Annales de l'institut Fourier

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Annales de l'institut Fourier, tome 25, nº 3-4 (1975), p. 163-176
<http://www.numdam.org/item?id=AIF_1975__25_3-4_163_0>

Ann. Inst. Fourier, Grenoble
25, 3 et 4 (1975), 163-176.

STOCHASTIC PROCESS MEASURABILITY CONDITIONS

by J. L. DOOB

Dédié à Monsieur M. Brelot à l'occasion de son 70^e anniversaire.

Introduction.

Separability, progressive measurability, well measurability, accessibility, predictability, are properties of a stochastic process introduced in order to make certain functions measurable. It is the purpose of this paper on the one hand to show the applicability and simplicity of separability in contexts where the other more recent and deeper concepts are commonly used, and on the other hand to show that the concept of separability can be extended to combine the old and new concepts. In the extension the points of the separability set of a stochastic process are replaced by optional times.

1.

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space. The outer measure P^* is defined on each set as the infimum of the measures of measurable supersets. Let $\{\mathcal{F}(t), 0 \leqslant t < \infty\}$ be an increasing right continuous family of σ algebras of measurable sets. It is supposed that $\mathcal{F}(0)$ contains the null sets. Unless some other convention is stated explicitly a « process X » means a stochastic process $X: \{x(t), t \geqslant 0\}$ with state space a compact metrizable Hausdorff space, adapted to $\mathcal{F}(\bullet)$. The process is separable if there is a « separability set » $s_\bullet = \{s_n, n \geqslant 1\}$, a countable dense subset of $[0, \infty)$

containing 0, with the property that the graph of each process sample function is in the closure of the graph of the restriction of the sample function to the separability set. A process X can be made separable by changing each random variable $x(t)$ on a null set.

An optional time will be called « discrete » if its range is countable. If T is a predictable optional time, a monotone increasing sequence $T_\bullet = \{T_n,\ n \geqslant 1\}$ of optional times « announces T » if $T_n < T$ where $T > 0$ and $\lim_{n\to\infty} T_n = T$.

If $T_1, \ldots, T_n$ are optional times, let T'_k be the *kth* value of $T_1(\omega), \ldots, T_n(\omega)$ when arranged in increasing order. Then T'_k is optional, and $T'_1, \ldots, T'_n$ will be called the arrangement of $T_1, \ldots, T_n$ in increasing order. If $T_1, \ldots, T_n$ are predictable, $T'_1, \ldots, T'_n$ are also predictable. The arrangement in increasing order is possible for a countable infinite set of optional times if T'_k is well defined. Arrangements in decreasing order are defined in the obvious way.

2. Cluster values of sample functions.

2.1. — If the state space is the extended real line we use the notation

$$x^*(t) = \limsup_{s\downarrow t} x(s), \qquad {}^*x(t) = \limsup_{s\uparrow t} x(s),$$
$$x_*(t) = \liminf_{s\downarrow t} x(s), \qquad {}_*x(t) = \liminf_{s\uparrow t} x(s)$$

except that ${}^*x(0)$ and ${}_*x(0)$ are defined as $x(0)$.

LEMMA. — *If X is an extended real valued separable process, ${}^*X$ and ${}_*X$ are predictable.*

It is sufficient to treat ${}^*X$. If $\delta > 0$ let $I(t, \delta)$ be the interval $[(t - \delta) \vee 0, t)$ for $t > 0$, the singleton $\{0\}$ for $t = 0$. Then if $s_\bullet$ is a separability set for X,

$${}^*X(t) = \lim_{\delta\to 0} \sup_{s_j \in I(t,\delta)} x(s_j).$$

Let $s_{\alpha_1}, s_{\alpha_2}, \ldots,$ be the successive members of $s_\bullet$ in $I(t, \delta)$ and let φ_{ki} be the indicator function on $[0, \infty)$ of the set where $\alpha_k = i$. Then

$$\sup_{s_j \in I(t,\delta)} x(s_j) = \lim_{k\to\infty} x(s_{\alpha_1}) \vee \ldots \vee x(s_{\alpha_k})$$
$$x(s_{\alpha_k}) = \sum_i x(s_i)\varphi_{ki}.$$

For each pair i, k, $x(s_i)\varphi_{ki}$ defines an adapted left continuous and therefore predictable process, so $^*\mathrm{X}$ is predictable.

