ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 23, nº 4 (1973), p. 105-143

<http://www.numdam.org/item?id=AIF_1973__23_4_105_0>

Ann. Inst. Fourier, Grenoble
23, 4 (1973), 105-143.

INTEGRAL REPRESENTATION FOR A CLASS OF MULTIPLY SUPERHARMONIC FUNCTIONS

by Kohur GOWRISANKARAN

Introduction.

Let $\Omega_1, \Omega_2, \ldots, \Omega_n$ be connected, locally compact, non-compact Hausdorff spaces with countable basis for open sets. Let there be a system of harmonic functions on each of the spaces satisfying the axioms I, II and III of Brelot [1]. Let us further suppose that each space has a base for the open sets consisting of completely determining regular domains. An extended real valued function φ on the product space is said to be n-superharmonic if (1) it is lower semicontinuous, (2) $\varphi \not\equiv +\infty$ and (3) φ is hyperharmonic in each variable for every fixed value of the others. Let $(n\mathrm{S})^+(\Omega_1 \times \cdots \times \Omega_n)$ or briefly $(n\mathrm{S})^+$ be the class of non-negative n-superharmonic functions on the product space.

In 1965, the author showed that there is an unique integral representation for the subclass of $(n\mathrm{S}^+)$ consisting of positive n-harmonic functions with the aid of Radon measures on the set of extreme elements belonging to a compact base, [5]. In 1968, R. Cairoli gave an unique integral representation for functions of two variables that are separately excessive and satisfy an additional condition, that is called condition (H) by him, [3]. Recently A. E. Drinkwater showed that with a natural topology, the analogue of Cartan-Brelot topology, $(n\mathrm{S})^+$ has a compact metrizable base and hence obtained an integral representation for elements of $(n\mathrm{S})^+$ with Radon measures on the extreme elements in the compact base. But she did not settle the question of uniqueness [4].

The purpose of this paper is to obtain an unique integral representation for a subclass $\mathscr{C}$ of $(n\mathrm{S})^+$. Motivated by the results of Cairoli [3], we define the class $\mathscr{C}$ in the case of product of two spaces Ω_1 and Ω_2 consisting of those elements ν in $(2\mathrm{S})^+$ for which $x \to \nu_x$ depends superharmonically on Ω_1 (see Definition 2.1) where ν_x is the canonical measure on the extreme elements $\mathscr{E}_2$ of a conveniently chosen compact base of $\mathrm{S}^+(\Omega_2)$ representing the function $\nu(x,.)$. With the topology as in [4] we show that $\mathscr{C}$ is closed in $(2\mathrm{S})^+$ and hence has a compact base. Hence every element in $\mathscr{C}$ is represented by a Radon measure on the base carried by the extreme elements. To show the uniqueness of the measure representing each element, we show that $\mathscr{C}$ is a lattice in its own order. Not surprisingly, a major part of the paper is devoted to prove this lattice property and to determine the extreme elements of the base. The key result used in proving the lattice property and to characterize the extreme elements is the Theorem 2.2 where we show an element ν of $(2\mathrm{S})^+$ belongs to $\mathscr{C}$ iff $\nu(x, y) = \int_{\mathscr{E}_2} \varphi\ (x, p)\nu\ (dp)$ where φ is a Borel function which is superharmonic in the first variable for every fixed p in $\mathscr{E}_2$. This result also enables us to show that $\mathscr{C}$ is precisely the set of elements of $(2\mathrm{S})^+$ satisfying the analytic analogue of Cairoli's condition (H) (cf. Theorem 2.8).

Finally, we define the class $\mathscr{C}$ contained in $(n\mathrm{S})^+$ by induction on n. Most of the proofs of the earlier case go through for the general situation and we obtain the integral representation in this case. We also give at the end a characterization of $\mathscr{C}$ in terms of the n-dimensional analogue of the Cairoli condition. We shall give elsewhere an application of the integral representation result to holomorphic functions of several variables.

1. Some measure theoretic results and a result about positive superharmonic functions.

LEMMA 1.1. — *Let* X *be a Lusin space and let* $\{\mathscr{C}_n, p_n, \varphi_n\}_{n=1 \text{ to } \infty}$ *be a strict subdivision of* X [9]. *Let* τ *be the topology on* X *with the base for open sets consisting* *of sets of the form* $\varphi_n(c)$, *c in* $\mathscr{C}_n$, $n = 1$ to ∞. *Then,* (X, τ) *is a Lusin space.*

Proof. — It is clear that the sets $\varphi_n(c)$, $c \in \mathscr{C}_n$, $n = 1$ to ∞, form a base for a topology. To show that (X, τ) is a Lusin space, it is enough to verify that the given strict subdivision is also a subdivision for (X, τ). For this, it is again enough to verify that any coherent family of sets $\varphi_n(c_n)$ τ-converges to an element x contained in all of them. But if $\{\varphi_n(c_n)\}$ is coherent then there is an element x contained in all of them such that the filter generated by $\{\varphi_n(c_n)\}$ converges in the original topology to x. Then, $\{\varphi_n(c_n)\}$ is precisely a base of τ-neighbourhoods of x and hence the filter generated by this sequence τ-converges to x. The lemma is proved.

COROLLARY. — *The Borel* σ*-algebras of* (X, τ) *and* X *with the initial topology are identical.*

Proof. — Clearly, τ is finer than the given topology on X. It follows that the open sets of the two topologies generate the same σ-algebra [9].

LEMMA 1.2. — *Let* X *be a Lusin space and* $\{\mathscr{C}_n, p_n, \varphi_n\}_{n=1 \text{ to } \infty}$ *a strict subdivision of* X. *Let* μ *and* ν *be two finite Borel measures* (1) *on* X *such that* $\mu[\varphi_n(c)] = \nu[\varphi_n(c)]$ *for every c in* $\mathscr{C}_n$ *and* $n = 1$ *to* ∞. *Then* $\mu \equiv \nu$.

Proof. — Let (X, τ) be the Lusin space associated with the given strict subdivision as in the above lemma. Since the Borel sets are the same for (X, τ) and X with the original topology, μ and ν are also finite Borel measures on (X, τ). Also, (X, τ) being a Lusin space, μ and ν are in fact Radon measures on (X, τ) [9]. Further for a base of τ-open sets, viz., $\varphi_n(c)$, c in $\mathscr{C}_n$, $n = 1$ to ∞, μ and ν have the same value. However, every τ-open set is a countable disjoint union of sets in the above base. Hence, μ and ν are the same on every τ-open set. Now, from the outer regularity of μ and ν, we deduce that μ and ν are identical measures. The proof is complete.

(1) We use the term measure for a set function which is countably additive, monotone, non-decreasing and equal to zero on the empty set.

THEOREM 1.1. — *Let* X *and* Y *be two Lusin topological spaces and* $\mathfrak{M}_1^+(X)$ *the set of finite Borel measures on* X. *Let* ν *be a mapping of* $Y \to \mathfrak{M}_1^+(X)$ *such that if* $\nu(y) = \nu_y$ *and* B *any Borel set of* X *then,* $y \to \nu_y(B)$ *is universally measurable for every finite Borel measure on* X. *Let* λ *and* ν *be finite Borel measures on* Y *and* X *respectively such that* (1) ν_y *is absolutely continuous relative to* ν *for every* y *in* Y, *and (2)* $\int \nu_y(X)\lambda\,(dy) < +\infty$. *Then, there exists a positive Borel function* f *on* $X \times Y$ *such that for* λ*-almost every* y *in* Y, $f(x, y)$ *is a Lebesgue-Radon-Nikodym derivative of* ν_y *relative to* ν; *i.e., for every Borel subset* B *of* X, *for* λ*-almost every* y *in* Y, *independent of* B,

$$\int_B f(x, y)\nu\,(dx) = \nu_y(B).$$

Proof. — Consider the class $\mathfrak{A}$ of Borel sets E contained in $X \times Y$ such that $y \to \int \chi_E(x, y)\nu_y\,(dx)$ is a universally measurable function on Y. Clearly, this class contains **R** where **R** is the algebra of subsets of $X \times Y$ consisting of finite disjoint unions of Borel rectangles. Further $\mathfrak{A}$ is easily seen to be a monotone class as a consequence of monotone convergence theorem. Hence, $\mathfrak{A}$ contains the monotone class generated by **R**. But the spaces X and Y being Lusin, in particular Lindelöf spaces, the monotone class generated by **R** is precisely the Borel σ-algebra $\mathbf{B}(X \times Y)$ of the product space $X \times Y$. Hence $\mathfrak{A}$ is precisely $\mathbf{B}(X \times Y)$. Now, once again by a repeated application of monotone convergence theorem it is easy to deduce that there is an unique finite Borel measure μ on $X \times Y$ such that for every non-negative Borel function g on $X \times Y$,

$$\int g\,d\mu = \int \lambda\,(dy) \int g(x, y)\nu_y\,(dx).$$

Let us now show that μ is absolutely continuous relative to the product measure $\nu \otimes \lambda$. Let E be any Borel set in the product space $X \times Y$ such that $(\nu \otimes \lambda)(E) = 0$. Then, there is a set A contained in Y of λ-measure zero, A may be assumed to be Borel, such that for every y in $Y \setminus A$, $\nu(E^y) = 0$ where E^y is the section through y of E. Since

ν_η is absolutely continuous relative to ν for every η in Y, it follows that $\nu_y(E^y) = 0$ for every y in $Y \setminus A$. Now,

$$\begin{aligned}\mu(E) &= \int \lambda\,(dy) \int \chi_E(x, y)\nu_y\,(dx) \\ &= \int_{Y/A} \nu_y(E^y)\lambda\,(dy) \\ &= 0.\end{aligned}$$

Let f be a non-negative valued Borel function on $X \times Y$ such that f is a Lebesgue-Radon-Nikodym derivative of μ relative to $\nu \otimes \lambda$. Then, for every Borel set E of $X \times Y$,

$$\int_E f(x, y)\nu\,(dx)\lambda\,(dy) = \mu(E).$$

Also,

$$\begin{aligned}\mu(X \times Y) &= \int \lambda\,(dy) \int 1\nu_y\,(dx) \\ &= \int \nu_y(X)\lambda\,(dy) \\ &< +\infty.\end{aligned}$$

Hence, f is $\nu \otimes \lambda$-integrable and so there is a Borel set F_1 of measure zero such that for every y in $Y \setminus F_1$, $f(x, y)$ is ν-integrable on X.

Let us now fix a Borel set E contained in X. For every Borel set F contained in Y, we have:

$$\begin{aligned}\int_{E\times F} f(x, y)\nu\,(dx)\lambda\,(dy) &= \mu(E \times F) \\ &= \int_F \lambda\,(dy) \int_E \nu_y\,(dx),\end{aligned}$$

i.e.,

$$\int_F \left[\nu_y(E) - \int_E f(x, y)\nu\,(dx)\right]\lambda\,(dy) = 0.$$

This is true for every Borel set F contained in Y. From this it is easy to see that there is a set A_E, depending on E, such that A_E is λ-measurable of λ-measure zero and for all y in $Y \setminus A_E$,

$$\nu_y(E) = \int_E f(x, y)\nu\,(dx).$$

Let $\mathfrak{F}$ be the countable collection of all Borel sets belonging to a strict subdivision of X. For every E in $\mathfrak{F}$, let A_E be the set of λ-measure zero determined as above. Let

$F = F_1 \cup \{A_E : E \in \mathfrak{F}\}$. Then, $\lambda(F) = 0$ and for every y in $Y \setminus F$ and all E in $\mathfrak{F}$ we have,

$$\nu_y(E) = \int_E f(x, y) \nu\,(dx).$$

Now, for every y in $Y\setminus F$, ν_y is a finite Borel measure on X and so is the indefinite integral $\int f(x, y)\nu\,(dx)$. It follows by Lemma I.2. that, for every y in $Y\setminus F$ and every Borel set B of X,

$$\nu_y(B) = \int_B f(x, y)\nu\,(dx).$$

The theorem is proved.

Remark. — The above result is still true if we alter the hypothesis as follows: X and Y are any topological spaces with countable basis for open sets; however, all the measures considered are Radon measures.

We need the following result to prove the lattice property of $\mathscr{C}$. Let Ω be a harmonic space of Brelot, i.e., with a system of harmonic functions satisfying the axioms I, II and III and with a positive potential > 0. Let, further, $\mathscr{B}$ be a countable base for open sets of Ω consisting of completely determining domains. Let S^+ denote the space of non-negative superharmonic functions on Ω.

