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HOMOGENEOUS ALGEBRAS ON THE CIRCLE : I. — IDEALS OF ANALYTIC FUNCTIONS

by Colin BENNETT and John E. GILBERT

1. Introduction.

A Banach algebra $\mathfrak{A}$ is said to be homogeneous on ∂D, the boundary of the open unit disk D, if

(H.0) *$\mathfrak{A}$ is a commutative semi-simple complex Banach algebra,*

(H.1) *the maximal ideal space of $\mathfrak{A}$ is ∂D,*

(H.2) *for each $e^{it} \in \partial$D and $f \in \mathfrak{A}$, $\mathfrak{A}$ contains the translate*

$$T(t)f \text{ of } f, \ (T(t)f)(\theta) = f(\theta - t),$$

(H.3) *the mapping of $e^{it} \to T(t)$ is a strongly continuous representation of ∂D in $\mathfrak{A}$*

(cf. [15, 18]). $\mathfrak{A}$ is thus an algebra (under pointwise multiplication) of continuous functions on ∂D containing all trigonometric polynomials as a dense subalgebra ; without loss of generality we may also assume T(t) is a contraction :

$$\|T(t) f\|_{\mathfrak{A}} \leq \|f\|_{\mathfrak{A}}, \ e^{it} \in \partial D, \ f \in \mathfrak{A}.$$

Only regular homogeneous algebras will be considered here.

Throughout this paper $\mathfrak{A}^+$ will denote the closed subalgebra of $\mathfrak{A}$ of all functions having analytic extensions into D. Since $\mathfrak{A}^+$ contains the characters $\{\chi^n : \chi(t) = e^{it}, n \geqslant 0\}$,$\mathfrak{A}^+$ is homogeneous on ∂D under the more general definition introduced by de Leeuw ([14, p. 375]). In particular, $\mathfrak{A}^+$ contains the trigonometric polynomials generated by $\{\chi^n : n \geqslant 0\}$ as a dense subalgebra and the maximal ideal space can be identified with $D \cup \partial D$. The Beurling-Rudin theorem ([8, p. 644]) when $\mathcal{A}$ is the Banach algebra $\mathcal{C}(\partial D)$ of all continuous functions on ∂D and $\mathcal{A}^+$ is then the usual disk algebra suggests the following conjecture :

CONJECTURE (Σ). – *Let I be closed ideal in $\mathcal{A}^+$ and $I_{\mathcal{A}}$ the closed ideal in $\mathcal{A}$ generated by I. Then I has the form*

$$I = qH^{\infty}(D) \cap I_{\mathcal{A}} \tag{1}$$

where q is the greatest common divisor of the inner functions in the factorization in $H^{\infty}(D)$ of the functions in I.

Frequently $I_{\mathcal{A}}$ can be described more explicitly. For if

$$Z(I) = \bigcap_{f \in I} f^{-1}(0) \quad , \quad Z(I_{\mathcal{A}}) = \bigcap_{g \in I_{\mathcal{A}}} g^{-1}(0)$$

are the zero-sets of I and $I_{\mathcal{A}}$ respectively, then $Z(I_{\mathcal{A}}) = Z(I) \cap \partial D$. When each such set $Z(I) \cap \partial D$ is a set of synthesis for $\mathcal{A}$ (for instance when every closed set in ∂D is a set of synthesis for $\mathcal{A}$) conjecture (Σ) simplifies to

(Σ)$'$ *each closed ideal I in $\mathcal{A}^+$ is of the form*

$$I = qH^{\infty}(D) \cap I_{\mathcal{A}}(K)\,, \; K = Z(I) \cap \partial D, \tag{2}$$

where $I_{\mathcal{A}}(K) = \{f \in \mathcal{A} : f^{-1}(0) \supseteq K\}$.

The Beurling-Rudin theorem confirms (Σ)$'$ for $\mathcal{A} = \mathcal{C}(\partial D)$. On the other hand, when $\mathcal{A}$ is the Wiener algebra $\mathcal{F}(\ell^1(Z))$, conjecture (Σ)$'$ fails (via a tensor algebra argument) though very likely (Σ) remains true with each closed ideal I in $\mathcal{A}^+$ being of the special form (2) whenever $Z(I) \cap \partial D$ is a set of synthesis for $\mathcal{F}(\ell^1(Z))$. Because single points in ∂D are sets of synthesis for $\mathcal{F}(\ell^1(Z))$, Kahane's result ([10], cf. also [7]) confirms (Σ) in the case

$$\mathcal{A} = \mathcal{F}(\ell^1(Z))$$

(and even a larger class of algebras) for closed ideals I in $\mathcal{A}^+$ for which $Z(I)$ is a single point in ∂D.

