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CLOSURES OF FACES OF COMPACT CONVEX SETS

by A.K. ROY

1. Introduction.

It is well-known that one of the disconcerting facts in the theory of infinite-dimensional compact convex sets is that the closure of a face need not be a face. The main purpose of this paper is to determine necessary and sufficient conditions which ensure that this pathology does not occur for a given face. It should be emphasised that our results are purely individual in character. We do *not* characterise the class of compact convex sets which have the property that the closures of all their faces are again faces. (As a matter of fact, this appears to be a very difficult problem.) By way of applications, it is shown that several results scattered in the literature can be proved in a rather economical and uniform manner by our method.

We conclude by giving several characterizations of cases when face (C) is closed in a compact convex set K, for any closed convex subset C of K without core points. This generalises a recent result in [11]. Our method of proof is quite different.

It is a pleasure to thank Dr. A.J. Ellis for showing some interest in this investigation and for providing me with the example at the end of § 3.

2. Definitions & Notations.

We will work with a fixed compact convex set K in a locally convex Hausderff topological vector space E defined over the reals

R. We assume throughout that K is "regularly embedded" in E in the sense defined in [1].

Following [1], we let $\partial_e K$ be the set of extreme points of K and let C(K), P(K) and A(K) denote, respectively, the space of continuous functions, the cone of continuous convex functions and the space of continuous affine functions, on K. Let $M_1^+(K)$ denote the convex set of probability measures on K equipped with the weak* topology induced on it by M(K), the dual of C(K).

For each $x \in K$, we write

$$M_x = \{\mu \in M_1^+(K) : \mu(a) = a(x), \forall\ a \in A(K)\}$$

which is a non-empty weak* compact convex subset of $M_1^+(K)$. Let Z_x denote the set of *maximal or boundary measures* [1] in M_x.

If $f \in C(K)$, we define

$$\hat{f}(x) = inf\{h(x) : h \in A(K), h \geq f\};$$

which is the least upper semicontinuous (u.s.c) concave majorant of f and, dually, we define $\check{f}$ as the greatest convex minorant of f.

If C is a proper compact convex subset of K, we define for each $\alpha \geq 1$,

$$D_\alpha(C) = (\alpha C - (\alpha - 1) K) \cap K$$

and by face (C) we mean the σ-compact set $\bigcup_{n=1} D_n(C)$. We recall [2 : page 99] that face (C) is the smallest, not necessarily closed, face of K containing C.

If f is a function defined on K and S is a subset of K, we consistently employ the notation $f(S) \leq \alpha$ to mean $f(x) \leq \alpha$ for all $x \in S$. A similar meaning should be given to $f(S) = 0$.

3. Conditions for the closure of a face to be a face.

Let $F \subset K$ be a face and let $a \in A(K)$ be such that $a \leq 0$ on F, and hence on $\overline{F}$. The theorem in this section is motivated by the following simple observation :

$$\widehat{0 \vee a}(x) = 0 \quad \text{for all} \quad x \in \mathrm{F}.$$

This follows from the fact (see [1]) that

$$\widehat{0 \vee a}(x) = \sup \{\mu(0 \vee a) : \mu \text{ discrete}, \mu \in \mathrm{M}_x\}.$$

However, since $\widehat{0 \vee a}$ is u.s.c. we cannot, in general, assert that $\widehat{0 \vee a}(x) = 0$ for all $x \in \overline{\mathrm{F}}$. But this is the case if and only if $\overline{\mathrm{F}}$ is also a face.

Let

$$\mathrm{F}^* = \{a \in \mathrm{A(K)} : a(\mathrm{F}) \leqslant 0\}$$

and

$$(\mathrm{F}^*)_* = \{x \in \mathrm{K} : a(x) \leqslant 0 \ \forall\, a \in \mathrm{F}^*\}.$$

Then we have the following.

LEMMA 3.1. $\overline{\mathrm{F}} = (\mathrm{F}^*)_*$

We omit the proof which is a simple application of the Hahn-Banach (separation) theorem. We will also need the following simple result.

LEMMA 3.2. – *Let* $f \in \mathrm{P(K)}$ *and let* $\{f_n\}$ *be a sequence of functions in* P(K) *converging uniformly to* f. *Then* $\{\hat{f}_n\}$ *converges uniformly to* $\hat{f}$.