2.2. Lemma. — *If* X *is an extended real valued separable process,* X^* *and* X_* *are progressively measurable.*

It is sufficient to treat X^*. Choose $b > 0$ and define

$$\begin{aligned} x_n(t) &= \sup\{x(s) : bj2^{-n} \leqslant s < b(j+1)2^{-n}\} \\ &\qquad\qquad \text{if } b(j-1)2^{-n} \leqslant t < bj2^{-n}, \quad j < 2^n \\ &= x^*(b) \text{ if } b(1-2^{-n}) \leqslant t \leqslant b. \end{aligned}$$

Then the function $(t, \omega) \rightsquigarrow x_n(t, \omega)$ on $[0, b] \times \Omega$ is measurable for the product of the σ-algebra of Borel subsets of $[0, b]$ by $\mathscr{F}(b)$. Since $\limsup_{n\to\infty} x_n(t) = x^*(t)$, X^* is progressively measurable.

2.3. — In the following and later theorems involving a separable process and discrete optional times, the discrete optional times will be chosen to have their values in the given separability set. This choice is not essential but will clarify the meaning of the corresponding theorems for optionally separable processes.

Theorem. — *Let* X *be separable and let* T *be a predictable time. There is then a sequence* $\mathrm{T}_\bullet$ *of discrete optional times announcing* T *such that for almost every* ω

$$\{x(\mathrm{T}_n(\omega), \omega),\ n \to \infty\},\ \{x(t, \omega),\ t \uparrow \mathrm{T}\}$$

have the same set of cluster values.

We first assume that the state space is a compact real interval and prove that there is a sequence $\mathrm{T}'_\bullet$ of discrete optional times announcing T for which

$$(2.3.1) \qquad \limsup_{n\to\infty} x(\mathrm{T}'_n) = {}^*x(\mathrm{T}) \text{ a.e.}$$

Let $U_\bullet$ announce T and let $s_\bullet$ be a separability set for X. According to a theorem of Chung [2] U_n can be chosen to be discrete with values in $s_\bullet$. Choose a_m large that

$$(2.3.2)\quad P\left\{\sup_{U_m<t\leqslant U_{m+1}} x(t) - \sup_{\substack{U_m<s_j\leqslant U_{m+1}\\ j\leqslant a_m}} x(s_j) > 1/m\right\} \leqslant 2^{-m}$$

and define the optional time T_{mj} as s_j if $U_m < s_j \leqslant U_{m+1}$ and otherwise as U_{m+1}. Arrange the set $\{T_{mj}: m \geqslant 1, j \leqslant a_m\}$ in increasing order to obtain a sequence $T'_\bullet$ announcing T and satisfying (2.3.1). Going back to a general compact metrizable state space let f be a real continuous function on the state space. Then $f(X)$ is a separable process so according to what has just been proved there is a sequence $T^{(f)}_\bullet$ of discrete optional times announcing T for which almost surely

$$(2.3.3)\qquad \limsup_{n\to\infty} f[x(T^{(f)}_n)] = \limsup_{t\uparrow T} f[x(t)].$$

Let $f_\bullet$ be a sequence of real continuous functions on the state space dense under the supremum norm in the class of all these functions. Let $T_\bullet$ be the arrangement of $\{T^{(f_k)}_n \vee U_k;\ n, k \geqslant 1$ in increasing order. Then $T_\bullet$ is a sequence of discrete optional times announcing T, and there is a null set independent of k for which

$$(2.3.4)\qquad \limsup_{n\to\infty} f[x(T_n)] = \limsup_{t\uparrow T} f[x(t)] \text{ a.e.}$$

for $f = f_k$, $k \geqslant 1$ and therefore simultaneously for every real continuous f. This fact implies the assertion of the theorem.