Clearly (Σ)$'$ must be modified when functions in $\mathcal{A}$ are all sufficiently smooth so that differentiation is allowed as, for instance, when $\mathcal{A} = \mathcal{C}^{(n)}(\partial D)$ the Banach subalgebra of $\mathcal{C}(\partial D)$ of functions with continuous derivatives up to order n. The ideal $I_{\mathcal{A}}(K)$ must then be the intersection of closed primary ideals rather than merely closed maximal ideals. We omit these modifications since (Σ) does not need to be modified. Taylor and Williams ([19]) have established (Σ) in the case $\mathcal{A} = C^{(n)}(\partial D)$ for closed ideals I in $\mathcal{A}^+$ for which $Z(I) \cap \partial D$ is finite ; more generally, a result of theirs ([19, theorem 5.3]) can be interpreted as completely confirming (Σ) for the Fréchet algebra $\mathcal{A} = \mathcal{C}^\infty(\partial D) = \bigcap_{n=0}^{\infty} \mathcal{C}^{(n)}(\partial D)$.

In this paper we shall consider conjecture (Σ) for the algebra $\mathcal{A}^+$ with $\mathcal{A}$ homogeneous on ∂D. Some restriction on the smoothness and spectral synthesis properties of $\mathcal{A}$ (or $\mathcal{A}^+$) seems to be necessary. We say that $\mathcal{A}$ satisfies the Ditkin Condition if for each $f \in \mathcal{A}$ with $f(0) = 0$ (see (1)) there is a sequence $\{\tau_n\} \subset \mathcal{A}$ such that

(a) $\tau_n = 1$ *in a neighborhood of* 1 $(= e^{i0})$,

(b) $\lim_n \|f\tau_n\|_{\mathcal{A}} = 0$

(cf. [11, p. 225]). If $\{\tau_n\}$ can be chosen independently of f, $\mathcal{A}$ is said to satisfy the Strong Ditkin Condition. For $\mathcal{A}^+$ the natural analogue is the Analytic Ditkin Condition : *for each* $f \in \mathcal{A}^+$ *with* $f(0) = 0$ *there is a sequence* $\{\tau_n\} \subset \mathcal{A}^+$ *such that*

(a)′ $\tau_n(0) = 1$, (b)′ $\lim_n \|f\tau_n\|_{\mathcal{A}} = 0.$

If $\{\tau_n\} \subset \mathcal{A}^+$ can be chosen independently of f we say $\mathcal{A}$ satisfies the Strong Analytic Ditkin Condition. Two conditions will be imposed on $\mathcal{A}$.

CONDITION (1) : *$\mathcal{A}$ contains $\mathcal{C}^\infty(\partial D)$, in particular, $\mathcal{A}$ is regular.*

CONDITION (2) : *$\mathcal{A}$ satisfies the Analytic Ditkin condition.*

The main theorem to be proved in this paper is the following :

THEOREM A. — *Suppose $\mathcal{A}$ is a homogeneous algebra on ∂D satisfying conditions (1) and (2). Then a closed ideal I in $\mathcal{A}^+$ is of the form*

(1) The value of f at $e^{it} \in \partial D$ is denoted by $f(t)$.

$$I = q\mathrm{H}^{\infty}(\mathrm{D}) \cap \mathrm{I}_{\mathfrak{A}}(\mathrm{K}) \quad , \quad \mathrm{K} = \mathrm{Z}(\mathrm{I}) \cap \partial\mathrm{D},$$

q an inner function, whenever $\mathrm{Z}(\mathrm{I}) \cap \partial\mathrm{D}$ *is at most countable.*

When differentiation is permitted in $\mathfrak{A}$ (condition (2) prohibits differentiation) a different version of the Analytic Ditkin Condition is needed. Denote by $\mathrm{M}_{\mathfrak{A}}$ the largest integer n for which $\mathfrak{A} \subseteq \mathcal{C}^{(n)}(\partial\mathrm{D})$. Then $\mathfrak{A}$ is said to satisfy the Analytic Ditkin Condition if for each $f \in \mathfrak{A}^{+}$ with $f^{(n)}(0) = 0$, $0 \leqslant n \leqslant \mathrm{M}_{\mathfrak{A}}$, there is a sequence $\{\tau_n\} \subseteq \mathfrak{A}^{+}$ such that

$$\text{(a)}' \quad \tau_n(0) = 1, \qquad\qquad \text{(b)}' \quad \lim_n \|f\tau_n\|_{\mathfrak{A}} = 0$$

The corresponding modifications in the other Ditkin Conditions are clear. A weaker version of theorem A holds when $\mathrm{M}_{\mathfrak{A}} \geqslant 1$.

