ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 22, nº 3 (1972), p. 95-130

<http://www.numdam.org/item?id=AIF_1972__22_3_95_0>

Ann. Inst. Fourier, Grenoble
22,3 (1972), 95-130

DUALITY AND THE MARTIN COMPACTIFICATION

by J.C. TAYLOR

Introduction.

Since P.A. Meyer showed, assuming the constants harmonic, that the cone of non-negative hyperharmonic functions defined by a Brelot sheaf coincided with the cone of functions excessive with respect to a Hunt process, axiomatic potential theory has been widely considered to be a special case of the probabilistic theory of potential.

In both potential theories Martin compactifications of the state space X can be constructed, given suitable assumptions, c.f. [8] and [14]. Except in the case of a Brownian motion it was not known whether the axiomatic Martin compactification could be obtained by the probabilistic procedure from a corresponding Hunt process. The aim of this article is to show that this is indeed possible.

The probabilistic Martin compactifications constructed by H. Kunita and T. Watanabe in [8] exist when the Hunt process has a suitable dual. It is shown in theorem 5.4 that, with certain assumptions, to each harmonic sheaf $\mathcal{H}$ there corresponds a Hunt process which has a dual in the sense of [8]. Further, if G is a suitable Green function for the given sheaf $\mathcal{H}$, then in proposition 6.6 it is shown that, for $x_0 \in X$, there exists a "normalizing" measure $\rho \in \mathfrak{M}^+(X)$ with $y \to \int G(x, y)\, \rho(dx) = G(x_0, y)$ outside a compact neighbourhood of x_0. By combining these two results it is then shown in theorem 6.9 that the corresponding axiomatic Martin compactification of X can be obtained by probabilistic methods.

In paragraph one an exposition is given of some results due to Sieveking in [13] which are used in paragraph two to give a refinement of the following result from [13] : for any strict Bauer sheaf $\mathcal{H}$ that possesses a Green function there exists a Hunt process with the property that whenever the adjoint sheaf $\mathcal{H}^*$ exists then the cone of non-negative *-hyperharmonic functions coincides with the corresponding cone of excessive functions.

In paragraphs three and four the relationship between this "adjoint" process and a "direct" process (corresponding to the sheaf $\mathcal{H}$) is axiomatized to cover a certain class of Hunt processes. If G is the intervening "Green" function it is shown in theorem 3.2 that $\hat{R}_E(x, G_y) = \hat{R}_E^*(y, G_x^*)$, where $G(x, y) = G_y(x) = G_x^*(y)$ and "*" indicates balayage with respect to the "adjoint" process.

This formula is then used to find a measure m for which the kernels $V(x, dy) = G(x, y)\, m(dy)$ and $V^*(y, dx) = G(x, y)\, m(dx)$ define dual Hunt processes with the "direct" process corresponding to the sheaf $\mathcal{H}$ and its dual being an "adjoint" process (see theorem 5.4).

Paragraph six concludes the article with a solution of the problem posed at the beginning of this introduction.

The author wishes to thank M. Sieveking for his kind permission to make use of [13] and also of another unpublished work in which the duality question was investigated. In this second work the sheaf $\mathcal{H}$ was essentially assumed to have an adjoint sheaf $\mathcal{H}^*$. The author has been able, by refining [13], to use the "adjoint" process in place of the adjoint sheaf and to extend, in paragraphs three and four, Sieveking's unpublished results on duality to a fairly extensive class of Hunt processes.

The subscripts "b", "c" and "o" are used to denote "bounded", "compact support" and "vanishing at infinity" in the following sense : $\mathfrak{M}_b^+(X)$ is the set of bounded positive Radon measures on X, etc.

The author thanks N.X. Loc for pointing out an error in an earlier proof of lemma 3.1.

1. Basic Lemmas.

The lemmas and proposition in this paragraph are due to Sieveking [13]. Let X denote a locally compact space with countable base and $\mathcal{H}$ a strict harmonic sheaf (in the sense of Bauer [1]) on X for which 1 is superharmonic. Let $\mathcal{E}$ denote the cone of non-negative hyperharmonic functions.

If $A \subset X$ and $u \in \mathcal{E}$, let $R_A u$ and $\hat{R}_A u$ denote respectively the *reduite* and *balayée* of u relative to A. For $\mu \in \mathfrak{M}_b^+(X)$ denote by μ^A the *balayée* of μ relative to A. Denote by $\hat{R}_A$ the map $f \to \hat{R}_A f$, where $\hat{R}_A f(x) = \langle \varepsilon_x^A , f \rangle$. The value of $\hat{R}_A f$ at x will also be denoted by $\hat{R}_A(x , f)$. Then $\hat{R}_A$ is a kernel on $(X , \mathcal{B})$, $\mathcal{B}$ the σ-field of universally measurable subsets of X and $\mu^A = \mu \hat{R}_A$.

Let $\mathcal{S} \subset \mathcal{E}$ be the cone of superharmonic functions. The T-topology on $\mathcal{S}$ is the weak topology defined by the linear maps $u \to R_\Phi u(x)$, with $\Phi \in \mathcal{C}_b^+(X)$ and $x \notin \text{supp } \Phi$. The operator R_Φ is defined by setting $R_\Phi u(x) = \int_0^\infty R_{(\Phi > t)} u(x)\, dt$, $u \in \mathcal{S}$. The basic properties of the T-topology can be found in [4] or [12].

For each $\Phi \in \mathcal{C}_b^+(X)$ and $x \in X$ there is a unique measure ε_x^Φ such that $R_\Phi u(x) = \langle \varepsilon_x^\Phi , u \rangle$, $\forall u \in \mathcal{S}$. The T-topology is metrizable and there exists $(\mu_n) \subset \mathfrak{M}_b^+(X)$, each μ_n of the form ε_x^Φ, $x \notin \text{supp } \Phi$, such that, for $(u_k) \subset \mathcal{S}$, $\lim_k u_k = u \in \mathcal{S}$ if and only if

$$\lim_k \langle \mu_n, u_k \rangle = \langle \mu_n , u \rangle$$

for all n.

LEMMA 1.1. – *Let* $\mu \in \mathfrak{M}^+(X)$ *be such that*

$$(1) \quad \langle \mu , u \rangle < \infty \ \forall u \in \mathcal{S} \quad \text{and} \quad (2) \quad u \to \langle \mu , u \rangle$$

is continuous. If $A \subset X$ *is semipolar then* $\mu(A) = 0$.

Proof. – It suffices to consider A compact and totally thin. Let p be a bounded, continuous, strict potential. Then $\hat{R}_A p(x) < p(x)$, $\forall x \in X$, and $A = \{x \mid \hat{R}_A p(x) \neq R_A p(x)\}$.

There is a decreasing sequence (O_n) of open sets with $A = \bigcap_n O_n$ and $R_A p = \inf_n R_{O_n} p$. Since $\hat{R}_A p = \lim_n R_{O_n} p$,

$$\langle \mu , R_A p \rangle = \langle \mu , \hat{R}_A p \rangle .$$

DEFINITION 1.2. – *A function* $G : X \times X \to \overline{\mathbf{R}}^+$ *will be called a Green function for* $\mathcal{H}$ *if the following conditions are satisfied :*

1) G *is lower semi-continuous, continuous off the diagonal ;*

2) $\forall\, y \in X$, $x \to G(x, y) = G_y(x)$ *is a potential with support* $\{y\}$;

3) *each potential* p *is of the form* $p(x) = \int G(x,y)\, \mu(dy)$, $\mu \in \mathcal{M}^+(X)$ (μ is *then unique* [12]) ; *and*

4) $y \to G_y$ *is continuous.*

Remark. – If $\mathcal{H}$ satisfies the axioms of Brelot and the hypothesis of proportionality, Mme. Hervé proved that a Green function for $\mathcal{H}$ exists (see [7], Proposition 18.1).

From now on it will be assumed that $\mathcal{H}$ has a Green function. If $\mu, \sigma \in \mathcal{M}^+(X)$ define $G\sigma(x) = \int G(x, y)\, \sigma(dy)$ and

$$G^*\mu(y) = \int G(x, y)\, \mu(dx) = \int G^*(y, x)\, \mu(dx)$$

where $G(x, y) = G^*(y, x)$. Then it follows that

$$<\mu, G\sigma> = <\sigma, G^*\mu>.$$

LEMMA 1.3. – *Let* $\mu \in \mathcal{M}^+(X)$ *and let* $\sigma \in \mathcal{M}_c^+(X)$ *not charge any semi-polar set. Then*

$$G^*\mu \geqslant G^*\sigma \ \text{ if } \ G^*\mu(y) \geqslant G^*\sigma(y)\,,\ \forall\, y \in A = \operatorname{supp} \sigma.$$

Proof. – Let p be a bounded strict potential. Then

$$<\sigma, \hat{R}_A p> = <\sigma, R_A p> = <\sigma, p>.$$

Hence $\sigma^A = \sigma \hat{R}_A = \sigma$.

$$\text{Now } G^*\mu(y) \geqslant \int \hat{R}_A(x, G_y)\, \mu(dx) = <\mu, G\nu> = <\nu, G^*\mu>$$

$$\geqslant <\nu, G^*\sigma> = <\sigma, G\nu> = \int \hat{R}_A(x, G_y)\, \sigma(dx)$$

$$= <\sigma^A, G_y> = G^*\sigma(y),$$

because if $\hat{R}_A G_y = G\nu$, then $\operatorname{supp} \nu \subset A$.

Let f, g be non-negative functions on X. Set $f \in o(g)$ if for any $\varepsilon > o$ there exists a compact set $K \subset X$ with $f(x) \leqslant \varepsilon g(x)$ on $X \backslash K$ (see [3]).

LEMMA 1.4. – *Let* $\sigma \in \mathcal{M}_c^+(X)$ *be finite and continuous on* $\mathcal{S}$. *Then there exists* $\mu \in \mathcal{M}^+(X)$ *such that* : (1) $G^*\sigma \in o(G^*\mu)$; (2) $G^*\mu \in \mathcal{C}(X)$; *and* (3) $\mu \geqslant \sigma$.

Proof. – There exists an increasing sequence (A_n) of compact subsets A_n of X with $A = \text{supp}\ \sigma \subset \mathring{A}_n \subset A_n \subset \mathring{A}_{n+1}, \forall_n$, and $X = \bigcup_n A_n$. Let $\varphi_n \in \mathcal{C}(X)$ be such that $1_{X \setminus A_{n+1}} \leqslant \varphi_n \leqslant 1_{X \setminus A_n}$.

