h_i(V_i \times C_{i,1})$. Let $C = N^2 \cap h_i(v \times C_{i,2k+2})$ for some point $v \in V_i$. Then the holonomy group of L consists of homeomorphisms preserving C.

Every leaf P in N^2 meets C in at most n points. Numbering these points gives us a homomorphism from the holonomy group G to the symmetric group $S(r) \subseteq S(n)$. Since G is finitely generated, there are only a finite number of such homomorphisms. Let A be the intersection of all these kernels. Then A acts trivially on C and is therefore the trivial subgroup of G. It follows that G is finite. This completes the proof of Theorem 4.2.

In order to prove that the conditions of Theorem 4.1 imply the conditions of Theorem 4.2 when X is a manifold we need the following result.

7.3. THEOREM. – *Let* P *be a connected manifold and let* G *be a group of homeomorphisms of* P, *such that the orbit of any point of* P *is finite. Then* G *is finite.* (*This is a mild generalization of the same result when* G *is cyclic, which was proved by Montgomery* [5]).

Proof. – We give the proof due to K.C. Millett, which is more elegant than the author's proof. Suppose G is infinite. Replacing G by a subgroup, we may assume without loss of generality that G is countable. Let $G = \{g_1, g_2, \ldots\}$.

For each pair (i, j) with $i \neq j$ define

$$A_{i,j} = \{x \in P : g_i x = g_j x\}.$$

Then $A_{i,j}$ is closed. Moreover $A_{i,j}$ has void interior. To see this note that $g_i^{-1} g_j$ has finite order by the theorem of Montgomery cited above, and so it is the identity on P if it is the identity on a non-empty open set (see [9]). The Baire category theorem produces a point which lies in no $A_{i,j}$ and this is a contradiction.

7.4. We now show that the equivalent conditions of Theorem 4.1 imply the equivalent conditions of Theorem 4.2 if X is a foliated manifold. By 4.1.3 there is a saturated neighbourhood N of the leaf L with

$$N \subset \cup_{i=1}^{k} h_i(V_i \times C_{i,4k+4}).$$

Let $C = N \cap h_1(v \times C_{1,1})$ for some $v \in V_1$. Then the holonomy group acts as a group of homeomorphisms of C. Let D be the connected component of the point $L \cap C$ in the open manifold C. By Theorem 7.3, the holonomy group acts as a finite group of homeomorphisms of D, and therefore the holonomy group is finite.

7.5. In order to prove Theorem 4.3, we observe that in 7.4 the holonomy group is a finite group of C^1 diffeomorphisms of D. Now any C^1-action of a compact Lie group on a manifold is equivalent to a C^∞-action [6], and so we may assume that a Riemannian metric on D is preserved.

The remainder of the proof of 4.3 is standard and is left to the reader. (A very general version has been given by Haefliger [4] p. 303).

8. Counterexamples.

8.1. If X is not a manifold, then the conditions of Theorem 4.1 need not imply the conditions of Theorem 4.2, and Theorem 7.3 is not necessarily true. Consider for example $\mathbf{C} \times I \times (\mathbf{R}/\mathbf{Z})$ with the flow which sends (z, x, s) at time t to $(ze^{2\pi ixt}, x, s+t$. Let X be the union of the subset where $z = 0$ with the subset where $x = 0$ and $x = 1$ and the subset

$$(z, x, s) : |z| \leqslant 1/q , x = p/q \text{ for coprime integers } p, q$$

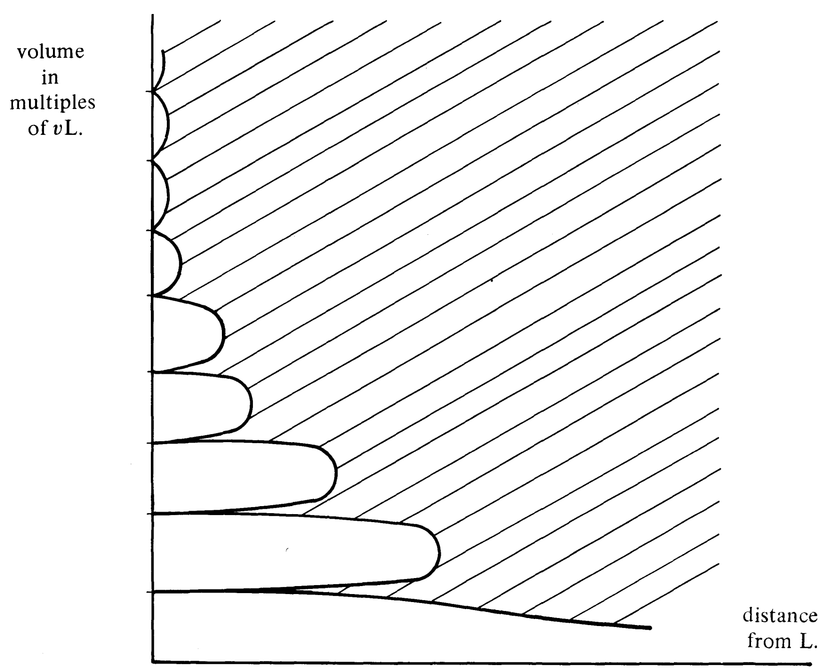
satisfies the conditions of Theorem 4.1 but not those of Theorem 4.2. Moreover X is a compact ANR of the nicest type.

8.2. If X is not a C^1-manifold but only a topological manifold, then no nice model of the kind given in Theorem 4.3 is possible.

For example the involution on 3-sphere defined by Bing [11] has a wild 2-sphere of fixed points. If x is a wild point on the 2-sphere, then there are no small invariant disk neighbourhoods of x. The mapping torus of this involution gives us a foliation of a 4-manifold by circles, some of which do not have nice neighbourhoods in the sense of Theorem 4.3.

9. The Diagram .

The following diagram is a pictorial representation of Proposition 7.1 which is very useful.



L is a fixed compact leaf. Each point x of M gives rise to a point $(d(x, L), v(L_x))$ where d is the distance function, L_x is the leaf through x and vL_x its volume (possibly infinite). Proposition 7.1 says that the point lies in the shaded area of the diagram.

We deduce immediately that the volume function is lower semi-continuous at a compact leaf, and that if X is any subset of M then the set of points at which $v|X$ is continuous is an open subset of X. These are important properties (see for example [2]).

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