THEOREM B. – *Let $\mathfrak{A}$ be a homogeneous algebra on* $\partial\mathrm{D}$ containing $\mathcal{C}^{\infty}(\partial\mathrm{D})$ *and such that* $\mathrm{M}_{\mathfrak{A}} \geqslant 1$. *Then, if $\mathfrak{A}$ satisfies the Analytic Ditkin Condition, a closed ideal* I *in* $\mathfrak{A}^{+}$ *is of the form*

$$\mathrm{I} = q\mathrm{H}^{\infty}(\mathrm{D}) \cap \mathrm{I}_{\mathfrak{A}},$$

q an inner function, whenever $\mathrm{Z}(\mathrm{I}) \cap \partial\mathrm{D}$ *is finite.*

The proof of theorem B will be omitted (a proof appears in [2]). A proof of theorem A is given in section 4 of this paper after the Carleman Transform has been introduced in section 2 and important estimates for the Carleman Transform obtained in section 3.

Part II of this series is devoted to the construction of two large classes of homogeneous algebras both of which satisfy the Strong Ditkin and Strong Analytic Ditkin Conditions. These two classes contain virtually all homogeneous algebras considered previously (as well as many new ones).

We wish to thank Professors Kahane, Taylor and Williams for showing us their papers in pre-publication form. However, the proof of theorem A given here is substantially the one used by one of us to characterize certain closed ideals in a Beurling algebra of functions analytic in a half-plane ; this latter result was obtained independently of [10] and [19] in 1967, 1968 and was announced in outline in [6].

2. Carleman Transform.

Until further notice (cf. (4.2)) we shall assume only that $\mathfrak{A}$ is a (Silov-) homogeneous algebra on ∂D satisfying condition (1). As a Banach space, $\mathfrak{A}$ is an essential $L^1(\partial D)$ - Banach module via convolution ; consequently

$$\lim_{\alpha} \|k_\alpha * f - f\|_{\mathfrak{A}} = 0, \qquad f \in \mathfrak{A}, \tag{3}$$

for any uniformly norm-bounded approximate identity $\{k_\alpha\}$ in $L^1(\partial D)$, say the Poisson Kernel

$$P_r = \mathcal{R}\ell\left(\frac{1 + r\chi}{1 - r\chi}\right), \qquad 0 \leqslant r < 1.$$

The Fourier series of any f in $C(\partial D)$ is given by $\sum_{-\infty}^{\infty} \hat{f}(n)\, \chi^n$ with $\hat{f}(n) = \frac{1}{2\pi} \int_{\partial D} f \chi^{-n}$. If such a function f has an analytic extension $f(z)$ into D then

$$f(z) = \sum_{0}^{\infty} \hat{f}(n)\, z^n = f * P_r, \qquad z = r\chi .$$

Since trigonometric polynomials are dense in $\mathfrak{A}$ a closed linear subspace J of $\mathfrak{A}$ is an ideal in $\mathfrak{A}$ if and only if $\chi^n f \in J$ for all integers n and f in J while a closed linear subspace I in $\mathfrak{A}^+$ is an ideal in $\mathfrak{A}^+$ if and only if $\chi^n f \in I$ for all $n \geqslant 0$ and f in I.

Condition (1) on $\mathfrak{A}$ is stronger than first appearances might suggest : in fact when $\mathcal{C}^{(\infty)}(\partial D) \subset \mathfrak{A}$ there is a continuous embedding

$$\mathcal{C}^{(N)}(\partial D) \to \mathfrak{A} \qquad , \qquad N = N_{\mathfrak{A}}, \tag{4}$$

of $\mathcal{C}^{(N)}(\partial D)$ into $\mathfrak{A}$ for some sufficiently large N (cf. [15, p. 57], [18, p. 54]). Hence there is a continuous embedding of the dual space $PM_{\mathfrak{A}}$ of $\mathfrak{A}$ into $(\mathcal{C}^{(N)}(\partial D))^*$, $N = N_{\mathfrak{A}}$. The elements of $PM_{\mathfrak{A}}$ will be referred to as $\mathfrak{A}$-pseudo-measures by analogy with the dual space of $\mathfrak{F}(\ell^1(Z))$. The bilinear form linking any space with its dual space will always be denoted by $\langle \cdot , \cdot \rangle$.