This is an obvious consequence of the fact that $f - \epsilon \leqslant g \leqslant f + \epsilon$ implies $\hat{f} - \epsilon \leqslant \hat{g} \leqslant \hat{f} + \epsilon$ for any $\epsilon > 0$.

Adopting the terminology cf [6], we say that F* is *perfect* if for any $a \in \mathrm{F}^*$ and $\epsilon > 0$, there exists $a_\epsilon \in \mathrm{F}^*$ such that 0, $a \leqslant a_\epsilon + \epsilon$.

We can now state the main result of this section as follows :

THEOREM 3.3. – *Let* $\mathrm{F} \subseteq \mathrm{K}$ *be a proper face. Then the following are equivalent :*

(1) $\overline{\mathrm{F}}$ *is a face.*

(2) F* is perfect.

(3) $\widehat{0 \vee a}(\overline{\mathrm{F}}) = 0 \ \forall\, a \in \mathrm{F}^*$.

(4) $\widehat{0 \vee f}(\overline{\mathrm{F}}) = 0 \;\forall f \in \mathrm{P(K)}$ *such that* $f(\mathrm{F}) \leqslant 0$.

(5) *If* $-g$, $f \in \mathrm{P(K)}$ *with* $f(\mathrm{F}) \leqslant g(\mathrm{F})$, *then* $\widehat{f \vee g} = g$ *on* $\overline{\mathrm{F}}$.

Comments 1. – If F is assumed to be closed, then the equivalence of (1) and a result similar to (2) has been proved in [4] by means of the "polar calculus". However, our proof, which is an adaptation to this context of an argument in [8], and formulation are somewhat different.

2. – We should note that the statements (2) – (5) have obvious "duals" : for example, the dual of (2) is $\widecheck{0 \wedge a}\,(\overline{\mathrm{F}}) = 0 \;\forall a \in \mathrm{F}^0$ where

$$\mathrm{F}^0 = \{a \in \mathrm{A(K)} : a(\mathrm{F}) \geqslant 0\}.$$

Proof of Theorem 3.3

(1) ⇒ (2). Suppose F* is not perfect. Then $\exists\, a_0 \in \mathrm{F}^*$ and $\epsilon_0 > 0$ such that $\forall\, b \in \mathrm{F}^*$,

$$\left.\begin{array}{ll} \text{either} & a_0 \nleqslant b + \epsilon_0 \\ \text{or} & 0 \nleqslant b + \epsilon_0 \end{array}\right\} . \quad (\alpha)$$

If $\mathrm{A(K)}^+$ denotes the positive cone in A(K), define

$$\mathrm{U} = \{a \in \mathrm{A(K)} \;:\; \|a\| \leqslant \epsilon_0\}$$

and

$$\mathrm{H} = \{(b - p\,,\, b - q) \;:\; b \in \mathrm{F}^*\,, p\,,\, q \in \mathrm{A(K)}^+\}.$$

Then (α) can be restated as

$$(a_0\,, 0) + (u_1\,, u_2) \notin \mathrm{H}, \forall\; u_1\,,\, u_2 \in \mathrm{U}.$$

This implies that $(a_0\,, 0) \notin \overline{\mathrm{H}}$ and hence by the Hahn-Banach theorem, $\exists\, \varphi \in (\mathrm{A(K)} \times \mathrm{A(K)})^*$ such that

$$\sup \varphi(\overline{\mathrm{H}}) < \varphi(a_0\,, 0). \quad (\beta)$$

H being a cone, (β) says that $\varphi \leqslant 0$ on H. Now, we can write

$$\varphi = \varphi_1 + \varphi_2 \quad \text{where} \quad \varphi_i \in \mathrm{A(K)}^* \; (i = 1,2)$$

and

$$\varphi_1(a) = \varphi(a\,, 0)\,, \varphi_2(b) = \varphi(0\,, b) \quad \text{for} \quad a\,,\, b \in \mathrm{A(K)}.$$

If $c \in A(K)^+$ then $(-c, 0) \in H$ and hence $\varphi_1(-c) = \varphi(-c, 0) \leqslant 0$, showing that $\varphi_1 \geqslant 0$. Similarly, $\varphi_2 \geqslant 0$ and thus (by [13]),

$\varphi_i(a) = \lambda_i a(x_i) \;\forall\; a \in A(K)$, for some $\lambda_i \in R^+$ and $x_i \in K$ $(i = 1,2)$.