2.4. Theorem. — *Let X be separable and let T be an optional time. There is then a decreasing sequence $T_\bullet$ of discrete optional times for which $T_n > T$ where $T < \infty$, $\lim_{n\to\infty} T_n = T$ and, for almost every ω with $T(\omega) < \infty$,*

$$\{x(T_n(\omega), \omega),\ n \to \infty\},\ \{x(t, \omega),\ t \downarrow T\}$$

have the same set of cluster values.

The proof of the preceding theorem shows that it is sufficient to show that if the state space is a compact real interval there is a decreasing sequence $T'_\bullet$ of discrete optional times for which $T'_n > T$ where $T < \infty$, $\lim_{n\to\infty} T'_n = T$ and

$$(2.4.1) \qquad \limsup_{n\to\infty} x(T'_n) = x^*(T) \text{ a.e.}$$

Let $s_\bullet$ be a separability set for X and let $U_\bullet$ be a decreasing sequence of optional times with all finite values in the separability set and $T < U_n \leqslant T + 1/n$ when $T < \infty$. Choose a_n so large that

$$(2.4.2) \qquad P\left\{T < \infty, \sup_{T<t<U_m} x(t) - \sup_{\substack{T<s_j<U_m \\ j\leqslant a_m}} x(s_j) > 1/m\right\} \leqslant 2^{-m}$$

and define T_{mj} as s_j if $T < s_j < U_m$ and otherwise as U_m. Arrange $\{T_{mj} : m \geqslant 1, j \leqslant a_m\}$ in decreasing order to get a sequence $T'_\bullet$ satisfying (2.4.1).

3. Local limit theorems.

3.1. — The following theorems will be stated for processes with state space a compact interval, that is for real bounded processes. The application to more general processes will be discussed after the statement of Theorem 3.2.

LEMMA. — *Let* $\{x_n, \mathscr{F}_n, -\infty < n < \infty\}$ *be an adapted process, with metrizable state space.*

(*a*) *If* x_∞ *is measurable and if for almost every* $\omega x_\infty(\omega)$ *is a cluster value of the sequence* $x_\bullet(\omega)$ *at* ∞ *then there is an increasing sequence* $\beta_\bullet$ *of bounded optional times, with*

$$\lim_{n\to\infty} \beta_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} x_{\beta_n} = x_\infty$$

almost everywhere.

(*b*) *If* $x_{-\infty}$ *is measurable and if for almost every* $\omega x_{-\infty}(\omega)$ *is a cluster value of the sequence* $x_\bullet(\omega)$ *at* $-\infty$ *then there is a decreasing sequence* $\alpha_\bullet$ *of bounded optional times, with*

$$\lim_{n\to\infty} \alpha_n = -\infty \quad \textit{and} \quad \lim_{n\to\infty} x_{\alpha_n} = x_{-\infty}$$

almost everywhere.

Part (a) was proved by Austin, Edgar and A. Tulcea in a trivially more special form. To prove (b) let d be a metric for the state space, choose $a_1 = -1$ and for $k > 1$ choose $a_k < a_{k-1}$ in such a way that

$$(3.1.1) \quad \mathrm{P}\left\{\omega : \min_{a_k \leqslant j < a_{k-1}} d[x_j(\omega), x_{-\infty}(\omega)] < 1/\mathrm{k}\right\} > 1 - 2^{-k}.$$

Define $\alpha_k(\omega)$ as the smallest j satisfying $a_k \leqslant j < a_{k-1}$ for which $d[x_j(\omega), x_{-\infty}(\omega)] < 1/k$, or $\alpha = a_{k-1}$ if there is no such j.

3.2. Theorem. — *Let* X *be a separable real bounded process and let* T *be a predictable time. If* $\lim_{n\to\infty} \mathrm{E}\{x(\mathrm{T}_n)\}$ *exists whenever* $\mathrm{T}_\bullet$ *is a sequence of discrete optional times announcing* T *then* X *almost surely has a left limit at* T.

