

IBRAHIM DIBAG

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of constant rank**

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## DECOMPOSITION IN THE LARGE OF TWO-FORMS OF CONSTANT RANK

by Ibrahim DIBAG

### 0. Introduction.

“Whether a vector-bundle admits a 2-form of constant rank” has been an important question in algebraic topology ; and a good deal of research (4, 5, 10) has been done on the subject. In this thesis we shall take, apriori, a vector-bundle that does admit such a 2-form,  $w$ , of constant rank  $2s$ . We shall then show that,  $w$ , locally decomposes into a sum :  $w = y_1 \wedge y_{s+1} + y_2 \wedge y_{s+2} + \dots + y_s \wedge y_{2s}$  of products of linearly-independent 1-forms  $(y_i)$  on  $E$ . The main task of the thesis is to find necessary and sufficient conditions for,  $w$ , to have a *global* such decomposition.

We shall define a  $2s$ -dimensional sub-bundle  $S_w$  of  $E$  on which,  $w$ , can be regarded as a 2-form of maximal rank ; and a necessary condition for,  $w$ , to decompose globally is that  $S_w$  is a trivial (product) bundle.

Using the triviality of  $S_w$  we shall represent  $w$ , as a map  $w_1 : B \rightarrow I_s$  ; where  $B$  is the base-space, and  $I_s = SO(2s)/U(s)$  is the homogenous space ; and,  $w$ , decomposes globally if and only if  $w_1$  lifts to  $SO(2s)$ .

We shall then investigate the integercohomology,  $H^*(I_s ; Z)$ , of  $I_s$  ; and the cohomology-mapping

$$p^* : H^*(I_s ; Z) \rightarrow H^*(SO(2s) ; Z)$$

induced by the projection  $p : SO(2s) \rightarrow I_s$ . We shall deduce that :

1)  $H^*(I_s ; Z)$  is, additively, generated by the duals of normal cells  $[2i_1 ; 2i_2 ; \dots ; 2i_k]$  for  $s > i_1 > i_2 > \dots > i_k \geq 1$  and the zero-cell  $[0]$ .

2)  $p^*[2i_1 ; 2i_2 ; \dots ; 2i_k]^*$  is of order 2 in  $H^*(SO(2s) ; Z)$ . From these two statements will follow the theorem that : “A necessary condition for the liftability of  $w_1$  is that Image of  $w_1^* \subset$  Subgroup of elements of  $H^*(B ; Z)$  of order 2” and the corollary that :

“If  $H^*(B ; Z)$  does not have any 2-torsion ; then a necessary condition for the liftability of  $w_1$  is  $w_1^* = 0$ .”

These results will then be applied to some special cases, and a full discussion will be given of the existence and decomposability of 2-forms of constant rank on i) spheres, ii) real, and iii) complex-projective spaces.

## 1. Fiber-bundle structures over two-forms of rank $2s$ .

### 1.1. *Définitions and notation :*

Let  $E$  be a real  $n$ -dimensional inner-product space ; and as usual, identify  $E$  with its dual  $E^*$  through the metric.

Then it is well known (e.g. refer to [9]) that :

i) Any 2-form,  $w$ , on  $E$  decomposes into

$$w = y_1 \wedge y_{s+1} + \dots + y_s \wedge y_{2s}$$

a sum of products of linearly-independent vectors ( $y_i$ ) of  $E$ .

ii) The number of terms in any such decomposition is unique ; and is called the “rank” of  $w$ .

Thus if  $\tilde{V}_{2s}(E) =$  manifold of ordered  $2s$ -tuplets of linearly-independent vectors in  $E$ .

$\tilde{A}_s(E) =$  Set of 2-forms on  $E$  of rank  $2s$ .

We can define  $\tilde{f}_s : \tilde{V}_{2s}(E) \rightarrow \tilde{A}_s(E)$  by

$$(y_1, y_2, \dots, y_{2s}) \mapsto y_1 \wedge y_{s+1} + \dots + y_s \wedge y_{2s}$$

and by the above,  $\tilde{f}_s$  is “onto”. Also, the real-symplectic group,  $Sp(s ; R)$  acts freely and transitively on the fibers of  $\tilde{f}_s$  ; and thus  $\tilde{f}_s$  factors through the orbit-space,  $\tilde{V}_{2s}(E)/Sp(s ; R)$ , in a bijective fashion.

1.2. *The Principal  $Sp(s ; R)$ -bundle :  $\tilde{V}_{2s}(E) (\tilde{A}_s(E) ; Sp(s ; R))$*

1.2.1. LEMMA. — *The map  $\tilde{f}_s : \tilde{V}_{2s}(E) \rightarrow \tilde{A}_s(E)$  admits a local cross-section.*

Note : In the following proof, we shall, for convenience of notation, take the definition of  $\tilde{f}_s$  to be :

$$\tilde{f}_s(y_1, \dots, y_{2s}) = y_1 \wedge y_2 + \dots + y_{2s-1} \wedge y_{2s}.$$

*Proof.* — Choose a basis  $(e_1, e_2, \dots, e_n)$  of  $E$ . Then any  $w \in \Lambda^2 E$  can be written as  $w = \sum_{i < j} a_{ij}(w) e_i \wedge e_j$  where  $a_{ij} : \Lambda^2 E \rightarrow R^1$  are continuous functions on  $\Lambda^2 E$ .

$Q_r = (w \in A_r(E) / a_{12}(w) \neq 0)$  is an open subset of  $\tilde{A}_r(E)$  for  $1 \leq r \leq s$ .

$$S_{2r} = \tilde{f}_r^{-1}(Q_r) \subset V_{2r}(E) \quad ; \quad \tilde{f}_r : S_{2r} \rightarrow Q_r$$

well-defined.