If $a \in F^*$, $(a, a) \in H$ and therefore

$$\begin{aligned} 0 \geqslant \varphi(a, a) &= \varphi_1(a) + \varphi_2(a0 \\ &= \lambda_1 a(x_1) + \lambda_2 a(x_2) \\ &= a(\lambda_1 x_1 + \lambda_2 x_2), \end{aligned}$$

showing, by lemma 3.1, that $\dfrac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} \in \overline{F}$. But by ($\beta$),

$$0 < \varphi(a_0, 0) = \varphi_1(a_0, 0) = \lambda_1 a_0(x_1)$$

and this shows, again by lemma 3.1, that $x_1 \notin \overline{F}$, so $\overline{F}$ is not a face.

(2) $\Rightarrow$ (3). If $a \in F^*$ and $\epsilon > 0$ then by assumption there exists $a_\epsilon \in F^*$ such that

$$0, a \leqslant a_\epsilon + \epsilon$$

which implies that

$$\widehat{0 \vee a}(x) \leqslant a_\epsilon(x) + \epsilon \leqslant \epsilon, \forall\; x \in \overline{F}$$

and since ϵ is arbitrary, we can conclude that $\widehat{0 \vee a}(\overline{F}) = 0$.

(3) $\Rightarrow$ (4). If $a_1, a_2 \in F^*$ then for each $x \in K$,

$$0 \vee a_1(x) \vee a_2(x) \leqslant 0 \vee a_1(x) + 0 \vee a_2(x)$$

and hence

$$\widehat{0 \vee a_1 \vee a_2} \leqslant \widehat{0 \vee a_1 + 0 \vee a_2} \leqslant \widehat{0 \vee a_1} + \widehat{0 \vee a_2}$$

by the subadditivity of the $\wedge$ function and it follows that $0 \vee a_1 \vee a_2$ $(\overline{F}) \leqslant 0$. By induction, this is true for any *finite* number of $a_i \in F^*$. If $f \in P(K)$ with $f(\overline{F}) \leqslant 0$, we know from [1] that f can be approximated uniformly by an increasing sequence of functions of the form $a_1^{(n)} \vee \ldots \vee a_k^{(n)}$ where $a_i^{(n)} \in F^*$ for $i = 1, \ldots, k$. Therefore, by lemma 3.2, $(0 \vee a_1^{(n)} \ldots \vee a_k^{(n)})^\wedge$ increases monotenically to $\widehat{0 \vee f}$ and it follows that $\widehat{0 \vee f}(\overline{F}) = 0$.

(4) $\Rightarrow$ (5). If $f(\mathrm{F}) \leqslant g(\mathrm{F})$ then $(f-g)\ (\mathrm{F}) \leqslant 0$ and since $f-g \in \mathrm{P(K)}$, by (4) $\widehat{0 \vee (f-g)} = 0$ on $\overline{\mathrm{F}}$. But

$$f \vee g = 0 \vee (f-g) + g$$

and therefore

$$\widehat{f \vee g} = \widehat{0 \vee (f-g) + g} \leqslant \widehat{0 \vee (f-g)} + \hat{g} = g \text{ on } \overline{\mathrm{F}}$$

and (5) follows.

(5) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). Let $x \in \overline{\mathrm{F}}$ and consider $\mu \in \mathrm{M}_x$. Suppose, if possible, that supp. $(\mu)\backslash\overline{\mathrm{F}} \neq \emptyset$, i.e. $\exists\, y \in$ supp. (μ), $y \notin \overline{\mathrm{F}}$. By the Hahn-Banach theorem, $\exists\, a \in \mathrm{A(K)}$ such that

$$a(\overline{\mathrm{F}}) \leqslant 0 < a(y).$$

So $a \in \mathrm{F}^*$. By continuity, there exists a neighbourhood U of y with $\mathrm{U} \cap \overline{\mathrm{F}} = \emptyset$ and $a(\mathrm{U}) \geqslant \alpha > 0$ for some α.

Now,

$$\begin{aligned}\widehat{0 \vee a}(x) &= \sup\{\lambda(0 \vee a) : \lambda \in \mathrm{M}_x\}\\ &\geqslant \int (0 \vee a)\, d\mu \geqslant \int_{\mathrm{U}} (0 \vee a)\, d\mu \geqslant \alpha\mu\ (\mathrm{U} \cap \text{supp.}(\mu)) > 0,\end{aligned}$$

which contradicts (3).