For each n, set $f_n(y) = \langle \sigma, R_{\varphi_n} G_y \rangle$. The function f_n is continuous. Let $x \in A$. The measures $\varepsilon_x^{\varphi_n}$ are carried by the compact set $K_n = A_{n+1} \setminus \mathring{A}_n$. The continuity of G implies, since $\|\varepsilon_x^{\varphi_n}\| \leqslant 1$, that on $X \setminus K_n$ the functions $y \to R_{\varphi_n}(x, G_y)$ are equicontinuous. Hence, f_n is continuous on $X \setminus K_n$. If $y_k \to y$ with $(y_k) \subset X \setminus A_{n-1}$, the functions $x \to R_{\varphi_n}(x, G_{y_k}) \leqslant G(x, y_k)$ and $x \to R_{\varphi_n}(x, G_y) \leqslant G(x, y)$ are uniformly bounded on the compact set A. Hence, since

$$R_{\varphi_n} G_{y_k} \to R_{\varphi_n} G_y \text{ on } \mathring{A}_n$$

the Lebesgue Dominated Convergence theorem implies that

$$\langle \sigma, R_{\varphi_n} G_{y_k} \rangle \to \langle \sigma, R_{\varphi_n} G_y \rangle .$$

Since G_y is a potential, $R_{\varphi_n} G_y \downarrow 0$ as $n \to \infty$. Hence, the functions $f_n \downarrow 0$ locally uniformly and so there exists $m(n) = m$ with $f_m(y) \leqslant 2^{-n}$ on A_n. Consequently, the function $f = \sum_n f_{m(n)} \in \mathcal{C}(X)$.

Let $\tau_n = \sigma R_{\varphi_n}$. Then since $f_n(y) = G^*\tau_n(y) = G^*\sigma(y)$ on $\complement A_{n+1}$ it follows that $G^*\sigma \in o(f)$. Set $\mu = \sigma + \sum_n \tau_{m(n)}$.

With the aid of this lemma it can be shown that there exists a Green function for $\mathcal{H}$ which decreases to zero at infinity in the variable y in a certain sense.

PROPOSITION 1.5. – *There exists a Green function* G *for* $\mathcal{H}$ *such that* :

1) $\forall\ \sigma \in \mathcal{M}_c^+(X)$ *finite, continuous on* $\mathcal{S}$

$G^*\sigma \in \mathcal{C}_0(X)$; *and*

2) $\exists\ \eta \in \mathcal{M}^+(X)$ *with* $G^*\eta = 1$.

Proof. – Let G be a Green function for $\mathcal{H}$ and let $(\rho_n) \subset \mathfrak{M}^+(X)$ be a sequence that defines the T-topology on $\mathcal{S}$. Let

$$X = \bigcup_m A_m \,,\ A_m \subset \mathring{A}_{m+1} \subset A_{m+1}$$

compact, $\forall m$, and set $\rho_{nm} = \rho_n \,|\, A_m$. Write the family (ρ_{nm}) as a sequence (τ_n). For each $y \in X$ there exists n with $< \tau_n , G_y > \neq 0$ as otherwise $G_y = 0$.

For each n, let $\mu_n \in \mathfrak{M}^+(X)$ be such that :

(1) $G^* \tau_n \in o(G^* \mu_n)$, (2) $G^* \mu_n \in \mathcal{C}^+(X)$, and (3) $\mu_n \geqslant \tau_n$.

By induction on q it can be shown that there exists a sequence of sequences $(b_{nq}) \subset R^+$ with the following properties :

1) $\forall q \,,\ \sum_n b_{nq} (G^* \mu_n \,|\, A_q)$ converges in the supremum norm to a continuous function on A_q ;

2) $\forall q \,,\ \sum_n b_{nq} (\mu_n \,|\, A_q) \in \mathfrak{M}^+(A_q)$; and

3) $\forall n, \forall q \ \ b_{n(q+1)} \leqslant b_{nq}$.

Set $a_n = b_{nn}$. Then $g = \sum_n a_n G^* \mu_n \in \mathcal{C}(X)$ and

$$\mu = \sum_n a_n \mu_n \in \mathfrak{M}^+(X).$$

Let $\sigma \in \mathfrak{M}_c^+(X)$ be finite and continuous on $\mathcal{S}$. Then $G^* \sigma \in o(g)$. There exists a finite sequence $n_1, \ldots, n_m$ and a constant $b > 0$ such that $b \left(\sum_{i=1}^m G^* \tau_{n_i} \right) \geqslant G^* \sigma$ on supp σ_m and hence by lemma 1.3 on X. Clearly, $\sum_{i=1}^m G^* \tau_{n_i} \in o \left(\sum_{i=1}^m G^* \mu_{n_i} \right)$ and so $G^* \sigma \in o(g)$.

Define $G_1(x, y) = G(x, y)/g(y)$. This is a Green function for $\mathcal{H}$ and has the desired properties.

2. Adjoint Resolvents.

In this paragraph let G be a fixed Green function for $\mathcal{H}$ such that (1) $\sigma \in \mathcal{M}_c^+(X)$, finite, continuous on $\mathcal{S}$ implies $G^* \sigma \in \mathcal{C}_0(X)$ and (2) there exists $\eta \in \mathcal{M}^+(X)$ with $G^* \eta = 1$.

Denote by $\mathcal{E}^*$ the convex cone of functions f which are the limit of an increasing sequence of functions of the form

$$G^* \mu \, , \mu \in \mathcal{M}^+(X).$$

DEFINITION 2.1. – *A submarkovian resolvent* $(V_\lambda)_{\lambda>0}$ *is said to be adjoint to* $\mathcal{H}$ *(relative to G) if* $\mathcal{E}^*$ *is the corresponding cone of excessive functions. A kernel* V *is said to be adjoint to* $\mathcal{H}$ *(relative to* G*) if it has a resolvent which is adjoint to* $\mathcal{H}$.

Remark. – Multiplication of $G(x, y)$ by $f(y)$, f continuous strictly positive leads to another Green function

$$G_1(x, y) = G(x, y) f(y).$$

The resolvent $(W_\lambda)_{\lambda>0}$ with $W_\lambda(y, g) = f(y) V_\lambda(y, g/f)$ is adjoint to $\mathcal{H}$ relative to G_1, and submarkovian if $1/f$ is supermedian relative to $(V_\lambda)_{\lambda>0}$.

DEFINITION 2.2. – *A measure* $\mu \in \mathcal{M}^+(X)$ *is said to be admissable if it has the following properties :*

1) $O = \emptyset$ *fine open, universally measurable* $\Rightarrow \mu(O) > 0$;

2) $A \subset X$ *semipolar* $\Rightarrow \mu(A) = 0$;

3) $G^* \mu \in \mathcal{C}_0(X)$;

4) *there exists* $(\rho_n) \subset \mathcal{M}^+(X)$ *which defines the T-topology and such that each* ρ_n *is absolutely continuous with respect to* μ.

PROPOSITION 2.3. – *Admissable measures exist.*

Proof. – Let $(\rho_n) \subset \mathcal{M}^+(X)$ be a sequence that defines the T-topology on $\mathcal{S}$ and let (A_m) be an increasing sequence of compact sets with $X = \bigcup_m A_m$.

Define $\rho_{nm} = \rho_n \mid A_m$ and write the family (ρ_{nm}) as a sequence (τ_n) Then there exists $(a_n) \subset \mathbf{R}^+$ such that

$$(1) \sum_n a_n(G^* \tau_n) \in \mathcal{C}_0(X) \quad \text{and} \quad (2) \sum_n a_n \tau_n(X) < +\infty.$$

Let $\mu = \sum a_n \tau_n$. Clearly μ satisfies (2), (3) and (4) of definition 2.2. Let $O \neq \phi$ be fine open and let p be a bounded, continuous, strict potential. Then $\hat{R}_{X\backslash O}\, p \neq p$ and so

$$<\mu , p> \; > \; <\mu , \hat{R}_{X\backslash O}\, p> \; = \; <\mu , R_{X\backslash O}\, p>.$$

Hence, $\mu(O) > 0$ if O is universally measurable.

PROPOSITION 2.4. – *Let* $\mu \in \mathcal{M}^+(X)$ *be an admissable measure and denote by* W^* *the kernel* $W^*(y , dx) = G^*(y , x)\, \mu(dx)$. *If* φ, $\psi \in \mathcal{C}_c^+$ *and*

$$1 + W^* \varphi \geqslant W^* \psi \text{ on } (\psi > 0),$$

then $1 + W^* \varphi \geqslant W^* \psi$.

Hence, W^* *satisfies the complete maximum principle.*

Proof. – Let $\psi_n = \max(\psi , 1/n) - 1/n$. By considering these functions it can be seen that it suffices to prove that $1 + W^* \varphi \geqslant W^* \psi$ given that the inequality holds on supp $\psi = A$.

Denote by σ the measure $\sigma(dx) = \psi(x)\, \mu(dx)$. Then $A = \operatorname{supp} \sigma$. If $\eta \in \mathcal{M}^+(X)$ is such that $G^* \eta = 1$ then

$$1 + W^* \varphi = G^* \tau , \tau = \eta + \psi . \mu .$$

Lemma 1.3 implies $G^* \tau \geqslant G^* \sigma$, i.e. $1 + W^* \varphi \geqslant W^* \psi$.

The remark (b) following theorem XT4 in [9] shows that W^* satisfies the complete maximum principle.

THEOREM 2.5. – *Let* W^* *be the kernel*

$$W^*(y , dx) = G^*(y , x)\, \mu(dx)$$

with μ *an admissable measure. Then* W^* *satisfies the hypotheses of Hunt's theorem,* i.e. *the following conditions :*

1) W^* *is a continuous dispersion kernel ;*

2) W^* *tends to zero at infinity ;*

3) $W^*(\mathcal{C}_c)$ *is dense in* $\mathcal{C}_0$;

4) W^* *satisfies the complete maximum principle.*

Further, $W^*1 \in \mathcal{C}_0(X)$ *and* $f \in \mathcal{B}^+$ *implies* W^*f *is lower semi-continuous.*

Proof. – (1) and (2) are clear and (4) has been proved.

Let $H = \{ W^*\varphi + \alpha \mid \varphi \in \mathcal{C}_c , \alpha \in \mathbf{R}\}$. Then if X_∞ is the one-point compactification of X, $H \subset \mathcal{C}(X_\infty)$. Suppose $m \in \mathcal{M}(X_\infty)$ is such that $< m , h > = 0 \ \forall \, h \in H$. Denote $m_\pm \mid X$ by $\nu_\pm$ and let $\rho_\pm = \nu_\pm W^*$. Then $\rho_+ = \rho_-$.

Let $f_n \in \mathcal{B}^+$ be such that $\rho_n(dx) = f_n(x)\, \mu(dx)$, where

$$(\rho_n) \subset \mathcal{M}^+(X)$$

is a sequence that defines the T-topology with each ρ_n absolutely continuous with respect to μ. Then it follows that

$$< \rho_\pm , f_n > = < \nu_\pm , W^* f_n > = \iint \nu_\pm(dy)\, G^*(y , z) f_n(z)\, \mu(dz)$$

$$= < \rho_n , G\nu_\pm > .$$

Hence, $\nu_+ = \nu_-$ if the functions $G\nu_\pm$ are superharmonic, i.e. finite on a dense set.

Let $D = \overline{\{x \mid G\nu_+(x) < \infty\}}$ and let O be open with $O \cap D = \emptyset$. Then $\mu(O)\,(+\infty) = \int_O \int \nu_+(dy)\; G(x , y)\; \mu(dx)$

$$= \int_O W^*(y , 1)\; \nu_+(dy) < + \infty.$$

Hence, $O = \emptyset$.