Let  $F$  be the subspace of  $E$  generated by  $(e_3, e_4, \dots, e_n)$

$$((y_1, y_2) \quad ; \quad (y_3, y_4, \dots, y_{2s})) \mapsto (y_1, y_2, \dots, y_{2s})$$

defines a continuous map

$$i : S_2 \times \tilde{V}_{2s-2}(F) \rightarrow S_{2s} \quad ; \quad (q ; w_0) \mapsto q + w_0$$

defines a continuous map  $B : Q_1 \times A_{s-1}(F) \rightarrow Q_s$  and that

$$\tilde{f}_s \circ i = B \circ (\tilde{f}_1 \times \tilde{f}_{s-1}).$$

Now, given  $w \in Q_s$ , we have :

$$w = \left( e_1 - \frac{a_{23}}{a_{12}} e_3 - \dots - \frac{a_{2n}}{a_{12}} e_n \right) \wedge (a_{12} e_2 + \dots + a_{1n} e_n) + w_0$$

where  $w_0 \in \tilde{A}_{s-1}(F)$ . Let

$$y_1(w) = e_1 - \frac{a_{23}}{a_{12}} e_3 - \dots - \frac{a_{2n}}{a_{12}} e_n$$

$$y_2(w) = a_{12} e_2 + a_{13} e_3 + \dots + a_{1n} e_n.$$

Then define continuous maps :

$$k_s : Q_s \rightarrow S_2 \quad \text{by} \quad k_s(w) = (y_1(w) ; y_2(w))$$

$$p_s : Q \rightarrow \tilde{A}_{s-1}(F) \quad \text{by} \quad p_s(w) = w_0.$$

By définition :  $B((\tilde{f}_1 \circ k_s) \times p_s) = l_d$ . We shall, now, prove by induction on  $s$  that  $\tilde{f}_s$  admits a local cross-section. For  $s = 1$ . Assume W.L.G. that  $w \in Q_1$ . Since  $\tilde{A}_{s-1}(F) = 0$  ;  $p_1(w) = 0$ .

Hence,  $k_1 : Q_1 \rightarrow S_2$  yields the desired lifting of  $\tilde{f}_1$ .

For  $s > 1$  ; again assume W.L.G. that  $w \in Q_s$ , and that the inductive hypothesis holds for  $s - 1$  ; i.e. there exists a neighbourhood  $U$  of  $p_s(w)$  in  $\tilde{A}_{s-1}(F)$  and a lifting  $L_{s-1}$  of  $\tilde{f}_{s-1}$  over  $U$ . Then  $N = p_s^{-1}(U) \subset Q_s$  is a neighbourhood for  $w$  in  $Q_s$  and hence in  $\tilde{A}_s(E)$  ; and

$$N \xrightarrow{k_1 \times (L_{s-1} \circ p_s)} S_2 \times V_{2s-2}(F) \xrightarrow{i} S_{2s} \subset \tilde{V}_{2s}(E)$$

yields the desired lifting  $L_s = i \circ (k_1 \times (L_{s-1} \circ p_s))$  of  $\tilde{f}_s$  over the neighbourhood  $N$  of  $w$ .

Q.E.D.

1.2.2. PROPOSITION. —  $\tilde{f}_s$  induces a principal  $Sp(s ; R)$ -bundle :  $\tilde{V}_{2s}(E) (\tilde{A}_s(E) ; Sp(s ; R))$ .

*Proof.* — The existence of a local cross-section to  $\tilde{f}_s$  implies that  $\tilde{A}_s(E)$  and the orbit-space  $\tilde{V}_{2s}(E)/Sp(s ; R)$  are homeomorphic ; and that  $\tilde{f}_s$  and the projection  $p : \tilde{V}_{2s}(E) \rightarrow \tilde{V}_{2s}(E)/Sp(s ; R)$  can be identified. The fact that the projection,  $p$ , induces a principal  $Sp(s ; R)$ -bundle follows from the fact that  $Sp(s ; R)$  is a closed subgroup of  $GL(2s ; R)$  ; and that the full-projection :

$$\tilde{V}_{2s}(E) \rightarrow \tilde{V}_{2s}(E)/GL(2s ; R) = G_{2s}(E)$$

= Grassmann-Manifold of  $2s$ -planes on  $E$ , induces a principal  $GL(2s ; R)$ -bundle.

1.3. The Principal Unitary-bundle :  $V_{2s}(E) (A_s(E) ; U(s))$ .

1.3.1. Let  $V_{2s}(E) =$  Stiefel Manifold of orthonormal  $2s$ -frames on  $E$ .  $A_s(E) = \tilde{f}_s(V_{2s}(E)) =$  Manifold of “normalized” 2-forms on  $E$  of

rank  $2s$ .  $f_s : V_{2s}(E) \rightarrow A_s(E)$  the “restriction” of  $\tilde{f}_s$  to  $V_{2s}(E)$ .

Then,  $U(s) = Sp(s; R) \cap O(2s)$  acts freely and transitively on the fibers of  $f_s$ ; and thus  $f_s$  factors through the orbit-space  $V_{2s}(E)/U(s)$  in a bijective-fashion.

LEMMA. — *There exists a retraction  $r : \tilde{V}_{2s}(E) \rightarrow V_{2s}(E)$  such such that  $\tilde{f}_s = f_s \circ r$  when restricted to  $\tilde{f}_s^{-1}(A_s(E))$ .*

*Sketch of Proof.* — Let  $y \in V_{2s}(E)$ ; and pick any orthonormal frame  $e$  in the plane of  $y$ . Then  $y = u \circ e$  for some  $u \in GL(2s; R)$ . Let  $u = tv$  be the polar decomposition of  $u$  into an orthogonal matrix  $t$  and a positive-definite symmetric matrix  $v$ . Put  $r(y) = t \circ e$ . Then, independence of the definition of  $r(y)$  on the frame used, and other properties of  $r$  can easily be verified.