THEOREM 1.2. — *The mapping* $S^+ \times S^+ \to S^+$ *given by* $(v, w) \longmapsto \sup(v, w)$, *the supremum in the specific order, is a Borel function; viz., the inverse image of every Borel set in* S^+ *under this mapping is Borel in* $S^+ \times S^+$. *The space* S^+ *is provided with the Cartan-Brelot topology and the product space with the product topology.*

Proof. — We may assume that the Cartan-Brelot topology on S^+ is defined using the regular domains ω in the base $\mathscr{B}$ and a countable dense subset belonging to Ω. Let ω be a fixed element of $\mathscr{B}$ and x in ω. We shall first show that the mapping $S^+ \times S^+ \to \mathbf{R}$ defined by

$$(v, w) \longmapsto \int \sup(v, w)\, d\rho_x^\omega,$$

is a lower semi-continuous function on $S^+ \times S^+$. Suppose (v_n, w_n) in $S^+ \times S^+$ converges to (v, w). We may find a positive number β such that $\int v \, d\rho_x^\omega \leqslant \beta$, $\int w \, d\rho_x^\omega \leqslant \beta$, and for all n, $\int v_n \, d\rho_x^\omega \leqslant \beta$ and $\int w_n \, d\rho_x^\omega \leqslant \beta$. Let $A_\beta = \{u \in S^+ : \int u \, d\rho_x^\omega \leqslant \beta\}$. Then A_β is a compact subset of S^+ and $A_\beta + A_\beta \subset A_{2\beta}$ is also a compact subset of S^+.

Let us consider v_n and w_n for a fixed n. We know that the specific supremum of v_n and w_n is the pointwise lower envelope of all the specific majorants of v_n and w_n. Let us assume that at least one of the sequences $\{v_n\}$, $\{w_n\}$ contains non-zero elements. Hence, we may find a sequence $\{u_k^n\}$ of elements in S^+ such that (1) u_k^n is a specific majorant of both v_n and w_n, (2) u_k^n decreases to a function $\tilde{u}_n$ as k tends to ∞ and (3) the lower semi-continuous regularization u'_n of $\tilde{u}_n$ is precisely the specific supremum of v_n and w_n. We observe that for every n, u_k^n converges to u'_n in the Cartan-Brelot topology as k tends to ∞ and further that $\int u'_n \, d\rho_x^\omega \leqslant 2\beta$ since $u'_n \leqslant v_n + w_n$. Hence, for every n, u'_n is an element of $A_{2\beta}$ and we may assume without loss of generality that u_k^n belongs to $A_{2\beta}$ for every k and every n. Now, given $\varepsilon > 0$, by the definition of Cartan-Brelot topology we may choose an integer k_n, for every n, such that

$$0 < \int u_{k_n}^n(\xi) \rho_x^\omega (d\xi) - \int u'_n(\xi) \rho_x^\omega (d\xi) < \varepsilon/2.$$

Let us now consider the sequence $\{u_{k_n}^n\}$ of positive superharmonic functions on Ω belonging to $A_{2\beta}$.

To prove the required lower semi-continuity, it is enough to show that the sequence $\left\{\int u'_n \, d\rho_x^\omega\right\}$ of real numbers has the property that from any subsequence of this sequence we may extract a further subsequence with a limit greater than or equal to $\int \sup(v, w) \, d\rho_x^\omega$. Accordingly, given any subsequence of $\{u'_n\}$ say $\{u'_m\}_{m \in M \subset N}$ consider the corresponding subsequence $\{u_{k_m}^m\}_{m \in M}$. This latter sequence of functions belong to $A_{2\beta}$, hence we may extract a further subsequence, say for $m \in M' \subset M$, which converges to a superharmonic function h in Cartan-Brelot-topology as m tends to ∞,

m in M′. Hence, for all m sufficiently large and m in M′

$$\int u^m_{k_m}(\xi)\rho^\omega_x\,(d\xi) > \int h(\xi)\rho^\omega_x\,(d\xi) - \varepsilon/2.$$

Hence, we get that for all m in M′, m sufficiently large,

$$\begin{aligned}\int u'_m(\xi)\rho^\omega_x\,(d\xi) &\geqslant \int u^m_{k_m}(\xi)\rho^\omega_x\,(d\xi) - \varepsilon/2\\ &> \int h(\xi)\rho^\omega_x\,(d\xi) - \varepsilon \quad (1)\end{aligned}$$

We shall now show that h is a specific majorant of v and w. We know that $u^m_{k_m}$ is a specific majorant of v_m and hence, $u^m_{k_m} = v_m + v'_m$ for some v'_m in S^+, in fact, in $A_{2\beta}$. Also as m tends to ∞ with m in M′, v_m converges to v, $u^m_{k_m}$ converges to h and now it is easy to deduce that v'_m converges to an element v' in S^+. Hence $h = v + v'$ and h is a specific majorant of v. It follows similarly that h is also a specific majorant of w; hence, h majorises $\sup(v, w)$ in the specific order. Hence, for all sufficiently large m in M′,

$$\begin{aligned}\int u'_m\,d\rho^\omega_x &\geqslant \int h\,d\rho^\omega_x - \varepsilon\\ &\geqslant \int \sup(v, w)\,d\rho^\omega_x - \varepsilon.\end{aligned}$$

This completes the proof of the lower semi-continuity of the function $(v, w) \longmapsto \int \sup(v, w)\,d\rho^\omega_x$.

Now, for any pair of real numbers a and b, $a < b$, any δ in $\mathcal{B}$ and x in δ, let

$$W(\delta, x, a, b) = \left\{u \in S^+ : a < \int u\,d\rho^\delta_x < b\right\}.$$

Then, there is a subbase for the Cartan-Brelot-topology consisting of a countable number of sets of the form $W(\delta, x, a, b)$. It is clear that the Borel σ-algebra $\mathbf{B}(S^+)$ of S^+ is the σ-algebra generated by this countable subbase. To prove that the mapping $(v, w) \longmapsto \sup(v, w)$ is Borel it is enough to show that the inverse image of the sets of the form $W(\delta, x, a, b)$ under this mapping are Borel sets of $S^+ \times S^+$. But,

$$\begin{aligned}&\{(v, w) \in S^+ \times S^+ : \sup(v, w) \in W(\delta, x, a, b)\}\\ &\qquad = \left\{(v, w) : a < \int \sup(v, w)\,d\rho^\delta_x < b\right\}\end{aligned}$$

and the set on the right side is Borel in $S^+ \times S^+$ in view of the lower semi-continuity proved above. The proof is complete.

COROLLARY. — *The mapping* $(v, w) \longmapsto \inf(v, w)$ *in the specific order is a Borel mapping of* $S^+ \times S^+ \to S^+$.

Proof. — The mapping $S^+ \times S^+ \to S^+$ given by $(v, w) \longmapsto v + w$ is jointly continuous and

$$v + w = \sup(v, w) + \inf(v, w).$$

Hence, $(v, w) \longmapsto \int \inf(v, w)\, d\rho_x^\omega$ is upper semi-continuous on $S^+ \times S^+$. The proof of the corollary is now completed as in the last part of the proof above.

LEMMA 1.3. — *Let* T *be a measurable space,* i.e. *a set with a* σ*-algebra* τ *of subsets of* T *and* $\vec{\varphi} : T \to S^+$. *The space* S^+ *is provided with the Cartan-Brelot-toplogy and* **B** *is the Borel* σ*-algebra of* S^+. *Then the following are equivalent conditions.*

(1) *The function* $\varphi(t, x) = (\vec{\varphi}(t))(x)$ *is* $\tau \times \mathbf{B}$ *measurable on* $T \times \Omega$.

(2) *The mapping* $\vec{\varphi} : T \to S^+$ *is Borel.*

Proof. — Suppose (2) is verified. In particular, the real valued function $t \to \int \vec{\varphi}(t)(\xi)\rho_x^\delta(d\xi)$ is τ-measurable on T for every fixed x in δ, where δ is any element of $\mathcal{B}$. However, $x \longmapsto \int \vec{\varphi}(t)(\xi)\rho_x^\delta(d\xi)$ is harmonic on δ for every t. We deduce that $(t, x) \longmapsto \int \vec{\varphi}(t)(\xi)\rho_x^\delta(d\xi)$ is $\tau \times (\mathbf{B} \cap \delta)$ measurable, [6, p. 487]. Define $f_\delta : T \times \Omega \to R^+$ as follows:

$$\begin{aligned} f_\delta(t, x) &= \int \vec{\varphi}(t)(\xi)\rho_x^\delta(d\xi) && \text{if} \quad (t, x) \in T \times \delta \\ &= 0 && \text{if} \quad (t, x) \in T \times (\Omega \setminus \delta). \end{aligned}$$

Then, clearly f_δ is a $\tau \times \mathbf{B}$-measurable function on $T \times \Omega$. Also, $\varphi(t, x) = \sup\{f_\delta(t, x) : \delta \in \mathcal{B}\}$. Since $\mathcal{B}$ is a countable set we conclude that (1) is satisfied.

Conversely, suppose $\varphi(t, x)$ is $\tau \times \mathbf{B}$-measurable on

$T \times \Omega$. Let δ be in $\mathcal{B}$ and x in δ. Then

$$t \longmapsto \int \varphi(t, \xi)\rho_x^\delta(d\xi)$$

is τ-measurable. Consider the subbase for the topology of S^+ consisting of sets of the form $W(\delta, x, a, b)$ as in the proof of Theorem 1.1. It is clear that $\{t \in T : \vec{\varphi}(t) \in W(\delta, x, a, b)\}$ is in τ. The proof of the Lemma is now completed easily.

THEOREM 1.2. — *Let* T *be a measurable space with a* σ*-algebra* τ *of subsets of* T. *Let* φ *and* ψ *be two non-negative valued* $\tau \times \mathbf{B}(\Omega)$ *measurable functions on* $T \times \Omega$ *where* $\mathbf{B}(\Omega)$ *denotes the Borel* σ*-algebra of* Ω. *Let further, for every* t *in* T, $\varphi(t, .)$ *and* $\psi(t, .)$ *be elements of* S^+. *Let for every* t *in* T, $\rho(t, .)$ *be the specific supremum of the superharmonic functions* $\varphi(t, .)$ *and* $\psi(t, .)$. *Then,* $\rho : T \times \Omega \to R$ *is a* $\tau \times \mathbf{B}(\Omega)$ *measurable function.*

Proof. — From the above lemma, we conclude that the inverse image of any Cartan-Brelot Borel set of S^+ under the mappings $\vec{\varphi}$ and $\vec{\psi}$ given by $\vec{\varphi}(t)(x) = \varphi(t, x)$ and $\vec{\psi}(t)(x) = \psi(t, x)$, belong to τ. Now let $\vec{\rho}$ be the composite of the mappings $T \to S^+ \times S^+$ given by $t \longmapsto (\vec{\varphi}(t), \vec{\psi}(t))$ and $S^+ \times S^+ \to S^+$ given by $(v, w) \longmapsto \sup\,(v, w)$ in the specific order. By Theorem 1.1 we know that the latter mapping is Borel. Also we conclude easily that the inverse image of any Borel set under the mapping $t \longmapsto (\vec{\varphi}(t), \vec{\psi}(t))$ is in τ using the following two facts : (1) this mapping is coordinatewise τ-measurable and (2) since S^+ is polish the Borel σ-algebra of the product of S^+ with itself is the product of the Borel σ-algebra $\mathbf{B}[S^+]$ with itself. Hence $\vec{\rho}$ is τ-measurable. It follows from the earlier lemma that the function ρ is $\tau \times \mathbf{B}(\Omega)$ measurable, completing the proof.

2. The class $\mathcal{C}$ of doubly superharmonic functions and the integral representation.

Let Ω_1 and Ω_2 be harmonic spaces of Brelot and we shall assume that $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) is a countable base for open sets of Ω_1 (resp. Ω_2) consisting of completely determining regular domains. Let S_k^+ be the space of non-negative superharmonic functions on Ω_k and Λ_k the Cartan-Brelot-compact base of S_k^+ consisting of the functions v such that for a fixed δ_0^k in $\mathscr{B}_k$ and a fixed point x_0^k in δ_0^k, $R_v^{\complement \delta_0^k}(x_0^k) = 1$; for $k = 1,2$. Let $\mathscr{E}_k$ be the set of extreme superharmonic functions belonging to the base Λ_k for $k = 1,2$. Let $(2\text{-S})^+$ denote the class of positive 2-superharmonic functions on $\Omega_1 \times \Omega_2$. Before we define the class $\mathscr{C}$ contained in $(2\text{-S})^+$ we need

DEFINITION 2.1. — *Let* $\nu : \Omega_1 \to \mathfrak{M}_1^+(\mathscr{E}_2)$ *be a mapping of* Ω_1 *into the space of finite Borel measures on* $\mathscr{E}_2$. *This mapping is said to depend superharmonically on* Ω_1 *if the folllowing hold good. Let us denote the measure* $\nu(x)$ *by* ν_x *for every* x *in* Ω_1.

(1) *For every Borel set contained in* $\mathscr{E}_2$, *the function* $x \to v_x(\mathrm{B})$ *is universally measurable for every finite Radon measure on* Ω_1.

(2) *For every regular domain* ω *contained in* Ω_1 *and any* x *in* ω *let* $\nu(\omega, x)$ *be the finite Borel measure on* $\mathscr{E}_2$ *defined by*

$$\nu(\omega, x)(\mathrm{B}) = \int \nu_\xi(\mathrm{B})\rho_x^\omega\,(d\xi).$$

Then, for every Borel set B *of* $\mathscr{E}_2$, $\nu(\omega, x)(\mathrm{B}) \leqslant \nu_x(\mathrm{B})$.