This contradiction implies that H is dense in $\mathcal{C}(X_\infty)$ since clearly $m_+(\omega) = m_-(\omega)$ if ω is the point at infinity.

Remarks. – 1) The proof that H is dense in $\mathcal{C}(X_\infty)$ is due to Sieveking [13].

2) Unless G is such that $1 \in \mathcal{E}^*$ the complete maximum principle appears false. At any rate W^* will satisfy the principle of domination. When $1 \in \mathcal{E}^*$ then for $\varepsilon > 0$ there exists $\tau \in \mathfrak{M}^+(X)$ with

$$G^* \tau \leqslant 1 \quad \text{and} \quad (1 + \varepsilon)\, G^* \tau + W^* \varphi \geqslant W^* \psi$$

on supp ψ (and hence on X) whenever $1 + W^* \varphi \geqslant W^* \psi$ on supp ψ. Consequently proposition 2.4 is still valid.

PROPOSITION 2.6. – *Let μ be an admissable measure. The kernel $W^*(y\,,dx) = G^*(y\,,x)\,\mu(dx)$ is adjoint to $\mathcal{H}$.*

Proof. – Since W^* is a Hunt kernel it is the potential kernel of a Feller semigroup. As a result it has a submarkovian resolvent $(W^*_\lambda)_{\lambda>0}$.

Any $(W^*_\lambda)_{\lambda>0}$ excessive function is the limit of an increasing sequence of functions of the form W^*f, $f \in \mathcal{B}^+$, and so is in $\mathcal{E}^*$.

Let $\mu \in \mathfrak{M}^+(X)$. Then since $G^* \mu$ is lower semi-continuous the proof of theorem XT4 in [9] shows that $G^* \mu$ is supermedian. The resolvent $(W^*_\lambda)_{\lambda>0}$ is such that $\lambda W^*_\lambda(x\,,\varphi) \to \varphi(x)$ as $\lambda \to \infty, \forall\, x \in X$ and $\varphi \in \mathcal{C}_c$, and hence (see proposition 6 in [2]) a lower semi-continuous supermedian function is excessive

The cone $\mathcal{E}^*$ is the cone of excessive functions relative to a Feller semigroup. Further, the hypothesis (L) of P.A. Meyer is satisfied and so, for any set $E \subset X$ and $u \in \mathcal{E}^*$, the reduite and balayée of u relative to E are defined. They will be denoted by $R^*_E u$ and $\hat{R}^*_E u$ respectively. The operator $\hat{R}^*_E$ is a kernel, usually denoted in probabilistic theory by P_E. The functions $R^*_E u$ and $\hat{R}^*_E u$, $u \in \mathcal{E}^*$, depend only on the cone $\mathcal{E}^*$ and not on the particular process which is used to define them. The prefix "*" will be used to indicate that the object is to be understood relative to $\mathcal{E}^*$ or $(W^*_\lambda)_{\lambda>0}$.

PROPOSITION 2.7. – *Let $O \subset X$ be open and $x\,, y \in O$. Then it follows that :*

1) $\hat{R}_O G_y = G_y$; *and*

1*) $\hat{R}^*_O\, G^*_x = G^*_x$;

where $G_y(x) = G^*_x(y) = G(x\,,y)$. Further, (1*) *holds if* O *is fine open and universally measurable.*

Proof. – (1) is well known. It is an immediate consequence of Korollar 2.4.3 in [1].

To prove (1*) for O fine open, universally measurable, let $A \subset O$ be universally measurable. Then

$$\int G(x, y)\, 1_A(x)\, \mu(dx) = W^*(y, A) = \hat{R}^*_O(y, W^* 1_A)$$

$$= \iint \hat{R}^*_O(y, dz)\, G(x, z)\, 1_A(x)\, \mu(dx).$$

Hence, $\forall\, y \in X$ fixed, $h_0(x) = G(x, y) \geqslant h_1(x) = \hat{R}^*_O(y, G^*_x)$ agree μ – a.e. on O. Condition (1) in definition 2.2 implies $h_0 = h_1$ on O.

Let μ denote a fixed admissable measure and denote by $\mathfrak{S}^* = \mathfrak{S}^*(\mu) \subset \mathcal{S}^*$, the cone of excessive functions (with respect to $W^*(y, dx) = G^*(y, x)\, \mu(dx)$) which are finite except on a set A of potential $W^* 1_A$ equal to zero.

Mokobodzki has shown (see for example [11] proposition 7) that there is a positive measure ν on X and an increasing sequence (X_n) of universally measurable sets X_n with the following properties :

1) for A universally measurable, $\nu(A) = 0 \Leftrightarrow W^* 1_A = 0$;

2) $\forall_n,\ u \in \mathfrak{S}^* \Rightarrow \int_{X_n} u\, d\nu < +\infty$;

3) $\nu(X \setminus \bigcup_n X_n) = o$;

4) the topology T on $\mathfrak{S}^*$ defined by the family (p_n) of semi-norms p_n, with $p_n(u - v) = \int_{X_n} |u - v|\, d\nu$, is such that $\mathfrak{S}^*$ is a union of metrizable caps K.

Further, the caps K referred to in (4) are such that $u \leqslant v$, $v \in K$ implies $u \in K$.

Let $(u_n) \subset \mathfrak{S}^*$ be monotone decreasing. Then, in view of the property of the caps, if $u = (\inf_n u_n)\hat{}$ it follows that $u = \lim_n u_n$, with respect to T. Let $w = \lim_n u_{n_k}$. Then $w = \inf_n u_{n_k} = u_n$ a.e. $d\nu$. Since $u = \inf u_n$ a.e. $d\nu$ it follows that $u = w$.

DEFINITION 2.8. – *Let $\mathcal{C}$ be a cone of non-negative measurable functions on a measure space* $(X, \mathcal{B})$. *An element $p \in \mathcal{C}$ is said to be strict if, for any two non-negative bounded measures μ, ν on $\mathcal{B}$, the following condition holds :*

$$<\mu , \mu> \leqslant <\nu , u> , \forall u \in \mathcal{C} \text{ and}$$

$$<\mu , p> = <\nu , p> < +\infty$$

implies $\mu = \nu$.

When $\mathcal{C}$ generates $\mathcal{B}$ and is the cone of excessive functions, relative to a submarkovian resolvent $(V_\lambda)_{\lambda > 0}$ with $V = V_0$ proper then there exist finite (even bounded) strict excessive functions of the form Vf (c.f. [15]). If V is bounded then V1 is strict. Here one assumes that 1 is excessive and that the minimum of two excessive functions is excessive.

PROPOSITION 2.9. – *Let* $\alpha \in \mathcal{M}^+(X)$ *be finite and T-continuous on* $\mathcal{S}^*$. *Then, if* A *is *-semipolar* (i.e. *wrt* W^*) $\alpha(A) = 0$.

Proof. – As before it suffices to consider A totally thin. Then if p is a finite strict *-excessive function, for example $p = W^* 1$, it follows that $A = \{x \in X \mid R_A^* p(x) > \hat{R}_A^* p(x)\}$.

Since the hypothesis (L) of P.A. Meyer is satisfied, there exists a decreasing sequence $(w_n) \subset \mathcal{S}^*$ with

$$\inf_n w_n \geqslant R_A^* p \geqslant (\inf_n w_n)^\wedge = \hat{R}_A^* p$$

(see [10]). The continuity of α implies

$$<\alpha , \inf w_n> = <\alpha , (\inf_n w_n)^\wedge> .$$

If $z \in X$, G_z is a potential and so $G_z(z) = +\infty$ implies $\{z\}$ is a polar set and hence $\mu(\{z\}) = 0$. Consequently, $\forall x \in X$, $G_x^* \in \mathcal{S}^*$.

PROPOSITION 2.10. – *Let* $O \neq \emptyset$ *be a *-fine open universally measurable set. Then there exists a nontrivial measure* $\eta \in \mathcal{M}^+(X)$ *such that :*

1) $G\eta < +\infty$ *everywhere ; and*

2) $\eta(X \backslash O) = 0$.

Proof. – Let p be a finite strict $*$-excessive function. Then $\hat{R}^*_{\complement O} p \neq p$ and so there exists $\alpha \in \mathfrak{M}^+(X)$ finite and continuous on $\mathcal{S}^*$ with $<\alpha , \hat{R}^*_{\complement O} p> \neq <\alpha , p>$. Since $<\alpha , \hat{R}^*_{\complement O} p> = <\alpha , R^*_{\complement O} p>$ this implies $\alpha(O) > 0$.

Let $\eta = \alpha \mid O$. Then $G\eta(x) \leqslant G\alpha(x) = <\alpha , G^*_x> + \infty$, $\forall\, x \in X$.

Remark. – The "dual" of this proposition is clearly true in view of lemma 1.1.

PROPOSITION 2.11. – *There exist Radon measures* η , σ *on* X *with the following properties :*

1) *the potential* $p = G\eta$ *is bounded, continuous and strict ;*

2) $G^*\eta$ *is finite on a dense set ;*

(1*) σ *is admissable ; and*

(2*) $G\sigma$ *is finite on a dense set.*

Proof. – Let $\nu \in \mathfrak{M}^+(X)$ be such that $G\nu$ is bounded, continuous and strict. Let $(\nu_n) \subset \mathfrak{M}^+_c(X)$ be such that $\nu = \sum\limits_n \nu_n$. Then there exists $(a_n) \subset (0 , 1)$ such that :

(i) $\sum\limits_n a_n \int (G\nu)\, d\nu_n < +\infty$; and

(ii) $\sum\limits_n a_n (G\nu_n) \in \mathcal{C}_b(X)$.

Let $\eta = \sum\limits_n a_n \nu_n$. Then $G\eta$ satisfies (1). Suppose $G^*\eta$ is infinite on an open set O. In view of (i) $<\eta , p> < +\infty$ and so

$$+\infty > <\eta , p> = \iint G(x , y)\, \eta(dx)\, \eta(dy) \geqslant$$

$$\geqslant \int_O G^*\eta(y)\, \eta(dy) = (+\infty)\, \eta(O)$$

The same argument applies when ν is taken to be an admissable measure.

This section concludes with a proposition that relates the two theories of balayage.

PROPOSITION 2.12. – *if* $E \subset X$ *is closed or open,*

$$\hat{R}_E(x, G_y) = \hat{R}^*_E(y, G^*_x).$$

Proof. – Assume E is open and fix $y \in X$. Then $x \to \hat{R}^*_E(y, G^*_x)$ is an excessive function which by proposition 2.7 (1*) agrees with $G(x, y)$ on E. Hence, $\hat{R}^*_E(y, G^*_x) \geqslant \hat{R}_E(x, G_y)$ and the result follows by symmetry.

Assume E is closed and $x \notin E$. Then G^*_x is continuous on a neighbourhood of E and so

$$R^*_E(y, G^*_x) = \inf_O R^*_O(y, G^*_x)$$

$$= \inf_O R_O(x, G_y)$$

$$\geqslant R_E(x, G_y),$$

where the infimum is taken over the family of open sets $O \supset E$.