This completes the proof of theorem 3.3.

COROLLARY 3.4. – *If* $\mathrm{F} \subseteq \mathrm{K}$ *is a face and* $\overline{\mathrm{F}}$ *is also a face, then*

$$\overline{\mathrm{F}} = \bigcap_{a \in \mathrm{F}^*} (\widehat{0 \vee a})^{-1}(0)$$

Remark. – Suppose that $\overline{\mathrm{F}}$ is a proper face where $\mathrm{F} \subseteq \mathrm{K}$ is a face. If $(\overline{\mathrm{F}})'$ denotes the *complementary set* of F, i.e. the union of all faces disjoint from F, then it is clear that $(\overline{\mathrm{F}})' \subseteq \mathrm{F}'$. It is natural to enquire whether this inclusion is an equality. That it is *not*, is shown by the following simple example :

$$\mathrm{K} = \mathrm{M}_1^+\,[0\,,1],\ \mathrm{F} = \{\mu \in \mathrm{M}_1^+\,[0\,,1] : \mu\,[0\,,a] = \mu\,[b\,,1] = 0\}$$

where $0 < a < b < 1$. It is clear that F is a face as is

$$\bar{F} = \{\mu \in M_1^+ [0, 1] : \text{supp.}(\mu) \subseteq [a, b]\}.$$

Now, $\epsilon_a \in F'$ but $\epsilon_a \notin (\bar{F})'$ as $\epsilon_a \in \bar{F}$, showing that $(\bar{F})' \subsetneqq F'$.

If $\partial_e K$ is closed then the necessary and sufficient condition for the closure of a face $F \subseteq K$ to be a face is expressed below in a different way. This has the advantage that it gives a rather explicit description of $\bar{F}$.

Let $S = \bar{F} \cap \partial_e K$ and define

$$T = \{f \in C(\partial_e K) : 0 \leqslant f \leqslant 1, f(S) = 1\}.$$

Then we have the following

THEOREM 3.5. – *Assume $\partial_e K$ closed. Then $\bar{F}$ is a face if and only if $\bigcap_{f \in T} [\check{f} = 1]$ is closed. When this condition is satisfied,*

$$\bar{F} = \bigcap_{f \in T} [\check{f} = 1].$$

This result has been extracted from [10] and since its proof is essentially the same as in [10], modulo some trivial details, we omit it.

An example. – There are of course several examples in the literature showing that the closure of a face may fail to be a face (see, for instance, [2]). However, to the best of our knowledge, all these examples deal with compact convex sets, usually Choquet simplexes, with non-closed boundaries. We now present an example to show that this pathology can occur even if the boundary is closed.

Let $\mathfrak{A}$ be the disc-algebra and let K be its state space. Following [7], we let $Z = \text{conv}(K \cup -iK)$. Then K is just the probability measures on the unit circle Γ, so that $\partial_e K$ and $\partial_e Z$ are both closed. Every point of $\partial_e Z$ is a split face of Z and so if $E \subseteq \partial_e Z$, the *norm-closed* convex hull of E (in $A(Z)^*$) is a norm-closed split face of Z (by the L-ideal theory of [3]). However, take $E = \gamma \cup -i\gamma$ where γ is a proper arc of Γ of length > 0. Using the fact that K is a simplex, it is easy to check that conv (E) is a face of Z. However, γ is not a peak set for $\mathfrak{A}$ since any function in $\mathfrak{A}$ which is constant on γ is necessarily constant on Γ, so that $\overline{\text{conv}}$ (E) (w^*-closure) is not a face of Z (by Theorem 2 of [7]).

4. Applications.

We will now give some applications of the results established in the last section.

PROPOSITION 4.1. – *If* K *has the property that* $\hat{f}$ *is continuous for all* $f \in \mathrm{P(K)}$, *then the closure of every face* F *in* K *is again a face.*

Proof. – Let $a \in \mathrm{F}^*$. As we have already remarked, $\widehat{0 \vee a}(\mathrm{F}) = 0$. But $0 \vee a \in \mathrm{P(K)}$ and therefore $\widehat{0 \vee a}$ is continuous by assumption. Thus $\widehat{0 \vee a}(\overline{\mathrm{F}}) = 0$ and we can use (3) of Theorem 3.3 to conclude that $\overline{\mathrm{F}}$ is a face.