(3) *For every* x *in* Ω_1 *and any sequence* $\{\omega_n\}$ *of regular domains of* Ω_1 *such that* $\{x\} = \cap\,\omega_n$ *and* $\overline{\omega}_{n+1} \subset \omega_n$ *for all* n, *and every Borel set* B *of* $\mathscr{E}_2$,

$$\lim_{n\to\infty} \int \nu_\xi(\mathrm{B})\rho_x^{\omega_n}\,(d\xi) = \nu_x(\mathrm{B}).$$

The above definition is really equivalent to stating that for every Borel set B of $\mathscr{E}_2$, $x \longmapsto \nu_x(\mathrm{B})$ is a superharmonic function. But we prefer to define it the longer way.

DEFINITION 2.2. — *A doubly superharmonic positive function on* $\Omega_1 \times \Omega_2$ *is said to belong to the class* $\mathscr{C}$ *if the mapping* $x \longmapsto \nu_x$, ν_x *being the canonical measure on* $\mathscr{E}_2$ *corresponding to the superharmonic function* $v(x, .)$ *on* Ω_2, *depends superharmonically on* x *in* Ω_1.

Remark. — The set of points x in Ω_1 for which $v(x, .) \equiv +\infty$ on Ω_2 is easily seen to be polar and this is a Borel set of Ω_1; for all x belonging to this set, we set ν_x equal to some arbitrary positive finite measure on $\mathscr{E}_2$.

Before proceeding further, we prove the following lemma.

LEMMA 2.1. — *Let v be a positive doubly superharmonic function on $\Omega_1 \times \Omega_2$. Then, for every Borel set* B *contained in $\mathscr{E}_2$, $x \longmapsto \nu_x(\mathrm{B})$ is a Borel function on Ω_1, where ν_x is the canonical measure on $\mathscr{E}_2$ corresponding to the superharmonic function $y \longmapsto v(x, y)$.*

Proof. — Let $\mathfrak{M}_1^+(\mathscr{E}_2)$ denote the set of finite Radon measures on $\mathscr{E}_2$. Let this set be given the narrow topology. Then, it is known that $\mathfrak{M}_1^+(\mathscr{E}_2)$ is a polish space, since $\mathscr{E}_2$ is a polish space, [9]. Now, consider the mapping $\varphi : \mathrm{S}_2^+ \to \mathfrak{M}_1^+(\mathscr{E}_2)$ given by the canonical measure on $\mathscr{E}_2$, corresponding to the element v in S_2^+. This is a one-one, onto mapping. And the inverse of this mapping is continuous as, if ν_n in $\mathfrak{M}_1^+(\mathscr{E}_2)$ converges to ν in $\mathfrak{M}_1^+(\mathscr{E}_2)$ then the superharmonic functions corresponding to ν_n converge to the function with the canonical measure ν in the Cartan-Brelot topology. However, $\mathfrak{M}_1^+(\mathscr{E}_2)$ being a polish space, this implies that the mapping $\varphi : \mathrm{S}_2^+ \to \mathfrak{M}_1^+(\mathscr{E}_2)$ is Borel [9]. Now, the mapping $x \longmapsto \nu_x$ is the composite of $\Omega_1 \to \mathrm{S}_2^+$ given by $x \longmapsto v(x, .)$ and φ. In view of Lemma 1.3, $\Omega_1 \to \mathrm{S}_2^+$ is Borel and it follows that $x \longmapsto \nu_x$ is a Borel function. In particular, $x \longmapsto \nu_x(f)$ is Borel for every positive continuous function f on $\mathscr{E}_2$. Now, by standard Baire class arguments, we can deduce that $x \longmapsto \nu_x(\mathrm{B})$ is Borel for every Borel set B in $\mathscr{E}_2$. The lemma is proved.

COROLLARY. — *The canonical measure corresponding to*

$$\int v(\xi, y)\rho_x^\omega\,(d\xi)$$

on $\mathscr{E}_2$ is precisely the measure

$$\int \nu_\xi \rho_x^\omega\,(d\xi)$$

on $\mathscr{E}_2$.

The corollary follows as a consequence of Fubini's Theorem and the uniqueness of the measure on $\mathcal{E}_2$ representing $\int v(\xi, y)\rho^{\omega}(d\xi)$.

Proposition 2.1. — *The set $\mathcal{C}$ is a convex cone for the canonical addition and scalar multiplication.*

The proof is obvious.

Let $\mathcal{C}_1$ be the vector space generated by the cone $\mathcal{C}$. That is, by the standard method of introducing the natural equivalence relation on $\mathcal{C} \times \mathcal{C}$ etc., $\mathcal{C}_1$ consists of differences of the form v-w, v and w in $\mathcal{C}$. The convex cone $\mathcal{C}$ is with vertex 0 and generates $\mathcal{C}_1$. Let « $\succ$ » be the order introduced by $\mathcal{C}$, i.e., with $\mathcal{C}$ as the positive cone. In this order v, w in $\mathcal{C}_1$ satisfy $v \succ w$ if $v - w$ is an element of $\mathcal{C}$. We shall now proceed to show that with this order $\mathcal{C}$ is a lattice.

Theorem 2.1. — *Let $\nu : \Omega_1 \to \mathfrak{M}_1^+(\mathcal{E}_2)$ depend superharmonically on Ω_1. Then the sets of $\nu_x = \nu(x)$ measure zero are independent of x in Ω_1. Also, if f is real valued Borel function on $\mathcal{E}_2$ then the ν_x-integrability of f is independent of x in Ω_1, except for a polar set E_f depending on f. Further, for a positive « integrable » Borel function f,*

$$\int f(p)\nu_x(dp)$$

is a superharmonic function on Ω_1.

Proof. — If B is a Borel set such that $\nu_x(\mathrm{B}) = 0$ for some x, then $\nu_\xi(\mathrm{B}) = 0$ for every ξ in Ω_1 since $\xi \longmapsto \nu_\xi(\mathrm{B})$ is a non-negative superharmonic function on Ω_1. Now, let E be a ν_x-measurable set for some x in Ω_1 such that $\nu_x(\mathrm{E}) = 0$. Then, there is a Borel set B containing E such that $\nu_x(\mathrm{B}) = 0$. Hence $\nu_\xi(\mathrm{B}) = 0$ for every ξ in Ω_1. It follows that E is ν_ξ-measurable for every ξ in Ω_1 and $\nu_\xi(\mathrm{E}) = 0$. Now, by standard measure theoretic arguments, it is clear that for any positive Borel function on $\mathcal{E}_2$,

$$x \longmapsto \int f(p)\nu_x(dp)$$

is identically zero or $+\infty$, or else superharmonic on Ω_1. The proof is now completed easily.

The following is the crucial result in getting the integral representation.

THEOREM 2.2. — *Let* v *be a positive* 2*-superharmonic function on* $\Omega_1 \times \Omega_2$ *belonging to the class* $\mathscr{C}$. *Let* ν *be any finite Borel measure on* $\mathscr{E}_2$ *such that for every* x *in* Ω_1, *the canonical measure* ν_x *on* $\mathscr{E}_2$ *corresponding to the positive superharmonic function* $v(x, .)$ *on* Ω_2 *is absolutely continuous relative to* ν.

Then, there is an extended real valued non-negative Borel function $f(x, p)$ *on* $\Omega_1 \times \mathscr{E}_2$ *such that*

(1) *for every* p *in* $\mathscr{E}_2$, $x \longmapsto f(x, p)$ *is superharmonic ou* Ω_1, *and*

(2) *for every* x *in* Ω_2,

$$v(x, y) = \int p(y) f(x, p) \nu(dp).$$

Conversely, let $f(x, p)$ *be a non-negative extended real valued Borel function on* $\Omega_1 \times \mathscr{E}_2$ *such that it is superharmonic in the first variable for every* p *in* $\mathscr{E}_2$. *Let* ν *be any finite Borel measure on* $\mathscr{E}_2$. *Then,*

$$v(x, y) = \int p(y) f(x, p) \nu(dp)$$

is either identical to 0 *or* ∞, *or else a positive* 2*-superharmonic function in the class* $\mathscr{C}$.

Proof. — Let δ_0 be the fixed domain in $\mathscr{B}_2$ defining the base Λ_2 (and the corresponding $\mathscr{E}_2$) with fixed y_0 in δ_0. Observe that

$$\int v(x, y) \rho_{y_0}^{\delta_0}(dy)$$

is a positive superharmonic function on Ω_1. Let x_n be an arbitrary element belonging to the regular domain ω_n in $\mathscr{B}_1$. Let $\alpha_n > 0$ be such that

$$\sum_{n=1}^{\infty} \alpha_n \rho_{x_n}^{\omega_n}(\partial \omega_n) < +\infty$$

and

$$\sum_{n=1}^{\infty} \alpha_n \iint v(x, y) \rho_{y_0}^{\delta_0}(dy) \rho_{x_n}^{\omega_n}(dx) < +\infty.$$

Let $\lambda = \Sigma \alpha_n \rho_{x_n}^{\omega_n}$. Then λ is a finite Radon measure on Ω_1. And,

$$\begin{aligned}\int \nu_x(\mathscr{E}_2)\lambda\,(dx) &= \int \lambda\,(dx) \int \nu(x, y)\rho_{y_0}^{\delta_0}\,(dy)\\ &= \sum_{n=1}^{\infty} \alpha_n \iint \nu(x, y)\rho_{y_0}^{\delta_0}\,(dy)\rho_{x_n}^{\omega_n}\,(dx)\\ &< +\infty.\end{aligned}$$

Further, both Ω_1 and $\mathscr{E}_2$ are polish topological spaces. Hence, by Theorem 1.1, there is a Borel function φ on $\Omega_1 \times \mathscr{E}_2$ such that $\varphi(x, p) \geqslant 0$ and except for a set F contained in Ω_1 of λ-measure zero, $\varphi(x, .)$ is the Radon-Nikodym derivative of ν_x relative to ν. It follows that F is of $\rho_{x_n}^{\omega_n}$-measure zero, for every n; i.e., F is $\mathscr{B}_1$-negligible.

Let ω be any regular domain in the base $\mathscr{B}_1$ and consider the integral

$$\int \varphi(\xi, p)\rho_x^{\omega}\,(d\xi) = \varphi^{\omega}(x, p).$$

As a function of x in ω, $\varphi^{\omega}(x, p)$ is harmonic or identically $+\infty$. Suppose for p in a set of positive ν-measure on $\mathscr{E}_2$, $\varphi^{\omega}(x, p) = +\infty$. Then,

$$\begin{aligned}+\infty &= \iint \varphi(\xi, p)\rho_x^{\omega}\,(d\xi)\nu\,(dp)\\ &= \int \rho_x^{\omega}\,(d\xi) \int \varphi(\xi, p)\nu\,(dp)\\ &= \int \rho_x^{\omega}\,(d\xi)\nu_{\xi}(\mathscr{E}_2)\\ &< +\infty.\end{aligned}$$

Hence, for ν-almost every p in $\mathscr{E}_2$, $\varphi^{\omega}(x, p)$ is harmonic on ω and is clearly a Borel function on $\mathscr{E}_2$. Hence, $\varphi^{\omega}(x, p)$ is jointly Borel on $\omega \times \mathscr{E}_2$ [6, p. 487].