Fix $x \notin E$. Let $h_1(y) = R^*_E(y, G^*_x)$ and

$$h_2(y) = R_E(x, G_y) = \hat{R}_E(x, G_y).$$

The function h_2 is *-excessive and $h_1 \geqslant h_2$. Hence, if

$$h_3(y) = \hat{R}^*_E(y, G^*_x)$$

then $h_3 \geqslant h_2$.

Let $A = (h_3 > h_2)$. Then $A \subset E$. If not there exists, as a consequence of proposition 2.10, a compact set $B \subset A \backslash E$ and a nontrivial measure $\eta \in \mathcal{M}^+(X)$ carried by B with $G\eta$ everywhere finite. The function $G\eta$ is continuous on a neighbourhood of E. Consequently,

$$< \eta, h_2 > = \hat{R}_E(x, G\eta) = \inf_O R_O(x, G\eta)$$

$$= \inf < \eta, R_O(x, G_\bullet) > = \inf_O < \eta, R^*_O(\bullet, G^*_x) >$$

$$\geqslant < \eta, \hat{R}^*_E G^*_x > = < \eta, h_3 >,$$

where the infimum is taken over the family of open sets $O \supset E$. This contradiction implies $A \supset E$.

The lemma 2.13 below implies $h_3 = G^*\alpha$ for some $\alpha \in \mathfrak{M}^+(X)$ and by definition $h_2 = G^*\beta$, $\beta(dz) = \hat{R}_E(x, dz)$. Let $p = Gm$ be a finite strict continuous potential for the sheaf $\mathcal{H}$. Then, if $A \neq \emptyset$, $m(A) > 0$. This follows from the observation that $h_3 \geqslant h_2$ implies $\beta \prec \alpha$ and $m(A) = 0$ implies

$$<\beta, p> = <m, h_2> = <m, h_3> = <\alpha, p> \leqslant <m, G_x^*> = p(x) < +\infty.$$

The type of argument used in the proof of proposition 2.7 to prove (1*) for a fine open set O shows that, for x fixed,

$$\{y \in E \mid G(x, y) > \hat{R}_E(x, G_y)\}$$

is a null set for m. Consequently, $A = \emptyset$ and $h_3 = h_2$.

Fix $y \in X$ and let $g_1(x) = \hat{R}_E^*(y, G_x^*)$ and $g_2(x) = \hat{R}_E(x, G_y)$. These two excessive functions agree on $X \backslash E$ and on the fine interior of E $g_2(x) = G(x, y) \geqslant g_1(x)$. Hence, $\hat{R}_E(x, G_y) \geqslant \hat{R}_E^*(y, G_x^*)$.

The dual argument proceeds in exactly the same way up to the moment where it is shown that $A \subset E$ implies $A = \emptyset$. In the dual case the set A is fine open and so by proposition 2.7 if $x \in A$,

$$h_2^*(x) = \hat{R}_E^*(y, G_x^*) \geqslant \hat{R}_A^*(y, G_x^*) = G(x, y) \geqslant h_3^*(x).$$

Hence, $A = \emptyset$. The remainder of the argument is formal.

Remark. – The basic idea of this proof is due to Sieveking (in the axiomatic context referred to in the introduction).

LEMMA 2.13. – *Let $u \leqslant G_x^*$ be a *-excessive function. Then there exists $\alpha \in \mathfrak{M}^+(X)$ with $u = G^*\alpha$.*

Proof. – There exists an increasing sequence $(G^*\eta_n)$ with $u = \sup_n G^*\eta_n$. The sequence of measures (η_n) is increasing for the balayage order $\prec$ defined by the sheaf and is dominated by ε_x (relative to $\prec$). Consequently, this sequence has a weak limit α. Since $G^*\alpha \leqslant \liminf G^*\eta_n = u$ it suffices to prove $\eta_n \prec \alpha$, for each n.

Since the continuous potentials with compact support determine $\prec$ it is enough to show that $<\eta_n , q> \leqslant <\alpha , q>$ if q is continuous and $\leqslant p$, where p is a finite continuous strict potential.

Let r be a continuous finite potential with $p \in o(r)$. Then, if $\varepsilon > 0$ there exists $\varphi \in \mathcal{C}_c(x)$ with (i) $0 \leqslant \varphi \leqslant 1$ and (ii) $p \leqslant \varepsilon r$ on $(\varphi < 1)$. Hence, if $0 \leqslant q \leqslant p$ is a continuous potential, for each n, $<\eta_n , q> \leqslant <\eta_n , q\varphi> + \varepsilon r(x)$ and so

$$<\eta_n , q> \leqslant <\alpha , q> + 2\varepsilon r(x)$$

in n is sufficiently great.

3. Balayage and *-balayage

Let X be locally compact with a countable base and let $\mathcal{B}$ be the σ-field of universally measurable sets. Denote by $(V_\lambda)_{\lambda>0}$ and $(V^*_\lambda)_{\lambda>0}$ the resolvent families of two Hunt semigroups on X, each of which satisfies the hypothesis (L) of P.A. Meyer. Denote by $\mathcal{S}$ and $\mathcal{S}^*$ the corresponding cones of excessive functions finite except on a set of potential zero. Assume $1 \in S \cap S^*$. For $E \subset X$, denote by R_E, $\hat{R}_E$ and R^*_E, $\hat{R}^*_E$ the corresponding operators defining the reduite and the balayée of an excessive function.

Let $G : X \times X \to \mathbf{R}^+$ be lower semi-continuous and satisfy the following conditions :

1) $G(x , y) < +\infty$ if $x \neq y$;

2) $\forall\, x , y \in X$, $G_y \in \mathcal{S}$ and $G^*_x \in \mathcal{S}^*$, where

$$G_y(x) = G^*_x(y) = G(x , y) = G^*(y , x) \;;$$

3) there exist $\nu , \nu^* \in \mathcal{M}^+(X)$ such that for $V = V_0$, $V^* = V^*_0$

$$V(x , dy) = G(x , y)\, \nu(dy) \text{ and}$$

$$V^*(y , dx) = G^*(y , x)\, \nu^*(dx) \;;$$

4) V and V^* are proper and $\forall x , y \in X$,

$$V(x , dy) \neq 0 , V^*(y , dx) \neq 0 \;;$$

5) Let $O \neq \emptyset$ be measurable and fine (resp. *-fine) open. Then there exists $\eta \neq 0$ in $\mathfrak{M}^+(X)$ such that

(i) $G^*\eta < +\infty$ everywhere (resp. $G\eta < +\infty$ everywhere) ;

and

(ii) $\eta(X \setminus O) = 0$.

6) $E \subset X$ closed, $x, y \in X$, imply $\hat{R}^*_E(y, G^*_x) = \hat{R}_E(x, G_y)$;

7) The σ-fields generated by the excessive and *-excessive functions both contain all Borel sets.

Note that "*-fine open" means "fine open relative to the resolvent $(V^*_\lambda)_{\lambda > 0}$". The prefix *-will be consistently used in this manner.

Remark. – Let $(V_\lambda)_{\lambda > 0}$ be the resolvent defined by a bounded continuous strict potential p. Let G be a Green function of the type considered in paragraph two. Let $(V^*_\lambda)_{\lambda > 0}$ be the resolvent defined by an admissable measure. Then the above hypotheses are satisfied.

LEMMA 3.1. – *Let O be a *-fine open K_σ-set. Then,* $\forall\, x, y \in X$,

$$R^*_O(y, G^*_x) = \hat{R}_O(x, G_y).$$

Proof. – Let $O = \cup E_n$, (E_n) an increasing sequence of closed sets. Then, (6) implies that

$$R^*_O(y, G^*_x) = \hat{R}^*_O(y, G^*_x) = \lim_n \hat{R}^*_{E_n}(y, G^*_x)$$

$$= \lim_n \hat{R}_{E_n}(x, G_y) = \hat{R}_O(x, G_y).$$

Since the excessive and *-excessive functions are lower semi-continuous the following result is a formal consequence of lemma 3.1. The proof, which is a variant of the proof of proposition 2.12, is given in full detail for the reader's convenience.

THEOREM 3.2. – $\forall\, E \subset X$,

$$\hat{R}_E(x, G_y) = \hat{R}^*_E(y, G^*_x).$$

Proof. – If $x \notin \mathrm{E}$,

$$\begin{aligned} \mathrm{R}^*_{\mathrm{E}}(y\,,\mathrm{G}^*_x) &= \inf_{\mathrm{O}} \mathrm{R}^*_{\mathrm{O}}(y\,,\mathrm{G}^*_x) \\ &= \inf_{\mathrm{O}} \hat{\mathrm{R}}_{\mathrm{O}}(x\,,\mathrm{G}_y) \\ &\geqslant \hat{\mathrm{R}}_{\mathrm{E}}(x\,,\mathrm{G}_y), \end{aligned}$$

where the infimum is taken over the family of *-fine open K_σ-sets $\mathrm{O} \supset \mathrm{E}$.

Fix $x \notin \mathrm{E}$. Let $h_1(y) = \mathrm{R}^*_{\mathrm{E}}(y\,,\mathrm{G}^*_x)$ and

$$h_2(y) = \hat{\mathrm{R}}_{\mathrm{E}}(x\,,\mathrm{G}_y) = \int \hat{\mathrm{R}}_{\mathrm{E}}(x\,,dz)\,\mathrm{G}(z\,,y).$$

The function h_2 is *-excessive and $h_1 \geqslant h_2$. Hence, if

$$h_3(y) = \hat{\mathrm{R}}^*_{\mathrm{E}}(y\,,\mathrm{G}^*_x)$$

it follows that $h_3 \geqslant h_2$.

Let $\mathrm{A} = (h_3 > h_2)$. If $\mathrm{A} \neq \emptyset$ then (5) implies there exists $\eta \in \mathfrak{M}^+(\mathrm{X})$, $\eta \neq 0$, with $\mathrm{G}\eta < +\infty$ and $\eta \neq 0$, with $\mathrm{G}\eta < +\infty$ and $\eta(\mathrm{X}\setminus\mathrm{A}) = 0$. Consequently,

$$\begin{aligned} <\eta\,,h_2> = \mathrm{R}_{\mathrm{E}}(x\,,\mathrm{G}\eta) &= \inf_{\mathrm{O}} \mathrm{R}_{\mathrm{O}}(x\,,\mathrm{G}\eta) = \inf_{\mathrm{O}} <\eta\,,\mathrm{R}_{\mathrm{O}}(x\,,\mathrm{G}_\bullet)> \\ &= \inf_{\mathrm{O}} <\eta\,,\hat{\mathrm{R}}^*_{\mathrm{O}}(\bullet,\mathrm{G}^*_x)> \geqslant <\eta\,,\hat{\mathrm{R}}^*_{\mathrm{E}}\,\mathrm{G}^*_x> = <\eta\,,h_3>, \end{aligned}$$

where the infimum is taken over the family of fine open K_σ-sets $\mathrm{O} \supset \mathrm{E}$ and the "dual" form of lemma 3.1 is used. Hence, $\mathrm{A} = \emptyset$ and so, if $x \notin \mathrm{E}$, $\hat{\mathrm{R}}^*_{\mathrm{E}}(y\,,\mathrm{G}^*_x) = \hat{\mathrm{R}}_{\mathrm{E}}(x\,,\mathrm{G}_y)$.