COROLLARY. — *Let  $B$  be a topological-space and  $w : B \rightarrow A_s(E)$  a continuous map; and  $\phi : B \rightarrow \tilde{V}_{2s}(E)$  a lifting of  $w$ . Then,  $r \circ \phi$  lifts  $w$  to  $V_{2s}(E)$ .*

1.3.2. PROPOSITION. —  $f_s$  induces a principal  $U(s)$ -bundle :

$$V_{2s}(E) (A_s(E) ; U(s)) .$$

*Proof.* — Let  $\phi$  be a cross-section to  $\tilde{f}_s$  over some compact neighbourhood  $\tilde{N}$  of  $\tilde{A}_s(E)$ . Put  $N = \tilde{N} \cap A_s(E)$  and  $\phi_1 = \phi/N$ . Then, by the preceding Corollary,  $r\phi_1$  is a cross-section to  $f_s$  over  $N$ . Define  $t : N \times U(s) \rightarrow f_s^{-1}(N)$  by  $t(n, u) = u((r\phi_1) n)$ . Then,  $t$ , is a homeomorphism (by compactness). Hence  $f_s$  is locally-trivial; and thus induces a principal  $U(s)$ -bundle.

1.4. Retraction of  $\tilde{A}_s(E)$  onto  $A_s(E)$ .

Let  $\tilde{W}_s =$  Set of non-singular and skew-symmetric  $2s \times 2s$  matrices.  $W_s =$  Set of orthogonal and skew-symmetric  $2s \times 2s$  matrices.

Then,  $GL(2s; R)$  acts on  $W_s$  by  $u \circ k = uk u^t$  for

$$u \in GL(2s; R) \text{ and } k \in \tilde{W}_s$$

and the subgroup,  $O(2s)$ , leaves  $W_s$  invariant under this action. If

$k = gv$  is the polar-decomposition of  $k \in \widetilde{W}_s$  ; then  $g \in W_s$  ; and thus  $k \rightarrow g$  defines a projection  $p : \widetilde{W}_s \rightarrow W_s$ .

LEMMA. – *There exists a continuous deformation retraction of  $\widetilde{W}_s$  onto  $W_s$  that commutes with the action of  $O(2s)$ .*

*Proof.* – Define a homotopy  $h_r : \widetilde{W}_s \rightarrow W_s$  by

$$h_r(gv) = g((1 - r)v + rl_d)$$

Then,  $h_0 = l_d$  ;  $h_1 = p$  ; and  $h_r$  commutes with the action of  $O(2s)$ .

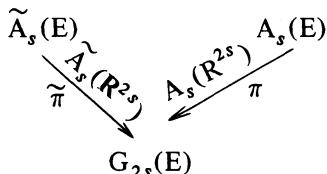
From this Lemma we recover the following :

PROPOSITION. – *There exists a retraction  $\theta : \widetilde{A}_s(E) \rightarrow A_s(E)$ .*

*Proof.* – Let's first assume that  $n = 2s$ . Then, an orthonormal frame  $e$  on  $E$  defines homeomorphisms ;  $t_e : \widetilde{W}_s \rightarrow \widetilde{A}_s(E)$  and  $t_e : W_s \rightarrow A_s(E)$  by  $t_e(k) = \sum_{i < j} k_{ij} e_i \wedge e_j$  and  $t_e = t_e / W_s$ .

A homotopy  $f_r : \widetilde{A}_s(E) \rightarrow A_s(E)$  can be defined by  $f_r = t_e \circ h_r \circ t_e^{-1}$  and it is, immediately, verified that this definition is independent of the orthonormal frame used. Thus,  $\theta = f_1$  yields the desired retraction.

For  $n \geq 2s$  ; we have the diagram :



a retraction  $\theta_p$  ; and a homotopy  $(f_r)_p : \widetilde{\pi}^{-1}(p) \rightarrow \pi^{-1}(p)$  over each  $2s$ -plane,  $p \in G_{2s}(E)$ . Then, the collections ,  $\theta = (\theta_p)_{p \in G_{2s}(E)}$  and  $f_r = (f_r)_p$  yield the desired retraction and the homotopy respectively.

Q.E.D.

2. Decomposability of two-forms of constant rank.

2.1. Notations and definitions :

Let  $E$  be an  $R^n$ -bundle (with a Riemannian-metric) over a connected base-space  $B$ . Let  $\tilde{V}_{2s}(E)$ ,  $V_{2s}(E)$ ,  $\tilde{A}_s(E)$ ,  $A_s(E)$  be the associated-bundles to  $E$  with fibers  $\tilde{V}_{2s}(R^n)$ ,  $V_{2s}(R^n)$ ,  $\tilde{A}_s(R^n)$ ,  $A_s(R^n)$  respectively. A 2-form,  $w$ , on  $E$  of constant rank  $2s$  is, by definition, a cross-section to  $A_s(E)$ . The maps  $\tilde{f}_s(E) : \tilde{V}_{2s}(E) \rightarrow \tilde{A}_s(E)$  and  $f_s(E) : V_{2s}(E) \rightarrow A_s(E)$  are defined and we have the following "global" versions of Propositions 1.2.2. and 1.3.2. :

PROPOSITION 1.2.2.\* –  $\tilde{f}_s(E)$  induces a principal  $Sp(s ; R)$ -bundle.

PROPOSITION 1.3.2.\* –  $f_s(E)$  induces a principal  $U(s)$ -bundle.

2.2. Local-Decomposability and the Sub-bundle  $S_w$  :

DEFINITION. – A 2-form,  $w$ , on  $E$  of constant rank  $2s$  is said to be locally-decomposable iff each point  $x \in B$  has a neighbourhood  $U_x$  and linearly-independent 1-forms  $(y_i)$   $i = 1, \dots, 2s$  on  $E$  over  $U_x$  s.t.  $w = y_1 \wedge y_{s+1} + \dots + y_s \wedge y_{2s}$  over  $U_x$ . (Or, alternatively, there exists a cross-section,  $y$ , to  $\tilde{V}_{2s}(E)$  over  $U_x$  such that  $w = \tilde{f}_s \circ y$ ).

LEMMA. – A 2-form,  $w$ , of constant rank  $2s$  on  $E$  is locally-decomposable.