Observation: This theorem can be applied as follows. Let X be separable, with a compact metrizable state space, and let T be a predictable time. Suppose that $\lim_{n\to\infty} \mathrm{E}\{f[x(\mathrm{T}_n)]\}$ exists whenever $\mathrm{T}_\bullet$ is a sequence of discrete optional times announcing T and f is a real continuous function on the state space. [Equivalently suppose that the distribution of $x(\mathrm{T}_n)$ has a limit $(n \to \infty)$ whenever $\mathrm{T}_\bullet$ is a sequence of discrete optional times announcing T.] Then according to Theorem 3.2 $f(\mathrm{X})$ has almost surely a left limit at T and it follows that X almost surely has a left limit à T. If X is extended real valued the condition is satisfied if $\lim_{n\to\infty} x(\mathrm{T}_n)$ exists in measure for $\mathrm{T}_\bullet$ as described. Corresponding observations for later theorems will be omitted.

We first prove that $\mathrm{L} = \lim_{n\to\infty} \mathrm{E}\{x(\mathrm{T}_n)\}$ does not depend on $\mathrm{T}_\bullet$. In fact if $\mathrm{T}'_\bullet$ and $\mathrm{T}''_\bullet$ are sequences of discrete optional times announcing T, giving expectation limits L′, L″, and if $\varepsilon > 0$, define a sequence $\mathrm{T}_\bullet$ announcing T by

$$\mathrm{T}_1 = \mathrm{T}'_1, \quad \mathrm{T}_2 = \mathrm{T}''_{a_2} \vee \mathrm{T}_1, \quad \mathrm{T}_3 = \mathrm{T}'_{a_3} \vee \mathrm{T}_2, \ \ldots$$

where a_n is so large that

$$\mathrm{P}\{\mathrm{T}_n = \mathrm{T}'_{a_n}\} > 1 - \varepsilon, \qquad \text{or} \qquad \mathrm{P}\{\mathrm{T}_n = \mathrm{T}''_{a_n}\} > 1 - \varepsilon$$

according as n is odd or even. But then $\lim_{n\to\infty} \mathrm{E}\{x(\mathrm{T}_n)\}$ is arbitrarily near both L′ and L″ for ε small. Hence $\mathrm{L}' = \mathrm{L}''$, as asserted. Now choose $\mathrm{T}_\bullet$ as in Theorem 2.3 and apply Lemma 3.1 to find a sequence of times $\beta_n \to \infty$ such that T_{β_n} is optional, discrete, $\mathrm{T}_{\beta_\bullet}$ announces T, and

$$\lim_{n\to\infty} x(\mathrm{T}_{\beta_n}) = {}^*x(\mathrm{T})$$

almost everywhere. Then $\mathrm{L} = \mathrm{E}\{{}^*x(t)\}$. Similarly

$$\mathrm{L} = \mathrm{E}\{{}_*x(\mathrm{T})\}$$

and the theorem follows.

3.3. THEOREM. — *Let* X *be a separable real bounded process and let* T *be an optional time. If* $\lim_{n\to\infty} \mathrm{E}\{x(\mathrm{T}_n)1_{\mathrm{T}_n<\infty}\}$ *exists whenever* $\mathrm{T}_\bullet$ *is a decreasing sequence of discrete optional times with almost sure limit* T *then* X *has an almost sure right limit at* T.

The proof is parallel to that of Theorem 3.2 and is omitted.