Fix $y \in \mathrm{X}$. Let $g_1(x) = \hat{\mathrm{R}}^*_{\mathrm{E}}(y\,,\mathrm{G}^*_x)$ and let $g_2(x) = \hat{\mathrm{R}}_{\mathrm{E}}(x\,,\mathrm{G}_y)$. These two excessive functions agree on $\mathrm{X}\setminus\mathrm{E}$ and on the fine interior of E $g_2(x) = \mathrm{G}(x\,,y) \geqslant g_1(x)$. Hence, $\hat{\mathrm{R}}_{\mathrm{E}}(x\,,\mathrm{G}_y) \geqslant \hat{\mathrm{R}}^*_{\mathrm{E}}(y\,,\mathrm{G}^*_x)$.

Remark. – This proof, but using open sets in place of fine open sets and making more use of continuity, was given by Sieveking in the axiomatic setting referred to in the introduction.

COROLLARY 3.3. – *Assume the hypotheses of paragraph two. The Hunt process defined by an admissable measure μ is a diffusion.*

Proof. – The theorem implies that for any open set O and $y \in O$ the measure $\hat{R}^*_{\complement O}(y, \cdot)$ is carried by ∂O. Hence by a lemma of Courrège and Priouret c.f. lemma 6.1 [16], the Feller semigroup $(P_t)_{t>0}$ for which $\int_0^\infty P_t dt = W^*$, where $W^*(y, dx) = G^*(y, x)\, \mu(dx)$, is a diffusion.

COROLLARY 3.4. – *Let $\mathcal{H}$ be a Brelot sheaf which has a positive potential and which satisfies the hypothesis of proportionality. Denote by G a Green function for $\mathcal{H}$ which satisfies the conditions in proposition 1.5. Then, $\forall\, y \in X, \forall\, E \subset X$, the measure $\hat{R}^*_E(y, dx) = \sigma^E_y(dx)$, where $\sigma^E_y(dx)$ is the measure defined by Mm. Hervé* (see [7]) *for which $\hat{R}_E G_y = G\sigma^E_y$.*

Proof. – In view of the remark preceding lemma 3.1 the measure $\hat{R}^*_E(y, \cdot)$, intrinsically defined by $\mathcal{S}^*$ and also by any admissable measure, satisfies $\hat{R}^*_E(y, G^*_x) = \hat{R}_E(x, G_y)$ where this second balayée is intrinsically defined by the sheaf $\mathcal{H}$.

COROLLARY 3.5. – *Let $\mathcal{H}$ and G satisfy the hypotheses of the previous corollary and let W^* be the kernel defined by means of an admissable measure μ and G. If G defines an adjoint sheaf $\mathcal{H}^*$* (see [7]) *then the cone of excessive functions, relative to W^*, coincides with the cone $\mathcal{C}^*$ of non-negative *-hyperharmonic functions.*

Proof. – As $W^*(\mathcal{C}_c)$ is dense in $\mathcal{C}_0$ corollaire 1* p. 552 in [7] implies that $\hat{R}^*_E$ is the corresponding balayage kernel for $\mathcal{H}^*$. Hence, $u \in \mathcal{C}^*$ implies $u + W^*\varphi \geqslant W^*\psi$ if it holds on $\overline{\{\psi > 0\}}$ and so $\mathcal{C}^* \subset \mathcal{S}^{*\prime}$, the cone of *-supermedian functions. Corollary 3.4 implies $\mathcal{S}^* \subset \mathcal{C}^*$ and as W^*1 is strict for $\mathcal{S}^*$, $\mathcal{S}^* = \mathcal{C}^*$ by [16] corollary 1.8.

COROLLARY 3.6. – *Let $A \subset X$ and let $s = G\mu$, $\mu \in \mathfrak{M}^+(X)$. Then, for all $n > 0$,*

$$(\hat{R}_A)^n (x, s) = \int G(x, y)\, (\mu\, [(\hat{R}^*_A)^n])\, (dy)$$

$$= G(\mu\, [\hat{R}^*_A)^n])\, (x).$$

Proof. – $\hat{R}_A(x, s) = \int\int \hat{R}_A(x, dz)\, G(z, y)\, \mu(dy)$

$$= \int \hat{R}^*_A(y \quad G^*_x)\, \mu(dy)$$

$$= \langle \mu \hat{R}^*_A, G^*_x \rangle = G(\mu \hat{R}^*_A)(x).$$

The general case is proved by an easy induction on n.

PROPOSITION 3.7. – *The semipolar and the *-semipolar sets coincide.*

Proof. – Since V is proper, a finite, strict, strictly positive, excessive function of the form $p = Va = G\mu$ exists (see [15] and (7)). In [15] it is shown that $A \subset X$ is semipolar if and only if $A = \bigcup_m A_m$, with $\lim_{n\to\infty} (\hat{R}_{A_m})^n u = 0 \ \forall m$, where u is a finite, strictly positive, excessive function.

Let A be semipolar. To prove that A is *-semipolar it suffices to show that $\lim_{n\to\infty} (\hat{R}_A)^n p = 0$ implies that A is *-semipolar.

For $\eta \in \mathfrak{M}^+(X)$,

$$\langle \eta, (\hat{R}_A)^n p \rangle = \langle \eta, G(\mu[(\hat{R}^*_A)^n]) \rangle = \langle \mu[(\hat{R}^*_A)^n], G^*\eta \rangle$$
$$= \langle \mu, (\hat{R}^*_A)^n G^* \eta \rangle.$$

Denote by $w = w(\eta)$ the infimum of the functions $(\hat{R}^*_A)^n G^*\eta$. Let $f \in B^+$ be such that $0 \leqslant V^* f \leqslant w$. Then,

$$0 = \langle \mu, V^* f \rangle = \langle f . \nu^*, p \rangle$$

and so $f . \nu^* = 0$. Hence, $V^* f = 0$ and $\hat{w} = 0$.

Choose η so that $G^*\eta$ is finite and strictly positive. Then

$$A = A' \cup A'' \text{ with}$$

A' *-semipolar and $(\hat{R}^*_A)^n G^*\eta \downarrow 0$ on A''. Let

$$A_n = \left\{ y \in A'' \mid (\hat{R}^*_A)^{n-1}(y, G^*\eta) \leqslant \frac{1}{2} G^*\eta(y) \right\}.$$

Then, $A'' = \bigcup_{n\geqslant 1} A_n$ and

$$(\hat{R}^*_{A_n})^{nk}\ G^*\eta \leqslant \left(\frac{1}{2}\right)^k G^*\eta.$$

Hence, each A_n is *-semipolar.

LEMMA 3.8. – *Let $\mu, \nu \in \mathfrak{M}^+(X)$ be such that $G\mu \leqslant G\nu$. Then $\mu \prec^* \nu$, where $\prec^*$ is the balayage order defined by the cone $\mathcal{E}^*$ of *-excessive functions u. That is, $u \in \mathcal{E}^*$ implies $<\mu, u> \leqslant <\nu, u>$.*

In particular, $G\mu = G\nu$ implies $\mu = \nu$.

Proof. – Let $\sigma \in \mathfrak{M}^+(X)$. Then $<\sigma, G\mu> \leqslant <\sigma, G\nu>$ and so $<\mu, G^*\sigma> \leqslant <\nu, G^*\sigma>$. Since each *-excessive function u is the limit of an increasing sequence of functions of the form $G^*\sigma$ the result is established.

If $G\mu = G\nu$ then μ and ν agree on *-excessive functions. In view of (7) $\mu = \nu$ since the cone of bounded *-excessive functions is infimum closed and contains 1 (see IT20 in [9]).

4. Regular Potentials.

In this paragraph several results of Constantinescu in [5] are obtained in a more general setting. In addition to the hypotheses of the preceding paragraph, the resolvents $(V_\lambda)_{\lambda \geqslant 0}$ are assumed to satisfy the following conditions :

(C) $\varphi \in \mathcal{C}_c$ implies $V\varphi$ and $V^*\varphi$ finite, continuous ;

(S) if $x \neq y$ then

$$V(x, \cdot) \neq V(y, \cdot) \text{ (resp. } V^*(x, \cdot) \neq V^*(y, \cdot))$$

(G) G is continuous off the diagonal.

Remarks. – 1) (S) is equivalent to saying that the cones of excessive and *-excessive functions both *separate the points of X.*

2) These hypotheses are satisfied by the resolvents considered in the remark preceding lemma 3.1.

DEFINITION 4.1. – *Let* $\mu \in \mathfrak{M}^+(X)$. *The potential* $G\mu$ *is said to be regular* (or *of class M in* [5]) *if there exists* $(\mu_n) \subset \mathfrak{M}^+(X)$ *such that*

1) $\mu = \sum_{n=0}^{\infty} \mu_n$; *and*

2) $\forall\, n,\ G\mu_n \subset \mathcal{C}_b$.

In what follows use will be made of the next lemma, which is a corollary of Bauer's Minimum Principle (c.f. [1]).

LEMMA 4.2. – *Let* K *be a compact space and let* g *be a finite non-negative upper semicontinuous function. Denote by* $\mathcal{F}$ *a set of lower semicontinuous functions* $f : K \to (-\infty, +\infty]$ *which separate the points of* K. *For each* $x \in K$ *let*

$$\mathfrak{M}_x = \{\mu \in \mathfrak{M}^+(K) \mid <\mu , g> = g(x) , <\mu , f> \leqslant f(x)\ \ \forall\, f \in \mathcal{F},\ \textit{and}\ \mu(1) \leqslant 1\}.$$

Then, if $g \neq 0$, *there exists* $x_0 \in K$ *with* $\mathfrak{M}_{x_0} = \{\varepsilon_{x_0}\}$.

Proof. – Let $\alpha = \sup_{x \in K} g(x)$. If $g \neq 0$ then $\alpha > 0$. Denote by L the set $\{g = \alpha\}$. If $x \in L$ and $\mu \in \mathfrak{M}_x$ then

$$\alpha = <\mu , g> = \int_{\{g \leqslant \alpha - \frac{1}{n}\}} g\,d\mu + \int_{\{g > \alpha - \frac{1}{n}\}} g\,d\mu$$

$$\leqslant \left(\alpha - \frac{1}{n}\right) a_n + \alpha(\mu(1) - a_n) = \alpha\mu(1) - a_n/n \leqslant \alpha\mu(1),$$

where $a_n = \mu\left\{g \leqslant \alpha - \frac{1}{n}\right\}$. Hence, $\alpha > 0$ implies $\mu(1) = 1$.

If $\mu\{g < \alpha\} > 0$, for some n, $a_n > 0$ and so $\alpha < \alpha\mu(1) \leqslant \alpha$. Consequently, $x \in L$ and $\mu \in \mathfrak{M}_x \Rightarrow \mu(K \setminus L) = 0$.