Proof. – Let  $x \in B$  ; and,  $c$ , a cross-section to  $\tilde{f}_s(E)$  :

$$\tilde{V}_{2s}(E) \rightarrow \tilde{A}_s(E)$$

over a neighbourhood  $N$  of  $w(x)$  in  $\tilde{A}_s(E)$ . Then, the composite  $w^{-1}(N) \xrightarrow{w} N \xrightarrow{c} V_{2s}(E)$  defines a cross-section  $y = cw$  to  $\tilde{f}_s(E)$  over  $w^{-1}(N)$  such that  $\tilde{f}_s \circ y = w$ . Q.E.D.

Given a 2-form,  $w$ , of constant rank  $2s$  ; then at each point  $x \in B$ ,  $w(x)$  determines a  $2s$ -dimensional subspace  $S_{w(x)}$  of  $E_x$  on which it is of maximal rank ; and local decomposability of  $w$ , immediately yields the following :



PROPOSITION. — *The union  $S_w = \bigcup_{x \in B} S_{w(x)}$  is a sub-bundle of  $E$  ; and,  $w$ , being a 2-form on  $S_w$  of maximal-rank determines a reduction of its structure group from  $GL(2s ; \mathbb{R})$  to  $Sp(s ; \mathbb{R})$ .*

This Proposition, clearly, demonstrates that the “existence of a 2-form of constant rank on  $E$ ” (which is assumed apriori in the thesis) is, already, a strong condition ; and will be useful in proving non-existence theorems about 2-forms of constant rank on spheres and projective-spaces in the last-chapter.

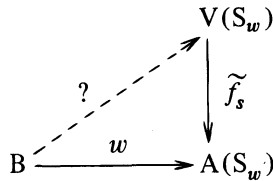
2.3. *Decomposition of 2-forms of constant rank :*

Let  $\tilde{V}(S_w)$ ,  $V(S_w)$ ,  $\tilde{A}(S_w)$ ,  $A(S_w)$  be the associated-bundles to  $S_w$  with fibers  $\tilde{V}(\mathbb{R}^{2s})$ ,  $V(\mathbb{R}^{2s})$ ,  $\tilde{A}(\mathbb{R}^{2s})$ ,  $A(\mathbb{R}^{2s})$  respectively.

DEFINITION. —  *$w$  is said to be decomposable iff*

$$w = y_1 \wedge y_{s+1} + \dots + y_s \wedge y_{2s}$$

*for linearly-independent 1-forms  $(y_i)$  on  $E$ . (Or, alternatively, the diagram : admits a lifting).*



An immediate consequence of this definition is the following :

*Observation.* — If,  $w$ , is decomposable ; then  $S_w$  is a trivial (product)-bundle.

Let  $r : \tilde{V}(S_w) \rightarrow V(S_w)$  and  $\theta : \tilde{A}(S_w) \rightarrow A(S_w)$  be the retractions of Sections 1.3. and 1.4. (respectively) defined globally on  $S_w$ .

DEFINITION. — *The “normalization” of,  $w$ , is defined to be the composite  $\theta w : B \xrightarrow{w} \tilde{A}(S_w) \xrightarrow{\theta} A(S_w)$  and is a “normalized” 2-form of rank  $2s$ . (i.e. a cross-section to  $A(S_w)$ ).*

DEFINITION. — A normalized 2-form,  $w$ , of rank  $2s$  decomposes orthogonally iff  $w = y_1 \wedge y_{s+1} + \dots + y_s \wedge y_{2s}$  for orthonormal-frame  $y = (y_1, \dots, y_{2s})$  on  $S_w$ .

PROPOSITION. — A 2-form,  $w$ , of constant rank  $2s$  decomposes iff its normalization decomposes orthogonally.

*Proof.* — Suppose,  $w$ , decomposes. i.e. there exists a continuous map  $L : B \rightarrow \tilde{V}(S_w)$  such that  $\tilde{f}_s \circ L = w$ . Since  $\theta$  is a retraction ;  $w \simeq \theta w$ , and thus  $\tilde{f}_s \circ L \simeq \theta w$ . Since  $\tilde{f}_s$  is a fibration ; by the covering-homotopy-theorem ; there exists a lifting  $T : B \rightarrow \tilde{V}(S_w)$  of  $\theta w$  to  $\tilde{V}(S_w)$  and by the “global-version” of Corollary 1.3.1.  $rT$  is a lifting of  $\theta w$  to  $V(S_w)$ . Thus,  $\theta w$  decomposes orthogonally.

Conversely, suppose  $\theta w$  decomposes orthogonally ; i.e. that there exists a lift  $k : B \rightarrow V(S_w)$  of  $\theta w$  to  $V(S_w)$ . Then,

$$f_s \circ k = \theta w \simeq w ;$$

and again, by the covering homotopy theorem, there exists a lifting of,  $w$ , to  $\tilde{V}(S_w)$ .

Q.E.D.

By Observation 2.3., a necessary condition for  $w$  to decompose is that  $S_w$  is a trivial (product)-bundle. Let's choose a particular product representation :  $S_w = B \times \mathbb{R}^{2s}$  which gives rise to further product representations : i)  $V(S_w) = B \times V(\mathbb{R}^{2s}) = B \times O(2s)$  and ii)  $A(S_w) = B \times A(\mathbb{R}^{2s}) = B \times O(2s)/U(s)$  and a representation of  $\theta w$  as a map  $w_1 : B \rightarrow O(2s)/U(s)$ .

$\theta w$  decomposes orthogonally iff  $w_1$  lifts to  $O(2s)$ . Since  $B$  is connected and  $w_1$  continuous ; we may, without loss of generality assume that  $w_1(B) \subset I_s = SO(2s)/U(s)$  ; and then lifting  $w_1$  to  $O(2s)$  is equivalent to lifting it to  $SO(2s)$ . We can summarize this in a single :

THEOREM. — A 2-form,  $w$ , of constant rank  $2s$  decomposes iff

- i)  $S_w$  is a trivial (product)-bundle.
- ii) The representation of its normalization as a map  $w_1$  :

$$B \rightarrow I_s = SO(2s)/U(s)$$

arising from any trivialization of  $S_w$  lifts to  $SO(2s)$ .