Now, let ω_1 and ω_2 be in the base $\mathscr{B}_1$ such that $\overline{\omega}_2$ is contained in ω_1 and let B be any Borel set of $\mathscr{E}_2$. Then, for x in ω_2,

$$\begin{aligned}\int_{\mathrm{B}} \varphi^{\omega_1}(x, p)\nu\,(dp) &= \int_{\mathrm{B}} \nu\,(dp) \int \varphi(\xi, p)\rho_x^{\omega_1}\,(dp)\\ &= \int \rho_x^{\omega_1}\,(d\xi) \int_{\mathrm{B}} \varphi(\xi, p)\nu\,(dp)\\ &= \int \nu_{\xi}(\mathrm{B})\rho_x^{\omega_1}\,(d\xi)\\ &\leqslant \int \nu_{\xi}(\mathrm{B})\rho_x^{\omega_2}\,(d\xi) \quad (\nu \text{ belongs to } \mathscr{C})\\ &= \int_{\mathrm{B}} \varphi^{\omega_2}(x, p)\nu\,(dp),\end{aligned}$$

i.e.,

$$\int_B [\varphi^{\omega_1}(x, p) - \varphi^{\omega_2}(x, p)]\nu\,(dp) \leqslant 0.$$

Hence, there is a set E_x of ν-measure zero in $\mathscr{E}_2$ such that for every p in the complement $\varphi^{\omega_1}(x, p) \leqslant \varphi^{\omega_2}(x, p)$. Let E be the union of the exceptional sets E_x corresponding to x in a countable dense subset of ω_2. Also we may suppose that E contains the set of p for which $\varphi^{\omega_2}(x, p) \equiv +\infty$. Now, E is of ν-measure zero and for every p in $\mathscr{E}_2$ and not in E, $\varphi^{\omega_1}(x, p)$ and $\varphi^{\omega_2}(x, p)$ are two harmonic functions on ω_2 and satisfy $\varphi^{\omega_1}(x, p) \leqslant \varphi^{\omega_2}(x, p)$ on a dense subset. Hence, for every p not in E, $\varphi^{\omega_1}(x, p) \leqslant \varphi^{\omega_2}(x, p)$ for all x in ω_2. The base $\mathscr{B}_1$ is countable and it is easy to see that it is possible to choose a set $\mathscr{E}'_2$ contained in $\mathscr{E}_2$ of full ν-measure such that for all p in $\mathscr{E}'_2$ and every pair of elements ω_1, ω_2 in $\mathscr{B}_1$ such that $\omega_1 \supset \overline{\omega_2}$,

$$\varphi^{\omega_1}(x, p) \leqslant \varphi^{\omega_2}(x, p) < +\infty$$

for all x in ω_2. Since a finite measure ν on the polish space $\mathscr{E}_2$ is Radon we may without loss of generality assume that $\mathscr{E}'_2$ is a K_σ-set, in particular, a Borel set. Also, for every Borel set B of $\mathscr{E}_2$ and for $\mathscr{B}_1$-nearly every x in Ω_1,

$$\begin{aligned}\int_B \varphi^{\omega}(x, p)\nu\,(dp) &= \int \nu_\xi(B)\rho_x^{\omega}\,(d\xi)\\ &\leqslant \nu_x(B)\\ &= \int_B \varphi(x, p)\nu\,(dp).\end{aligned}$$

Let for every ω in $\mathscr{B}_1$, $\varphi_1^{\omega}(x, p) = 0$ if x is not in ω and $= \varphi^{\omega}(x, p)$ if x is in ω. Then φ_1^{ω} is a Borel function on $\Omega_1 \times \mathscr{E}'_2$, so is the $\sup\{\varphi_1^{\omega} : \omega \in \mathscr{B}_1\} = f$.

We claim that for every p in $\mathscr{E}'_2$, the function $x \longmapsto f(x, p)$ is lower semi-continuous on Ω_1. To prove this, let p be fixed in $\mathscr{E}'_2$. Consider $\{x : f(x, p) > \alpha\}$. If x_0 is in this set then, there is a ω in $\mathscr{B}_1$, ω contains x_0, such that $\varphi^{\omega}(x_0, p) > \alpha$. Since φ^{ω} is continuous in x, there is an open neighbourhood U of x_0 in which $\varphi^{\omega}(x, p) > \alpha$. Now, $f(x, p) \geqslant \varphi^{\omega}(x, p) > \alpha$ for all x in U. This shows that $\{x : f(x, p) > \alpha\}$ is open, proving the lower semi-continuity.

Let x be in Ω_1 and $\{\omega_n\}$ a sequence of regular domains in $\mathscr{B}_1$ such that $\overline{\omega}_{n+1}$ is contained in ω_n for every n and $\cap\omega_n = \{x\}$. It is easily seen that $\lim\limits_{n\to\infty} \varphi^{\omega_n}(x, p) = f(x, p)$ for every p in $\mathscr{E}_2'$. In particular, by monotone convergence theorem, for all Borel sets B of $\mathscr{E}_2$,

$$\int_{\mathrm{B}} f(x, p)\nu\,(dp) = \lim_{n\to\infty} \int_{\mathrm{B}} \varphi^{\omega_n}(x, p)\nu\,(dp).$$

However, except for x in F, the set which is $\mathscr{B}_1$-negligible introduced at the beginning of the proof,

$$\int_{\mathrm{B}} \varphi(x, p)\nu\,(dp) = \lim_{n\to\infty} \int_{\mathrm{B}} \varphi^{\omega_n}(x, p)\nu\,(dp).$$

It follows that for all x in $\Omega_1\setminus\mathrm{F}$, $f(x, p)\nu\,(dp)$ and $\varphi(x, p)\nu\,(dp)$ are the same measures on $\mathscr{E}_2'$. Hence,

$$\begin{aligned}\int_{\mathrm{B}} \nu\,(dp) \int f(\xi, p)\rho_x^\omega\,(d\xi) &= \int \rho_x^\omega\,(d\xi) \int_{\mathrm{B}} f(\xi, p)\nu\,(dp)\\ &= \int \rho_x^\omega\,(d\xi) \int_{\mathrm{B}} \varphi(\xi, p)\nu\,(dp)\\ &= \int_{\mathrm{B}} \nu\,(dp) \int \varphi(\xi, p)\rho_x^\omega\,(d\xi)\\ &= \int_{\mathrm{B}} \varphi^\omega(x, p)\nu\,(dp).\end{aligned}$$

The above equation is true whatever be the Borel set B of $\mathscr{E}_2'$ and all elements ω of $\mathscr{B}_1$. It follows that

$$\int f(\xi, p)\rho_x^\omega\,(d\xi) = \varphi^\omega(x, p)$$

for ν-almost every p in $\mathscr{E}_2'$, the exceptional set depends on x in ω. Now exactly by the same argument as before, using the continuity of $\varphi^\omega(x, .)$ and $\int f(\xi, p)\rho_x^\omega\,(d\xi)$ we may find a Borel set $\mathscr{E}_2''$ of $\mathscr{E}_2$, contained in $\mathscr{E}_2'$ such that (1) $\nu(\mathscr{E}_2' - \mathscr{E}_2'') = 0$ and (2) for every p in $\mathscr{E}_2''$ and every regular domain ω in Ω_1, ω in $\mathscr{B}_1$, $\int f(\xi, p)\rho_x^\omega\,(dp) = \varphi^\omega(x, p)$. Now,

$$\begin{aligned}\int f(\xi, p)\rho_x^\omega\,(dp) &= \varphi^\omega(x, p)\\ &\leqslant f(x, p)\end{aligned}$$

for all x in ω and every p in $\mathscr{E}_2''$, and this shows that $x \longmapsto f(x, p)$ is superharmonic on Ω_1 for every p in $\mathscr{E}_2''$. We may change f to take the value 0 for all x in Ω_1 and all p in $\mathscr{E}_2\setminus\mathscr{E}_2''$, then f is defined on the whole of $\Omega_1 \times \mathscr{E}_2$

and for every x in Ω_1, $f(x, p)\,d\nu$ is unaltered. Further, $(x, y) \longmapsto \int p(y) f(x, p)\nu(dp)$ is a positive 2-superharmonic function on $\Omega_1 \times \Omega_2$ and for all x in $\Omega_1 \setminus F$ and y in Ω_2, where F is $\mathscr{B}_1$-negligible,

$$\int p(y) f(x, p)\nu(dp) = \int p(y)\varphi(x, p)\nu(dp) = v(x, y)$$

It is now clear that for all (x, y) in $\Omega_1 \times \Omega_2$

$$v(x, y) = \int p(y) f(x, p)\nu(dp).$$

Conversely, let f be as in the hypothesis and ν any finite Borel measure on $\mathscr{E}_2$. It follows by straightforward application of Fatou's lemma and Fubini's theorem that

$$v(x, y) = \int f(x, p)p(y)\nu(dp)$$

is a doubly duperharmonic function on $\Omega_1 \times \Omega_2$. Again we deduce using Fatou's lemma that for any Borel set B of $\mathscr{E}_2$, $x \longmapsto \int_B f(x, p)\nu(dp)$ is lower-semi-continuous. Also, for any regular domain ω,

$$\int \rho_x^\omega(d\xi) \int_B f(\xi, p)\nu(dp) = \int_B \nu(dp) \int f(\xi, p)\rho_x^\omega(d\xi) \leqslant \int_B f(x, p)\nu(dp).$$

This shows that the mesaures $\{f(x, p)\nu(dp)\}$ depend superharmonically on x. But since the measure on $\mathscr{E}_2$ representing $v(x, .)$ is unique, $f(x, p)\nu(dp)$ is precisely the canonical measure corresponding to $v(x, .)$. Hence, v is in the class $\mathscr{C}$. The proof is complete.

THEOREM 2.3. — *The cone $\mathscr{C}$ is a lattice in its own order, that is, in the order « $\succ$ ».*

Proof. — Let v and w be any two positive 2-superharmonic functions belonging to the class $\mathscr{C}$. Let λ be the canonical measure on $\mathscr{E}_2$ corresponding to the superharmonic positive function $(v + w)(x_0, .)$ on Ω_2 for a fixed x_0 in Ω_1. Clearly, the canonical measures ν_x (resp. μ_x) corresponding to the superharmonic function $v(x, .)$ (resp. $w(x, .)$)

on Ω_2 are absolutely continuous relative to λ for all x in Ω_1. Hence, by Theorem 2.2, we may find non-negative Borel functions φ and ψ on $\Omega_1 \times \mathscr{E}_2$ such that for every p in $\mathscr{E}_2$, $\varphi(.,p)$ and $\psi(.,p)$ are superharmonic on Ω_1 and

$$\begin{aligned} v(x, y) &= \int \varphi(x, p)p(y)\lambda\,(dp) \\ w(x, y) &= \int \psi(x, p)p(y)\lambda\,(dp). \end{aligned}$$

Let us define $\rho(x, p) = \sup\,[\varphi(x, p), \psi(x, p)]$ for every p in $\mathscr{E}_2$ where the supremum is taken in the specific order on positive superharmonic functions on Ω_2. By Theorem 1.2, we know that ρ is a Borel function on $\Omega_1 \times \mathfrak{E}_2$. Let

$$h(x, y) = \int \rho(x, p)p(y)\lambda\,(dp).$$

Then, h is a positive 2-superharmonic function on $\Omega_1 \times \Omega_2$ belonging to the class $\mathscr{C}$. It is clear from the way ρ is defined that $h - v$ and $h - w$ are both elements of $\mathscr{C}$, that is, $h \succ v$ and $h \succ w$. We shall show that h is precisely the supremum.

Let λ' be any finite Borel measure on $\mathscr{E}_2$ such that $\lambda' \geqslant \lambda$. Then by Theorem 2.2, v and w can be represented by Borel functions φ' and ψ' on $\Omega_1 \times \mathscr{E}_2$ with respect to the measure λ', viz.,

$$\begin{aligned} v(x, y) &= \int \varphi'(x, p)p(y)\lambda'\,(dp) \\ w(x, y) &= \int \psi'(x, p)p(y)\lambda'\,(dp), \end{aligned}$$

where $\varphi'(.,p)$ and $\psi'(.,p)$ are non-negative and superharmonic on Ω_1. Let g be a Radon-Nikodym derivative of λ relative to λ'. Then,

$$\begin{aligned} v(x, y) &= \int \varphi(x, p)p(y)g(p)\lambda'\,(dp) \\ w(x, y) &= \int \psi(x, p)p(y)g(p)\lambda'\,(dp). \end{aligned}$$

Now, as in the proof of Theorem 2.2, we can conclude that for all p in $\mathscr{E}_2$ except for a Borel set of λ'-measure zero,

$$g(p)\int \varphi(\xi, p)\rho_x^\omega\,(d\xi) = \int \varphi'(\xi, p)\rho_x^\omega\,(d\xi)$$

and $$g(p)\int \psi(\xi, p)\rho_x^\omega\,(d\xi) = \int \psi'(\xi, p)\rho_x^\omega\,(d\xi)$$

for every ω in $\mathscr{B}_1$ and all x in ω; and this set of λ'-measure zero is independent of ω in $\mathscr{B}_1$ and x in ω. However, $\varphi(.,p)$, $\varphi'(.,p)$, $\psi(.,p)$ and $\psi'(.,p)$ are all superharmonic on Ω_1 and hence, $\varphi(x,p)g(p) = \varphi'(x,p)$ and

$$g(p)\psi(x,p) = \psi'(x,p)$$

λ'-almost everywhere. Hence,

$$\sup\,[\varphi'(x,p),\psi'(x,p)] = g(p)\sup\,[\varphi(x,p),\psi(x,p)]$$

λ'-almost everywhere, the supremum being taken in the specific order. It follows that

$$\begin{aligned} h(x,y) &= \int \sup\,[\varphi(x,p),\psi(x,p)]p(y)\lambda\,(dp) \\ &= \int g(p)\sup\,[\varphi(x,p),\psi(x,p)]p(y)\lambda'\,(dp) \\ &= \int \sup\,[\varphi'(x,p),\psi'(x,p)]p(y)\lambda'\,(dp). \end{aligned}$$

Now, let u be an element of $\mathscr{C}$, majorising both v and w, i.e., $u - v$ and $u - w$ are both elements of $\mathscr{C}$. Let λ' be the canonical measure on $\mathcal{E}_2$ corresponding to the positive superharmonic function $v(x_0,y) + w(x_0,y) + u(x_0,y)$ where x_0 is a fixed point of Ω. Let φ', ψ', ρ' be non-negative Borel functions on $\Omega_1 \times \mathscr{E}_2$, superharmonic on Ω_1 for every p in $\mathscr{E}_2$ and represent in the above sense respectively the functions v, w and u with respect to the measure λ'. Also, let φ_1' and ψ_1' be similar functions representing $u - v$ and $u - w$ respectively. That is,

$$(u - v)(x,y) = \int \varphi_1'(x,p)p(y)\lambda'\,(dp)$$

and

$$(u - w)(x,y) = \int \psi_1'(x,p)p(y)\lambda'\,(dp).$$

An easy computation along with the uniqueness of the measure on $\mathscr{E}_2$ representing any positive superharmonic function on Ω_2 shows that

$$\rho'(x,p) = \varphi'(x,p) + \varphi_1'(x,p)$$

and

$$\rho'(x,p) = \psi'(x,p) + \psi_1'(x,p),$$

for λ'-almost every p in $\mathscr{E}_2$. Hence, for λ'-almost every p in $\mathscr{E}_2$,

$$\rho'(x,p) = \sup\,[\varphi'(x,p),\psi'(x,p)] + \rho_1'(x,p)$$

where the supremum is taken in the specific order; and ρ_1' is non-negative and superharmonic in the first variable and ρ_1' jointly Borel. Let

$$h'(x, y) = \int \rho_1'(x, p)p(y)\lambda'(dp).$$

By Theorem 2.2, h' is an element of $\mathscr{C}$. Further, $u = h + h'$. This proves that h is the supreum of φ and ψ in the order « $\succ$ ».