The Fourier series of a distribution $S \in \mathcal{D}'(\partial D) = (\mathcal{C}^\infty(\partial D))^*$ is defined by $\sum_{-\infty}^{\infty} \hat{S}(n)\, \chi^n$ with $\hat{S}(n) = \langle S, \chi^{-n} \rangle$, the convolution $S * f$ by $(S * f)(e^{it}) = \langle S, T(t)f \rangle$, $f \in \mathcal{C}^\infty(\partial D)$. Denote by H_+ the functions analytic in D satisfying (for some N)

$$S(0) = 0, \quad S(re^{it}) = 0(|r - 1|^{-(N+1)}), \; r \to 1-,$$

and by H_- those functions s analytic in $C \setminus (D \cup \partial D)$ satisfying (for some H)

$$s(re^{it}) = 0(1), \quad r \to \infty, \quad s(re^{it}) = 0(|r - 1|^{-(N+1)}), \quad r \to 1+.$$

It is known that S belongs to H_+ if and only if $S(re^{it}) = S_+ * P_r$ for some $S_+ \in \mathcal{D}'(\partial D)$ with $\hat{S}_+(n) = 0$, $n \leqslant 0$, while $s \in H_-$ precisely when $s\left(\frac{1}{r}e^{it}\right) = S_- * P_r$ for some $S_- \in \mathcal{D}'(\partial D)$ with $\hat{S}_-(n) = 0, n > 0$. Furthermore, for fixed t,

$$\lim_{r \to 1-} S(re^{it}) = S_+, \qquad \lim_{r \to 1+} s(re^{it}) = S_-$$

weakly, and hence strongly, in $\mathcal{D}'(\partial D)$.

Suppose now that $\phi \in PM_{\alpha}$ $(\subset \mathcal{D}'(\partial D))$. Then functions

$$\phi_+(z) = \sum_{1}^{\infty} \phi(n) z^n, \qquad \phi_-(z) = \sum_{-\infty}^{0} \phi(n) z^n$$

belong respectively to H_+, H_-. The distributional boundary values of $\phi_+(z)$ and $\phi_-(z)$ on ∂D will be denoted by ϕ_+, ϕ_- respectively (the 'Riesz projections' of ϕ). One consequence of (4) is that

$$|\hat{\phi}(n)| \leqslant \text{const.} \quad \|\phi\|_{PM} \; (1 + |n|^{N_\alpha}), \quad -\infty < n < \infty,$$

for some constant independant of $\phi \in PM_\alpha$, $\|\phi\|_{PM}$ being the norm of ϕ in PM_α. The Carleman Transform of ϕ as the term is used in this paper is an extension of ϕ_- to a function meromorphic in D. Our definition incorporates modifications introduced by Nyman ([16, chap. 2]) and used explicitly or implicitly by many subsequent authors (cf. [9], [13] or [19]).

Given $f \in \mathfrak{A}^+$ and $\phi \in \mathrm{PM}_{\mathfrak{A}}$, then ϕ is orthogonal to the ideal I_f generated in $\mathfrak{A}^+$ by f if and only if $< \phi , \chi^n f > = 0$, $n \geqslant 0$. Hence, if ϕ is orthogonal to I_f and $\psi \in \mathrm{PM}_{\mathfrak{A}}$ is given by

$$< \psi , h > = < \phi , hf > , \qquad h \in \mathfrak{A}, \tag{5}$$

then $\hat{\psi}(n) = 0$, $n \leqslant 0$, and $\psi(z) = \sum_1^\infty \hat{\psi}(n) z^n$ belongs to H_+.

(2.1) DEFINITION. – *The Carleman Transform* Φ *of* ϕ *is the function*

$$\Phi(z) = \phi_-(z), \quad |z| > 1 \quad ; \quad \Phi(z) = \psi(z)/f(z) - \phi_+(z) \ , \quad |z| < 1.$$

Since $\psi(z)\, g(z) = <\phi , gf\mathrm{T}(t)\,\mathrm{P}_r>$ for all $g \in \mathfrak{A}^+$, it is not hard to check that $\psi(z)/f(z)$, hence $\Phi(z)$, is independent of the particular pair f, ψ in (5).

(2.2) REMARK. – $\Phi \equiv 0$ *if and only if* $\phi \in \mathfrak{B} = \{\beta \in \mathrm{PM}_{\mathfrak{A}} : \hat{\beta}(n) = 0,\ n \leqslant 0\}$. *Thus* Φ *determines* ϕ *uniquely not as an element of* $\mathrm{PM}_{\mathfrak{A}}$ *but as an element of* $\mathrm{PM}_{\mathfrak{A}}/\mathfrak{B}$, *the dual space of* $\mathfrak{A}^+$.

For the remainder of this section I will denote a closed ideal in $\mathfrak{A}^+$ and Φ the Carleman Transform of an $\mathfrak{A}$-pseudo-measure ϕ orthogonal to I. The cospectrum cosp(I) of I is the set of common zeros (counted according to multiplicity) of functions in I ; Z(I) is thus the set of distinct elements in cosp(I).

(2.3) THEOREM. – *If* $z_0 \in \partial\mathrm{D}$ *does not belong to* Z(I) *then* Φ *is analytic in some neighborhood of* z_0.

