ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 23, nº 3 (1973), p. 155-185

<http://www.numdam.org/item?id=AIF_1973__23_3_155_0>

Ann. Inst. Fourier, Grenoble
23,3 (1973), 155-185

HOLOMORPHIC FUNCTIONS ON LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

II. Pseudo convex domains (*) (**)

by Sean DINEEN

In this article we investigate the problem of when pseudo-convex domains are domains of holomorphy. This problem was originally posed for infinite dimensional locally convex spaces by Bremmerman ([4], [5], [6], [7], [8]). The problem has been solved in the affirmative for tube domains [5], for open subsets of C^n (n = countable infinity) [18], for Riemann domains over C^n [27] and for open subsets of a Banach space E with a basis whose intersection with each finite dimensional subspace of E is Runge [16]. We use the method of Hirschowitz [18] to circumvent the problem of having no continuous norm and show that if $U \subset E$ is pseudo convex and p is a continuous semi norm on E such that $\{y , p(y) < \delta\} \subset U$ then $U + \{y , p(y) = 0\} = U$. In this way the characterisation problem on U can be transferred to a space on which there exists a continuous norm. By this method we are able to prove generalisations of the Cartan-Thullen-Oka-Norguet-Bremmerman theorem in a variety of cases which include the following :

a) U open in $\prod_{n=1}^{\infty} E_n$, E_n a Frechet space with a basis for each n and $U \cap F$ Runge for each finite dimensional subspace F of E.

b) U open in $\sum_{i=1}^{\infty} E_i$ where each E_i is a Frechet space with a basis and $U \cap F$ is Runge for each finite dimensional subspace F of E.

(¹) A number of the results contained in this paper were announced in C.R. Acad. Sc., Paris, t. 274, 544-547 (1972).

(²) Part I has appeared in Ann. Inst. Fourier, t. 23, 1 (1973).

c) U an open subset of a nuclear space and $U \cap F$ Runge for each finite dimensional subspace F of E.

It has been shown in [19] that the theorem does not hold in its full generality for all locally convex topological vector spaces. Although complete results on this problem are not yet available an examination of the results obtained here suggest that the following properties of the locally convex space E will have some bearing on the final solution :

1) Countability conditions on E (e.g. is E separable, Lindelof).

2) Geometric position of $E/p^{-1}(0)$ in E (p is a continuous semi-norm on E) (e.g. is $E/p^{-1}(0)$ complemented topologically in E ?)

3) Geometric properties of $(E/p^{-1}(0), p)$ (p is a continuous semi-norm on E) (e.g. does $(E/p^{-1}(0), p)$ have a basis ?).

Unless otherwise stated our notation is the same as [15]. For background information on pseudo-convexity and plurisubharmonic functions we refer to [18], [9], [24], [26], [31], [33].

I thank Phillip Boland and Phillip Noverraz for many helpful conversations and correspondance.

1. *The Cartan-Thullen-Oka-Bremmerman-Norguet theorem.*

We consider the following properties on an open subset of a locally convex space E.

(H1) U *is a domain of existence of a holomorphic function (i.e. there exists* $f \in \mathcal{H}(U)$ *such that there is no* U_1, U_2 *connected open in* E, $\phi \neq U_2 \subset U_1 \cap U$, $U_1 \not\subset U$, *and* $f_1 \in \mathcal{H}(U_1)$ *such that*

$$f|_{U_2} = f_1|_{U_2}).$$

(H2) *For each sequence,* $(\xi_n)_{n=1}^{\infty}$, *of elements of* U, $\xi_n \to \xi$, $\xi \in \delta U$ (δU *is the boundary of* U *in* E) *there exists* $f \in \mathcal{H}(U)$ *such that* $\sup_n |f(\xi_n)| = \infty$.

(H3) U *is a domain of holomorphy (i.e. there exists no* U_1, U_2 *open connected in* E *such that* $\phi \neq U_2 \subset U_1 \cap U$, $U_1 \not\subset U$ *and for each* $f \in \mathcal{H}(U)$ *there exists* $f_1 \in \mathcal{H}(U_1)$ *such that* $f_1|_{U_2} = f|_{U_2}$.

(H4) U *is holomorphically convex (i.e. the holomorphic hull* (1).

(1) $\hat{A}_{\mathcal{H}(U)}$ will denote the holomorphic hull of A with respect to $\mathcal{H}(U)$.

of each compact subset of U *is a precompact subset of* U(1).

(H5) U *is holomorphically convex and if* K *is a compact subset of* U *and* V *is a balanced open neighbourhood of* 0 *in* E *such that* $K + V \subset U$, *then* $\hat{K}_{\mathscr{H}(U)} + V \subset U$.

(H6) U *is a plurisubharmonically convex (i.e. the plurisubharmonic hull of each compact subset of* U *is a precompact subset of* U)

(H7) U *is pseudo convex (i.e.* $U \cap F$ *is pseudo convex for each finite dimensional subspace* F *of* E).

(H8) U *is polynomially convex (i.e. the polynomial hull of each compact subset* K *of* U, $\hat{K}_{\mathscr{P}(E)}$, *is a precompact subset of* U).

(H9) U *is finitely polynomially convex if* $U \cap F$ *is polynomially convex for each finite dimensional subspace* F *of* E).

It shall be necessary to consider locally convex spaces which may not be Hausdorf, however, if F is the closure of 0 in E and π denotes the quotient mapping of E onto $\frac{E}{F}$ then (H_i) $(i = 1, \ldots, 8)$ is true for an open subset U of E if and only if the same is true of $\pi(U)$(2). Thus we restrict ourselves to Hausdorff locally convex spaces in the proofs but use the fact that the results proved are valid for non-Hausdorff spaces.

If E is a locally convex space such that each open subset of E which satisfies H(7) (resp H(9)) also satisfies H(i) ($i = 1,2,3,4,5$) (resp satisfies H(i), $i = 1,2,3,4,5,6,8$) then we say E is a CTONB(i) (resp CTONBR(i)) space. The following proposition can be combined with CTONB(i) (resp CTONBR(i)) spaces to prove the usual Cartan-Thullen-Oka-Norguet-Bremmerman (. . . . . Runge) theorem.

PROPOSITION 1.1. – *For any* l.c.s. E *and any open subset* U *of* E *we have the following :*

(1) K is a precompact subset of U if it is a precompact subset of E and there exists p a continuous semi-norm on E such that $\inf_{\substack{x \in K \\ y \in \complement U}} p(x - y) > 0$.

(2) This can easily be proved by using the techniques we develop in the remainder of this section.

$$(\mathrm{H1}) \Rightarrow (\mathrm{H2}) \Rightarrow (\mathrm{H3}) \Rightarrow (\mathrm{H4}) \Leftrightarrow (\mathrm{H5}) \Rightarrow (\mathrm{H6}) \Rightarrow (\mathrm{H7})$$

$$(\mathrm{H8}) \Rightarrow (\mathrm{H4}) \text{ and } (\mathrm{H9}) \Rightarrow (\mathrm{H7}).$$

Proof. – (H1) ⇒ (H2) can easily be proved by using Cauchy's inequalities (see [12] for Banach spaces).

$$(\mathrm{H5}) \Rightarrow (\mathrm{H4}) \Rightarrow (\mathrm{H6}) \Rightarrow (\mathrm{H7}),\ (\mathrm{H8}) \Rightarrow (\mathrm{H4}) \text{ and } (\mathrm{H9}) \Rightarrow (\mathrm{H7})$$

are either trivial or well known ([9], [31]).

Suppose (H3) is not true then there exists U_1, U_2 open connected in E such that (H3) fails to hold for the pair U_1, U_2.