It is now easy to see that the infimum of φ and ψ is precisely $(\varphi + \psi) - \sup(\varphi, \psi)$. It can also be seen directly that the infimum of φ and ψ is precisely

$$\int \inf[\varphi(x, p), \psi(x, p)]p(y)\lambda(dp)$$

where the infimum is taken in the specific order.

The proof of the theorem is complete.

Let us consider the vector spaces (2S) generated by $(2S)^+$ in the standard way and the vector space $\mathscr{C}_1$ generated by $\mathscr{C}$. Clearly $\mathscr{C}_1$ can be identified with a vector subspace of (2S). The vector space (2S) and so $\mathscr{C}_1$ can be given the locally convex Hausdorff topology τ under which all the linear functionals $\varphi \longmapsto \iint \varphi\, d\rho_x^\omega\, d\rho_y^\delta$ are continuous for ω in $\mathscr{B}_1$, δ in $\mathscr{B}_2$ and (x, y) in $\omega \times \delta$. It is shown in [4] that $(2S)^+$ with this topology is locally compact and metrizable and has the compact base

$$\{\varphi : \iint \varphi\, d\rho_{x_0}^{\omega_0}\, d\rho_{y_0}^{\delta_0} = 1\}$$

for fixed ω_0, δ_0, x_0 and y_0. We shall assume that the ω_0 and x_0 (resp. δ_0 and y_0) are the domain and the point respectively used in getting the base Λ_1 of S_1^+ (resp. Λ_2 of S_2^+). Clearly, $\mathscr{C}_1$ with τ is a locally convex Hausdorff topological vector space generated by the cone $\mathscr{C}$. Also, $\mathscr{C}_0 = \{\varphi \in \mathscr{C} : \iint \varphi\, d\rho_{x_0}^{\omega_0}\, d\rho_{y_0}^{\delta_0} = 1\}$ is a base for the cone. We shall now show that $\mathscr{C}$ is closed in $((2S), \tau)$.

Theorem 2.4. — *The topological space* $(\mathscr{C}, \tau)$ *is locally compact and metrizable.*

Proof. — $(\mathscr{C}, \tau)$ is a topological subspace of $(2S)^+$ and is hence metrizable. To complete the proof, it is enough to show that $\mathscr{C}$ is τ-closed in $(2S)^+$. Accordingly, suppose φ_n belongs to $\mathscr{C}$ nda converges in τ to φ in $(2S)^+$. Let ω be any regular domain in Ω_1 contained in the base $\mathscr{B}_1$. Let ν_x and $\nu(\omega, x)$ be the canonical measures on $\mathscr{E}_2$ corresponding to the functions $y \longmapsto \varphi(x, y)$ and

$$y \longmapsto \int \varphi(\xi, y)\rho_x^\omega\,(d\xi)$$

respectively: in the second case x is in ω. We recall that $\nu(\omega, x)$ is precisely the measure $\int \nu_\xi \rho_x^\omega\,(d\xi)$ (Lemma 2.1). Now the fact that φ_n converges in τ implies that

$$\int \varphi_n(\xi, y)\rho_x^\omega\,(d\xi)$$

converges in the Cartan-Brelot topology to $\int \varphi(\xi, y)\rho_x^\omega\,(d\xi)$. This is true whatever be the regular domain ω and x in ω.

Let us fix an x in Ω_1, and, let $\omega_1, \omega_2, \ldots, \omega_m, \ldots$ be regular domains containing x such that $\cap \omega_m = \{x\}$ and $\overline{\omega}_{m+1}$ is contained in ω_m for every m. Now, φ_n is an element of $\mathscr{C}$ and hence, $\nu_n(\omega_{m+1}, x) \geqslant \nu_n(\omega_m, x)$ for every n and every m where $\varphi_n(\omega_k, x)$ is the canonical measure on $\mathscr{E}_2$ corresponding to $\int \varphi_n(\xi, y)\rho_x^{\omega_k}\,(d\xi)$. Hence, fixing m,

$$\int \varphi_n(\xi, y)\rho_x^{\omega_{m+1}}\,(d\xi) = \int \varphi_n(\xi, y)\rho_x^{\omega_m}\,(d\xi) + w_n^m \tag{1}$$

where w_n^m is a non-negative superharmonic function on Ω_2. Let us proceed to the limit as n tends to infinity. Since $\int \varphi_n(\xi, .)\rho_x^{\omega_{m+1}}\,(d\xi)$ converges to $\int \varphi(\xi, .)\rho_x^{\omega_{m+1}}\,(d\xi)$ and $\int \varphi_n(\xi, .)\rho_x^{\omega_m}\,(d\xi)$ converges to $\int \varphi(\xi, .)\rho_x^{\omega_m}\,(d\xi)$ and since S_2^+ is closed we conclude that

$$\int \varphi(\xi, .)\rho_x^{\omega_{m+1}}\,(d\xi) = \int \varphi(\xi, .)\rho_x^{\omega_m}\,(d\xi) + u_m$$

where u_m is the element of S_2^+ which is the limit of the sequence w_n^m. It follows that $\nu(\omega_{m+1}, x) \geqslant \nu(\omega_m, x)$. This is true for every positive integer m. Let μ_x be the measure which is the limit of the increasing sequence of measures

$\nu(\omega_m, x)$ in the sense that for every $f \geqslant 0$ and Borel on $\mathscr{E}_2$,

$$\mu_x(f) = \lim_{m\to\infty} \int f \, d\nu(\omega_m, x).$$

In particular it follows that for every y in Ω_2,

$$\begin{aligned}\int p(y)\mu_x\,(dp) &= \lim_{m\to\infty} \int p(y)\nu(\omega_m, x)\,(dp)\\ &= \lim_{m\to\infty} \int \varphi(\xi, y)\rho_x^{\omega_m}\,(d\xi)\\ &= \varphi(x, y).\end{aligned}$$

It follows that $\nu_x = \mu_x$ by the uniqueness of the measure representing $\varphi(x, .)$. And in particular we deduce that $\nu(\omega_m, x) \leqslant \nu_x$ for every positive integer m and that $\nu(\omega_m, x)$ increases to ν_x. This shows that the measures ν_x depend superharmonically on x in Ω_1, i.e. the function φ belongs to the class $\mathscr{C}$. The proof is complete.

Corollary. — $\mathscr{C}_0 = \{\varphi : \iint \varphi \, d\rho_{x_0}^{\omega_0} \, d\rho_{y_0}^{\delta_0} = 1\}$ *is compact.*

Proof. — The proof of the above theorem shows that $\mathscr{C}_0$ is a closed subset of the corresponding base $(2S)_0^+$ of $(2S)^+$. The proof is complete.

The next result characterizes all the extreme elements of the compact base $\mathscr{C}_0$.

Theorem 2.5. — *An element φ of $\mathscr{C}$ is an extremal generator iff there are extremal generators p and p' in S_1^+ and S_2^+ respectively such that $\varphi(x, y) = p(x)p'(y)$ for all (x, y) in $\Omega_1 \times \Omega_2$.*

Proof. — It is known [4, page 37] that every function of the form $p(x)p'(y)$, where p and p' are positive extreme superharmonic functions on Ω_1 and Ω_2 respectively, is an extremal generator of even $(2S)^+$. Conversely, we shall show that every extremal generator of $\mathscr{C}$ is of this form. Let φ be an extremal generator of $\mathscr{C}$. Let ν be a finite Radon measure on $\mathscr{E}_2$ and ϕ a non-negative Borel function on $\Omega_1 \times \mathscr{E}_2$, superharmonic on Ω_1 for every p in $\mathscr{E}_2$ such that $\phi(x, p)\nu\,(dp)$ is the canonical measure on $\mathscr{E}_2$ corresponding to $\varphi(x, .)$.

Suppose now there is a Borel set E of $\mathscr{E}_2$ such that ν_x, for every x in Ω_1, is carried by the set E and $E = \bigcup_{n=1}^{\infty} E_n$ where E_n are all Borel subsets of E and $E_m \cap E_n = \emptyset$ if $m \neq n$. Then, we claim that there is an unique integer m such that ν_x is carried by E_m for every x in Ω_1, i.e., $\nu_x(E_n) = 0$ for all $n \neq m$. Suppose this is not true, and say $\nu_x(E_1) \neq 0$ and $\nu_x(E/E_1) \neq 0$. Then,

$$w_1(x, y) = \int_{E_1} p(y)\varphi(x, p)\nu\,(dp)$$

and

$$w_1'(x, y) = \int_{E/E_1} p\,(y)\varphi(x, p)\nu\,(dp)$$

are both elements of $\mathscr{C}$ (Theorem 2.2) and $w_1 \neq 0$, $w_1' \neq 0$. However, $w_1 + w_1' = v$. Since v is an extremal generator there is an α between 0 and 1 such that $w_1 = \alpha v$ and $w_1' = (1 - \alpha)v$. It is clear that α is neither zero nor one. Hence, $(1/\alpha)w_1 = v = \left(\frac{1}{1-\alpha}\right) w_1'$. Looking at the canonical measures of the positive superharmonic functions w_1, w_1' and v, we conclude, in view of the uniqueness of integral representation for elements of S_2^+, that the measures,

$$\varphi(x, p)\nu\,(dp), \frac{1}{\alpha}\,\chi_{E_1}(p)\varphi(x, p)\nu\,(dp)$$

and $\frac{1}{(1-\alpha)}\,\chi_{E/E_1}(p)\varphi(x, p)\nu\,(dp)$ are identical. This implies in particular, that the measure $\varphi(x, p)\nu\,(dp)$ is carried by both E_1 and $E \setminus E_1$. This in turn implies that $\varphi(x, p)\nu\,(dp)$ is the zero measure, i.e. $v \equiv 0$. This contradicts the assumption that v is *a* non-zero element. Hence, we conclude that there is at the most one integer m such that $\nu_x(E_m) \neq 0$. However there is at least one since ν_x is a non-trivial measure. Again by Theorem 2.1, this set is independent of x in Ω_1.

Now, consider a strict subdivision of the polish space $\mathscr{E}_2$ [9, part II]. Let $\{B_n^0\}$ be the sequence of mutually disjoint Borel sets in the first stage of the subdivision, i.e., $\cup B_n^0 = \mathscr{E}_2$. By the above, we can choose an unique $B_{k_0}^0$ which carries the measures ν_x. It is clear now, than we can choose by induction a coherent sequence of sets $B_{k_0}^0$, $B_{k_1}^1$, $\ldots$, B_{kn}^n, $\ldots$, $B_{k_n}^n$ a

Borel set in the n^{th} stage of subdivision carrying the measure ν_x for every x. But, this coherent sequence of sets generate a filter which converges to an unique element p' in $\mathscr{E}_2$. We claim that ν_x is precisely a constant multiple of the Dirac measure at p'. Suppose p_1 in $\mathscr{E}_2$ and $p_1 \neq p'$. Then, there are neighbourhoods W of p' and W_1 of p_1 such that $W \cap W_1 = \emptyset$. Then, $B^n_{k_n}$ is contained in W for all sufficiently large n. In particular, $\nu_x(W_1) = 0$ for all x in Ω_1. It follows that the support of ν_x is $\{p'\}$. Hence, $v(x, y) = w(x)p'(y)$ for every (x, y) in $\Omega_1 \times \Omega_2$; where clearly $w(x)$ is a positive superharmonic function on Ω_1. Suppose now that $w = w_1 + w_2$ where w_1 and w_2 are positive superharmonic functions on Ω_1. Then, it is easily seen that w_1p' and w_2p' are both elements of $\mathscr{C}$ and,

$$v = w_1p' + w_2p'.$$

Hence w_1p' and w_2p' are proportional to each other. Hence, by fixing y at a point η other than the support of p', so that $0 < p'(\eta) < +\infty$, we see that w_1 and w_2 are proportional to each other. This shows that w is an extremal generator of S_1^+. The proof is complete.