Let $\mathcal{E} = \{h \mid h = f \mid L,\ f \in \mathcal{F}\}$. Then $\mathcal{E}$ satisfies the hypotheses of Bauer's Minimum Principle. Consequently, there is a point $x_0 \in L$ with $\mathcal{N}_{x_0} = \{x_0\}$, where

$$\mathcal{N}_x = \{\mu \in \mathfrak{M}_1^+(L) \mid <\mu , h> \leqslant h(x)\ \ \forall\, h \in \mathcal{E}\}.$$

Now $x \in L \Rightarrow \mathcal{N}_x = \mathfrak{M}_x$ which completes the proof.

PROPOSITION 4.3. – *Let $\mu \in \mathfrak{M}^+(X)$ be such that* A *semipolar implies* $\mu(A) = 0$. *Let* $W(x, dy) = G(x, y)\, \mu(dy)$. *Then,*

$$\hat{R}_B(W1_B) = W1_B, \ \forall B \in \mathcal{B}.$$

Proof. – Let $B_0 = \{y \in B \mid \hat{R}^*_B(y, \cdot) \neq \varepsilon_y\}$. This measurable set is semipolar and so $\mu(B_0) = 0$ implies

$$\begin{aligned}\hat{R}_B(x, W1_B) &= \iint \hat{R}_B(x, dz)\, G(z, y)\, 1_B(y)\, \mu(dy)\\ &= \int \hat{R}_B(x, G_y)\, 1_B(y)\, \mu(dy)\\ &= \int_{X\setminus B_0} \hat{R}_B(x, G_y)\, 1_B(y)\, \mu(dy)\\ &= \int_{X\setminus B_0} G_y(x)\, 1_B(y)\, \mu(dy) = W1_B(x).\end{aligned}$$

PROPOSITION 4.4. – *Let ν be the measure for which*

$$V(x, dy) = G(x, y)\, \nu(dy).$$

Then, A *semipolar implies* $\nu(A) = 0$.

Proof. – It suffices to assume A is compact and totally thin. Then $V1_A$ is a continuous finite function with $\hat{R}_A(V1_A) = V1_A$.

Let $K \supset A$ be compact and let $g = (V1_A) \mid K$. Denote by $\mathfrak{F}$ the set $\{f \mid f = u \mid K, u \in \mathcal{S}\}$.

Lemma 4.2 implies $g = 0$ since, $\forall x \in K, \varepsilon^A_x \in \mathfrak{M}_x$ and $\varepsilon^A_x \neq \varepsilon_x$. Hence, $V1_A = 0$ and so if $\sigma = \nu \mid A$ it follows that $G\sigma = 0$. Lemma 3.8 implies $\sigma = 0$.

The following sequence of lemmas will be used to relate the conclusion of proposition 4.3 to the notion of a regular potential.

LEMMA 4.5. – *Let $\mu \in \mathfrak{M}^+_c(X)$ and let $s = G\mu$. If $A \subset X$ is such that $\hat{R}_A s = s$ and if $U \supset \operatorname{supp} \mu$ is open, then $\hat{R}_{A\cap U}\, s = s$.*

Proof. – Proposition 7.3 in the appendix implies that

$$\mu^{*A} \leqslant \mu^{*A\cap U} + \mu^{*A\setminus U},$$

where $\mu^{*E} = \mu \hat{R}^*_E$. The measure $\mu^{*A\setminus U}$ is carried by $X \setminus U$ and hence

$$\int G(x, y)\, 1_U(y)\, \mu^{*A}(dy) \leqslant \int G(x, y)\, 1_U(y)\, \mu^{*A\cap U}(dy)$$

$$\leqslant \int G(x, y)\, \mu^{*A\cap U}(dy) = \hat{R}_{A\cap U}\, s(x) \leqslant s(x)$$

(see corollary 3.6).

$\hat{R}_A s = s$ implies $G\mu = G\mu^{*A}$ and lemma 3.8 implies $\mu = \mu^{*A}$. Hence, $s(x) = \int G(x, y)\, 1_U(y)\, \mu^{*A}(dy)$ and so $\hat{R}_{A\cap U}\, s = s$.

LEMMA 4.6. – *Let* $s = G\mu$ *and let* $A \subset X$ *compact be such that*:

1) supp $\mu \subset A$;

2) $s \mid A$ *is finite and continuous ; and*

3) $\hat{R}_A s = s$.

Then s is continuous.

Proof. – (see [5] lemma 3.2). Since G is continuous outside the diagonal, s is continuous and finite on $X \setminus A$. Hence, the result is true if $\partial A = \emptyset$.

Let $x_0 \in \partial A$ and consider the family $\mathcal{G}$ of functions $\varphi \in \mathcal{C}_c^+$ with $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on a neighbourhood of x_0. Set

$$W(x, dy) = G(x, y)\, \mu(dy)$$

and let $\mathcal{F} = \{W\varphi \mid \varphi \in \mathcal{G}\}$. Let $s' = \inf \mathcal{F}$. Then, by corollary 7.2 in the appendix, s' is excessive and $s' \prec s$, i.e. $s - s'$ is excessive. Hence, s' and also each $W\varphi$, $\varphi \in \mathcal{G}$, are continuous on A.

Dini's theorem implies, $\varepsilon > 0$ given, that there exists $\varphi_0 \in \mathcal{G}$ with $W\varphi < s' + \varepsilon$ on A if $\varphi \in \mathcal{G}$ and $\varphi \leqslant \varphi_0$. Let $\lambda \in \mathbf{R}$ be such that $s'(x_0) < \lambda < s'(x_0) + \varepsilon$ and let U be a neighbourhood of x_0 with $s' < \lambda < s' + \varepsilon$ on $U \cap A$. Then $\varphi \in \mathcal{G}$, and $\varphi \leqslant \varphi_0$ implies $W\varphi < \lambda + \varepsilon$ on $A \cap U$ and so $\hat{R}_{A\cap U}(W\varphi) \leqslant \lambda + \varepsilon$.

Choose $\varphi \in \mathcal{G}$, $\varphi \leqslant \varphi_0$ with supp $\varphi \subset U$. Since $W\varphi \prec s$, $\hat{R}_A s = s$ implies $\hat{R}_A(W\varphi) = W\varphi$. Lemma 4.5 implies $\hat{R}_{A\cap U}(W\varphi) = W\varphi$. Hence,

$$\limsup_{x \to x_0} W\varphi(x) \leqslant \lambda + \varepsilon < s'(x_0) + 2\varepsilon \leqslant W\varphi(x_0) + 2\varepsilon.$$

From this it follows that

$$\lim_{x \to x_0} \sup s(x) \leqslant \lim_{x \to x_0} \sup W\varphi(x) + \lim_{x \to x_0} \sup W(1 - \varphi)(x)$$

$$\leqslant s(x_0) + 2\varepsilon.$$

since $W(1 - \varphi)$ is continuous on a neighbourhood of x_0.

PROPOSITION 4.7. – (see Lemma 3.3 in [5]). *Let*

$$s = G\mu \, , \mu \in \mathcal{M}^+(X)$$

be such that :

1) *s is finite on a dense subset of* X ; *and*

2) $\forall$ A $\subset$ X *compact,* $\hat{R}_A(W1_A) = W1_A$, *where*

$$W(x, dy) = G(x, y)\, \mu(dy).$$

Then $G\mu$ *is a regular potential.*

Proof. – (Constantinescu [5]). As G is lower semi-continuous $E = \{s = +\infty\}$ is a polar set. Then (2) implies $W1_E = 0$ and so $\mu(E) = 0$.

Let $A \subset X$ be compact and such that $s \mid A$ is finite and continuous. Then $W1_A$ is finite and continuous on A. Lemma 4.6 implies $W1_A$ continuous.

Let $X = \bigcup_n A_n$, A_n compact with $A_n \subset A_{n+1}$. Lusin's theorem implies that, for each n, there is a sequence (B_{nm}) of disjoint compact subsets $B_{nm} \subset A_n \setminus (E \cup A_{n-1})$ such that

$$\text{(i) } \mu(A_n \setminus (E \cup A_{n-1})) = \sum_m \mu(B_{nm}) \quad \text{and} \quad \text{(ii) } s \mid B_{nm}$$

is finite and continuous for each m.

Let $\mu_{nm} = \mu \mid B_{nm}$. Then $G\mu_{nm} = W1_{B_{nm}}$ is a bounded continuous function and $\mu = \sum_{n,m} \mu_{n,m}$.

COROLLARY 4.8. – *Let* $\mu \in \mathcal{M}^+(X)$ *be such that :*

1) $\mu(A) = 0$, $\forall A \subset X$ *semipolar ; and*

2) $G\mu$ *and* $G^*\mu$ *are finite on dense sets.*

Then $G\mu$ *and* $G^*\mu$ *are both regular potentials. Further, there exists* $(\mu_n) \subset \mathfrak{M}^+(X)$ *with* $\mu = \sum_n \mu_n$ *and, for each n,* $G\mu_n$ *and,* $G^*\mu_n \in \mathcal{C}_b$.

Proof. – Propositions 4.3 and 4.7 imply that both $G\mu$ and $G^*\mu$ are regular potentials.

Let $\mu = \sum_n \mu_n$ with $G\mu_n \in \mathcal{C}_b$, $\forall n$. Then if $0 \leqslant \nu \leqslant \mu$, $\nu = \sum_n \nu_n$ with $\nu_n \leqslant \mu_n$, $\forall n$ and hence each $G\nu_n \in \mathcal{C}_b$ (since $G\sigma$ is lower semi-continuous for any $\sigma \in \mathfrak{M}^+(X)$). Let $\mu = \sum_n \mu'_n$ with $G^*\mu'_n \in \mathcal{C}_b$, $\forall n$. Then, $\forall n$, there exists $(\mu'_{nm}) \subset \mathfrak{M}^+(X)$ with $\mu'_n = \sum_m \mu'_{nm}$ and $\forall m$, $G\mu'_{nm} \in \mathcal{C}_b$. Since $G^*\mu'_{nm} \in \mathcal{C}_b$ $\forall n, m$, the result follows.

5. Application to Duality Theory.

The hypotheses made in paragraphs one and two are assumed to hold.

DEFINITION 5.1. – *A resolvent* $(V_\lambda)_{\lambda > 0}$ *is said to correspond to a sheaf* $\mathcal{H}$ *if the cone of excessive functions coincides with the cone of non-negative hyperharmonic functions.*

If $\mathcal{H}$ is a strict Bauer sheaf then it is well known that there exist submarkovian resolvents that correspond to $\mathcal{H}$ which are the resolvents of Hunt semigroups (theorem 2, Kapitel III in [6] or [16]).

DEFINITION 5.2. – *Let* $m \in \mathfrak{M}^+(X)$ *and let* V, V^* *be two kernels on* $(X, \mathcal{B})$. *They are said to be in duality with respect to m if,* $\forall f, g \in \mathcal{B}^+$.

$$\langle Vf, g\rangle_m = \langle f, V^*g\rangle_m, \quad \text{where} \quad \langle h, k\rangle_m = \int h(x)\,k(x)\,m(dx).$$

Two resolvents $(V_\lambda)_{\lambda>0}$ *and* $(V_\lambda^*)_{\lambda>0}$ *are said to be in duality with respect to* m if, $\forall\, \lambda > 0$, V_λ *and* V_λ^* *are in duality with respect to* m.