Let $V_1 = \{x \in U \cap U_1,\ f(x) = f_1(x)$ for all $f \in \mathcal{H}(U)$,

$$f_1 \in \mathcal{H}(U_1) \text{ such that } f|_{U_2} = f_1|_{U_2}\}$$

By definition $U_2 \subset V_1$. Since all holomorphic functions are continuous V_1 is a closed subset of $U \cap U_1$. By using the Taylor series expansion and the fact that analytic continuation is unique we find that V_1 is an open subset of $U \cap U_1$. Let V be the connected component of V_1 which contains U_2. We now consider V as an open subset of U_1. V is not a component of U_1 since U_1 is connected and $U_1 \not\subset U$. Since E is a l.c.s. this implies that V is not sequentially closed in U_1. Let $(x_n)_{n=1}^{\infty}$ be a sequence in V,

$$x_n \to x \in U_1 \cap \complement U$$

Since $x \notin V \subset U \cap U_1$, $x \notin U$. Hence $x \in \delta U$. Now f_1 is continuous at x (since $x \in U_1$). Hence

$$\text{i.e.} \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f_1(x_n) = f_1(x)$$

exists and is finite

$$\sup_n |f(x_n)| < \infty \text{ for all } f \in \mathcal{H}(U).$$

Thus (H2) ⇒ (H3).

(H3) ⇒ (H4). Suppose (H4) is not true. Let K be compact in U and $x_\alpha \in \hat{K}_{\mathcal{H}(U)}$, $x_\alpha \to \delta U$(1) as $\alpha \to \infty$ (this is possible since $\hat{K}$ is

(1) $x_\alpha \to \delta U$ if $(x_\alpha)_{\alpha\in A}$ is a precompact subset of E and $\inf_{\substack{\alpha\in A \\ y\in U)}} p(x_\alpha - y) = 0$ for each continuous semi-norm on E.

contained in the convex hull of K and hence is always a precompact subset of E). Let p be a continuous semi-norm on E such that

$$d_p(\mathrm{K}, \mathcal{C}\mathrm{U}) = \inf_{\substack{x \in \mathrm{K} \\ y \in \mathcal{C}\mathrm{U}}} p(x-y) = \delta > 0$$

Let $y \in$ E, $p(y) < \delta$. Choose $\delta_1 > 1$ such that

$$\mathrm{K}_1 = \mathrm{K} + \{\lambda y |\, |\lambda| \leqslant \delta_1\} \subset \mathrm{U}$$

(hence K_1 is also a compact subset of U). Let W be a convex balanced neighbourhood of 0 such that

$$\|f\|_{\mathrm{K}_1 + \delta_1 \mathrm{W}} = \mathrm{M} < \infty$$

for a given preassigned $f \in \mathcal{H}(\mathrm{U})$. For any $\alpha, \lambda \in \mathbf{C}$, $|\lambda| \leqslant 1$ and $\omega \in \mathrm{W}$ we have

$$\left| \frac{\hat{d}^n f(x_\alpha)}{n!} (\lambda y + \omega) \right| \leqslant \left\| \frac{\hat{d}^n f(x)}{n!} (\lambda y + \omega) \right\|_{x \in \mathrm{K}}$$

$$= \sup_{x \in \mathrm{K}} \left| \frac{1}{2\pi} \int_{|\lambda_1| = \delta_1} \frac{f(x + \lambda_1(\lambda y + \omega))}{\lambda_1^{n+1}} \, d\lambda_1 \right|$$

Now $x + \lambda_1 \lambda y + \lambda_1 \omega \in \mathrm{B\,K} + \{\alpha y, |\alpha| \leqslant \delta_1\} + \delta_1 \mathrm{W}$ and this implies

$$\left| \frac{\hat{d}^n f(x_\alpha)}{n!} (\lambda y + \omega) \right| \leqslant \sup_{x \in \mathrm{K} + \{\lambda y, |\alpha| \leqslant \delta_1\} + \mathrm{W}} |f(x)| \cdot \delta_1^{-n} \leqslant \frac{\mathrm{M}}{\delta_1^n}$$

Hence $$\sum_{n=0}^{\infty} \left| \frac{\hat{d}^n f(x_\alpha)}{n!} (\lambda y + \omega) \right| \leqslant \mathrm{M} \cdot \sum_{n=0}^{\infty} \frac{1}{\delta_1^n} < \infty$$

Since α, λ and ω were arbitrary we have

$$\sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(x_\alpha)}{n!} \right\|_{\{\lambda y, |\lambda| \leqslant 1\} + \mathrm{W}} < \infty$$

which means that the Taylor series of f at x_α converges in a W-neighbourhood of y. Choose x_α such that $d_p(x_\alpha, \mathcal{C}\mathrm{U}) < \delta/2$ then if

$$\tilde{f}(x + x_{\alpha_1}) = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(x_{\alpha_1})}{n!} (x)$$

$\widetilde{f}$ is holomorphic in a p_δ-neighbourhood of x_{α_1}.

Let V be a convex neighbourhood of x_α contained in

$$U \cap \{x , p(x - x_\alpha) < \delta\}.$$

Taking $V = U_2$ and $U_1 = \{x , p(x - x_\alpha) < \delta\}$ we find that (H3) is not true. Hence (H3) $\Rightarrow$ (H4), (H4) $\Rightarrow$ (H5).

If (H5) were not true then there would exist K compact in U and $y \in E$ such that $K_1 = K + \{\lambda y_1 \mid |\lambda| \leqslant 1\} \subset U$ and is compact there but $\hat{K} + \{\lambda y_1 \mid |\lambda| \leqslant 1\} \not\subset U$. The method used to show (H3) $\Rightarrow$ (H4) can now be used to show $\hat{K}_1$ is not a precompact subset of U. This contradicts (H4) and hence (H4) $\Rightarrow$ (H5).

Remark. – If E is quasi-complete then we can replace the condition "precompact set" by "compact set" in proposition 1.1.

The remainder of this section will be devoted to showing certain spaces are CTONB(i) or CTONBR(i) spaces. We first consider locally convex spaces for which we can prove the required result directly and then proceed to investigate certain kinds of projective limits of CTONB(i) and CTONBR(i) spaces.

LEMMA 1.1. – *Let* U *be a connected pseudo convex open subset of* E, E *an arbitrary* l.c.s. *Let* p *be a continuous semi-norm on* E *such that* $\{y , p(y) \leqslant \delta\} \subset U$ *for some* $\delta > 0$ then

$$U = U + \{y , p(y) = 0\}$$

Proof. – Let $x \in U$ and $y \in E , p(y) = 0$, be arbitrary. Since U is connected there exists a finite dimensional subspace F of E such that

i) $x , y \in F$

ii) 0 and x belong to the same connected component of $U \cap F$.

F is finite dimensional and hence every semi-norm on F is continuous. By definition $U \cap F$ is pseudo-convex since U is pseudo convex and hence the connected component of $U \cap F$ containing 0 say U_1, is again a pseudo convex subset of F.

Let $\widetilde{p}$ denote the restriction of p to F.

Hence $$U_1 \supset \{\omega \in F , p(\omega) \leqslant \delta\} \cdot$$

The method used in [18] can now be applied to show

$$U_1 + \{\omega \in F , \widetilde{p}(\omega) = 0\} = U_1$$

Hence $$x + \lambda y \in U_1 \subset U \quad \text{for all} \quad \lambda \in C.$$

Since x and y were arbitrarily chosen this implies

$$U + \{y , p(y) = 0\} = U.$$

Remark. – If π denotes the quotient mapping from E onto $E/p^{-1}(0)$ then the proceeding lemma is equivalent to showing

$$U = \pi^{-1}(\pi(U)).$$

LEMMA 1.2. – *Let π be a linear mapping from the vector space* E *onto* F. *Let* U *be an open*(1)*subset of E such that* $U = \pi^{-1}(\pi(U))$ *then* $\pi(U)$ *is pseudo convex (resp.* $\pi(U) \cap F_1$ *is Runge*(2)*. for each finite dimensional subspace* F_1 *of* F), *if* U *is pseudo convex (resp.* $U \cap G$ *is Runge for each finite dimensional subspace* G *of* E).