4. Global limit theorems.

4.1. THEOREM. — *Let* X *be a separable real bounded process. Suppose that whenever* $\mathrm{T}_\bullet$ *is an increasing bounded sequence of discrete optional times* $\lim_{n\to\infty} \mathrm{E}\{x(\mathrm{T}_n)\}$ *exists. Then* X *almost surely has left limits.*

(The language of the conclusion means as usual that almost every sample function has a left limit at every strictly positive parameter value.) It is sufficient to show that the predictable processes ${}^*\mathrm{X}$, ${}_*\mathrm{X}$ are indistinguishable. Since they are almost surely equal at any bounded predictable time by Theorem 3.2, these processes must be indistinguishable, as shown by a section argument [3]. One of the purposes of this paper however is to show that in many contexts the use of deep section arguments is unnecessary. We therefore give a second proof of the theorem using only elementary measure theory. Let a, b be real numbers with $a < b$ and define

$$\mathrm{T} = \inf\{s : {}^*x(s) > b > a > {}^*x(s)\}.$$

To prove the theorem we prove that $T = \infty$ almost everywhere. If not, there is a pair (a, b) and a number k so that $P^*\{T \leqslant k\} = \delta > 0$. If S is a positive random variable define

$$[S, n] = \bigcup_{j=1}^{n} \{x(s_j) > b, \ S \leqslant s_j < T \leqslant k\} \cap \bigcup_{j=1}^{n} \{x(s_j) < a, \ S \leqslant s_j < T \leqslant k\}.$$

Choose c_1 so large that $P^*\{[0, c_1]\} > \delta/2$ and define

$$T_1 = k \wedge \min(s_j : \ j \leqslant c_1, \ x(s_j) > b\}.$$

Then T_1 is optional, $x(T_1) > b$ when $T_1 < k$ and

$$P\{T_1 < k\} > \delta/2.$$

Choose c_2 so large that

$$P^*\{[0, c_1] \cap [T_1, c_2]\} > \delta/2$$

and define

$$T_2 = k \wedge \min\{s_j > T_1 : j \leqslant c_2, x(s_j) < a\}.$$

Then T_2 is optional, $x(T_2) < a$ when $T_2 < k$ and

$$P\{T_1 < T_2 < k\} > \delta/2.$$

Continuing in this way we obtain an increasing sequence $T_\bullet$ of optional times, bounded by k, for which $\lim_{n\to\infty} x(T_n)$ does not exist almost everywhere, a contradiction to the conclusion of Theorem 3.2, so $T = \infty$ almost everywhere, as was to be proved.

Note that our hypotheses are not strong enough to imply the measurability of $x(T)$ for T bounded and optional.

4.2. THEOREM. — *Let* X *be a separable real bounded process. Suppose that whenever* $T_\bullet$ *is a decreasing sequence of bounded discrete optional times* $\lim_{n\to\infty} E\{x(T_n)\}$ *exists. Then* X *almost surely has right limits.*

Let ε be strictly positive and define the following optional times by induction:

$$T_0 = 0, T_{n+1} = \inf\{t > 0 : [\text{osc.}\, x(.) \text{ in } (T_n, T_n + t)] > \varepsilon\}$$

for n a countable ordinal, and $T_n = \sup_{m<n} T_m$ if n is a limit countable ordinal. According to a standard argument for almost every ω either $T_n \to \infty$ or there is a first countable ordinal m with $T_m = T_{m+1} = \cdots$. If $T_n \to \infty$ almost everywhere for every ε X almost surely has right limits. Otherwise choose ε, m so that $P\{T_m = T_{m+1} < \infty\} > 0$. But then, contrary to Theorem 3.3, X does not almost surely have a right limit at $T_m \wedge k$ for large k and the proof is complete.

5. Optional separability sets.

5.1. — If X is a process, a sequence $S_\bullet$ of finite optional times will be called an optional separability set for X if for each ω the set $S_\bullet(\omega)$ contains 0 and is dense in $[0, \infty)$ and the graph of the sample function $x(\bullet, \omega)$ is in the closure of the graph restricted to the parameter set $S_\bullet(\omega)$. If each S_n is predictable $S_\bullet$ will be called a predictable separability set. A process X having an optional [predictable] separability set will be called optionally [predictably] separable. In either case the set $\{S_n \wedge k : n, k \geqslant 1\}$ is a separability set for X of the same type whose times are bounded. Since the graph of an accessible time is a subset of a countable union of graphs of predictable times [3] there is no reason to consider separately optional separability sets whose times are accessible.