DEFINITION 5.3. – *Let* $(V_\lambda)_{\lambda>0}$ *be a resolvent on* $(X, \mathcal{B})$. *If* $(V_\lambda^*)_{\lambda>0}$ *is a resolvent on* $(X, \mathcal{B})$ *and* $m \in \mathfrak{M}^+(X)$ *then* $((V_\lambda^*)_{\lambda>0}, m)$ *is called* a *Kunita-Watanabe (or* KW) *dual of* $(V_\lambda)_{\lambda>0}$ *if the following conditions are satisfied :*

(KW1) $\forall\, x \in X$, $\varepsilon_x V_0$ *is absolutely continuous with respect to* m *;*

(KW2) $(V_\lambda)_{\lambda>0}$ *and* $(V_\lambda^*)_{\lambda>0}$ *are in duality with respect to* m*; and*

(KW3) *the resolvent* $(V_\lambda^*)_{\lambda>0}$ *is such that :*

1) $\lim_{\lambda\to\infty} \lambda\, V_\lambda^*(y, \varphi) = \varphi(y)$, *uniformly on the compact subsets of* X, $\forall\, \varphi \in \mathcal{C}_c$ *; and*

2) $\forall\, \lambda \geq 0$, $\forall\, f \in \mathcal{B}_b$ *with* $\overline{\{f > 0\}}$ *compact,* $V_\lambda^* f \in \mathcal{C}_b$.

One of the principal results of this article is the following theorem which shows that KW-duals exist in the setting of axiomatic potential theory.

THEOREM 5.4. – *Let* $\mathcal{H}$ *be a strict Bauer sheaf on* X *which has a Green function. Denote by* G *a Green function for* $\mathcal{H}$ *that satisfies the conditions in proposition* 1.5.

Then there exists a positive Radon measure m *on* X *such that the kernels*

$$V(x, dy) = G(x, y)\, m(dy) \quad \textit{and}$$

$$V^*(y, dx) = G^*(y, x)\, m(dx)$$

have the following properties :

1) *the resolvents* $(V_\lambda)_{\lambda>0}$ *and* $(V_\lambda^*)_{\lambda>0}$ *of* V *and* V^* *both exist;*

2) *the resolvent* $(V_\lambda)_{\lambda>0}$ *corresponds to* $\mathcal{H}$ *and the resolvent* $(V_\lambda^*)_{\lambda>0}$ *is adjoint to* $\mathcal{H}$ *(relative to* G) ;

3) $((V^*_\lambda)_{\lambda>0}, m)$ *is a* KW *dual of* $(V_\lambda)_{\lambda>0}$ *and* $((V_\lambda)_{\lambda>0}, m)$ *is a* KW *dual of* $(V^*_\lambda)_{\lambda>0}$; *and*

4) *both resolvents are the resolvents associated with the transition semigroups of diffusions.*

Further, if $\mathcal{H}$ *is a Brelot sheaf ofr which* G *defines an adjoint sheaf* $\mathcal{H}^*$, *then* $(V^*_\lambda)_{\lambda>0}$ *corresponds to* $\mathcal{H}^*$

Proof. – Let $\nu \in \mathcal{M}^+(X)$ be such that (a) $G\nu(x)$ is bounded continuous and strict and (b) $G^*\nu$ is finite on a dense set (see proposition 2.11). The kernel $W(x, dy) = G(x, y)\,\nu(dy)$ has a resolvent which corresponds to $\mathcal{H}$ and is the resolvent of a Hunt semigroup (see [16]).

Let ν^* be an admissable measure with $G\nu^*$ finite on a dense set (proposition 2.11).

Corollary 4.8 implies that there exists $(\eta_n) \subset \mathcal{M}^+_c(X)$ with (i) $\forall\, n$, $G\eta_n$ and $G^*\eta_n$ continuous bounded, and (ii) $\nu + \nu^* = \sum_n \eta_n$.

Let $(a_n) \subset (0, 1)$ be such that

$$\text{(a)} \sum_n a_n G\eta_n \in \mathcal{C}_b \quad \text{and} \quad \text{(b)} \sum_n a_n G^*\eta_n \in \mathcal{C}_0$$

(proposition 3.7 and lemma 1.3 imply $G^*\eta_n \in \mathcal{C}_0$).

Denote by m the measure $\sum_n a_n \eta_n$. It is clearly admissable. Further, the potential $p = Gm$ can be seen to be strict in the sense of [16].

Clearly, V and V* are in duality with respect to m if

$$V(x, dy) = G(x, y)\, m(dy) \quad \text{and} \quad V^*(y, dx) = G^*(y, x)\, m(dx).$$

It follows that the corresponding resolvents $(V_\lambda)_{\lambda>0}$ and $(V^*_\lambda)_{\lambda>0}$, which exist in view of the choice of m, are in duality [8].

It remains to consider condition (KW3). For the resolvent $(V^*_\lambda)_{\lambda>0}$ it is clearly satisfied since V* is the potential kernel of a Feller semigroup and V* is strong Feller in the sense of [2]. In the case of $(V_\lambda)_{\lambda>0}$ for (KW3) (1) it suffices to note that each $\varphi \in \mathcal{C}_c$ is the uniform limit of a sequence of differences of continuous bounded superharmonic functions (see the approximation theorem of Mme Hervé in [7]). The second condition, (KW3) (2), holds because V is a strong Feller kernel.

The last statement follows from corollary 3.5 while the fourth statement follows from corollary 3.3 and proposition 4.1 in [16].

6. Application to the Martin Compactification.

The sheaf $\mathcal{H}$ will now be supposed to be a Brelot sheaf possessing a positive potential, with 1 superharmonic, and such that the hypothesis of proportionality is satisfied. For such a sheaf Green functions exist. Let G be a Green function for $\mathcal{H}$ satisfying the conditions in proposition 1.5.

Let $(K_\alpha)_{\alpha \in A}$ be a family of continuous functions $K_\alpha : X \setminus D_\alpha \to \overline{\mathbf{R}}$, D_α compact $\forall\, \alpha \in A$. Then there is a unique compactification $\overline{X}$ of X such that (1) all the functions K_α have continuous extensions to $\overline{X} \setminus D_\alpha$, and (2) their extensions separate the points of the boundary $\overline{X} \setminus X$ (c.f. proposition 1 in [14]).

DEFINITION 6.1. *The compactification* X *is said to be defined by* $(K_\alpha)_{\alpha \in A}$.

Let $x_0 \in X$ and set $K(x, y) = 1$ if

$$x = y = x_0 \quad \text{and} \quad = G(x, y)/G(x_0, y)$$

otherwise. Define $K_x^*(y) = K(x, y)$.

DEFINITION 6.2. — *The Martin compactification of* X *(corresponding to* $\mathcal{H}$*) is the compactification defined by the family* $(K_x^*)_{x \in X}$*. It will be denoted by* $\overline{X} = X \cup \Delta$.

It is not hard to show that this compactification is independent of the point x_0 and that it can be identified with the subspace $\overline{\mathcal{E}(\Lambda)}$, Λ a base of the cone $\mathcal{S}$ (c.f. [14]).

Denote by n^* a continuous, finite and strictly positive function which coincides with $G(x_0, \text{-})$ outside a compact neighbourhood A of x_0.

PROPOSITION 6.3. – *The Martin compactification of* X *is the compactification of* X *defined by the functions* $(1/n^*)\,(V^*\varphi)$, $\varphi \in \mathcal{C}_c$, *if* $V^*(y\,,\,dx) = G^*(y\,,\,x)\,\mu(dx)$ *is a kernel that maps* $\mathcal{C}_c$ *into* $\mathcal{C}$.

Proof. – If $y \notin A$ then

$$(1/n^*(y))\; V^*(y\,,\varphi) = \int K(x\,,\,y)\,\varphi(x)\,\mu(dx).$$

Let $\bar{y} \in \Delta$ and let $(y_a)_a$ be a net in X which converges to $\bar{y}$. The functions $K(-\,,y_a)$ converge in the topology of uniform convergence on compact sets to a harmonic function $h = \overline{K}(-\,,\bar{y})$. Hence.

$$\lim\,(1/n^*(y_a))\; V^*(y_a\,,\varphi) = \int \overline{K}(x\,,\,\bar{y})\,\varphi(x)\,\mu(dx).$$

The extensions of the functions $(1/n^*)\;(V^*\varphi)$ separate the points of $\Delta = \overline{X}\setminus X$. If $x_1 \in X$ there exists $(\varphi_m) \subset \mathcal{C}_c$ such that

i) $\operatorname{supp}\varphi_{m+1} \subset \operatorname{supp}\varphi_m$, $\forall m$,

ii) $\{x_1\} = \cap \operatorname{supp}\varphi_m$, and

iii) $<\mu\,,\varphi_m> = 1\ \forall\, m$.

The measures $\varphi_m\,.\,\mu$ converge weakly to ε_{x_1} and since their supports are contained in a fixed compact set,

$$\overline{K}(x_1\,,\bar{y}) = \lim_{m\to\infty}\,(1/n^*)\,(V^*\varphi_m)\,(\bar{y}),$$

where $(1/n^*)\,(V^*\varphi_m)$ also denotes its extension to $\overline{X}$.

Let $(V_\lambda)_{\lambda>0}$ be the closed resolvent of a Hunt semigroup that corresponds to $\mathcal{H}$ and let $((V^*_\lambda)_{\lambda>0}\,, m)$ be a KW dual of $(V_\lambda)_{\lambda>0}$. Then there is a unique measurable function $G : X \times X \to \overline{\mathbf{R}}^+$ such that :

1) $\forall y \in X, x \to G(x\,,\,y)$ is excessive

$\forall x \in X, y \to G(x\,,\,y)$ is *-excessive

2) $V(x\,,\,dy) = G(x\,,\,y)\,m(dy)$ and $V^*(y\,,\,dx) = G(x\,,\,y)\,m(dx)$,

(see [8]). It is called the *o-potential kernel.*

PROPOSITION 6.4. – *With the above hypotheses, the o-potential kernel* G *is a Green function for* $\mathcal{H}$.

Proof. – (KW3) (1) implies that any *-supermedian lower semi-continuous function is *-excessive. Hence, 1 is *-excessive. Since by (KW3) (2) any *-excessive function is lower semi-continuous, this implies that the minimum of two *-excessive functions is *-excessive.

Let $y_0 \in X$ and $\varphi \in \mathcal{C}_c^+(E)$ be such that $\varphi \leqslant 1 = \varphi(y_0)$. Let $O = \{\varphi > 0\}$. Since $(V_\lambda)_{\lambda > 0}$ is closed an arguement used in proposition 2.7 implies that, for all $x \in X$, if

$$D(x) = \{y \mid G(x, y) > R_O(x, G_y)\}$$

then $m(D(x) \cap O) = 0$. The set $D(x)$ is *-fine open and measurable and so by proposition 1.2 in [15], $\lim\limits_{\lambda \to \infty} \lambda V_\lambda^*(y_0, D(x)) = 1$ whenever $y_0 \in D(x)$.