(2.4) THEOREM. – *If* $z_0 \in \mathrm{D}$ *belongs to cosp(I) with multiplicity* k *then* Φ *has a pole at* z_0 *of order at most* k.

Remarks. – (a) Some form of (2.3) always is true for the Carleman Transform whatever the definition (cf. [16, p. 50], [11, p. 180], [13, lemma 8]). (b) In the special examples with which he was concerned Nyman completed the characterization of the singularities of Φ by proving that Φ analytic in some neighborhood of $z_0 \in \partial D$ implies $z_0 \in Z(I)$. However, there is some doubt about his proof ([12, pp. 121-124]).

Proof of (2.3). – We can assume $z_0 = 1$. There is a function f in I and a neighborhood $\mathfrak{N}$ of 1 such that $f(z) \neq 0$, $z \in \overline{\mathfrak{N}} \cap \overline{D}$; Φ is thus analytic in $\mathfrak{N}\backslash\partial D$. Since $re^{i\theta} \to \theta + i \log r$ maps $\mathfrak{N}$ onto a neighborhood of 0 in the complex plane, (2.3) follows by exhibiting a sequence $\{\Phi_j\}$ of functions all analytic in some neighborhood $\mathfrak{M}$ of 1, $\mathfrak{M} \subset \mathfrak{N}$, such that $\Phi_j \to \Phi$ pointwise in $\mathfrak{M}\backslash\partial D$ and

$$|\Phi_j(re^{it})| \leqslant \text{const. } |r - 1|^{-(N+1)}, \quad re^{it} \in \mathfrak{M}\backslash\partial D, \tag{6}$$

uniformly in j (cf. [11, p. 180]).

Let $\{\mathcal{K}_j\}$ be a nested sequence of ∂D-neighborhoods (all suitably small) shrinking to 1 as $j \to \infty$ and $\{\sigma_j\} \subset \mathcal{C}^\infty(\partial D)\ (\subset \mathfrak{A})$ functions satisfying

$$\sigma_j \geqslant 0,\ \sigma_j(-\theta) = \sigma_j(\theta),\ \frac{1}{2\pi}\int_{\partial D} \sigma_j = 1,\ \text{supp}\,(\sigma_j) \subseteq \mathcal{K}_j.$$

This family $\{\sigma_j\}$ is thus a positive summability kernel on ∂D ([11, p. 10]). Set

$$\Phi_j(re^{it}) = \frac{1}{2\pi}\int_{\partial D} \Phi(re^{i\theta})\,\sigma_j(t - \theta)\,d\theta \quad , \quad r \neq 1.$$

Clearly all Φ_j are analytic in $\mathfrak{M}\backslash\partial D$ for $\mathfrak{M} \subset \mathfrak{N}$, and $\Phi_j \to \Phi$ pointwise in $\mathfrak{M}\backslash\partial D$. Also the estimates (6) hold since ϕ_+ belongs to H_+, ϕ_- belongs to H_- and $f(z) \neq 0$ when $z \in \overline{\mathfrak{M}} \cap \overline{D}$. To check the analyticity of Φ_j on $\mathfrak{M} \cap \partial D$, fix $e^{it} \in \mathfrak{M} \cap \partial D$ and fix j. Then

$$\lim_{r\to 1+} \Phi_j(re^{it}) = (\phi_- * \sigma_j)(e^{it}),$$

$$\lim_{r\to 1-} \frac{1}{2\pi}\int_{\partial D} \phi_+(re^{i\theta})\,\sigma_j(\theta - t)\,d\theta = (\phi_+ * \sigma_j)(e^{it}).$$

On the other hand, in view of the regularity of $\mathfrak{A}$ and the choice of σ_j, the functions $g_r = (\mathrm{T}(t)\,\sigma_j)/(f * \mathrm{P}_r)$ belong to $\mathfrak{A}$ provided, say $r \geqslant r_0$, and satisfy

$$\|g_r\|_{\mathfrak{A}} = O(1) \quad , \quad r \to 1- ;$$

with this it is easy to see that $g_r \to g_1$ as $r \to 1-$ (use also (3) and (4)). But then

$$\frac{1}{2\pi} \int_{\partial \mathrm{D}} (\psi/f)\,(re^{i\theta})\; \sigma_j\,(t-\theta)\,d\theta = \langle \psi\,, g_r * \mathrm{P}_r \rangle$$

$$\to \langle \psi\,, g_1 \rangle = \langle \phi\,, \mathrm{T}(t)\,\sigma_j \rangle = (\phi * \sigma_j)\;(e^{it})$$

as $r \to 1-$. Consequently,

$$\lim_{r \to 1-} \Phi_j(re^{it}) = (\varphi_- * \sigma_j)\;(e^{it}) = \lim_{r \to 1+} \phi_j\,(re^{it}).$$

Since $\phi_- * \sigma_j$ is continuous, Morera's theorem shows that Φ_j analytic in $\mathfrak{M} \cap \partial \mathrm{D}$ completing the proof.