Proof. – It suffices to show that if F_1 is a finite dimensional subspace of F there exists a subspace G of E such that π ;

$$U \cap G \to \pi(U) \cap F_1$$

is a linear isomorphism.

Suppose F_1 is spanned by $\xi_1 , \ldots , \xi_n$. Choose $\eta_1 , \ldots , \eta_n \in E$ such that $\pi(\eta_i) = \xi_i$ for $i = 1 , \ldots , n$. Hence π ; $G \to F_1$ is an isomorphism where G is the n-dimensional subspace of E spanned by $\eta_1 , \ldots , \eta_n$ (linear mapping of one n dimensional subspace onto another is always an isomorphism).

(1) For vector spaces we say U is open if and only if $U \cap F$ is open in F for each finite dimensional subspace F of E.

(2) U is Runge if the polynomials on E are dense in $\mathcal{H}(U)$ when $\mathcal{H}(U)$ is endowed with the compact open topology.

Now $U \cap G \subset U$ and $U \cap G \subset G$, hence

$$\pi(U \cap G) \subset \pi(U) \cap \pi(G) = \pi(U) \cap F_1 .$$

Conversly if $y \in \pi(U) \cap F_1$, $y = \sum_{i=1}^{n} \alpha_i \xi_i$. Let $x = \sum_{i=1}^{n} \alpha_i \eta_i$ then $x \in G$ and $\pi(x) = y$. Hence $x \in \pi^{-1}(\pi(U)) = U$ and thus $x \in U \cap G$. This means $\pi(U \cap G) = \pi(U) \cap F_1$ and we have completed the proof.

We now restrict ourselves to specific locally convex spaces.

DEFINITION. – *A basis*(1), $(U_n)_{n=1}^{\infty}$, *in* L.C.S. E *is a strong basis if there exists a set of continuous semi-norms on* E, $(p_\alpha)_{\alpha \in A}$ *which defines the topology of* E *such that*

$$p_\alpha \left(\sum_{n=1}^{\infty} \alpha_n U_n \right) = \operatorname{Sup}_n \, p_\alpha \left(\sum_{j=1}^{n} \alpha_j U_j \right)$$

for each $\alpha \in A$.

Any basis in a barrelled locally convex space is a strong basis. In particular any basis in a Frechet space is a strong basis. We let E^n (resp. E_n) denote the closed vector space spanned by $U_1, \ldots, U_n$ (resp $(U_j)_{j=1}^{\infty}$).

PROPOSITION 1.2. – *A metrizable locally convex space* E *which has a strong basis is a CTONBR(1) space.*

The proof is rather long and is divided into a number of lemmas.

Let U denote a finitely polynomially convex subset of E. Let $(U_n)_{n=1}^{\infty}$ be a strong basis in E and let $(p_n)_{n=1}^{\infty}$ denote the corresponding family of semi-norms. Without loss of generality we can suppose $(p_n)_{n=1}^{\infty}$ is an increasing family of semi-norms and by lemma 1.2 we can also assume that

$$U = \pi^{-1}(\pi(U))$$

where

$$\pi ; E \to E/p_1^{-1}(0) \rightleftharpoons \pi(E).$$

(1) A basis is always taken to be a Schauder basis (i.e. the projection onto each coordinate is continuous).

if p is a continuous semi-norm on E we denote by $\widetilde{p}$ the corresponding semi-norm on $\pi(\mathrm{E})$. (Note that $\widetilde{p}_j$ is a continuous norm on $\pi(\mathrm{E})$ for each j and $(\widetilde{p}_j)_{j=1}^{\infty}$ defines the quotient topology on $\pi(\mathrm{E})$.

LEMMA 1.3. – *Let* $x = \alpha_1 + \alpha_2 \in \pi(\mathrm{E})$ *where* $\alpha_1 \in \pi(\mathrm{E}^n)$ *and*

$$\alpha_2 \in \pi(\mathrm{E}_n)$$

then $\widetilde{p}_j(\alpha_1) \leqslant \widetilde{p}_j(x)$ *for each positive integer* j.

Proof. – Choose $\beta_1 \in \mathrm{E}^n$, $\beta_2 \in \mathrm{E}_n$ such that $\pi(\beta_j) = \alpha_j$ for

$$j = 1,2.$$

Let $\gamma \in \mathrm{E}$ be chosen such that

$$p_1(\beta_1 - \gamma) = 0$$

Hence $\gamma - \beta_1 = \eta_1 + \eta_2$ where $\eta_1 \in \mathrm{E}^n$ and $\eta_2 \in \mathrm{E}_n$,

$$p_1(\eta_1) = 0 \quad \text{and} \quad p_1(\eta_2) = 0.$$

This means that $p_1(\gamma - \eta_2 - \beta_1) = 0$ and

$$p_j(\gamma - \eta_2) = p_j(\beta_1 + \eta_1) \leqslant p_j(\beta_1 + \eta_1 + \eta_2) = p_j(\gamma)$$

Hence $\widetilde{p}_j(\alpha_1) = \inf\limits_{\gamma \in \pi^{-1}(\alpha_1) \cap \mathrm{E}^n} p_j(\gamma)$.

Now suppose $\gamma \in \mathrm{E}$, $\gamma = \gamma_1 + \gamma_2$, $\gamma_1 \in \mathrm{E}^n$, $\gamma_2 \in \mathrm{E}_n$ and

$$p_1(\beta_1 + \beta_2 - \gamma_1 - \gamma_2) = 0.$$

We then have $p_1(\beta_1 - \gamma_1) = 0$ and $p_1(\beta_2 - \gamma_2) = 0$

Hence $\widetilde{p}_j(\alpha_1) \leqslant p_j(\gamma_1) \leqslant p_j(\gamma_1 + \gamma_2)$ which implies

$$\widetilde{p}_j(\alpha_1) \leqslant \inf_{\substack{\gamma = \gamma_1 + \gamma_2 \in \mathrm{E} \\ \gamma \in \pi^{-1}(\beta_1 + \beta_2)}} p_j(\gamma_1 + \gamma_2) = \widetilde{p}_j(\pi(\beta_1 + \beta_2)) = \widetilde{p}_j(x).$$

LEMMA 1.4. – *Let* P *be a continuous polynomial on* $\pi(\mathrm{E}^n)$ *then there exists a* p_1 continuous polynomial on E, Q, *such that*

$$\mathrm{Q}(x + y) = \mathrm{Q}(x) \text{ for all } x \in \mathrm{E}^n,\ y \in \mathrm{E}_n \tag{1}$$

$$\mathrm{P} \circ \pi|_{\mathrm{E}^n} = \mathrm{Q}|_{\mathrm{E}^n} \tag{2}$$

Proof. – Define $\widetilde{Q}$ on E^n by $\widetilde{Q} = P \circ \pi|_{E^n}$. Now extend $\widetilde{Q}$ to E to get Q by

$$Q\left(x = \sum_{i=1}^{\infty} \alpha_i U_i\right) = \widetilde{Q}\left(\sum_{i=1}^{\infty} \alpha_i U_i\right)$$

(1) and (2) are immediately satisfied. It remains to show Q is a p_1-continuous polynomial on E.

Let $(x_m)_{m=1}^{\infty} \in E$ be arbitrary and suppose $p_1(x_m) \to 0$ as $m \to \infty$ For each m, $x_m = y_m + z_m$, $y_m \in E^n$, $z_m \in E_n$. And $p_1(y_m) \to 0$ as $m \to \infty$.

By construction

$$Q(x_m) = Q(y_m) = (P \circ \pi)\ (y_m) = P(\pi(y_m))$$

Now $\widetilde{p}_1$ is a continuous norm on $\pi(E)$ and hence its restriction to $\pi(E^n)$ is also continuous. Hence P is $\widetilde{p}_1$ continuous on $\pi(E^n)$ and since $\widetilde{p}_1(\pi(x)) \leqslant p_1(x)$ for all $x \in E$ this implies $P(\pi(y_m)) \to 0$ as $m \to \infty$ whenever $p_1(y_m) \to 0$ as $m \to \infty$. Hence Q is p_1 continuous on E. This completes the proof.