If $y_0 \in D(x)$ then

$$1 = \lim_{\lambda \to \infty} \lambda V_\lambda^*(y_0, \varphi) \leqslant \overline{\lim_{\lambda \to 0}}\, \lambda V_\lambda^*(y_0, D(x) \cap O) + \lim_{\lambda \to \infty} \lambda V_\lambda^*(y_0, \complement D((x)) = 0.$$

Consequently, for all $x \notin X$, $y_0 \in D(x)$, i.e. $G_{y_0} = R_O G_{y_0}$. Hence, for all $y \in X$, G_y is a potential of support $\{y\}$.

Since $\mathcal{H}$ satisfies the hypothesis of proportionality there exists a strictly positive finite measurable function f with

$$G_0(x, y) = G(x, y) f(y)$$

a Green function for $\mathcal{H}$. Let $\varphi \in \mathcal{C}_c$. Then, since $G(x, y) > 0 \ \forall\ (x, y)$ and G_0 is a Green function the condition (KW3) (2) implies f is continuous on $X \setminus \operatorname{supp} \varphi$ in view of the formula,

$$f(y) = \left[\int G_0(x, y)\, \varphi(x)\, m(dx)\right] \Big/ \left[\int G(x, y)\, \varphi(x)\, m(dx)\right].$$

Consequently, f is continuous on X and so G is a Green function.

COROLLARY 6.5. – *With the above hypotheses, $(V_\lambda^*)_{\lambda > 0}$ is adjoint to $\mathcal{H}$ (relative to the o-potential kernel G).*

PROPOSITION 6.6. – *Let $x_0 \in X$ and let G be a Green function for $\mathcal{H}$ that satisfies the conditions in proposition 1.5. There exists $\rho \in \mathfrak{M}^+(X)$ such that if $n^* = G^*\rho$ then :*

1) $0 < n^*(y) < +\infty$;

2) n^* *is continuous ; and*

3) $n^* = G(x_0, -)$ *outside a compact neghbourhood of* x_0.

Proof. – In view of the results of paragraphs two and three, $A \subset X$ implies $\hat{R}_A(x_0, G_y) = \hat{R}^*_A(y, G^*_{x_0})$, where $\hat{R}^*_A$ is the operator defining balayage for the cone $\mathcal{E}^*$ and $G_y(x) = G^*_x(y) = G(x, y)$.

If $x_0 \in O$, an open relatively compact set with $A = \overline{O}$, then proposition 2.7 implies $G^*_{x_0} = \hat{R}^*_A G^*_{x_0}$. The measure $\varepsilon^{*A}_y = \hat{R}^*_A(y, -)$ is carried by ∂O if $y \notin \overline{O} = A$ since $\hat{R}_A G_y = G\varepsilon^{*A}_y$, which is harmonic on $O \cup (X \setminus A)$ if $y \notin \overline{O}$.

Let $\sigma \in \mathcal{M}^+_c(X)$ be continuous and finite on the cone $\mathcal{S}$ and such that $G^*\sigma \geqslant 1$ on ∂O. Let $a = \sup_{y \in \partial O} G(x_0, y) < +\infty$. Then, if $y \notin \overline{O}$, it follows that

$$G^*_{x_0}(y) \leqslant a\hat{R}^*_A(y, G^*\sigma) \leqslant aG^*\sigma(y)$$

and hence $G^*_{x_0}$ vanishes at infinity.

Consequently, if $0 < \lambda < G(x_0, x_0)$ it follows that

$$n^* = \inf (G^*_{x_0}, \lambda)$$

is a finite, continuous function which coincides with $G^*_{x_0}$ outside a compact neighbourhood of x_0.

Since $n^* \leqslant G^*_{x_0}$ the result follows from lemma 2.13.

Let $(V_\lambda)_{\lambda>0}$ be a resolvent on $(X, \mathcal{B})$ and let $((V^*_\lambda)_{\lambda>0}, m)$ be a KW dual of $(V_\lambda)_{\lambda>0}$. Denote by G the o-potential kernel.

DEFINITION 6.7. – *A measure* ρ *is called a normalizing measure if* $G^*\rho$ *is finite, continuous and strictly positive.*

DEFINITION 6.8. – *The Kunita-Watanabe compactification of* X *defined by* $(V^*_\lambda)_{\lambda>0}$ *and a normalizing measure* ρ *is the compactification defined by the functions* $(1/G^*\rho)\,(V^*\varphi)$, $\varphi \in \mathcal{C}_c$.

THEOREM 6.9. – *Let* $\mathcal{H}$ *be a Brelot sheaf which has a positive potential, satisfies the hypothesis of proportionality, and for which 1 is superharmonic.*

Then there exists a resolvent $(V_\lambda)_{\lambda>0}$ *that corresponds to* $\mathcal{H}$ *which has a* KW *dual* $((V^*_\lambda)_{\lambda>0}, m)$ *and a normalizing measure* ρ *such that the Martin compactification of* X *corresponding to* $\mathcal{H}$ *is the Kunita-Watanabe compactification of* X *defined by* $(V^*_\lambda)_{\lambda>0}$ *and* ρ.

Proof. – By theorem 5.4 there exists a KW dual $((V^*_\lambda)_{\lambda>0}, m)$ for $(V_\lambda)_{\lambda>0}$. Further, proposition 6.6 implies that if, as may be assumed, the o-potential kernel G is a Green kernel for $\mathcal{H}$ satisfying the conditions in proposition 1.5, then a normalizing measure ρ exists such that $G^*\rho$ coincides with $G^*_{x_0}$ outside a compact neighbourhood of x_0. Hence, the result follows from proposition 6.3.

7. Appendix

The hypothesis (L) is assumed.

LEMMA 7.1. – *Let* u, v *be excessive. The following conditions are equivalent :*

1) $u \prec v$ (i.e. $v - u$ *is excessive*) ; *and*

2) $\forall\, \lambda > 0$, $\lambda V_\lambda(x, v) + u(x) \leqslant \lambda V_\lambda(x, u) + v(x)$ *on* $(u < +\infty)$.

Proof. – Clearly, $v = u + w$, w excessive implies (2).

Assume (2) and set $s(x) = v(x) - u(x)$ if $u(x) < +\infty$ and $s(x) = +\infty$ if $u(x) = +\infty$. Then, s is surmedian and $v = u + s$. Let $w = \hat{s}$.

COROLLARY 7.2. – *Let* $\mathcal{F}$ *be a family of excessive functions such that* $u_1, u_2 \in \mathcal{F}$ *implies there exists* $u \in \mathcal{F}$ *with* $u \leqslant u_i$. *Let* u_0 *be excessive and such that* $u \in \mathcal{F}$ *implies* $u \prec u_0$. *Then,* $\widehat{\inf}\, \mathcal{F} \prec u_0$ *and* $\inf \mathcal{F}(x) = \widehat{\inf}\, \mathcal{F}(x)$ *on* $(u_0 < +\infty)$. *Further, if* $v_0 \prec u\ \forall\, u, \forall u \in \mathcal{F}$, *then* $v_0 \prec \widehat{\inf}\, \mathcal{F}$.

Proof. – Let $\widehat{\inf}\, \mathcal{F} = (\inf w_n)\hat{}\ $, $(w_n) \subset \mathcal{F}$ decreasing. If $w = \inf w_n$ then,

$$\forall\, \lambda > 0,\ \lambda V_\lambda(x\,,u_0) + w(x) \leqslant \lambda V_\lambda(x\,,w) + u_0(x) \quad \text{on} \quad (u_0 < \infty).$$

Since these inequalities hold for $\widehat{\inf}\ \mathfrak{F} = \hat{w}$ the first result follows.

Also $u_0 = w_n + t_n$, $\forall n$ and so $u_0 = w + t$, t excessive. Hence, $w(x) = \hat{w}(x)$ on $(u_0 < \infty)$.

If $v_0 \prec u$, $\forall u \in \mathfrak{F}$, then $v_0 + v_n = w_n$, v_n excessive. Hence, $v_0 + v = w$, $v = \inf\limits_n v_n$ and so $v_0 + \hat{v} = \hat{w}$.

PROPOSITION 7.3. – *Let* $(V_\lambda)_{\lambda>0}$ *be the resolvent of a Hunt semigroup on* X *that satisfies* (L) *and is such that the σ-field generated by the excessive functions contains all the Borel sets. Then, if* $E\,,F \subset X$

1) $\hat{R}_{E\cup F}\, u \prec \hat{R}_E u + \hat{R}_E u$, $\forall u$ *excessive ; and*

2) $\mu^{E\cup F} \leqslant \mu^E + \mu^F$, $\forall\, \mu \in \mathfrak{M}^+(X)$.

Proof. – Let G, H be fine open sets. Then, on $G \cup H$,

$$R_{G\cup H}\, u + \min\,\{R_G u\,, R_H u\} = R_G u + R_H u.$$

Since the cone of excessive functions is closed under min, it follows that $R_{G\cup H}\, u + u(G\,,H) = R_G u + R_H u$ on X, if

$$u(G\,,H) = R_{G\cup H}\,(\min\,\{R_G u\,, R_H u\}).$$

Let u be finite on X and let $E \subset G$, $F \subset H$ be fine open neighbourhoods. It follows (from the fact that $R_E u = \inf R_G u$, G fine open $\supset$ E) that

$$R_{E\cup F}\, u + \inf_{G\,,H}\, u(G\,,H) = R_E u + R_F u.$$

Denote by $u(E\,,F)$ the regularisation of $\inf\limits_{G\,,H} u(G\,,H)$. Then $\hat{R}_{E\cup F}\, u + u(E\,,F) = \hat{R}_E u + \hat{R}_F u$. This proves (1) for u finite. Note that $u \leqslant u'$ implies $u(E\,,F) \leqslant u'(E\,,F)$. Hence, (1) follows for arbitrary u by passing to the limit, using $u_n = \min\,(u\,,n)$.

Let $x \in X$ and let $\ell(u - v) = u(E\,,F)(x) - v(E\,,F)(x)$, where u, v are bounded excessive functions. Then ℓ is a positive linear form on E, the vector space of differences of bounded excessive functions.

In view of the above, $\varepsilon_x^{E \cup F} + \ell = \varepsilon_x^E + \varepsilon_x^F$ on E and so ℓ satisfies the Daniell condition : $(f_n) \subset \mathcal{E}$ and $f_n \downarrow 0$ implies $\ell(f_n) \downarrow 0$. Since $\mathcal{E}$ is a subvector lattice of $\mathcal{B}_b$ it follows that there is a unique measure $\varepsilon_x^{(E,F)}$ which represents ℓ on $\mathcal{E}$. Clearly,

$$\varepsilon_x^{E \cup F} + \varepsilon_x^{(E,F)} = \varepsilon_x^E + \varepsilon_x^F$$

and further, the family $(\varepsilon_x^{(E,F)})_{x \in X}$ is a kernel $\hat{R}_{(E,F)}$.

Let $\mu \in \mathcal{M}^+(X)$. Then $\mu^{E \cup F} + \mu \hat{R}_{(E,F)} = \mu^E + \mu^F$.

Remark. – The result is true without the hypothesis (L) if E and F are taken to be nearly Borel.

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Manuscrit reçu le 10 décembre 1971
accepté par M. BRELOT

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