Remark. – This result was first proved in [10] by a more elaborate method.

DEFINITION 4.2 (after [12]). – *A compact convex set K has the strong equal support property (s.e.s.p. for short) if* (i) $\partial_e \mathrm{K}$ *is closed and* (ii) *for any* $x \in \mathrm{K}$ *and* $\mu, \nu \in Z_x$ *we have* supp. (μ) = supp. (ν).

We now prove

PROPOSITION 4.3. – *If* K *has the strong equal support property, then the closure of every face* F *in* K *is again a face.*

Proof. – If $\mathrm{S} = \overline{\mathrm{F}} \cap \partial_e \mathrm{K}$ then it is rather easy to show from the defining property of a face that $\overline{\mathrm{F}} = \overline{\text{conv}}$ (S). Let $a \in \mathrm{F}^*$. Then $\widehat{0 \vee a}(\mathrm{F}) = 0$. If $x \in \overline{\mathrm{F}}$, there exists a probability measure μ on S representing x (See [13]) : μ is obviously maximal since $\mathrm{S} \subseteq \partial_e \mathrm{K}$, hence all $\lambda \in Z_x$ have their supports in S by the s.e.s.p. But

$$\widehat{0 \vee a}(x) = \lambda_1 (0 \vee a) \quad \text{for some} \quad \lambda_1 \in Z_x$$
$$= 0$$

and hence $\overline{F}$ is a face by Theorem 3.3.

PROPOSITION 4.4. – *Let* K *be a Choquet simplex. If* $F \subseteq K$ *is a face then* $\overline{F}$ *is a face if and only if* $\partial_e \overline{F} \subseteq \partial_e K$.

Proof. – If $\overline{F}$ is a face then by the Krein-Milman theorem, $\partial_e \overline{F}$ is non-empty and it is clear that $\partial_e \overline{F} \subseteq \partial_e K$.

On the other hand, suppose F is a face with the property that $\partial_e \overline{F} \subseteq \partial_e K$. Consider $a \in F^*$. If $x \in \partial_e \overline{F} \subseteq \partial_e K$, by Herve's criterion [1],

$$\widehat{0 \vee a}(x) = (0 \vee a)(x) = 0.$$

But K being a simplex, $\widehat{0 \vee a}$ is an u.s.c. affine function (by the Choquet-Meyer theorem), hence by the Bauer maximum principle, $\widehat{0 \vee a}(\overline{F}) = 0$ and we see that $\overline{F}$ is a face.

COROLLARY 4.5. – *If* K *is a Choquet simplex and if* S *is a compact subset of* $\partial_e K$ *then* conv (S) *is a face of* K.

Proof. – This is immediate from the preceding Proposition, once we observe that F = conv (S) is a face in K and that $\partial_e \overline{F} = S \subseteq \partial_e K$.

Remarks 1. – It would be interesting to know whether Prop. 4.4 extends to compact convex sets with the equal support property [12].

2. – In [15], the following result of Mokobodzki is proved : *If* K *is a Choquet simplex and if* B *is a compact convex subset of* K *with* $\partial_e B \subseteq \partial_e K$ *then* B *is a face of* K.

This again follows immediately from the preceding discussion once we note that F = conv $(\partial_e B)$ is a face and that $\partial_e \overline{F} = \partial_e B \subseteq \partial_e K$.

5. Compactness of face (C).

Considerations of subsets of K of the form *face* (x), $x \in K$, have proved useful in several contexts : for example, they are important in the local version of Choquet's Uniqueness Theorem [9] and in Wil's proof of the existence and uniqueness of central measures for points of K (see [1]). Their usefulness is also suggested by the following simple result :

PROPOSITION 5.1. – *If* K *is metrisable then every closed face* F *of* K *has the form* face (x) *for some* $x \in$ K.

Proof. – The metrisability implies that K, and hence F, is separable. Let $\{k_n\}$ be a dense subset of F and define

$$x = \sum_{n=1}^{\infty} \frac{1}{2^n} k_n$$

It is clear that this series defines an element x of F, and that $k_n \in$ face $(x) \subseteq$ F ; it follows that $\overline{\mathrm{F}} =$ face (x).