Now suppose P is a continuous polynomial on $\pi(E^n)$. Let Q be the p_1-continuous polynomial on E associated with P in the previous lemma.

We define $\widetilde{P}$ on $\pi(E)$ in the following manner,

$$\widetilde{P}(x) = Q(\pi^{-1}(x)) \quad \text{for all } x \in \pi(E).$$

$\widetilde{P}$ is well defined for if $y, \omega \in \pi^{-1}(x)$ then $\pi(y) = \pi(\omega) = x$ i.e. $\pi(y - \omega) = 0$.

Since Q is p_1-continuous on E we have $Q(z + (y - \omega)) = Q(z)$ for all $z \in E$.

Hence $Q(y) = Q(\omega)$.

Also if $y_n \in \pi(E)$, $\widetilde{p}_1(y_n) \to 0$ as $n \to \infty$ then there exists $z_n \in E$, $\pi(z_n) = y_n$ and $p_1(z_n) \to 0$ as $n \to \infty$. Hence $Q(z_n) \to Q(0)$ as $n \to \infty$, this implies that $\widetilde{P}(y_n) = Q(z_n) \to Q(0) = (0)$ as $n \to \infty$ and we have shown that $\widetilde{P}$ is $\widetilde{p}_1$ continuous on $\pi(E)$.

LEMMA 1.5. – *An open subset* U *of a separable metrizable* l.c.s. E *is the domain of existence of a holomorphic function if and* *only if there exists* $(V_n)_{n=1}^{\infty}$ *an increasing sequence of open subsets of* U *which covers* U *and is such that* $\hat{V}_n$ *is bounded away from the boundary of* U *for each n.*

Proof. – (see also [12]). Let $(p_n)_{n=1}^{\infty}$ be an increasing sequence of continuous semi-norms on E which defines the topology of E. Suppose U is the domain of existence of f. For each K compact in U choose j_K a positive integer and $\alpha_K > 0$ such that

$$\|f\|_{K+2\alpha_K\{x\,;\,p_{j_K}(x)\leqslant 1\}} < \infty .$$

Since E is separable we can choose $(K_n)_{n=1}^{\infty}$ a sequence of compact subsets of U such that $\bigcup_{n=1}^{\infty} (K_n + \alpha_{K_n}\{x\,,\,p_{j_{K_n}}(x) \leqslant 1\}) \supset U$. By using Cauchy's inequality and the fact that U is the domain of existence of f we find that

$$\inf_{\substack{x\in \hat{W}_N \\ y\in \complement U}} p_N(x-y) \geqslant \inf_{i=1,\ldots,N} \alpha_{K_i}$$

where
$$W_N = \bigcup_{i=1}^{N} K_i + \{x\,,\,p_{j_{K_i}}(x) \leqslant \alpha_{K_i}\}$$

and
$$N = \sup_{i=1,\ldots,N} j_{K_i}$$

The sequence $(W_n^0)_{n=1}^{\infty}$ has all the required properties. Conversely suppose $(V_n)_{n=1}^{\infty}$ was an increasing sequence of open subsets of U which covered U and was such that $\hat{V}_n$ was bounded away from the boundary of U for each n.

Let M be a countable dense subset of U and let $(\xi_n)_{n=1}^{\infty}$ be a sequence of elements in M containing each point of M infinitely often. For each ξ_n let $A_n = \{x \in E\,,\, d(x\,,\,\xi_n) < d(\xi_n\,,\,\complement U)$

where
$$d(x\,,\,y) = \sum_{n=1}^{\infty} \frac{1}{n^2}\,\frac{p_n(x-y)}{1+p_n(x-y)} .$$

Let $C_2 = V_2$.

Choose $z_2 \in A_2 \cap \complement V_2$. Choose k_2 such that $\hat{V}_{k_2} \supset C_2 \cup \{z_2\}$ and let $C_3 = \hat{V}_{k_2}$ and so on by means of an obvious inductive process. For each n there exists $f_n \in \mathcal{H}(U)$ such that

$$\|f_n\|_{\hat{V}_{k_n}} < \frac{1}{2^n} \text{ and } |f_n(z_{n+1})| > 2^n + |\sum_{i=2}^{n-1} f_i(z_{n+1})|$$

The function $f = \sum_{n=2}^{\infty} f_n \in \mathcal{H}(U)$ and has U as its natural domain of existence [1].

Lemma 1.6. – *Let* U *be an open subset of a metrizable locally convex space* E *and let* F *be a closed subspace of* E. *If* π_1 *denotes the quotient mapping of* E *onto* E/F *and* $U = \pi^{-1}(\pi_1(U))$ *then* U *is the domain of existence of a holomorphic function if the same is true of the open set* $\pi_1(U)$.

Proof. – Now E/F is metrizable and lemma 1.5. implies that if $\pi_1(U)$ is the domain of existence of a holomorphic function then there exists $f \in \mathcal{H}(\pi_1(U))$ such that $\|f\|_{V \cap U} = \infty$ for any open subset V which intersects $\delta(\pi(U))$.

Let $g = f \circ \pi$, then $g \in \mathcal{H}(U))$.

Now if $\xi \in \partial U$ and W is a neighbourhood of ξ in E then $\pi_1(W)$ is an open subset of $\frac{E}{F}$ which contains $\pi_1(\xi) \in \delta(\pi_1(U))$.

Hence $$\|g\|_{W \cap U} = \infty.$$

This implies that U is the domain of existence of g.

Proof of proposition 1.2. – Let U be a connected open pseudoconvex subset of E. By lemma 1.1. we can suppose $U = \pi^{-1}(\pi(U))$ where π is the quotient mapping from E onto $E/p_1^{-1}(0)$. By lemma 1.6. it suffices to show $\pi(U)$ is the domain of existence of a holomorphic function.

For each compact subset K of $\pi(U)$ contained in $\pi(E^n)$ for some n and each integer j such that

(1) An examination of the construction we have used shows that if U_1 is any open subset of E such that

$$U_1 \cap \delta U \neq \phi \text{ then } \|f\|_{U \cap U_1} = \infty.$$

$$d_j(\mathrm{K}) = \inf_{\substack{x \in \mathrm{K} \\ y \in \mathcal{C}(\pi(\mathrm{U}))}} \widetilde{p}_j(x - y) > 0$$

we let $$\mathrm{K}_{(j)} = \mathrm{K} + \{y\,,\, \widetilde{p}_j(y) < \frac{1}{4}\, d_j(\mathrm{K})\}$$

The set of all such $\mathrm{K}_{(j)}$'s forms an opening covering of $\pi(\mathrm{U})$. Since E has a basis it is separable and hence we can subtract from the set of all $\mathrm{K}_{(j)}$'s a countable open covering of $\pi(\mathrm{U})$, say $(\mathrm{K}_{(j_n)})_{n=1}^{\infty}$.

By lemma 1.3 it suffices to show $\widehat{\bigcup_{n=1}^{m} \mathrm{K}_{j_n}}$ is bounded away from the boundary of $\pi(\mathrm{U})$ for each m. If this were not so there would exist $(\xi_l)_{l=1}^{\infty} \in \pi(\mathrm{U})$, $\xi_l \to \xi \in \delta(\pi(\mathrm{U}))$

and $$\sup |f(\xi_l)| \leqslant \|f\|_{\bigcup_{n=1}^{m} \mathrm{K}_{j_n}}$$

for all $$f \in \mathcal{H}(\pi(\mathrm{U})).$$

By the same procedure as used in [16] we can suppose $\xi = 0$ and $\xi_j \in \bigcup_{n=1}^{\infty} \mathrm{E}^n$ for each integer j.

Let

$$\theta = \sup_{i=1,\dots,m} j_i$$

Choose M a positive integer such that

(a) $\mathrm{K}_i \subset \mathrm{E}^{\mathrm{M}}$ for $i = 1, \dots, m$.