Remark 1. — The above theorem gives a one-one, onto mapping of $\mathscr{E}_1 \times \mathscr{E}_2 \to \mathscr{E}(\mathscr{C}_0)$ viz., $(p, p') \to pp'$ where $\mathcal{E}(\mathscr{C}_0)$ is the set of extreme elements of $\mathscr{C}_0$. It is easily seen that this mapping is a homeomorphism. Hence the finite Borel measures, which are the finite Radon measures of the two spaces, may be identified. Also, by Krein-Milman theorem, we get that the convex combinations of elements of the form pp', $p \in \mathscr{E}_1$ $p' \in \mathscr{E}_2$ are dense in $\mathscr{C}_0$.

Theorem 2.6. — *Let v be a positive doubly superharmonic function on $\Omega_1 \times \Omega_2$ belonging to the class $\mathscr{C}$. Then, there is an unique finite Radon measure μ on $\mathscr{E}_1 \times \mathscr{E}_2$ such that for all (x, y) in $\Omega_1 \times \Omega_2$,*

$$v(x, y) = \int p(x)p'(y)\mu\,(dp\,dp').$$

Proof. — All the conditions to apply the integral representation theorem of Choquet are verified and we conclude that there is one and only one Radon measure μ on $\mathscr{C}_0$ carried by

$\mathscr{E}(\mathscr{C}_0)$ such that for every continuous linear functional l on $\mathscr{C}_1$,

$$\int l(\mathrm{P})\mu\,(d\mathrm{P}) = l(\varphi).$$

Now, we observe that this measure μ can be considered as a finite Radon measure on $\mathscr{E}_1 \times \mathscr{E}_2$ and, for every ω in $\mathscr{B}_1$ and δ in $\mathscr{B}_2$ any (x, y) in $\omega \times \delta$,

$$\iint \varphi\, d\rho_x^\omega\, d\rho_y^\delta = \int \mu\,(dp\, dp') \iint pp'\, d\rho_x^\omega\, d\rho_y^\delta.$$

Fixing (x, y) in $\Omega_1 \times \Omega_2$, let us choose regular domains ω_n in $\mathscr{B}_1$, δ_n in $\mathscr{B}_2$ satisfying $\overline{\omega}_{n+1} \subset \omega_n$, $\overline{\delta}_{n+1} \subset \delta_n$, $\{x\} = \cap\,\omega_n$ and $\{y\} = \cap\,\delta_n$. An easy application of monotone convergence theorem shows that

$$\varphi(x, y) = \int p(x)p'(y)\mu\,(dp\, dp').$$

Conversely, let ν be any finite Radon measure on $\mathscr{E}_1 \times \mathscr{E}_2$ such that

$$\varphi(x, y) = \int p(x)p'(y)\nu\,(dp\, dp').$$

The measure ν can be considered as a Radon measure on $\mathscr{C}_0$ carried by the set $\mathscr{E}(\mathscr{C}_0)$. For any pair of regular domains ω and δ,

$$\begin{aligned}\iint \varphi\, d\rho_y^\delta\, d\rho_x^\omega &= \iint d\rho_x^\omega\, d\rho_y^\delta \int pp'\nu\,(dp\, dp')\\ &= \int \nu\,(dp\, dp') \iint pp'\, d\rho_x^\omega\, d\rho_y^\delta \quad \text{(Fubini)}.\end{aligned}$$

And now it is easy to deduce that $\varphi = \int pp'\nu\,(dp\, dp')$ vectorially. It follows that $\mu \equiv \nu$. The proof is complete.

Corollary. — *Every element φ in $\mathscr{C}$ can be written in the form $\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ where φ_1 is 2-harmonic, φ_2 (resp. φ_3) is harmonic on Ω_1 (resp. Ω_2) for every fixed y in Ω_2 (resp. x in Ω_1) and φ_4 is a potential in each variable for every fixed value of the other.*

Proof. — Let Δ_1^1 (resp. Δ_1^2) be the set of extreme harmonic functions contained in $\mathscr{E}_1$ (resp. $\mathscr{E}_2$). Let ν_1, ν_2, ν_3 and ν_4 be respectively the restriction of μ to the Borel sets $\Delta_1^1 \times \Delta_1^2$, $\Delta_1^1 \times (\mathscr{E}_2 \setminus \Delta_1^2)$, $(\mathscr{E}_1 \setminus \Delta_1^1) \times \Delta_1^2$ and $(\mathscr{E} \setminus \Delta_1^1) \times (\mathscr{E}_2 \setminus \Delta_1^2)$.

Let v_1, v_2, v_3 and v_4 be respectively the non-negative 2-superharmonic functions on $\Omega_1 \times \Omega_2$ belonging to $\mathscr{C}$ with the corresponding canonical measures ν_1, ν_2, ν_3 and ν_4. Then, it is clear that v_1, v_2, v_3 and v_4 satisfy the required conditions, completing the proof.

Remark. — The example in [4] shows that the class $\mathscr{C}$ is in general not equal to $(2S)^+$. We believe that the class $\mathscr{C}$ is in every case strictly contained in $(2S)^+$.

Before we proceed to the case of functions of more than two variables, we give below a characterization of the class $\mathscr{C}$ and a consequence. We note that there is an apparent asymmetry in our definition of the class $\mathscr{C}$. The following result shows that the class is indeed symmetric in the sense that a corresponding condition in the other variable is automatically satisfied. Also, we believe that the equivalent condition is more suited for application.

THEOREM 2.7. — *Let v be a positive doubly superharmonic function on $\Omega_1 \times \Omega_2$. Then, the following are equivalent.*

(1) *v belongs to the class $\mathscr{C}$.*

(2) (Cairoli) *for every pair of regular domains ω and δ, ω in $\mathscr{B}_1$ δ in $\mathscr{B}_2$ and any (x, y) in $\omega \times \delta$,*

$$\int v(\xi, y)\rho_x^\omega\,(d\xi) + \int v(x, \eta)\rho_y^\delta\,(d\eta) \leqslant v(x, y) + \iint v\, d\rho_x^\omega\, d\rho_y^\delta.$$

Proof. — Suppose v satisfies (2). Fix a regular domain ω in $\mathscr{B}_1$. Then, $v(x, y) - \int v(\xi, y)\rho_x^\omega\,(d\xi)$ is the difference of two lower semi-continuous functions on Ω_2. Also, the given inequality can be written in the form,

$$\int \left[v(x, y) - \int v(\xi, y)\rho_x^\omega\,(d\xi)\right]\rho_y^\delta\,(d\eta) \leqslant v(x, y) - \int v(\xi, y)\rho_x^\omega\,(d\xi).$$

This is true for any regular domain δ in the base $\mathscr{B}_2$ and any y in δ. It follows that $y \longmapsto v(x, y) - \int v(\xi, y)\rho_x^\omega\,(d\xi)$ is positive and an $S_{\mathscr{B}_2}$-function on Ω_2. However, it is the difference of two positive superharmonic functions on Ω_2 and is hence itself superharmonic on Ω_2. Let ν_x (resp.

$\nu(\omega, x))$ be the canonical measure on $\mathscr{E}_2$ corresponding to the superharmonic functions $v(x, .)$ $\left(\text{resp.} \int v(\xi, y)\rho_x^\omega(d\xi)\right)$. It follows that $\nu(\omega, x) \leqslant \nu_x$ for all x in ω. Now, as in the earlier situations it is easy to show that $x \longmapsto \nu_x$ depends superharmonically on x. This shows that v belongs to the class $\mathscr{C}$.

Conversely, let $\mathscr{C}_0'$ be the set of v in the compact base $\mathscr{C}_0$ which satisfy the condition (2). To prove the theorem it is enough to show that $\mathscr{C}_0' = \mathscr{C}_0$. Clearly, $\mathscr{C}_0'$ is a convex set. Suppose w is in $\mathscr{C}_0$ and is of the form $w_1 w_2$ where w_1 (resp. w_2) is positive and superharmonic on Ω_1 (resp. Ω_2). Then, for any pair of regular domains ω and δ,

$$\left[-\int w_1(\xi)\rho_x^\omega(d\xi) + w_1(x)\right]\left[w_2(y) - \int w_2(\eta)\rho_y^\delta(d\eta)\right] \geqslant 0$$

i.e.,

$$w_1(x)\int w_2(\eta)\rho_y^\delta(d\eta) + w_2(y)\int w_1(\xi)\rho_x^\omega(d\xi) \leqslant w_1(x)w_2(y) + \int w_1\, d\rho_x^\omega \int w_2\, d\rho_y^\delta.$$

This implies that $w_1 w_2$ is in $\mathscr{C}_0'$. In particular all the extreme elements of the base $\mathscr{C}_0$ belong to $\mathscr{C}_0'$. Hence, it is sufficient to prove that $\mathscr{C}_0'$ is closed in $\mathscr{C}_0$. Let v_n belong to $\mathscr{C}_0$ and converge to v in $\mathscr{C}_0$. We have to show that v belongs to $\mathscr{C}_0'$. Let ω and δ be fixed regular domains belonging respectively to $\mathscr{B}_1$ and $\mathscr{B}_2$. Let (x_0, y_0) be fixed in $\omega \times \delta$. Choose regular domains δ_n, ω_n for $n = 1$ to ∞ such that (1) $\omega_n \in \mathscr{B}_1$, $\overline{\omega}_{n+1} \subset \omega_n$ and $\cap \omega_n = \{x_0\}$ and (2) $\delta_n \in \mathscr{B}_2$, $\overline{\delta}_{n+1} \subset \delta_n$ and $\cap \delta_n = \{y_0\}$. We may assume without loss of generality that $\overline{\omega}_1 \subset \omega$ and $\overline{\delta}_1 \subset \delta$. We have, for every m, every (x, y) in $\omega \times \delta$,

$$\int v_m(\xi, y)\rho_x^\omega(d\xi) + \int v_m(x, \eta)\rho_y^\delta(d\eta) \leqslant v_m(x, y) + \iint v_m\, d\rho_x^\omega\, d\rho_y^\delta.$$

Integrating this inequality we obtain,

$$\iint \rho_{x_0}^{\omega_n}(dx)\rho_{y_0}^{\delta_n}(dy)\left[\int v_m(\xi, y)\rho_x^\omega(d\xi) + \int v_m(x, \eta)\rho_y^\delta(d\eta)\right] \leqslant \iint v_m\, d\rho_{x_0}^{\omega_n}\, d\rho_{y_0}^{\delta_n} + \iint v_m\, d\rho_{x_0}^\omega\, d\rho_{y_0}^\delta.$$

This inequality reduces to

$$\iint v_m \, d\rho^{\omega}_{x_0} \, d\rho^{\delta_n}_{y_0} + \iint v_m \, d\rho^{\delta}_{y_0} \, d\rho^{\omega_n}_{x_0} \leqslant \iint v_m \, d\rho^{\omega_n}_{x_0} \, d\rho^{\delta_n}_{y_0} + \iint v_m \, d\rho^{\omega}_{x_0} \, d\rho^{\delta}_{y_0}.$$

Let us fix the n and let m tend to ∞. Since v_m converges to v we conclude that,

$$\iint v(\xi, \eta)\rho^{\omega}_{x_0}(d\xi)\rho^{\delta_n}_{y_0}(d\eta) + \iint v(\xi, \eta)\rho^{\omega_n}_{x_0}(d\xi)\rho^{\delta}_{y_0}(d\eta) \leqslant \iint v \, d\rho^{\omega_n}_{x_0} \, d\rho^{\delta_n}_{y_0} + \iint v \, d\rho^{\omega}_{x_0} \, d\rho^{\delta}_{y_0}.$$

Now, let n tend to infinity and in view of the superharmonicity of the functions involved, we get

$$\int v(\xi, y_0)\rho^{\omega}_{x_0}(d\xi) + \int v(x_0, \eta)\rho^{\delta}_{y_0}(d\eta) \leqslant v(x_0, y_0) + \iint v \, d\rho^{\omega}_{x_0} \, d\rho^{\delta}_{y_0}.$$

It is now easy to deduce that v belongs to $\mathscr{C}'_0$. The proof is complete.

COROLLARY. — *Every positive* 2*-superharmonic function* v *on* $\Omega_1 \times \Omega_2$ *such that* v *is harmonic on* Ω_1 (resp. Ω_2) *for every fixed* y *in* Ω_2 (resp. Ω_1) *belongs to* $\mathscr{C}$. *In particular, every positive* 2*-harmonic function on* $\Omega_1 \times \Omega_2$ *is in the class* $\mathscr{C}$.

Proof. — To be precise let us assume that v is harmonic in the first variable. Let ω and δ be any pair of regular domains in $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively. Then,

$$\int v(\xi, y)\rho^{\omega}_{x}(d\xi) + \int v(x, \eta)\rho^{\delta}_{y}(d\eta) = v(x, y) + \iint v(\xi, \eta)\rho^{\omega}_{x}(d\xi)\rho^{\delta}_{y}(d\eta)$$

and this completes the proof.