The method used above was to assume the existence apriori, of a metric on  $E$  (and thus on  $S_w$ ) ; and to show that,  $w$ , decomposes iff its normalization (with respect to this metric) decomposes orthogonally.

A more and invariant approach does not pre-suppose the existence of a metric on  $S_w$ .  $w$ , determines a reduction of the structure-group of  $S_w$  to  $Sp(s; R)$  ; and since  $U(s)$  is a maximal compact subgroup of  $Sp(s; R)$  ; it undergoes a further reduction to  $U(s)$  ; and thus  $S_w$  admits a unique Hermitian metric. Then,  $w$ , becomes normalized with respect to the corresponding real-metric, and thus decomposes iff it decomposes orthogonally. The rest of the theory goes as before ; and one, again, obtains the above theorem with obvious modifications.

### 3. Cohomology of $I_s$ .

#### 3.1. Preliminaries :

Let  $x \in P^{n-1}$  ; and  $\phi_x$  be the "reflection" through the hyperplane perpendicular to  $x$  ; and  $\phi_0$  the reflection corresponding to the initial point  $(1, 0, \dots, 0)$ . Then, we imbed  $P^{n-1} \subset SO(n)$  by  $x \rightarrow \phi_x \phi_0$ . We, now, list the following standard results ; and for proofs we refer the reader to [8] pp. 40-45.

*Observation* : i)  $P^{n-1} \cap SO(n-1) = P^{n-2}$ . ii)  $P^i \circ P^j = P^j \circ P^i$  and iii)  $P^i \circ P^i = P^i \circ P^{i-1}$  in  $SO(n)$ .

Let  $P^{n-1}/P^{n-2}$  be the space obtained by collapsing  $P^{n-2}$  to a point ; and  $SO(n)/SO(n-1)$  the left coset-space.

LEMMA. — *The natural-map*  $T : P^{n-1}/P^{n-2} \rightarrow SO(n)/SO(n-1)$  *is a "homeomorphism"*.

PROPOSITION. — *The matrix-multiplication*

$$m : (P^n \times SO(n) ; P^{n-1} \times SO(n)) \rightarrow (SO(n+1) ; SO(n))$$

*is a relative-homeomorphism.*

THEOREM. —  $SO(n)$  is a cell-complex with normal cells

$$[i_1 ; i_2 ; \dots ; i_k] \quad \text{for} \quad n > i_1 > i_2 > \dots > i_k \geq 1$$

given by

$$E^{i_1} \times E^{i_2} \times \dots \times E^{i_k} \rightarrow P^{i_1} \times P^{i_2} \times \dots \times P^{i_k} \xrightarrow{m} SO(n)$$

and the zero-cell  $[0]$  ; and matrix-multiplication  $m$  :

$$SO(n) \times SO(n) \rightarrow SO(n)$$

is a cellular-map.

### 3.2. Cellular Structure of $I_s$ :

Observation :  $I_s = SO(2s)/U(s) = SO(2s - 1)/U(s - 1)$ .

Proof. — Obviously,  $SO(2s - 1) \cap U(s) = U(s - 1)$  and

$$SO(2s - 1) \circ U(s) = SO(2s)$$

by a dimension argument. Thus,

$$I_s = SO(2s - 1) \circ U(s)/U(s) = SO(2s - 1)/U(s - 1).$$

Q.E.D.

Let  $\bar{P}^{2s+1}$  and  $\bar{P}^{2s}$  denote the images of  $P^{2s+1}$  and  $P^{2s}$  under the projections  $SO(2s + 2) \rightarrow I_{s+1}$  and  $SO(2s + 1) \rightarrow I_{s+1}$  respectively. We, then, have the following :

LEMMA. —  $\bar{P}^{2s+1} = \bar{P}^{2s}$

Proof. — It is an immediate consequence of the fact that the “composite”  $P^{2s+1} \subset SO(2s + 2) \rightarrow I_{s+1}$  factors through  $P_s(C)$  ; and that  $P^{2s} \subset P^{2s+1} \rightarrow P_s(C)$  is “onto”.

Q.E.D.

Let  $v : SO(2s) \times I_s \rightarrow I_s$  be the action of  $SO(2s)$  on  $I_s$ . Then, we obtain the analogue of Proposition 3.1. for  $I_s$  :

PROPOSITION. —  $v : (P^{2s} \times I_s ; P^{2s-1} \times I_s) \rightarrow (I_{s+1} ; I_s)$  is a relative-homeomorphism

which in turn becomes the key in the proof of the following

**THEOREM.** —  $I_s$  is a cell-complex consisting of even-dimensional normal-cells  $[2i_1 ; 2i_2 ; \dots ; 2i_k]$  for  $s > i_1 > i_2 > \dots > i_k \geq 1$ , given by

$$E^{2i_1} \times \dots \times E^{2i_k} \rightarrow P^{2i_1} \times \dots \times P^{2i_k} \xrightarrow{m} SO(2s) \xrightarrow{\text{proj}^n} I_s$$

and the zero-cell  $[O]$ ; and the action-map  $v : SO(2s) \times I_s \rightarrow I_s$  is cellular.

*Proof.* — We prove the theorem by induction on  $s$ .

For  $s = 1$ ;  $I_1$  is just the zero-cell  $O$ ; and thus  $v : SO(2) \times I_1 \rightarrow I_1$  is, obviously, cellular. By the preceding proposition,  $I_{s+1}$  is the adjunction-space:  $I_{s+1} = I_s \vee_v (P^{2s} \times I_s)$ . We, now, apply the following standard Lemma: "If  $K$  and  $L'$  are cell-complexes;  $L$  a subcomplex of  $K$  and  $v : L \rightarrow L'$  a cellular-map; then the adjunction-space,  $K \vee_v L'$  is a cell-complex having  $L'$  as a subcomplex; and the images of the cells of  $(K - L)$  as the remaining cells" with