(b) there exists $\xi_k \in \left(\widehat{\bigcup_{n=1}^{m} \mathrm{K}_{j_n}}\right) \cap \mathrm{E}^{\mathrm{M}}$

for which $\widetilde{p}_\theta(\xi_k) < \frac{1}{4} \inf_{i=1,\dots,m} d_{j_i}(\mathrm{K}_i) = \alpha/4$.

Let P be a continuous polynomial on $\pi(\mathrm{E}^{\mathrm{M}})$ and let P denote the extension to $\pi(\mathrm{E})$ of P which we have previously discussed.

Hence $$|\mathrm{P}(\xi_k)| = |\widetilde{\mathrm{P}}(\xi_k)| \leqslant \sup_{n=1,\dots,m} \|\widetilde{\mathrm{P}}\|_{\mathrm{K}_{j_n}}$$

Now if $\omega_0 \in \mathrm{K}_i$, $\omega_1 \in \pi(\mathrm{E}^{\mathrm{M}})$, $\omega_2 \in \pi(\mathrm{E}_{\mathrm{M}})$ and

$$\widetilde{P}_{j_i}(\omega_1 + \omega_2) < \frac{1}{4} d_{j_i}(K_i)$$

then $$\widetilde{P}(\omega_0 + \omega_1 + \omega_2) = P(\omega_0 + \omega_1)$$

and by lemma 1.3

$$\widetilde{P}_{j_i}(\omega_1) \leqslant \widetilde{P}_{j_i}(\omega_1 + \omega_2) \leqslant \frac{1}{4} d_{j_i}(K_i)$$

Hence

$$|P(\xi_k)| \leqslant \sup_{i=1,\ldots m} \|P\|_{K_{j_i} \cap \pi(E^M)}.$$

Now

$$d_\theta \left(\bigcup_{i=1}^{\infty} K_{j_i} \cap \pi(E^M)\right) (^1)$$

$$\geqslant \inf_{i=1,\ldots m} d_\theta (K_{j_i} \cap \pi(E^M))$$

$$\geqslant \inf_{i=1,\ldots m} d_{ji} (K_{j_i} \cap \pi(E^M)) = \frac{3}{4} \alpha.$$

Since $\widetilde{p}_\theta(\xi_k) < \alpha/4$ this contradicts the fact that $\pi(U) \cap \pi(E^M)$ is pseudo convex and hence holomorphically convex (we need the fact that $\widetilde{p}_\theta$ was a norm on $\pi(E)$ in order to insure that $K_{j_i} \cap \pi(E^M)$ was a compact subset of a finite dimensional space)

Remark. – If there existed a continuous norm on E or if we knew that $\pi(E)$ had a strong basis then the proof of proposition 1.2 could be considerably shortened and would in fact be more or less the same as that given for Banach spaces with a basis in [16].

An examination of the final part of the proof of proposition 1.2 shows that we have in fact proved the following result.

PROPOSITION 1.3. – *A metrizable space with a strong basis is a* CTONBR (8) *space.*

LEMMA 1.1. – *If* U *is an open subset of a Lindelof*(²) L.C.S. *with a strong basis* E *then* E *can be endowed with the structure of*

(¹) $d_\theta(x) = d_\theta(x, 0) = \widetilde{p}_\theta(x)$

(²) A topological space X is Lindelof if every open cover of X contains a countable subcover.

a locally convex semi-metrizable space with a strong basis weaker than the original topology on E *such that* U *is open with respect to the new structure.*

Proof. – Let $(p_\alpha)_{\alpha \in A}$ be the set of continuous semi-norms on E associated with a strong basis. For each $x \in U$ choose $\alpha_x \in A$ such that $V_x = x + \{y\,,\, p_{\alpha_x}(y) < \delta_x\} \subset U$ for some $\delta_x > 0$. $\bigcup_{x \in U} V_x$ is an open covering of U.

Since E is Lindelof U is also Lindelof and hence we can choose $(x_n)_{n=1}^\infty$ a sequence of elements of U such that $\bigcup_{n=1}^\infty V_{x_n} = U$. Now $(E\,, (p_{x_n})_{n=1}^\infty)$ is a semi-metrizable l.c.s. with a strong basis and and U is open in $(E\,, (p_{x_n})_{n=1}^\infty)$. This completes the proof.

PROPOSITION 1.4. – *A Lindelof space with a strong basis is a* CTONBR(2) *and a* CTONBR(8) *space.*

Proof. – Let U denote a pseudo convex open subset of E such that $U \cap F$ is Runge for each finite dimensional subspace F of E. Let m denote the topology of E and let $i(m)$ denote the semi-metrizable topology on E as constructed in the previous lemma. It is immediate by proposition 1.3. that U is polynomially convex and hence E is a CTONBR(8) space.

By proposition 1.2 U is the domain of existence of a holomorphic function when U is endowed with the $i(m)$ topology. Now if $x_n \in U, x_n \overset{m}{\to} x \in \delta U$ then $x_n \overset{i(m)}{\to} x$ and hence there exists $f \in \mathcal{H}(U)$ such that $\sup |f(x_n)| = \infty$. Hence E is a CTONBR(2) space.

Examples of spaces which satisfy the condition of proposition 1.4.

1) E Lindelof, barrelled and possessing a basis.

2) $E = \sum_{i=1}^\infty E_i$, where E_i is a Frechet space with a basis for each i.

We now consider CTONBR(4) spaces.

PROPOSITION 1.5. – *If* E *is a* l.c.s. *such that for each compact subset* K *of* E *there exists a closed complemented subspace of* E *which contains* K *and is a* CTONBR(4) *space then* E *is a* CTONBR(4) *space.*

Proof. – Let U be an open finitely polynomially convex subset of E. For K compact in U choose E(K) a closed complemented subspace of E which is a CTONBR(4) space and which contains K. Hence K is a compact subset of $U \cap E(K)$ and thus

$$\hat{K}_{\mathfrak{P}(E(K))} = \hat{K}_{\mathfrak{P}(E)}$$

is a precompact subset of U. This completes the proof.

Examples of spaces which satisfy the criterion of proposition 1.5 :

1) E an arbitrary L^p space, $(1 \leqslant p < \infty)$ (note, E need not be complete nor separable)

2) $E = \sum_{i \in A}^{\infty} E_i$, where A is an arbitrary indexing set and E_i is a Frechet space with a basis for each i.

We now consider projective limits of various kinds.

Let $(E_i)_{i \in A}$ be a set of l.c.s. spaces.

Let E be a vector space and let π_i be a linear mapping from E onto E_i for each $i \in A$. *We say* E *is the projective limit of* $(E_i)_{i \in A}$ *by means of the mappings* $(\pi_i)_{i \in A}$ *if* E *has the weakest locally convex topology for which all the functions* π_i *are continuous. We write* $E = \varprojlim_{i \in A} (E_i, \pi_i)$. *The projective limit is said to be directed if* A *is directed and for each* $i, j \in A$ *there exists* $k \in A, k \geqslant i, k \geqslant j$ *and continuous linear mappings* π_i^k, π_j^k *such that the following diagram is well defined and commutative*

$$\begin{tikzcd}
 & & E_i \\
E \arrow[r, "\pi_k"] \arrow[urr, bend left] \arrow[drr, bend right, "\pi_j"'] & E_k \arrow[ur, "\pi_i^k"] \arrow[dr, "\pi_j^k"] & \\
 & & E_j
\end{tikzcd}$$

$\varinjlim_{i \in A} (E_i, \pi_i)$ *is said to be an* N-*projective limit if it is directed and for each i the mapping π_i*, $E \to E_i$ *is open.* The following lemma can be proved easily.