3. Estimates for the Carleman transform.

Denote by $\mathfrak{G}$ the set of all functions G analytic except possibly on ∂D and satisfying :

$\mathfrak{G}$(i). *outside* ∂D, *G belongs to* H_- ,

$\mathfrak{G}$(ii). *inside* ∂D, G *has a representation* $\mathrm{G} = g/f$ *with* $g \in \mathrm{H}_+$ *and* $f \in \mathrm{H}^\infty(\mathrm{D})$.

The Carleman Transform Φ belongs to $\mathfrak{G}$ provided Φ is analytic in D.

For each $\mathrm{G} \in \mathfrak{G}$, $u(z) = \log|\mathrm{G}(z)|$ is subharmonic except possibly on ∂D and, by $\mathfrak{G}$ (*i*),

$$u\,(re^{i\theta}) = O\,(|r-1|^{-1}), \quad r \to 1+.$$

To estimate G inside ∂D let $\mathrm{G} = g/f$. Then, for some disk D_ρ centered at $z_0 = r_0\, e^{i\theta_0}$ in D of sufficiently small radius ρ

$$u(z_0) \leqslant \frac{1}{\pi\rho^2} \iint_{D_\rho} \{\log|g(z_0 + \sigma e^{i\theta})| - \log|f(z_0 + \sigma e^{i\theta})|\} \sigma d\sigma d\theta.$$

Since we may assume $|f(z)| \leqslant 1$, $z \in D$,

$$u(z_0) \leqslant \frac{1}{\pi\rho^2} \iint_{D_\rho} \{\log^+ |g(z_0 + \sigma e^{i\theta})| + |\log|f(z_0 + \sigma e^{i\theta})||\} \sigma d\sigma d\theta.$$

Changing to polar coordinates with origin at z_0 we deduce

$$u(z_0) \leqslant \frac{1}{\pi\rho^2} \iint_{S_\rho} \{\log^+ |g(re^{i\theta})| + |\log|f(re^{i\theta})||\} r dr d\theta = I_1 + I_2$$

where S_ρ is the sector

$$r_0 - \rho \leqslant r \leqslant r_0 + \rho \quad , \quad \theta_0 - \pi\rho/r_0 < \theta < \theta_0 + \pi\rho/r_0$$

(ρ small). Then, for some constant independent of ρ,

$$I_1 \leqslant \frac{\text{const.}}{\rho} \int_{r_0-\rho}^{r_0+\rho} \log(1/1 - r)\, dr$$

since $g \in H_+$; and so with $\rho = \frac{1}{2}(1 - r_0)$.

$$I_1 = O(|1 - r_0|^{-1}) \quad , \quad r_0 \to 1-.$$

On the other hand, since f has bounded characteristic,

$$I_2 < \frac{\text{const.}}{\rho} \sup_{0<r<1} \int_0^{2\pi} |\log|f(re^{i\theta})|| \, d\theta = O(|1 - r_0|^{-1})$$

as $r \to 1-$. Hence $u(re^{i\theta}) = O(|1 - r|^{-1})$, $r \to 1+$. Application of results of Domar ([5]) in the form given in [19] (lemma (5.3)) gives :

(3.1) THEOREM. – *When* $u(z) = \log|G(z)|$, $G \in \mathcal{G}$, *is subharmonic except possibly on a closed subset* E *of* ∂D, then

$$u(z) \leqslant A/d(z, E) \quad , \quad z \notin E,$$

for some constant A, *where* $d(z, E)$ is the *distance from* z *to* E.

If z_0 is an isolated point of E, (3.1) gives

$$G(z) = O(\exp[\alpha |z - z_0|^{-1}]) \quad , \quad z \to z_0,$$

for some constant α. This estimate is used to derive the more precise estimates for G needed later. Let $f = B_f\, S_f\, 0_f$ be the usual factorization of f in $H^\infty(D)$ ([9, p. 67]) and let $\mu = \mu_c + \mu_d$ be the decomposition into continuous and discrete parts of the unique positive singular measure defining S_f. We write (with obvious meaning)

$$\mu_d = \sum_{\partial D} m_f(t)\, \delta_t,$$

where $m_f(t) \geqslant 0$ and δ_t is the point mass at $e^{it} \in \partial D$; we then define $m_G(t)$ by

$$m_G(t) = \inf m_f(t) \quad , \quad e^{it} \in \partial D,$$

the infimum being taken over all $f \in H^\infty(D)$ occuring in representations $G = g/f$ of G as an element of $\mathfrak{G}$. The following estimate will be vital.