Remark 1. — It can be shown that $\mathscr{C}$ is precisely the closure of $S_1^+ \otimes S_2^+$ in the projective topology.

As a consequence of the above characterization of the class $\mathscr{C}$ we get the following interesting property of $\mathscr{C}$.

THEOREM 2.8. — *Let* φ_n *be elements of* $\mathcal{C}$ *such that* $\{\varphi_n\}$ *decreases pointwise. Let* $\varphi = \lim_{n\to\infty} \varphi_n$ *pointwise. Then* $\hat{\varphi}$ *belongs to* $\mathcal{C}$.

Proof. — It is clear that $\varphi \geqslant 0$ and a Borel function on $\Omega_1 \times \Omega_2$. It follows by monotone convergence theorem that

$$\int \varphi(\xi, y)\rho_x^\omega\,(d\xi) + \int \varphi(x, \eta)\rho_y^\delta\,(d\eta) \leqslant \varphi(x, y) + \iint \varphi\, d\rho_x^\omega\, d\rho_y^\delta,$$

for every pair of regular domains ω in $\mathcal{B}_1$ and δ in $\mathcal{B}_2$. It is known [7] that $\hat{\varphi}$ is a 2-superharmonic function and $\hat{\varphi}(x, y)$ is the limit of $\iint \varphi\, d\rho_x^\omega\, d\rho_y^\delta$ as the regular domains ω shrink to x and the regular domains δ shrink to y, as in the case of functions of a single variable. Now, since $\hat{\varphi} \leqslant \varphi$, we have,

$$\int \hat{\varphi}(\xi, y)\rho_x^\omega\,(d\xi) + \int \hat{\varphi}(x, \eta)\rho_y^\delta\,(d\eta) \leqslant \varphi(x, y) + \iint \varphi\, d\rho_x^\omega\, d\rho_y^\delta.$$

This is true whatever be (x, y) in $\omega \times \delta$. Let us fix (x_0, y_0) in $\omega \times \delta$ and take the lim inf as (x, y) in $\omega \times \delta$ tends to (x_0, y_0). Since the left side is clearly the sum of two 2-superharmonic functions, we get

$$\int \hat{\varphi}(\xi, y_0)\rho_{x_0}^\omega\,(d\xi) + \int \hat{\varphi}(x_0, \eta)\rho_{y_0}^\delta\,(d\eta) \leqslant \liminf_{(x, y)\to(x_0, y_0)} \left[\varphi(x, y) + \iint \varphi\, d\rho_x^\omega\, d\rho_y^\delta\right].$$

But on the right side $\iint \varphi\, d\rho_x^\omega\, d\rho_y^\delta$ being a doubly harmonic function on $\omega \times \delta$, the limit of $\iint \varphi\, d\rho_x^\omega\, d\rho_y^\delta$ as (x, y) converges to (x_0, y_0) exists and is equal to $\iint \varphi\, d\rho_{x_0}^\omega\, d\rho_{y_0}^\delta$. Hence, the right side of the last inequality is equal to

$$\hat{\varphi}(x_0, y_0) + \iint \varphi\, d\rho_{x_0}^\omega\, d\rho_{y_0}^\delta.$$

This shows that φ belongs to $\mathcal{C}$, proving the theorem.

Remark 1. — It can be shown that if φ and ψ are elements of $\mathcal{C}$ then the pointwise lower envelope of all « $\succ$ » majorants of φ and ψ is the precisely the $\succ$ supremum of φ and ψ. Then, it is possible to find a pointwise decreasing sequence of elements u_n in $\mathscr{C}$ such that $u_n \succ v$ and $u_n \succ w$ and if $u = \lim_{n \to \infty} u_n$ then, $\hat{u}$ is precisely the above supremum.

Remark 2. — It is possible to show as above that every upper bounded (resp. lower bounded) family of elements of $\mathscr{C}$ has the least upper bound (resp. greatest lower bound).

Remark 3. — It can be shown as in the proof of Theorem 1.2, with the aid of the first remark above that $(v, w) \longmapsto \sup(v, w)$ and $(v, w) \longmapsto \inf(v, w)$, the supremum and infimum in the order $\succ$, are Borel mappings of $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$.

LEMMA 2.2. — *Let v be a positive doubly superharmonic function on $\Omega_1 \times \Omega_2$. Let for every x in Ω_1, $u(x, y)$ be the greatest harmonic minorant of the superharmonic function $y \longmapsto v(x, y)$ on Ω_2. Then, u is a Borel function on $\Omega_1 \times \Omega_2$ such that for every y in Ω_2, $x \longmapsto u(x, y)$ is nearly superharmonic on Ω_1.*

Proof. — Let U be an open subset of Ω_2 and let y in Ω_2 be fixed. Let λ be the swept out measure corresponding to sweeping out on $\complement U$ and the Dirac measure ε_y [8, Théorème 10.1], viz., for every w in S_2^+,

$$\hat{R}_w^{\complement U}(y) = \int w(\eta)\lambda(d\eta).$$

In particular, for every x in Ω_1,

$$\hat{R}_{v(x,\cdot)}^{\complement U}(y) = \int v(x, \eta)\lambda(d\eta).$$

It follows that as x_m converges to x_0 in Ω_1,

$$\begin{aligned}\liminf_{x_m \to x_0} \hat{R}_{v(x_m,\cdot)}^{\complement U}(y) &= \liminf_{x_m \to x_0} \int v(x_m, \eta)\lambda(d\eta) \\ &\geqslant \int v(x_0, \eta)\lambda(d\eta) \quad \text{(Fatou's lemma)} \\ &= \hat{R}_{v(x_0,\cdot)}^{\complement U}(y).\end{aligned}$$

Hence, $x \longmapsto \hat{R}_{v(x,\cdot)}^{\complement U}(y)$ is a lower semi-continuous function on Ω_1. Now, let ω be any regular domain of Ω_1 and x in ω. Then,

$$\begin{aligned}\int \hat{R}^{\complement U}_{v(\xi,\,.)}(y)\rho_x^{\omega}(d\xi) &= \int \rho_x^{\omega}(d\xi)\int v(\xi,\eta)\lambda(d\eta)\\ &= \int \lambda(d\eta)\int v(\xi,\eta)\rho_x^{\omega}(d\xi)\\ &\leqslant \int v(x,\eta)\lambda(d\eta)\\ &= \hat{R}^{\complement U}_{v(x,\,.)}(y).\end{aligned}$$

Hence, $x \longmapsto \hat{R}^{\complement U}_{v(x,\,.)}(y)$ is superharmonic on Ω_1. This is true for every y in Ω_2.

Now, let $U_1, U_2, \ldots, U_n, \ldots$ be a sequence of relatively compact open sets of Ω_2 forming a covering of Ω_2 such that for every n, $\overline{U}_n$ is contained in U_{n+1}. Then, $\hat{R}^{\complement U_n}_{v(x,\,.)}$ decreases with n and the limit as n tends to infinity is precisely the greatest harmonic minorant of the superharmonic function $y \longmapsto v(x, y)$. Hence, we conclude that u is a Borel function on $\Omega_1 \times \Omega_2$ and that for every y in Ω_2, $x \longmapsto u(x, y)$ is a nearly superharmonic function on Ω_1. The lemma is proved.

Remark. — It can be shown that $\hat{u}$ the lower semi-continuous regularization of u on $\Omega_1 \times \Omega_2$ is 2-superharmonic on $\Omega_1 \times \Omega_2$, is harmonic in y and hence belongs to the class $\mathscr{C}$. Also then it is seen that for every y, $\hat{u}(x, y) = u(x, y)$, $\mathscr{B}_1$-nearly everywhere on Ω_1 and this exceptional set is independent of y in Ω_2.

THEOREM 2.9. — *Let v be a positive 2-superharmonic function on $\Omega_1 \times \Omega_2$. Then the following are equivalent:*

(1) *v belongs to the class $\mathscr{C}$.*

(2) *Let Δ_2' denote the set of harmonic functions in $\mathscr{E}_2$. For every compact set K contained in $\mathscr{E}_2 \setminus \Delta_2'$ the function $x \longmapsto \nu_x(K)$ is superharmonic on Ω_1, where ν_x is the canonical measure on $\mathscr{E}_2$ representing $v(x, .)$.*

Proof. — It is obvious that if v belongs to $\mathscr{C}$ then the condition (2) is fulfilled. Conversely, let v be an element of $(2S)^+$ satisfying the condition (2). Let B be any Borel subset of $\mathscr{E}_2 \setminus \Delta_2'$. Since ν_x is a Radon measure on $\mathscr{E}_2$, we get that $\nu_x(B)$ is the increasing directed limit of the numbers $\nu_x(K)$ for all compact sets K contained in B. It follows that $x \longmapsto \nu_x(B)$ is a superharmonic function on Ω_1. Hence, we deduce by standard measure theoretic arguments that $P(x, y) = \int_{\mathscr{E}_2 \setminus \Delta_2'} p(y) \nu_x (dp)$ is a positive 2-superharmonic function on $\Omega_1 \times \Omega_2$ and further it is clear that P belongs to the class $\mathscr{C}$. Now,

$$\begin{aligned} \varphi(x, y) &= \int_{\mathscr{E}_2 \setminus \Delta_2'} p(y) \nu_x (dp) + \int_{\Delta_2'} p(y) \nu_x (dp) \\ &= P(x, y) + u(x, y). \end{aligned}$$

But, clearly $u(x, y)$ is precisely the greatest harmonic minorant of $y \longmapsto \varphi(x, y)$ on Ω_2. By the earlier lemma, we know that $x \rightarrow u(x, y)$ is a nearly superharmonic function on Ω_1. Further, $u(x, y)$ is the difference of two positive superharmonic functions $\varphi(x, y)$ and $P(x, y)$ for every y in Ω_2. Hence, $x \longmapsto u(x, y)$ is, in fact, superharmonic on Ω_1. But since $y \longmapsto u(x, y)$ is harmonic, it follows by the corollary to Theorem 2.7, that u is an element of $\mathscr{C}$. Since $\mathscr{C}$ is a convex cone it follows that φ is in $\mathscr{C}$, completing the proof.

3. Multiply superharmonic functions in the class $\mathcal{C}$ and the integral representation.

We shall give now the extensions of the proofs to get the integral representation of the elements of $\mathscr{C}$ in the case of functions of more than two variables. Let $\Omega_1, \ldots, \Omega_n$ be harmonic spaces of Brelot and let S_k^+ be the cone of superharmonic functions $\geqslant 0$ on Ω_k for $k = 1$ to n. Let δ_0^j be a fixed regular domain belonging to a base $\mathscr{B}_j$ of completely determining domains of the space Ω_j. Let for $j = 1$ to n,

$$\Lambda_j = \left\{ \varphi \in S_j^+ : \int \varphi \, d\rho_{y_0^j}^{\delta_0^j} = 1 \right\}$$

for a fixed element y_0^j in δ_0^j; and let $\mathscr{E}_j$ be the set of extreme elements of this compact base Λ_j.

We shall define the class $\mathscr{C}$ of n-superharmonic functions by induction on n. Let us assume that the class is defined for all integers up to n. Let us further assume that for all $k = 3$ to $n - 1$, a k-superharmonic function belonging to the class $\mathscr{C}$ has a unique integral representation similar to the case of 2-superharmonic functions as shown earlier. We shall further assume as part of the induction hypothesis that the class $\mathscr{C}$ in k-variables with the corresponding topology is locally compact and metrizable.

Definition 3.1. — *A non-negative n-superharmonic function* v *on* $\Omega_1 \times \cdots \times \Omega_n$ *is said to belong to the class* $\mathscr{C}$ *if* (1) *for every* x *in* Ω_1, $v(x, .)$ *is a function belonging to the class* $\mathscr{C}$ *as a* $(n - 1)$*-superharmonic function on*

$$\Omega_2 \times \cdots \times \Omega_n$$

and (2) *the canonical measure* $\lambda(x)$ *on* $\mathscr{E}_2 \times \cdots \times \mathscr{E}_n$ *corresponding to the function* $v(x, .)$ *depends superharmonically on* x *in* Ω_1.

Let us now recall [4] that the space $(n\mathrm{S})^+$ of non-negative n-superharmonic functions on $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ can be provided with a topology τ such that this space is (1) metrizable and (2) is locally compact. The topology τ is the weakest such that the mappings $v \to \int v \, d\rho_{\mathbf{x}}^{\boldsymbol{\omega}}$ are continuous where $\rho_{\mathbf{x}}^{\boldsymbol{\omega}}$ denotes the product measure $\rho_{x_1}^{\omega_1} \otimes \cdots \otimes \rho_{x_n}^{\omega_n}$ where ω_j belongs to $\mathscr{B}_j$ and x_j is an element of ω_j. We now have the following:

Theorem 3.1. — *The set* $\mathscr{C}$ *with the induced* τ *topology is locally compact and metrizable.*

Proof. — $(\mathscr{C}, \tau)$ being the subspace of a metrizable space is clearly metrizable and to prove the local compactness, it is enough to show that $\mathscr{C}$ is a closed subspace of $(n\mathrm{S})^+$. Suppose v_m belongs to $\mathscr{C}$ and $\{v_m\}$ converges in τ to an element v in $(n\mathrm{S})^+$. We shall show that v belongs to $\mathscr{C}$.