Lemma 1.6. – *The directed projective limit* $\varprojlim_{i \in A} (E_i, \pi_i)$ *is an* N-*projective limit if and only if the mapping π_i^k is open for each $k, i \in A, k \geqslant i$.*

A semi-metric d ou E *a* L.C.S. *which has the form*

$$d(x, y) = p_1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \frac{p_n(x-y)}{1 + p_n(x-y)}$$

where p_n is a continuous semi-norm on E *for each n is said to be a suitable semi-metric.*

The following properties of a suitable semi-metric d are easily checked ;

1) d is continuous on E and generates the same locally convex structure on E as the sequence of semi-norms $(p_n)_{n=1}^{\infty}$.

2) $\{x \in E, d(x) = 0\}$ is a closed vector subspace of E.

Lemma 1.7. – *Let* U *be a pseudo convex open subset of (a)* l.c.s. E. *Suppose d is a suitable semi-metric on* E *such that*

$$\{y, d(y) < \varepsilon\} \subset U$$

for some $\varepsilon > 0$ *then* $U \supset \{y, d(y) < \beta\}$ *where* $\beta = \sup_y \{d(y), y \in E$ *and there exists* $x \in E, d(x-y) = 0, \lambda x \in U$ for $|\lambda| \leqslant 1\}$.

Proof. – Let $\omega \in E, d(\omega) < \beta, \omega_y \in E, d(\omega - \omega_y) = 0$ and

$$\lambda\omega_y \in U \text{ for all } \lambda \in C, |\lambda| \leqslant 1.$$

If ω and ω_y are linearly dependent then $\omega \in U$ trivially and the lemma is proved. Hence we suppose that the vector space spanned by ω and ω_y, V, is 2-dimensional. Let

$\delta_{U \cap V}(z, \omega_y) = \sup \{\lambda, z + \beta\omega_y \in U \cap V \text{ for } |\beta| \leqslant \lambda\}$

Then $-\log \delta_{U\cap V}(z, \omega_y)$ is a plurisubharmonic function on $U \cap V$ and hence $-\log \delta_{U\cap V}(\lambda(\omega - \omega_y), \omega_y)$ is a subharmonic function of λ for $\lambda \in C$.

Now $d(\omega - \omega_y) = 0$ and hence $\gamma\omega_y + \lambda(\omega - \omega_y) \subset U$ for all sufficiently small γ and all $\lambda \in C$. This implies that

$$- \log \delta_{U\cap V}(\lambda(\omega - \omega_y)$$

is bounded above and hence constant. For $\lambda = 0$ we have $\lambda\omega_y \in U$ for $|\lambda| \leqslant 1$. Hence $-\log \delta_{U\cap V}(0, \omega_y) < 0$ and so $(\omega - \omega_y) + \omega_y \in U$ i.e. $\omega \in U$. This completes the proof.

LEMMA 1.8. – *Let* U *be a pseudo convex open subset of the* N-*projective limit* $\varprojlim_{i \in A} (E_i, \pi_i)$ *then* U *is* π_i-*open for some* $i \in A$. (i.e. *for each* $x \in U$ *there exists* V_x *open in* E_i *such that*

$$x \in \pi^{-1}(V_x) \subset U).$$

Proof. – Without loss of generality we can suppose $0 \in U$ and that V open in E_i is such that $\pi_i^{-1}(V) \subset U$ (use the fact that we have a directed projective limit).

Let $W = \{x \in U |$ there exists V_x open in E_i and

$$x + \pi^{-1}(V_x) \subset U\}.$$

W is non empty and open in U. Suppose $W \neq U$ then since E is locally convex there exists $x_n \in W, x_n \to y \in U \cap \complement W$. Choose $j \in A, j \geqslant i$ and q a continuous semi-norm on E_j such that

$$y + \{x \in E, q(\pi_j(x)) < \delta\} \subset U \text{ for some } \delta > 0.$$

Let $(p_n)_{n=2}^{\infty}$ be a sequence of continuous semi-norms on E_i such that

$$x_n + \{x \in E, p_n(\pi_i(x)) < \delta_n\} \subset U.$$

Under the open mapping π_i^j there exists p_1 a continuous semi-norm on E_i such that

$$\pi_i^j(\{x \in E_j, q(x) < \delta/4\}) \supset \{x \in E_i, p_1(x) < \alpha\}$$

Let d be the suitable semi-metric defined on E_i by $(p_n)_{n=1}^{\infty}$.

We now define the suitable semi-metric $\widetilde{d}$ on E by

$$d(x\,,y) = p_1(\pi_i(x)\,,\pi_i(y)) + \sum_{n=2}^{\infty} \frac{1}{n^2} \frac{p_n(\pi_i(x) - \pi_i(y))}{1 + p_n(\pi_i(x) - \pi_i(y))}$$

$$= d(\pi(x)\,,\pi(y))$$

By construction we see that each x_n contains a $\widetilde{d}$ ball in U and $d \geqslant p_1$. We now have

$$\pi_i^j(x \in E_j\,, q(x) < \delta/4) \supset \{x \in E_i\,, p(x) < \alpha\} \supset \{x \in E_i\,, d(x) \subset \alpha\}$$

Let n be a positive integer such that

$$q(\pi_j(x_n - y)) < \delta/4 \tag{1}$$

$$\widetilde{d}(y - x_n) < \alpha/2 \tag{2}$$

Hence $$x_n + \{x \in E\,, q(\pi_j(x)) < 3\delta/4\} \subset U \tag{3}$$

If $\omega \in E$ and $\widetilde{d}(\omega) < \alpha$ then $d(\pi_i(\omega) < \alpha$. This implies that

$$\pi_i(\omega) \in \pi_i^j\{x \in E_j\,, q(x) < \delta/4\}$$

and hence there exists $\omega_1 \in E$ such that

$$\pi_i^j(\pi_j(\omega)) = \pi_i(\omega) \text{ and } q(\pi_j(\omega)) < \delta/4.$$

Hence $$\pi_i(\omega_1) = \pi_i(\omega).$$

By (3) $$x_n + \omega_1 \in U.$$

Since π_i is linear $\pi_i(\omega - \omega_1) = 0$ and hence $\widetilde{d}(\omega - \omega_1) = 0$
By lemma 1.1.

$$U + \{y\,, \widetilde{d}(y) = 0\} = U$$

since x_n contains a $\widetilde{d}$ neighbourhood in U. Since $x_n + \omega_1 \in U$ this implies $x_n + \omega \subset U$ i.e. $x_n + \{x \in E\,, \widetilde{d}(x) < \alpha\} \subset U$. By (2)

$$y + \{x \in E\,, \widetilde{d}(x) < \alpha/2\} \subset U$$

which implies $y \in W$ i.e. $U = W$.

This completes the proof.

PROPOSITION 1.6. – *The* N-*projective limit of* CTONB(i) (resp. CTONBR(i)) *spaces is a* CTONB(i) (resp. CTONBR(i)) *space for* $i = 1,2,3$.

Proof. – Our method of proof is the same for CTONB(i) spaces as for CTONBR(i) spaces so we restrict ourselves to the former. Let U be a pseudo convex open subset of E. By lemma 1.8 there exists $i \in A$ such that $U = \pi_i^{-1}(\pi_i(U))$.

Now suppose E_i is a CTONB(1) space for each i. Hence $\pi_i(U)$ is the domain of existence of a holomorphic function f. This implies that $f \circ \pi_i$ is defined on U. The Riemann domain, $\widetilde{U}$, associated with $f \circ \pi$ spread over E can now be constructed and shown to be pseudo convex. Since this domain contains U it must also be E_i open. If $\widetilde{U} \neq U$ then there exists $\alpha > 0$ such that

$$\widetilde{U}_\alpha = \{x \, , |\widetilde{f}(x)| < \alpha\} \not\subset U$$

where $\widetilde{f}$ denotes the extension of $f \circ \pi_i$ to $\widetilde{U}$. Hence f can be factored through a domain spread over $\pi(U)$ and this contradicts the fact that f has $\pi(U)$ as its domain of existence([1]). The same method can be applied to show that the N-projective limit of CTONB(3) spaces is a CTONB(3) space. Now suppose E_i is a CTONB(2) space for each i. If $\xi_j \in U$, $\xi_j \to \xi$ $\xi \in \delta U$ then $\pi(\xi_j) \in \pi(U)$ and $\pi(\xi_j) \to \pi(\xi)$

$$\pi(\xi) \in \delta(\pi(U))$$

hence there exists $f \in \mathcal{H}(\pi(U))$ such that $\sup_n |f(\pi(\xi_n))| = \infty$. Since $f \circ \pi \in \mathcal{H}(U)$ this implies that E is a CTONB(2) space.