In view of the preceding remarks, it is natural to look for conditions which ensure that face (x) is closed. This was recently done in [11] where it is proved, among other things, that face (x) is closed iff face $(x) = \mathrm{D}_n(x)$ for some n (see § 2 for the definition of $\mathrm{D}_n(x)$). We propose to generalize this result in the theorem below. It should be pointed out that the proof of the implication (1) ⇒ (2) in this theorem follows an argument used in [14] in a different situation. We denote by $\mathrm{P(K)}^+$ the cone of non-negative continuous convex functions on K.

THEOREM 5.2. – *Let* C *be a proper compact convex subset of* K *without core points. Then the following are equivalent :*

1) *face* (C) *is closed.*

2) *face* (C) $= \mathrm{D}_n(\mathrm{C})$ *for some* n.

3) *If* $f_m \in \mathrm{P(K)}^+$ *and* $\lim_{m \to \infty} f_m(u) = 0$ *uniformly for* $u \in$ C *then* $\lim_{m \to \infty} \hat{f}_m(y) = 0$ *uniformly for* $y \in$ *face* (C).

4) *face* (C) *is norm-closed in the space* A(K)*.

Moreover, all the above statements are implied by
(*) : lin M_c^+ is a norm-closed (or weak*-closed) subspace of M(K), where $\mathrm{M}_c^+ = \{\mu \in \mathrm{M}_1^+(\mathrm{K}) : \text{resultant } (\mu) \in \mathrm{C}\}$.

Proof. – (1) ⇒ (2). Suppose face (C) is closed and face (C) $\neq D_n(C)$ for all n. This means that given any $n \in N^+$ (= set of positive integers), $\exists y_n \in$ face (C) such that $y_n \nleqslant n2^n C$. (By this is meant that

$$y_n \nleqslant n2^n u, \ \forall \ u \in C).$$

Define $y = \sum_{n=1}^{\infty} 2^{-n} y_n$. Now $y \in$ face (C) as this set is closed by assumption. But then $y \nleqslant mC$ for all $m \in N^+$ which is a contradiction as face (C) $= \bigcup_{m=1}^{\infty} D_m(C)$.

(2) ⇒ (3). By (2), if $y \in$ face (C) then $y \leqslant nu$ for some $u \in C$, hence $\hat{f}_m(y) \leqslant n \hat{f}_m(u)$ as $\hat{f}_m$ is a concave function and thus $\lim_{m \to \infty} \hat{f}_m(y) = 0$ uniformly on face (C) if $\lim_{m \to \infty} \hat{f}_m(u) = 0$ uniformly on C.

(3) ⇒ (2). Suppose face (C) $\neq D_n(C) \ \forall \ n \in N^+$. This means that given $n \in N^+$, $\exists \, y_n \in$ face (C) such that $y_n \nleqslant nC$ but $y_n \leqslant n^{m(n)} u_n$ for some $u_n \in C$ and for some sufficiently large $m(n) \in N^+$. Since $(nC - y_n) \cap \tilde{K} = \emptyset$, where $\tilde{K}$ is the (closed) cone generated by K, a standard Hahn-Banach argument shows that $\exists \ a_n \in A(K)^+$ such that

$$a_n(y_n) > n \ a_n(u) \ \forall \ u \in C$$

and $$a_n(y_n) \leqslant n^{m(n)} a_n(u_n)$$

These inequalities imply that $\sup \{a_n(u) : u \in C\} > 0$

and $$a_n(y_n) > n \ \sup \{a_n(u) : u \in C\}.$$

Let $$b_n = a_n / n \ \sup \{a_n(u) : u \in C\}$$

Then $$b_n \in A(K)^+ \ \text{ and if } \ u \in C,$$

$$b_n(u) = a_n(u)/n \ \sup \{a_n(u) : u \in C\} \leqslant \frac{1}{n}$$

showing that $\lim_{n \to \infty} b_n(u) = 0$ uniformly for $u \in C$: however, $b_n(y_n) > 1$ and so b_n does not tend to zero uniformly on face (C), contradicting (3).

(2) ⇒ (1). Obvious.

(2) ⇒ (4). Obvious.