$$K = P^{2s} \times I_s ; \quad L = P^{2s-1} \times I_s ; \quad L' = I_s$$

By the inductive hypothesis,  $v : SO(2s) \times I_s \rightarrow I_s$ ; and hence its restriction to the subcomplex,  $P^{2s-1} \times I_s$ , is cellular; and thus we deduce that,  $I_{s+1}$ , is a cell-complex having  $I_s$  as a subcomplex; and the  $v$ -images of the cells of  $(P^{2s} - P^{2s-1}) \times I_s$  as the remaining cells. By the inductive-hypothesis, the cells of  $I_s$  are normal cells  $[2i_1 ; \dots ; 2i_k]$  for  $s > i_1 > \dots > i_k \geq 1$ , and the zero-cell  $[O]$ ; and the  $v$ -images of the cells of  $(P^{2s} - P^{2s-1}) \times I_s$  are normal-cells  $[2s ; 2i_2 ; \dots ; 2i_k]$  for  $s > i_2 > \dots > i_k \geq 1$ . The proof will be complete once we prove that:  $v : SO(2s + 2) \times I_{s+1} \rightarrow I_{s+1}$  is cellular; and this is done in five steps:

- i)  $v : P^{2s} \times I_s \rightarrow I_{s+1}$  is cellular.
- ii)  $v : SO(2s + 1) \times I_s \rightarrow I_{s+1}$  is cellular.
- iii)  $v : SO(2s + 1) \times I_{s+1} \rightarrow I_{s+1}$  is cellular.
- iv)  $v : P^{2s+1} \times I_{s+1} \rightarrow I_{s+1}$  is cellular.
- v)  $v : SO(2s + 2) \times I_{s+1} \rightarrow I_{s+1}$  is cellular.

Only iv) has a non-trivial proof which can be outlined as follows :

*Proof of iv).* — By iii) the restriction of,  $v$ , to the subcomplex,  $P^{2s} \times I_{s+1}$  of,  $P^{2s+1} \times I_{s+1}$ , is cellular ; and thus it suffices to prove that :

$$v(P^{2s+1} ; (I_{s+1})^{2q}) \subset (I_{s+1})^{2(s+q)}$$

Let  $s + 1 > i_1 > i_2 > \dots > i_k \geq 1$  and  $i_1 + i_2 + \dots + i_k = q$

$$\begin{aligned} v(P^{2s+1} ; \overline{P^{2i_1} \times \dots \times P^{2i_k}}) &= \overline{P^{2s+1} \times P^{2i_1} \times \dots \times P^{2i_k}} \\ &= \overline{P^{2i_1} \times \dots \times P^{2i_k} \times P^{2s+1}} = v(P^{2i_1} \times \dots \times P^{2i_k} ; \overline{P^{2s+1}}) \\ &= v(P^{2i_1} \times \dots \times P^{2i_k} ; \overline{P^{2s}}) = \overline{P^{2s} \times P^{2i_1} \times \dots \times P^{2i_k}} \\ &= v(P^{2s} ; \overline{P^{2i_1} \times P^{2i_2} \times \dots \times P^{2i_k}}) \subset v((SO(2s + 1))^{2s} ; (I_{s+1})^{2q}) \\ &\subset (I_{s+1})^{2(s+q)} \text{ by Part iii).} \end{aligned}$$

Q.E.D.

**COROLLARY.** — *The projection  $p : SO(2s) \rightarrow I_s$  is cellular ; and maps normal cells  $[2i_1 ; 2i_2 ; \dots ; 2i_k]$  of  $SO(2s)$  onto normal cells  $[2i_1 ; 2i_2 ; \dots ; 2i_k]$  of  $I_s$ . The images of the remaining cells, i.e.  $[j_1 ; j_2 ; \dots ; j_k]$  where  $j_t$  is odd for some  $1 \leq t \leq k$  are contained in a skeleton of lower dimension.*

### 3.3. Integer-Cohomology of $I_s$ and the Lifting Problem :

Since  $I_s$  is a cell-complex consisting of even dimensional cells only ; the co-boundary operator is identically zero ; and hence the  $2q^{th}$ -cohomology group  $H^{2q}(I_s ; Z)$  coincides with  $2q^{th}$ -cochains,  $C^{2q}(I_s ; Z)$ , which is the free abelian group generated by the duals  $[2i_1 ; \dots ; 2i_k]^*$  of normal cells  $[2i_1 ; \dots ; 2i_k]$  for  $q = i_1 + \dots + i_k$ .

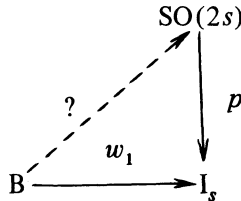
**PROPOSITION.** — *Image  $p^* \subset$  Subgroup of elements of*

$$H^*(SO(2s) ; Z)$$

*of order 2.*

*Proof.* — By the above ;  $p^*[2i_1 ; \dots ; 2i_k]^* = [2i_1 ; \dots ; 2i_k]^*$  and  $2[2i_1 ; \dots ; 2i_k]^* = \delta[2i_1 - 1 ; \dots ; 2i_k]^*$  in  $SO(2s)$ .

THEOREM. — *A necessary condition for the lifting of the diagram :*



is that :

*Image  $w_1^* \subset$  Subgroup of elements of  $H^*(B ; Z)$  of order 2.*

COROLLARY. — *If  $H^*(B ; Z)$  contains no 2-torsion ; then a necessary condition for the liftability of  $w_1$  is that  $w_1^* = 0$ .*

#### 4. Applications.

##### 4.1. Lower-Dimensional Spaces :

We now, combine Theorems 2.3. and 3.4. with elementary obstruction theory to obtain the following :

PROPOSITION. — *Let,  $w$ , be a 2-form of constant rank  $2s(s > 1)$  on an  $R^n$ -bundle  $E$  over a connected base-space  $B$  whose cohomology vanishes in dimensions greater than or equal to four. Necessary and sufficient conditions for,  $w$ , to decompose are i)  $S_w$  is a trivial (product)-bundle ; and ii)  $2w_1^* = 0$  in  $H^2(B ; z)$  where*

$$i \in H^2(I_s ; Z) = Z$$

*is the generator and  $w_1$  is the representation of,  $w$ , arising from any trivialization of  $S_w$ .*

When  $B$  is an orientable 3-manifold, the tangent-bundle  $T(B)$  of  $B$  is trivial ; and  $S_w$  is the pull-back of the tangent-bundle  $T(S^2)$  of the 2-sphere by the Gauss-Map  $P : B \rightarrow S^2$  ; and thus the first Chern-Class,  $c_1(S_w) = 2P^*(i)$ , where  $i \in H^2(S^2 ; Z)$  is the generator. Also by Alexander Duality,  $2P^*(i) = 0$  iff  $P^*(i) = 0$ . Applying Theorem 2.3. yields the observation — A nowhere-vanishing 2-form,  $w$ , on an orientable 3-manifold decomposes iff  $P^*(i) = 0$ .