Examples

1) $\prod_\omega C$ is a CTONB(1) space for any cardinal number ω.

2) $\prod_\omega E_i$ is a CTONBR(i) space for $i = 1,2,3$ whenever E_i is a CTONBR(i) space and ω is arbitrary. (in particular if E_i is Frechet with a basis)

([1]) Since $U \subset E$ it is also possible to complete this proof for CTONB(1) spaces without constructing a Riemann domain (similarly for CTONB(3) spaces).

PROPOSITION 1.7. — *A directed projective limit of CTONBR(8) spaces is a CTONBR(8) space.*

Proof. — Let $E = \varprojlim_{i \in A} (E_i, \pi_i)$ where each E_i is a CTONBR(8) space and let U be a finitely polynomially convex open subset of E. For each $i \in A$ let V_i be the largest i-open subset of U.

Let $\hat{V}_i = \bigcup_{K \in \mathcal{K}(V_i)} \hat{K}_{\mathfrak{P}(E)}$ where $\mathcal{K}(V_i)$ denotes the set of all compact finite dimensional subsets of V_i and $\hat{K}_{\mathfrak{P}(E)}$ is the polynomial hull of K in E. Since U is finitely polynomially convex it is immediate that $V_i \subset \hat{V}_i \subset U$. Now suppose $x \in \hat{V}_i$ then there exists $K \in \mathcal{K}(V_i)$ and p a continuous semi-norm on E_i such that

$$x \in \widehat{K + \{y \in E, p(\pi_i(y)) < \alpha\}} \subset V_i$$

for some $\alpha > 0$.

For $$y \in E, p(\pi_i(y)) < \alpha$$

$$K_y = K + \{\omega y, |\omega| \leqslant 1\} \subset \mathcal{K}(V_i)$$

Hence $$\hat{K}_{\mathfrak{P}(E)} + \{\omega y, |\omega| \leqslant 1\} \subset \hat{K}_{y\mathfrak{P}(E)} \subset \hat{V}_i$$

Thus $x + \{y, y \in E, p(\pi_i(y)) < \alpha\} \subset V_i$ which means $\hat{V}_i = V_i$. By construction V_i is finitely polynomially convex.

By hypothesis V_i is a polynomially convex subset of E.

Let K be compact in U. Hence K is a compact subset of some V_i. Hence $\hat{K}_{\mathfrak{P}(E)}$ is bounded away from the boundary of V_i. This implies immediately that $\hat{K}_{\mathfrak{P}(E)}$ is a precompact subset of U. Hence U is polynomially convex.

Example 1. — A nuclear space is a CTONBR(8) space (if E is nuclear then E is the directed projective limit of semi pre-Hilbert spaces).

COROLLARY 1.1. — *A finitely polynomially convex open subset of a Frechet-Nuclear space is the limit of an increasing sequence of domains of existence of holomorphic functions.*

Example 2. — A l.c.s. with a strong basis is a CTONBR(8) space.

2. Holomorphic functions on open subsets of $\sum_{i=1}^{\infty} C$.

It can easily be seen by means of the methods of the last section that $\sum_{i=1}^{\infty} C$ is a CTONB(2) space. In this section we give an alternate proof of this fact and we also show that if E is an infinite dimensional l.c.s. on which every G-holomorphic fonction is holomorphic then E is linearly isomorphic to $\sum_{i=1}^{\infty} C$. (In I) we showed that every G-holomorphic function on $\sum_{i=1}^{\infty} C$ was holomorphic).

PROPOSITION 2.1. – *Let* U *be a pseudo convex open subset of* $\sum_{i=1}^{\infty} C$ $(= E)$ *and let* E_1 *be a finite dimensional subspace of* E *then each* $f \in \mathcal{H}(U \cap E_1)$ *can be extended to a holomorphic function on* U.

Proof. – Immediate by using the corresponding finite dimensional result and extending the function to all of U by induction. Since the extension is G-holomorphic it is holomorphic.

PROPOSITION 2.2. – $\sum_{i=1}^{\infty} C$ *is a* CTONB(2) *space.*

Proof. – Let U be a pseudo-convex open subset of $\sum_{i=1}^{\infty} C$. If $x_n \in U, x_n \to x \in \delta U$ then there exists E_1 a finite dimensional subspace of $\sum_{i=1}^{\infty} C$ such that $x_n \in E_1$ for all n. Since $U \cap E_1$ is pseudo convex there exists $f \in \mathcal{H}(U \cap E_1)$ such that $\sup_n |f(x_n)| = \infty$. An application of proposition 2.1. completes the proof.

PROPOSITION 2.3. – *Let* E *be an infinite dimensional* L.C.S. *such that each G-holomorphic function on* E *is holomorphic then* E *is linearly isomorphic to* $\sum_{i=1}^{\infty} C$.

Proof. – Let $(x_\alpha)_{\alpha\in A}$ be a Hamel basis for E.

Then E is algebraically isomorphic to $\sum\limits_{\alpha\in A} C$. Suppose now that the cardinality of A was uncountable. Let $\{x_\alpha\}$ be the 1-dimensional space spanned by x_α. For each n let

$$f_n = n^{2n} \cdot \sum_{\substack{i_1,\ldots,i_n\in A\\ i_j\neq i_k \text{ for } j\neq k}} y_{i_1}, \ldots, y_{i_n}$$

where $y_{i_1}, \ldots, y_{i_n}$; $\sum\limits_{\alpha\in A} \{x_\alpha\} \to C$ is the n-homogeneous polynomial

$$y_{i_1}, \ldots, y_{i_n}\Big(\sum_{i\in A} \alpha_i x_i\Big) = \alpha_{i_1}, \ldots, \alpha_{i_n}$$

Let $f = \sum\limits_{n=1}^{\infty} f_n$. It is easily checked that f is a G-holomorphic function on E and hence it is holomorphic on E. Since the direct sum topology on E induced by the algebraic isomorphism form $\sum\limits_{i\in A} \{x_i\}$ is finer than the original topology on E we see that f is holomorphic on $\sum\limits_{i\in A} \{x_i\}$ with the direct sum topology. Hence there exists V open convex and balanced in $\sum\limits_{i\in A} \{x_i\}$ such that

$$\|f_n\|_V \leqslant 1$$

for each n $\left(\text{since } \dfrac{d^n f(0)}{n!} = f_n\right)$.

Since A is uncountable there exists $\delta > 0$ and $(\alpha_n)_{n=0}^{\infty}$ a sequence of distinct elements of A such that

$$\{\lambda x_{\alpha_n}, |\lambda| \leqslant \delta\} \subset V \qquad \text{for each } n.$$

Hence $\sum\limits_{i=1}^{n} \delta/n \; x_{\alpha_i} \in V$ for each n.

This implies that

$$\left| n^{2n} \cdot \frac{\delta^n}{n^n} \right| \leqslant 1 \qquad \text{for each } n$$

Since this is impossible we have that A is countable. We order A as an increasing sequence. Let W be a convex balanced open subset of $\sum_{i=1}^{\infty} \{x_i\}$. There then exists $(\delta_n)_{n=1}^{\infty}$ a decreasing sequence of positive numbers such that $\{\lambda x_n, |\lambda| \leqslant \delta_n\} \subset W$. For each n let

$f_n ; \sum_{i=1}^{\infty} \{x_i\} \to \mathbf{C}$ be the mapping such that

$$f_n\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \frac{\alpha_n^n \cdot (2^n)^n}{\delta_n^n}$$

Then $f = \sum_{n=1}^{\infty} f_n$ is a G-holomorphic function on E and hence holomorphic. Let V be a neighbourhood of 0 in E such that $\|f_n\|_V \leqslant 1$ for each n. We show $V \subset W$ and this completes the proof.