Let ω be any regular domain of Ω_1 and x an element of ω. Consider the function $y \longmapsto \int v_m(\xi, y) \rho_x^{\omega} (d\xi)$. It is clear that this function $v_m^{\omega}(x, .)$ is $(n - 1)$-superharmonic and $\geqslant 0$ on $\Omega_2 \times \cdots \times \Omega_n$. But we assert that this is in $\mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$. This is seen as follows. As in the previous cases (cf. Lemma 1.3), it is easy to see that $\vec{\varphi} : \Omega_1 \to \mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$, $\vec{\varphi}(x) = \wp_m(x, .)$ is a Borel function. Since the right side is a polish space, $\vec{\varphi}$ can be approximated uniformly on Ω_1 by a sequence of functions

$$\vec{s}_n : \Omega_1 \to \mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$$

such that $\vec{s}_n$ takes at the most countably many values and each of them on a Borel subset of Ω_1 [9, Ch. 1]. Hence to show that $\int \wp_m(\xi, .)\rho_x^\omega(d\xi)$ belongs to $\mathscr{C}$, it is enough to prove that $\int \vec{s}_n(\xi)\rho_x^\omega(d\xi)$ belongs to $\mathscr{C}$. But this latter integral is simply the sum of a convergent series of elements in $\mathscr{C}$ and hence belongs to $\mathscr{C}$. The same is true for all m. Also, as in Lemma 2.1, it can be shown that $x \longmapsto \nu_x^m(\mathrm{B})$ is Borel on Ω_1 for all Borel sets B of $\mathscr{E}_2 \times \cdots \times \mathscr{E}_n$.

Now, it is clear that τ-convergence of $\wp_m$ to $\wp$ implies that $\wp_m^\omega(x, .)$ converges as a $(n-1)$-superharmonic function to $\wp^\omega(x, .)$. However, $\mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$ is closed in $((n-1)\mathrm{S})^+$ by the induction hypothesis and $\wp_m^\omega(x, .)$ is in this set for every m and we conclude that $\wp^\omega(x, .)$ is in $\mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$.

Let us now fix x in Ω_1. Let ω_k be regular domains of Ω_1 such that $\overline{\omega}_{k+1} \subset \omega_k$ for every integer k and $\cap \omega_k = \{x\}$. Since $\wp_m$, for each m, belongs to $\mathscr{C}(\Omega_1 \times \cdots \times \Omega_n)$,

$$\wp_m^{\omega_{k+1}}(x, .) = \wp_m^{\omega_k}(x, .) + w_m^k$$

where w_m^k is an element of $\mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$, in fact the element with the canonical measure $\nu_m^{\omega_{k+1}} - \nu_m^{\omega_k}$ on $\mathscr{E}_2 \times \cdots \times \mathscr{E}_n$. It is clear that for fixed k, w_m^k converges as m tends to ∞ and the limit u_k necessarily belongs to $\mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$. It follows

$$\wp^{\omega_{k+1}}(x, .) = \wp^{\omega_k}(x, .) + u_k.$$

Hence the corresponding canonical measures on

$$\mathscr{E}_2 \times \cdots \times \mathscr{E}_n$$

satisfy the equation

$$\nu(\omega_{k+1}, x) = \nu(\omega_k, x) + a \text{ positive Radon measure.}$$

In particular, $\nu(\omega_{k+1}, x) \geqslant \nu(\omega_k, x)$. This is true for every k. Let $\nu(x)$ be the limit of this increasing sequence of measures; for every non-negative Borel function f on $\mathcal{E}_2 \times \cdots \times \mathcal{E}_n$, $\int f\, d\nu(x)$ is the increasing limit of $\int f\, d\nu(\omega_k, x)$. Now, since v is n-superharmonic, we get that

$$\lim_{k \to \infty} \int v(\xi, y)\rho_x^{\omega_k}(d\xi) = v(x, y),$$

and the limit is an increasing limit. Clearly, the limit of an increasing sequence of elements in $\mathcal{C}(\Omega_2 \times \cdots \times \Omega_n)$ also belongs to the same class and we conclude that $v(x, .)$ belongs to $\mathcal{C}(\Omega_2 \times \cdots \times \Omega_n)$ for every x in Ω_1. Now, for every y in $\Omega_2 \times \cdots \times \Omega_n$,

$$\begin{aligned} v(x, y) &= \lim_{k \to \infty} \int v(\xi, y)\rho_x^{\omega_k}(d\xi) \\ &= \lim_{k \to \infty} \int \mathbf{P}(y)\nu(\omega_k, x)\,(d\mathbf{P}), \\ &= \int \mathbf{P}(y)\nu(x)\,(d\mathbf{P}), \end{aligned}$$

since the measures $\nu(\omega_k, x)$ increase to $\nu(x)$. It follows by the uniqueness of the measure on $\mathcal{E}_2 \times \cdots \times \mathcal{E}_n$ representing the elements of $\mathcal{C}(\Omega_2 \times \cdots \times \Omega_n)$ that $\nu(x)$, for every x in Ω_1, is precisely the canonical measure corresponding to the function $v(x, .)$. Now it is easy to show that $x \longmapsto \nu_x(\mathrm{B})$ is lower semi-continuous in x, for every Borel set of $\mathcal{E}_2 \times \cdots \times \mathcal{E}_n$. This shows that $x \longmapsto \nu_x$ depends superharmonically, concluding the proof.

We conclude as before,

Corollary. — *$\mathcal{C}$ has a compact, metrizable base.*

For instance, we can take for such a base

$$\mathcal{C}_0 = \mathcal{C} \cap \left\{ v \in (n\mathrm{S})^+ : \int v \, d\rho_{\mathbf{x}_0}^{\boldsymbol{\omega}} = 1 \right\}$$

Let us now consider the vector space $\mathcal{C}_1$ generated by $\mathcal{C} = \mathcal{C}(\Omega_1 \times \cdots \times \Omega_n)$. It is clear that with the τ-topology defined above $\mathcal{C}_1$ is a locally convex topological vector space and $\mathcal{C}$ is a convex cone with vertex at the origin generating $\mathcal{C}_1$. Consider the partial order « $\succ$ » introduced by $\mathcal{C}$; viz., v and w in $\mathcal{C}_1$ satisfy $v \succ w$ if $v - w$ is an element of $\mathcal{C}$. It is clear that the order is compatible with the topology and that $\mathscr{C}$ is closed. We now have the following sequence of three results establishing the unique integral representation. The proofs are very similar to the corresponding results in the case of two variables (cf. Theorems 2.3, 2.5, 2.6). We shall not give the proofs.

THEOREM 3.2. — *$\mathscr{C}(\Omega_1 \times \cdots \times \Omega_n)$ is a lattice for the order « $\succ$ ».*

THEOREM 3.3. — *An element v in $\mathscr{C}(\Omega_1 \times \cdots \times \Omega_n)$ is an extremal generator if and only if there are extreme superharmonic functions p_k in S_k^+ such that $v = p_1 p_2 \ldots p_n$.*

THEOREM 3.4. — *Corresponding to every element v in $\mathscr{C}(\Omega_1 \times \cdots \times \Omega_n)$ there is an unique finite Radon measure ν on $\mathscr{E}_1 \times \cdots \times \mathscr{E}_n$ such that*

$$v(\boldsymbol{x}) = \int \boldsymbol{p}(\boldsymbol{x})\nu\,(d\boldsymbol{p}),$$

for every $\boldsymbol{x}$ in $\Omega_1 \times \cdots \times \Omega_n$.

Finally, we give below equivalent characterization of the class $\mathscr{C}$. Let ω_i be any regular domain in Ω_i and x_i any element of ω_i. Let us introduce the operator $T(\omega_i, x_i)$ acting on any measurable positive or real valued integrable function f on Ω_i by setting

$$T(\omega_i, x_i)(f) = f(x_i) - \int f(\xi)\rho_{x_i}^{\omega_i}(d\xi).$$

Then, we have,

THEOREM 3.5. — *Let v be a positive n-superharmonic function on the product space $\Omega_1 \times \cdots \times \Omega_n$. Then the following are equivalent.*

(1) *v is an element of $\mathscr{C}(\Omega_1 \times \ldots \times \Omega_n)$.*

(2) *for every $\omega_1, \ldots, \omega_n$ regular domains belonging to $\mathscr{B}_1, \ldots, \mathscr{B}_n$ respectively, $(x_1, \ldots, x_n)$ in $\omega_1 \times \cdots \times \omega_n$, $2 \leqslant k \leqslant n$, and any permutation π of the integers 1 through k, the following inequality is verified:*

$$T(\omega_{\pi(1)}, x_{\pi(1)}) \circ T(\omega_{\pi(2)}, x_{\pi(2)}) \circ \cdots \circ T(\omega_{\pi(k)}, x_{\pi(k)})(v) \geqslant 0$$

when the other variables are kept fixed.

Proof. — Consider the set $\mathscr{C}_0'$ of all elements of $\mathscr{C}_0$ for which condition (2) is verified. Clearly this set $\mathscr{C}_0'$ is a convex set and it contains all the extreme elements of $\mathscr{C}_0$, i.e. elements of the form $p_1 p_2 \ldots p_n$. Now as in the proof of Theorem 2.7, it can be shown that $\mathscr{C}_0'$ is closed and is hence compact. It follows that $\mathscr{C}_0 = \mathscr{C}_0'$. Hence every φ in $\mathscr{C}$ satisfies the condition (2).

The converse is proved by induction. The result is true for $n = 2$. (Cf. Theorem 2.7). Let us assume that it is true for all integers upto and including $(n - 1)$. We shall show that it is true for ωf. Accordingly let φ be a positive multiply superharmonic function on $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ such that (2) is verified. It follows that for any x in Ω_1, $y \longmapsto \varphi(x, y)$ is an element of $\mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$. Now, for any regular domain ω of Ω_1 and x in ω, as in the proof of Theorem 3.1, it can be shown that $\int \varphi(\xi, y) \rho_x^\omega (d\xi)$ is an element of $\mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$. Let π be any permutation of the integers $2, 3, 4, \ldots, j$, $j \leqslant n$. Then,

$$T(\omega_{\pi(2)}, x_{\pi(2)}) \circ T(\omega_{\pi(3)}, x_{\pi(3)}) \circ \cdots \circ T(\omega_{\pi(j)}, x_{\pi(j)}) \left[\varphi - \int \varphi(\xi, .) \rho_x^\omega (d\xi)\right] = T(\omega_{\pi(2)}, x_{\pi(2)}) \circ \cdots \circ T(\omega_{\pi(j)}, x_{\pi(j)}). T(\omega, x)[\varphi]$$

and the right side is $\geqslant 0$ by assumption. It follows that $\varphi - \int \varphi(\xi, .) \rho_x^\omega (d\xi)$ belongs to $\mathscr{C}(\Omega_2 \times \cdots \times \Omega_n)$. We can now conclude that $x \longmapsto \nu_x$ where ν_x is the canonical measure on $\mathscr{E}_2 \times \cdots \times \mathscr{E}_n$ corresponding to $\varphi(x, .)$ depends superharmonically on x in Ω_1. Hence φ belongs to $\mathscr{C}(\Omega_1 + \cdots \times \Omega_n)$ completing the proof.

Finally we would like to make a remark concerning the assumption of the existence of a base of completely determining domains. As can be observed easily most of the results of this paper do not make use of the existence of such domains. All the results except Theorem 2.4 and Theorem 3.1 are true in general. Naturally the T-topology of M^me^ Hervé [8] replaces the Cartan-Brelot topology on S^+ and very minor modifications are needed to see that the proofs carry over to the general case. Without the completely determining domains, we have to give a different topology on $\mathscr{C}_1$ and we have to prove the local compactness and metrizability of $\mathscr{C}$. This can be done as follows. For any continuous function of the form $f.g$ where $f \in C(\overline{\Omega}_1)$, $g \in C(\overline{\Omega}_2)$ [8], we could define

$$\nu_{fg} = [\nu(x, .)_g]_f(x, y)$$

and extend the definition by linearity to ν_φ for $\varphi = \Sigma\alpha_i f_i g_i$. The topology on $\mathscr{C}$ would be the coarsest one for which $\nu \longmapsto \nu_\varphi(x, y)$ is continuous for all φ and (x, y) not in the support of φ. Using heavily the results in [8] and with proofs similar to that in [4] the required local compactness can be proved. The metrizability of $\mathscr{C}$ is really easy to verify. In this paper, we preferred to assume the existence of completely determining domains and use the results of [4]. Lastly, we remark that the topology suggested above is only for the class $\mathscr{C}$ or $\mathscr{C} - \mathscr{C}$ and could not be carried over to all positive multiply superharmonic functions.

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Manuscrit reçu le 19 octobre 1972.

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