If X is separable and if T is optional the process X^T with $x^T(t) = x(T + t)$ with associated σ algebra family $\mathscr{F}^T(.)$, $\mathscr{F}^T(t) = \mathscr{F}(T + t)$, is not necessarily separable. If X is optionally separable with optional separability set S., however, X^T is also optionally separable, with optional separability set $S_\bullet^T$ given by $S_n^T = (S_n - T) \vee 0$. Moreover if $S_\bullet$ is a predictable separability set for X, $S_\bullet^T$ is a predictable separability set for X^T.

It is an old result, recalled in the Introduction, that every process whose state space is compact and metrizable has a standard modification which is separable. That is, in the present terminology, the modification has an optional separability set each of whose times is identically constant. The following theorem, whose proof unfortunately uses section theorems, shows that the situation is simpler in the context of optional separability sets, at least if the processes are somewhat restricted.

5.2. Theorem. — *A well measurable process is indistinguishable from some optionally separable process. An accessible or predictable process is indistinguishable from some predictably separable process.*

Let X be well measurable and suppose first that X is real and bounded. Let I be a left closed right open subinterval of $[0, \infty)$ with right endpoint $b < \infty$. The set

$$A_r = \{(t, \omega) : t \in I, \ x(t, \omega) > r\}$$

is well measurable so by a section theorem of Meyer there is an optional time $T_{nr}(I)$ whose values lie in $I \cup \{b\}$, whose graph, except for points with $T_{nr}(I)(\omega) = b$, is in A_r and for which

$$P\{T_{nr}(I) \in I\} > P\{A_r'\} - 1/n,$$

where A_r' is the projection of A_r on Ω. If the set of all optional times $T_{nr}(I) : n \geqslant 1$, r rational, I with rational endpoints, is rewritten as $\{S_n', n \geqslant 1\}$ then

$$\sup_{t \in I} x(t) = \sup_n \{x(S_n') : S_n'(\omega) \in I\} \text{ a.e.}$$

simultaneously for every such I. We now go to a compact metrizable state space and apply the result just obtained to the process $f(X)$, where f is continuous from the state space to the reals. We obtain a countable family $S_\bullet^{(f)}$ of optional times for which

$$\sup_{t \in I} f[x(t)] = \sup_n \{f[x(S_n^{(f)})] : S_n^{(f)}(\omega) \in I\} \text{ a.e.} \tag{5.2.1}$$

If this is done for every f in a sequence $f_\bullet$ dense (uniform norm) in the space of real continuous functions on the state space we obtain a sequence $S_\bullet$, the collection of all $S_n^{(f_k)}$, for which (5.2.1) is true with S_n instead of $S_n^{(f)}$ simultaneously for all f and I. The sequence $S_\bullet$ is an optional separability set for X, neglecting a null set, that is an optional separability set for a suitably chosen process indistinguishable from X. If X is predictable, $T_{mn}(I)$ can be taken predictable so S_n

becomes predictable. If X is accessible, $T_{mn}(I)$ can be taken accessible and in turn each $T_{mn}(I)$ can be replaced by countably many predictable times whose graphs have union the graph of $T_{mn}(I)$, so again S_n is predictable.

5.3. — With the help of Theorem 5.2 the results obtained in preceding sections for separable processes have easily proved analogues for well measurable and predictable processes. The point is that the proofs for separable processes need no formal change, only a change in the interpretation of the symbols used. The principle involved is illustrated in the following proof of a simple known result, the well measurable version of Lemma 2.1 : *If* X *is an extended real valued well measurable process,* $^*X$ *and* $_*X$ *are predictable.* To prove this we can assume that X is optionally separable and then in the proof of Lemma 2.1 if the separability sequence $s_\bullet$ is replaced by the optional separability sequence $S_\bullet$ the proof yields the present result. The well measurable version of Lemma 2.2 is proved in the same way.