If we further specialize by taking  $B$  to be an open connected domain in  $R^3$  and use the Hopf-Classification Theorem that  $[P] \rightarrow P^*(i)$  is an isomorphism :  $[B ; S^2] \rightarrow H^2(B ; Z)$  ; we obtain :

COROLLARY. — *A nowhere-vanishing 2-form,  $w$ , on an open connected domain  $B$  of  $\mathbb{R}^3$  decomposes iff the Gauss-Map  $P : B \rightarrow S^2$  for  $S_w$  is null-homotopic.*

4.2. *Methods of Constructing p-forms on Spheres :*

i) *“From constant  $(p + 1)$ -forms on  $\mathbb{R}^n$ ”.*

Let  $w \in \Lambda^{p+1} \mathbb{R}^n$  ; and define  $t : S^{n-1} \rightarrow \Lambda^p \mathbb{R}^n$  by  $t(x) = \delta_x(w)$  for all  $x \in S^{n-1}$ , where  $\delta_x$  is the “adjoint” of the wedge-product map,  $d_x : \Lambda \mathbb{R}^n \rightarrow \Lambda \mathbb{R}^n$  given by  $d_x(y) = x \wedge y$ . Then

$$\delta_x t(x) = \delta_x \circ \delta_x(w) = 0 ;$$

and thus,  $t$ , is a differentiable  $p$ -form on  $S^{n-1}$ .

ii) *“From constant  $p$ -forms on  $\mathbb{R}^n$ ”*

Let  $w \in \Lambda^p \mathbb{R}^n$ . Then  $t(x) = \delta_x \circ d_x(w) = w - d_x \circ \delta_x(w)$  for  $x \in S^{n-1}$  defines a differentiable  $p$ -form,  $t$ , on  $S^{n-1}$  which is called the “tangential component” of  $w$ .

PROPOSITION. — *The tangential-component of a normalized 2-form of maximal-rank on  $\mathbb{R}^{2n}$  is a 2-form on  $S^{2n-1}$  of constant rank  $(2n - 2)$ .*

*Proof.* —  $w = x \wedge \delta_x(w) + t(x)$  for all  $x \in S^{n-1}$ . The transformation on  $\mathbb{R}^{2n}$  given by  $x \rightarrow \delta_x(w)$  has square equal to minus identity; and thus  $\delta_{\delta_x(w)}(t(x)) = 0$  which implies that  $t(x) \in \Lambda^2 U_x$  for

$$U_x = (x ; \delta_x(w)) ;$$

and hence  $\text{rank}(w) = \text{rank}(x \wedge \delta_x(w)) + \text{rank } t(x)$ .

*Note.* —  $t(-x) = t(x)$  ; and thus,  $t$ , also defines a 2-form on  $S^{2n-1}$  of constant rank  $(2n - 2)$ .

4.3. *Existence and decomposability of 2-forms of constant rank on spheres :*

PROPOSITION. —  $S^{4n+3}$  admits a 2-form of constant rank  $4n$ .



*Proof.* – Represent  $S^{4n+3} = Sp(n+1)/Sp(n)$  ; and let

$$w_0 = e_1 \wedge e_{2n+1} + \cdots + e_{2n} \wedge e_{4n}$$

be a “normalized” 2-form at the distinguished point  $e_{4n+3}$ . For  $x \in S^{4n+3}$ , take any  $u \in Sp(n+1)$  such that  $u(e_{4n+3}) = x$  ; and define  $w(x) = (\Lambda^2 u) w_0$ . Since,  $Sp(n) \subset U(2n)$  leaves  $w_0$ -invariant ;  $w$  is a well defined 2-form on  $S^{4n+3}$  of constant rank  $4n$ . Q.E.D.

*Note.* – i)  $w(e^{i\theta} x) = e^{2i\theta} w(x)$  and ii)  $\delta_{J(x)}(w(x)) = 0$  where  $J$  is multiplication by  $i = \sqrt{-1}$  ; and thus,  $w$ , defines a 2-form on  $P_{2n+1}(C)$  (and hence on  $P^{4n+3}$ ) of constant rank  $4n$ .

Combining Proposition 2.2 with the Standard Theorem of [7] pp. 144 ; we obtain the following :

*Statement.* – *The existence of a 2-form of constant rank  $2s$  on  $S^n$  implies :*

- i) *the existence of a field of  $2s$ -frames on  $S^n$  for  $4s \leq n$ .*
- ii) *the existence of a field of  $(n - 2s)$ -frames on  $S^n$  for  $4s > n$ .*

and using Adams' results on Vector Fields on Spheres ; we deduce :

**COROLLARY 1.** –  $S^{4n+1}$  does not admit a 2-form of constant rank  $2s$  for  $0 < s < 2n$ .

**COROLLARY 2.** –  $S^{2n}$  does not admit a 2-form of constant rank  $2s$  for  $0 < s < n$ .

It is also a consequence of Adams' results and Kirchoff's Theorem (Refer to [7] pp. 217) that  $S^2$  and  $S^6$  are the only even dimensional spheres which are almost-complex, i.e. admit 2-forms of maximal rank. We can, now, summarize all these results in the following :

**THEOREM.** – 1) *The only even dimensional spheres which admit 2-forms of constant rank are  $S^2$  and  $S^6$  which admit 2-forms of maximal rank. None of these forms can be decomposed.*