Let $x = \sum_{i=1}^{\infty} \alpha_i x_i \in V$ then for each n,

$$\left| \frac{\alpha_n^n \cdot (2^n)^n}{\delta_n^n} \right| \leqslant 1.$$

Hence $|\alpha_n| \leqslant \frac{\delta_n}{2^n}$. Now $\beta_n x_n \in W$ for each n when $|\beta_n| \leqslant \delta_n$. Since W is convex

$$\sum_{i=1}^{\infty} \frac{1}{2^n} \beta_n x_n \in W$$

(where all but a finite number of β_n's are zero)

Letting $\qquad \beta_n = \alpha_n \cdot 2^n$

we have that $\qquad \sum_{i=1}^{\infty} \frac{1}{2^n} \cdot \alpha_n \cdot 2^n \cdot x_n \in W$

i.e. $x \in W$.

Hence $V \subset W$ and so the direct sum topology on E is weaker than the original topology. This means they are equal.

Hence $E \approx \sum_{i=1}^{\infty} C$ algebraically and topologically.

3. Runge's theorem in locally convex spaces.

DEFINITION. – *A* l.c.s. E *is said to be p-Runge if the polynomials on* E *are dense in* $\mathcal{H}(U)$ *with the compact open topology for every open polynomially convex subset* U *of* E.

PROPOSITION 3.1. – *A* l.c.s. E *with a strong basis is p-Runge.*

Proof. – Let U be a polynomically convex open subset of E. Let K be an arbitrary polynomially convex compact subset of U and f be an arbitrary element of $\mathcal{H}(U)$. Choose p a continuous semi-norm on E associated with a strong basis such that

1) $d_p(K, \complement U) > 0$

2) there exists a p-neighbourhood of K, ω, contained in U such that f/ω is p continuous. We can now apply the method used for Banach spaces with a basis to get the required result ([13]).

PROPOSITION 3.2. – *The* N*-projective limit of p-Runge spaces is a p-Runge space.*

Proof. – Let $E = \varprojlim_{i \in A} (E_i, \pi_i)$ and let U be a polynomially convex subset of E. Since U is pseudo convex and open in E lemma 1.8 implies that U is π_i-open in E for some $i \in A$. Now let $f \in \mathcal{H}(U)$ then there exists $x \in U$ and V a neighbourhood of 0 in E_j for some $j \geqslant i$ such that

$$\|f\|_{x+\pi_j^{-1}(V)} = M < \infty$$

and $$x + \pi_j^{-1}(V) \subset U.$$

For each positive integer greater than or equal to M let $$V_n = \{x \in U , |f(x)| < n\}$$

V_n is a pseudo convex open subset of U and an application of lemma 1.8 implies that V is π_j-open in E for each n.

Since $\bigcup_{n>M} V_n = U$ this means that f is bounded in a E_j-neighbourhood of each point of U. For each $x \in \pi_j(U)$ let $\widetilde{f}(x) = f(\xi)$ where $\pi_j(\xi) = x$. If $\pi_j(\xi) = \pi_j(\eta) = x$ then $\pi_j(\xi - \eta) = 0$ and hence

$$f(\xi) = f(\xi + (\eta - \xi)) = f(\eta)$$

which implies that f is well defined. Now if $\omega \in \pi_j(U)$ then there exists W open in $\pi_j(U)$ containing ω such that

$$\|f\|_{\pi_j^{-1}(W)} < \infty$$

Hence

$$\|f\|_W = \sup_{x \in W} |\widetilde{f}(x)| = \sup_{\substack{\pi_j(\xi)=x \\ x \in W}} |f(\xi)| = \|f\|_{\pi^{-1}(W)} < \infty$$

We thus have that $\widetilde{f} \in \mathcal{H}(\pi_j(U))$ and $\widetilde{f} \circ \pi_j = f$.

Now E_j is a p-Runge space and $\pi_j(U)$ is polynomially convex in E_j. Let K be compact in E and $j > i$ be arbitrary then $\pi_j(K)$ is a compact subset of E_j.

Hence there exists $P \in \mathcal{P}(E_j)$ such that

$$\|\widetilde{f} - P\|_{\pi_j(K)} \leqslant \varepsilon$$

This implies that $$\|\widetilde{f} \circ \pi_j - P \circ \pi_j\|_K \leqslant \varepsilon$$

i.e. $$\|f - P \circ \pi_j\|_K \leqslant \varepsilon$$

This completes the proof since π_i is linear and hence $P \circ \pi_j$ is a continuous polynomial on E.

COROLLARY 3.1. – *A Frechet space with a basis is p-Runge.*

An examination of the proof of the previous proposition shows that we have recovered a generalisation of a result of Nachbin [48].

PROPOSITION 3.3. – *Let* U *be an open subset of the* N*-projective limit* $E = \varprojlim_{i \in A} (E_i , \pi_i)$ *then each element of* $\mathcal{H}(U)$ *can be factored through some* E_i, $i \in A$.

(i.e. if $f \in \mathcal{H}(\mathrm{U})$, $\exists\, i \in \mathrm{A}$, U_i *open in* E_i *and* $f_i \in \mathcal{H}(\mathrm{U}_i)$ *such that* $f = f_1 \circ \pi_i|_{\mathrm{U}_i}$ *and* $\pi_i^{-1}(\mathrm{U}_i) \supset \mathrm{U}$).

Proof. – In proposition 3.2 we have proved this result for U pseudo convex. If U is not pseudo convex it is possible to find the envelope of holomorphy of U (which may not be univalent) and to apply the same method of proof.

The following result is also immediate by means of the methods we have developed in § 1 and in this section.

PROPOSITION 3.4. – *Let* U *be an open subset of the* N*-projective limit of metrizable* L.C.S. *with a stong basis such that* $\mathrm{U} \cap \mathrm{F}$ *is Runge for each finite dimensional subspace* F *of* E. *Then*

$$\mathrm{V} = \{x \in \mathrm{E}\,, x \in \hat{\mathcal{K}}_{\mathfrak{F}(\mathrm{E})}\}$$

where K *ranges over all compact finite dimensional subspaces of* E} *is the envelope of holomorphy of* U.

4. Concluding remarks.

In generalising to various l.c.s. the known results on Banach spaces with a basis and on open subsets of $\prod_{i=1}^{\infty} \mathrm{C}$ we have concentrated for the most part in removing countability requirements on E and in overcoming the frequent lack of a continuous norm on E. A different approach has been taken by M. Pr. Noverraz who concentrates for the most part in removing the basis requirements on E and with replacing them by various "approximation requirements". He also generalizes various sections of the CTONB theorem to particular kinds of l.c.s. (eg. to the duals of Frechet-Schwartz spaces).

While his approach is different from ours, his results are complimentary and many of his results may be applied (and conversely) to strengthen the results obtained in this work.

Added in proof.

Since completing this article the following papers have been written and are intimately related to the problems we have discussed in I and II :

1. S. DINEEN, Holomorphically complete locally convex topological vector spaces, Séminaire Lelong, 1972-73.
2. S. DINEEN, Sheaves of holomorphic functions on infinite dimensional vector spaces (Math. Ann., 202, 337-345, 1973.
3. L. GRUMAN, The Levi problem in certain infinite dimensional vector spaces (to appear).
4. L. GRUMAN and C.O. KISELMAN, Le problème de Levi dans les espaces de Banach à base, (C.R. Acad. Sc., Paris, t. 214, 1972, 1296-1298.
5. Ph. NOVERRAZ, Sur la pseudo-convexité et la convexité polynomiale en dimension infinie, (to appear in *Annales de l'Institut Fourier*).

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Manuscrit reçu le 11 septembre 1972
accepté par M. Brelot

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