(4) ⇒ (2). By the regular embedding of K in A(K)*, we can regard each $x \in K$ as a member of the unit ball of A(K)*. As a norm-closed subset of the complete metric space A(K)*, face (C) is complete and hence by the Baire Category Theorem, some $D_{n_0}(C)$ must have non-empty relative interior, i.e. there exists some $y_0 \in D_{n_0}(C)$ such that for some neighbourhood of the origin,

$$U = \{u \in A(K)^* : \|u\| < \eta\}$$

$$(y_0 + U) \cap \text{face } (C) \subseteq D_{n_0}(C).$$

Let $y \in$ face (C) and define

$$z = \frac{\eta}{2+\eta} y + \frac{2}{2+\eta} y_0 \in \text{face } (C)$$

Then $$\| z - y_0 \| = \frac{\eta}{2+\eta} \|y - y_0\| \leqslant \frac{2\eta}{2+\eta} < \eta$$

and hence $$z \in (y_0 + U) \cap \text{face } (C) \subseteq D_{n_0}(C).$$

Therefore, for some $c \in C'$ and $k \in K$,

$$\frac{\eta}{2+\eta} y + \frac{2}{2+\eta} y_0 = n_0 c - (n_0 - 1) k$$

or, $$c = \frac{1}{n_0} \cdot \frac{\eta}{2+\eta} y + \left(1 - \frac{1}{n_0} \cdot \frac{\eta}{2+\eta}\right) k'$$

where $$k' = \frac{\left(1 - \frac{1}{n_0}\right) k + \frac{1}{n_0} \cdot \frac{2}{2+\eta} y_0}{1 - \frac{1}{n_0} \cdot \frac{\eta}{2+\eta}} \in K.$$

Thus, $$y \in D_\alpha (C) \quad \text{where} \quad \alpha = \left(1 + \frac{2}{\eta}\right) n_0$$

and this implies (2).

This completes the proof of the equivalence of (1), (2), (3) and (4). As far as statement (*) is concerned, first note that since

M_c^+ is a w^* compact subset of $M_i^+(K)$, by a known result [1 : page 112] lin M_c^+ is w^* closed iff lin M_c^+ is norm-closed. Now the proof follows exactly the argument used in [11] to prove (vi) $\Rightarrow$ (i) in Theorem 1.9 of that paper.

Remarks 1. – The use of the Baire Category Theorem above was suggested by the proof of a similar result in [5]. The argument proving (1) $\Rightarrow$ (2) could also be used here.

2. – We have not been able to decide whether any of the first four statements in Theorem 5.2 implies (*).

BIBLIOGRAPHY

[1] E.M. ALFSEN, Compact convex sets and boundary integrals, *Ergebnisse der Mathematik,* Springer-Verlag, Berlin, 1971.

[2] E.M. ALFSEN, On the geometry of Choquet simplexes, *Math. Scand.,* 15 (1964), 97-110.

[3] E.M. ALFSEN & E.G. EFFROS, Structure in real Banach spaces, Part I & II, *Annals of Math.,* 96, No. 1 (1972), 98-173.

[4] L. ASIMOW, Exposed faces of dual cones and peak-set criteria for function spaces, *Journal of Function Analysis,* vol. 12, No. 4 (1973).

[5] F. DEUTSCH & R.J. LINDAHL, Minimal extremal subsets of the unit sphere, *Math. Annalen,* 197 (1972).

[6] A.J. ELLIS, On faces of compact convex sets and their annihilators, *Math. Annalen,* 184 (1969).

[7] A.J. ELLIS, Split faces in function algebras, *Math Annalen,* 195 (1972).

[8] G. JAMESON, Nearly directed subspaces of partially ordered linear spaces, *Proc. Edinburgh Math. Soc.,* (2) 16 (1968).

[9] J. KOHN, Barycentres of unique maximal measures, *J. of Funct. Analysis,* 6 (1970).

[10] A. LIMA, On continuous convex functions and split faces, *Proc. London Math. Soc.*, (3) 25 (1972).

[11] A. LIMA, Closed faces with internal points, *Preprint series – Matematisk institutt,* Universiteteti Oslo (1972).

[12] J.N. MCDONALD, Compact convex sets with the equal support property, *Pac. J. of Math.*, vol. 37, No. 2 (1971).

[13] R. PHELPS, Lectures on Choquet's Theorem, Van Nostrand, Princeton (1960).

[14] M. RAJAGOPALAN & A.K. ROY, Maximal core representing measures and generalized polytopes, *Quart. J. of Math.*, Oxford, vol. 25, no. 99 (1974).

[15] M. ROGALSKI, Etude du quotient d'un simplexe par une face fermée... relation d'équivalence, *Seminaire Brelot – Choquet – Deny* (Theorie du Potentiel), 1967/68, No. 2.

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