(3.2) THEOREM. – *Suppose* $G \in \mathfrak{G}$ *has an isolated singularity at* $z_0 = e^{it_0} \in \partial D$. *Then*

(i) *for every* $\varepsilon > 0$

$$G(z) = O(\exp[(m_G(t_0) + \varepsilon)\, |z - z_0|^{-1}]), \quad z \to z_0,$$

(ii) *if* $m_G(t_0) = 0$

$$G(z) = O(|z - z_0|^{-N}), \quad z \to z_0,$$

for some integer N.

It is enough to consider $z_0 = 1$. Let ω be the mapping ω : $\zeta \to z = (\zeta - 1)/(\zeta + 1)$ which, in particular, takes the right hand half plane onto D ; define functions $\Gamma, \gamma, \phi, \ldots$ in this half-plane corresponding to functions $G, g, f, \ldots$ in D by $\Gamma = G \circ \omega$, $\gamma = g \circ \omega$, $\phi = f \circ \omega, \ldots$. When G satisfies the hypotheses of (3.2) with $z_0 = 1$, $\Gamma(\zeta)$ is analytic outside some disk $D_v = \{\zeta : |\zeta| < v\}$, v large, and by the preliminary estimate for G

$$\Gamma(\zeta) = O(e^{\beta|\zeta|}) \quad , \quad |\zeta| \to \infty,$$

for some $\beta \geqslant 0$. In the left hand half plane

$$\Gamma(\xi + i\eta) \leqslant \text{const.} \left[\frac{1}{|\xi|}(1 + |\zeta|)\right]^{N} \quad , \quad \xi < 0,$$

N a suitable integer, and, if $G = g/f$ in D so that $\Gamma = \gamma/\phi$ in $\mathcal{R}\ell(\zeta) > 0$, the function ϕ belongs to $H^{\infty}(R)$ ([9, p. 130]) while

$$\gamma(\xi + i\eta) \leqslant \text{const.} \left[\frac{1}{|\xi|}(1 + |\zeta|)\right]^{N} \quad , \quad \xi > 0, \tag{8}$$

for some integer N. As a function in $H^{\infty}(R)$,

$$\phi(\zeta) = e^{-m_f(0)\zeta} B_{\phi} S_{\phi} 0_{\phi} \tag{9}$$

([9, p. 133]), B_{ϕ}, S_{ϕ} and 0_{ϕ} having the obvious meanings.

(3.3) THEOREM. – *When $\phi \in H^{\infty}(R)$ has the factorization* (9),

$$\lim_{\rho \to \infty} \frac{\log|\phi(\rho e^{i\theta})|}{\rho} = -m_f(0)\cos\theta \tag{10}$$

for almost all θ, $-\pi/2 < \theta < \pi/2$.

Proof. – Assertion (10) is the content of the Ahlfors-Heins theorem applied to $\log|\phi|$ ([1]). The proofs of Boas ([3, pp. 114-123]) contain all the difficulties.

Proof of (3.2). – (i) Given $\varepsilon > 0$ choose a representation $G = g/f$ of G in D such that $m_f(0) < m_G(0) + \frac{1}{2}\varepsilon$. We prove that

$$\Gamma(\zeta) = O(e^{(m_f(0) + 2\varepsilon)|\zeta|}) \quad , \quad |\zeta| \to \infty . \tag{11}$$

Choose θ_0 and θ_1 with $-\pi/2 < \theta_0 < 0$, $0 < \theta_1 < \pi/2$ for which (10) exists, ϕ being the function $\phi = f \circ \omega$ corresponding to the special choice of f. Then by (8) and (10),

$$\Gamma(\rho e^{i\theta_j}) = O(\exp[(m_f(0) + \varepsilon)\rho\cos\theta_j]), \qquad \rho \to \infty,$$

$j = 0,1$. Now divide $C \setminus D_\nu$ into four sectors :

$S_0 = \{\zeta : \theta_0 \leqslant \arg(\zeta) \leqslant \theta_1\}$, $S_1 = \{\zeta : \theta_1 \leqslant \arg(\zeta) \leqslant \theta_1 + \pi/2\}$,

$S_2 = \{\zeta : \theta_0 - \pi/2 \leqslant \arg(\zeta) \leqslant \theta_1 + \pi/2\}$,

$S_3 = \{\zeta : \theta_0 - \pi/2 \leqslant \arg(\zeta) \leqslant \theta_0\}$.