2) *The only non-zero 2-forms of constant rank on  $S^{4n+1}$  are those of rank  $4n$ , and none of these forms can be decomposed.*

3)  $S^{4n+3}$  admits 2-forms of constant ranks 2,  $4n$ ,  $4n + 2$ . Those of constant rank 2 always decompose ; whereas those of constant rank  $4n$  and  $4n + 2$  cannot be decomposed for  $n \geq 2$ . A 2-form,  $w$ , on  $S^7$  of constant rank 4 decomposes iff i)  $S_w$  is a trivial bundle ; and ii)  $\partial[w_1] \in \pi_6 U(2)$  vanishes, where  $w_1$  is the representation of the normalization of  $w$  (with respect to the canonical Riemannian-Metric on  $S^7$ ) arising from any trivialization of  $S_w$  as a map

$$w_1 : S^7 \rightarrow I_2 ; \quad \text{and} \quad \partial : \pi_7 I_2 \rightarrow \pi_6 U(2)$$

is the boundary-operator of the exact homotopy sequence of the fibration  $SO(4) \rightarrow I_2$ .

A 2-form,  $w$ , on  $S^7$  of constant rank 6 decomposes iff i)

$$\partial[P] \in \pi_6 SO(6)$$

vanishes ; where  $P : S^7 \rightarrow S^6$  is the Gauss-Map for  $S_w$ , and

$$\partial : \pi_7 S^6 \rightarrow \pi_6 SO(6)$$

is the boundary-operator of  $SO(7) \rightarrow S^6$ . ii)  $\partial[w_1] \in \pi_6 U(3)$  vanishes ; where  $w_1 : S^7 \rightarrow I_3$  is the representation of the normalization of  $w$ , and  $\partial : \pi_7 I_3 \rightarrow \pi_6 U(3)$  is the boundary-operator of  $SO(6) \rightarrow I_3$ .

*Remark.* — The above theorem solves completely the existence and decomposability problem of 2-forms of constant rank for  $S^{2n}$ ,  $S^{4n+1}$ , and for  $S^{4n+3}$  up to  $S^{15}$ . The first unsolved case is the existence question of 2-forms of constant rank 10 on  $S^{15}$ . The next is the existence question of 2-forms of constant rank 16 and 18 on  $S^{23}$ .

#### 4.4. Existence and Decomposability of 2-forms of constant rank on Projective Spaces :

Parts 1 and 2 and most of 3 of the preceding Theorem go through unchanged for real-projective spaces. The only changes in Part 3 are i) 2-forms,  $w$ , on  $P^{4n+3}$  of constant rank 2 decompose iff  $c_1(S_w) \in H^2(P^{4n+3}; Z) = Z_2$  vanishes. ii) The discussions for 2-forms on  $S^7$  do not have their analogues for  $P^7$  ; since,  $w$ , can no longer be represented as an element of  $\pi_7 I_2$  or  $\pi_7 I_3$ . A necessary condition for the decomposability of such forms is the decomposability of

the corresponding forms on  $S^7$  (which can be determined by the previous Theorem). However, whether this is sufficient is not known.

The case of the complex projective spaces can be best summarized in the following :

**PROPOSITION.** —  $\mathbf{P}(\mathbb{C})$ , being a complex analytic manifold, admits a 2-form of constant rank  $2n$ .

The only non-zero 2-forms on  $\mathbf{P}_{2n}(\mathbb{C})$  of constant rank are those of constant rank  $4n$  which cannot be decomposed.

$\mathbf{P}_{2n+1}(\mathbb{C})$ , admits 2-forms of constant ranks  $4n + 2$  and  $4n$  which cannot be decomposed for  $n \geq 2$ .

#### 4.5. Translation-Invariant 2-forms on Lie-Groups :

**PROPOSITION.** — A Lie-Group,  $G$ , admits translation-invariant 2-forms of constant rank  $2s$  for  $2s \leq \dim G$  ; and any translation-invariant 2-form on  $G$  decomposes.

### Appendix

The analogous problem of decomposing a 2-form of constant rank on a complex vector-bundle is attacked in exactly the same way ; and is reduced to the lifting-problem of the diagram :

$$\begin{array}{ccc}
 & & \nearrow U(2s) \\
 & ? & \downarrow p \\
 B & \xrightarrow{w_1} & U(2s)/Sp(s)
 \end{array}$$

One then investigates integer-cohomology of the homogenous-space,  $U(2s)/Sp(s)$  ; and the Kernel of the map,  $p^*$  :

$$H^*(U(2s)/Sp(s)) \rightarrow H^*(U(2s))$$

## BIBLIOGRAPHY

- [1] A. BOREL, Sur La Cohomologie des Espaces Fibre Principaux ..., *Ann. Math.*, 57(1953), 115-207.
- [2] A. BOREL, F. HIRZEBRUCH, Characteristic Classes and Homogenous Spaces I, *Amer. J. Math.*, 80(1958), 459-538.
- [3] J. MARTINET, Sur Les Singularités des Formes Differentiables, Thesis Grenoble (1969).
- [4] W.S. MASSEY, Obstructions to the Existence of Almost-Complex Structures, *Bull. Amer. Math. Soc.*, 67 (1961), 559-564.
- [5] C.E. MILLER, The Topology of Rotation Groups, *Ann. Math.*, 57 (1953), 91-114.
- [6] MOSHER-TANGORA, Cohomology Operations and Application in Homotopy Theory, Harper-Row Publishers (1968).
- [7] N.E. STEENROD, The Topology of Fibre-Bundles, Princeton Univ. Press (1951).
- [8] N.E. STEENROD, Cohomology Operations, *Annals of Math Studies*, n° 50.
- [9] S. STERNBERG, Lectures on Differential Geometry, Prentice Hall Edition (1964).
- [10] E. THOMAS, Complex-Structures on Real Vector-Bundles, *Amer. J. Math.*, 89 (1967), 887-907.

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Ibrahim DIBAG,  
Department of Mathematics  
Middle-East Technical University  
Ankara (Turquie).