Since the well measurable versions of the results obtained in the previous sections are proved by replacing separability sets by optional separability sets we omit discussion of proofs in the following remarks. Theorems 2.3 and 2.4 have obvious well measurable analogues except for one change : in each case we no longer can say that T_n is discrete, merely that T_n is optional. In fact the discrete nature of the optional time T_n becomes in the well measurable context the property that $T_n(\omega)$ has its values in the set $S_\bullet(\omega)$, where $S_\bullet$ is the given optional separability set. This fact is not interesting in the well measurable context, in which S_n is not identically constant. Similarly in the well measurable versions of Theorems 3.2, 3.3, 4.1, 4.2, we can no longer restrict T_n to be discrete. Finally, there is also a predictable version of Theorem 4.1. In fact *if* X *is a predictable real bounded process and if*

$$\lim_{n\to\infty} E\{x(T_n)\}$$

exists whenever $T_\bullet$ *is an increasing bounded sequence of predictable times, then* X *almost surely has left limits.* To prove this note that in the proof of Theorem 4.1 if a predictable separability set $S_\bullet$ replaces the separability set $s_\bullet$ the optional times $T_1, T_2, \ldots$ found are predictable.

For a different approach to some of these theorems see [3], [4], [5].

6. Application.

6.1. — The results on separable processes will be applied after proving a lemma having independent interest. In this lemma $\{\mathscr{F}_n, -\infty < n < \infty\}$ is an increasing family of σ algebras of measurable sets of a probability space and

$$\mathscr{F}_{-\infty} = \bigcap_n \mathscr{F}_n, \qquad \mathscr{F}_\infty = \bigcup_n \mathscr{F}_n.$$

LEMMA. — *Let x, x_n be integrable real random variables, $-\infty \leqslant n \leqslant \infty$. Suppose that x_n is $\mathscr{F}_n$ measurable and that*

$$\lim_{n\to-\infty} x_n = x_{-\infty}, \qquad \lim_{n\to\infty} x_n = x_\infty \ a.e.$$

Then

$$\lim_{n\to-\infty} \mathrm{E}\{|x - x_n| \mid \mathscr{F}_n\} = \mathrm{E}\{|x - x_{-\infty}| \mid \mathscr{F}_{-\infty}\} \ a.e.$$

$$(6.1.1) \qquad \lim_{n\to\infty} \mathrm{E}\{|x - x_n| \mid \mathscr{F}_n\} = \mathrm{E}\{x - x_\infty| \mid \mathscr{F}_\infty\} \ a.e.$$

Observation: If $\{x_n, \mathscr{F}_n, n \geqslant 1\}$ is an L^1 bounded supermartingale, for example, with $\lim_{n\to\infty} x_n = x_\infty$ almost everywhere, this theorem gives the apparently new result

$$\lim_{n\to\infty} \mathrm{E}\{|x_\infty - x_n| \mid \mathscr{F}_n\} = 0$$

almost everywhere. This result is false as an L^1 limit.

To prove the lemma note first that by Fatou's lemma for conditional expectations

$$(6.1.2) \qquad \begin{aligned} \liminf_{n\to-\infty} \mathrm{E}\{|x - x_n| \mid \mathscr{F}_n\} &\geqslant \mathrm{E}\{|x - x_{-\infty}| \mid \mathscr{F}_{-\infty}\} \text{ a.e.} \\ \liminf_{n\to\infty} \mathrm{E}\{|x - x_n| \mid \mathscr{F}_n\} &\geqslant \mathrm{E}\{|x - x_\infty| \mid \mathscr{F}_\infty\} \text{ a.e.} \end{aligned}$$

In the other direction

$$(6.1.3) \qquad \begin{aligned} &\limsup_{n\to-\infty} \mathrm{E}\{|x - x_n| \mid \mathscr{F}_n\} \\ &\quad \leqslant \limsup_{n\to-\infty} [\mathrm{E}\{|x - x_{-\infty}| \mid \mathscr{F}_n\} + |x_{-\infty} - x_n|] \\ &\quad = \mathrm{E}\{|x - x_{-\infty}| \mid \mathscr{F}_{-\infty}\} \text{ a.e.} \end{aligned}$$

and

$$(6.1.4) \quad \limsup_{n\to\infty} \mathrm{E}\{|x - x_n| \mid \mathscr{F}_n\}$$
$$\leqslant \limsup_{n\to\infty} [\mathrm{E}\{|x - x_\infty| \mid \mathscr{F}_n\} + \mathrm{E}\{|x_\infty - x_k| \mid \mathscr{F}_n\} + |x_k - x_n|]$$
$$= \mathrm{E}\{|x - x_\infty| \mid \mathscr{F}_\infty\} + 2|x_\infty - x_k| \text{ a.e.}$$

for every k. Hence (6.1.1) is true.

This lemma is easily generalized in various directions. For example if almost everywhere convergence in the hypotheses is replaced by convergence in measure the conclusion is true with convergence in measure. If all random variable concerned are pth power integrable for some $p > 1$ a trivial rewording of the proof yields

$$\lim_{n\to-\infty} \mathrm{E}\{|x - x_n|^p | \mathscr{F}_n\} = \mathrm{E}\{|x - x_{-\infty}|^p | \mathscr{F}_{-\infty}\} \text{ a.e.},$$

with a corresponding result when $n \to \infty$.

6.2. — Going back to the conventions of the early sections, let X be a real right continuous process whose random variables are integrable, let x be an integrable random variable and define

$$(6.2.1) \qquad y_0(t) = \mathrm{E}\{|x - x(t)| \mid \mathscr{F}(t)\},$$

choosing versions of the conditional expectations to make the process Y_0 separable. If T is a finite discrete optional time,

$$(6.2.2) \qquad y_0(\mathrm{T}) = \mathrm{E}\{|x - x(\mathrm{T})| \mid \mathscr{F}(\mathrm{T})\} \text{ a.e.}$$

because

$$(6.2.3) \quad y_0(\mathrm{T}) = \sum_a \mathrm{E}\{|x - x(a)| \mid \mathscr{F}(a)\}1_{\{\mathrm{T}=a\}}$$
$$= \sum_a \mathrm{E}\{|x - x(\mathrm{T})| \mid \mathscr{F}(\mathrm{T})\}1_{\{\mathrm{T}=a\}} \text{ a.e.}$$

Now suppose in addition that $x(\mathrm{T})$ is integrable whenever T is optional and bounded. If $\mathrm{T}_\bullet$ is a decreasing sequence of bounded discrete optional times, $> \mathrm{T}$, with limit T, Lemma 6.1 yields

$$(6.2.4) \qquad \lim_{n\to\infty} y_0(\mathrm{T}_n) = \mathrm{E}|\{x - x(\mathrm{T})| \mid \mathscr{F}(\mathrm{T})\} \text{ a.e.}$$

According to Theorem 4.2 the process Y_0 must almost surely have right limits. Hence if we define $y(t) = y_0(t_+)$ we obtain an almost surely right continuous modification Y of Y_0 satisfying

$$(6.2.5) \qquad y(T) = E\{|x - x(T)| \mid \mathscr{F}(T)\} \text{ a.e.}$$

for each bounded optional time T (or each finite optional time if we had allowed T to be unbounded in the hypothesis on X).

If X is also supposed to have left limits and if $x(T_-)$ is supposed integrable for every bounded predictable time T an application of Lemma 6.1 shows that for T bounded and predictable Y almost surely has a left limit at T, with

$$(6.2.6) \qquad y(T^-) = E\{|x - x(T^-)| \mid \mathscr{F}(T_-)\} \text{ a.s.}$$

Hence according to Theorem 4.1 Y_0 almost surely has left limits. The corresponding discussion for X supposed left continuous is omitted.

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Manuscrit reçu le 10 décembre 1974.

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