Since Γ is bounded on ∂D_ν and (7) holds, the Phragmen-Lindelöf theorem shows that (11) holds in S_0. In the remaining sectors the function $\Lambda(\zeta) = \left(\zeta - \frac{1}{2}\nu\right)^{-N} \Gamma(\zeta)$, N sufficiently large, is uniformly bounded on all boundaries except $\arg(\zeta) = \theta_0$ and θ_1 and on these

$$\Lambda(\rho e^{i\theta_j}) = O(\exp[(m_f(0) + \varepsilon)\rho\cos\theta_j]) \quad , \quad \rho \to \infty,$$

$j = 0, 1$. Thus Λ is uniformly bounded throughout S_2 while in S_1 and S_3,

$$\Lambda(\zeta) = O(\exp[(m_f(0) + \varepsilon)|\zeta|]) \quad , \quad |\zeta| \to \infty,$$

again using the Phragmen-Lindelof theorem ([3, theorems 1.4.2, 6.2.3]). This proves (11).

(ii) If $m_G(0) = 0$, then Γ is of minimal type as part (i) shows. But $\Lambda(\zeta)$ is bounded on $\mathcal{R}\ell(\zeta) = \nu$. Hence Λ is bounded throughout $C \setminus D_\nu$ ([3, theorem 6.2.4]) and so

$$\Gamma(\zeta) = O(1 + |\zeta|^N) \quad , \quad |\zeta| \to \infty. \tag{12}$$

Converting both (11) and (12) to estimates for G we obtain (3.1).

4. Proof of theorem A.

Let I be a closed ideal in $\mathcal{A}^+$, q the greatest common divisor of the inner functions in the factorization in $H^\infty(D)$ of the functions in I. Set $K = Z(I) \cap \partial D$ (no restrictions on K for the moment). Now certainly

$$I \subset qH^\infty(D) \cap I_{\mathcal{A}}(K). \tag{13}$$

To prove the reverse inclusion choose any $g \in qH^\infty(D) \cap I_{\mathcal{A}}(K)$ and define $\phi_g \in PM_{\mathcal{A}}$ by

$$<\phi_g, h> = <\phi, hg> \quad , \quad h \in \mathcal{A}.$$

Then $g \in I$ if and only if $\phi_g \in \mathcal{B}$ (cf. (2.2)) for all $\phi \in PM_{\mathcal{A}}$ orthogonal to I. Now, if Φ_g denotes the Carleman Transform of ϕ_g, automatically $\Phi_g(0) = 0$ and $\Phi_g = O(1)$ at infinity. Hence $g \in I$ if and only Φ_g is entire whenever, as we shall henceforth assume, ϕ is an $\mathcal{A}$-pseudo-measure orthogonal to I.

(4.1) Lemma. – *The Carleman* Transform Φ_g of ϕ_g *is given by*

$$\phi_1(z),\ |z| > 1,\ \ g(z)\,(\psi(z)/f(z)) - \phi_2(z),\ |z| < 1, \tag{14}$$

for some $\phi_1 \in H_-$ *and* $\phi_2 \in H_+$, *where* $(\psi/f - \phi_+)$ *is a representation of* Φ *in D. Furthermore, the singularities of* Φ_g *must lie in* $Z(I) \cap \partial D$.

Proof. – To establish (14) define $\psi_g \in PM_{\mathcal{A}}$ by

$$<\psi_g, h> = <\phi_g, fh> = <\phi, gfh> \quad , \quad h \in \mathcal{A},$$

f fixed in I. Then, with $\psi \in PM_{\mathcal{A}}$ defined by (5),

$$\psi_g(z) = <\phi, gf\,T(t)\,P_r>$$

$$<\phi, f\,T(t)\,P_r> g(z) = \psi(z)\,g(z)$$

for $z \in D$. Hence (14) holds.

By (2.4) the function $q(\psi/f)$ is analytic in D. Thus $g(\psi/f)$, hence Φ_g, is analytic in D since q divides g in $H^\infty(D)$. Consequently, the only possible singularities of Φ_g must lie in $Z(I) \cap \partial D$ (cf. (2.3)).

Representing Φ_g in D by $(g\psi - f\phi_2)/f$ we see from (4.1) that Φ_g belongs to the class $\mathcal{G}$. In addition, since q divides both g and

f in $H^\infty(D)$, i.e. $\bar{q}\, g$ and $\bar{q}\, f \in H^\infty(D)$, Φ_g admits a representation

$$\Phi_g = (\bar{q}\, g\psi - \bar{q}\, f\phi_2)/\bar{q}\, f, \qquad z \in D,$$

as a function in $\mathcal{G}$. Now, with the notation of § 3,

$$m_q(t) = \inf_{f \in I} m_f(t) \quad , \quad e^{it} \in \partial D.$$

Thus

$$m_{\Phi_g}(t) = 0 \quad , \quad e^{it} \in \partial D.$$

Estimate (3.2) (ii) shows, therefore, that any isolated singularity of Φ_g necessarily is a pole.

(4.2) *Remark.* – Only now is the Analytic Ditkin condition imposed. Its effect will be to rule out the possiblity of Φ_g having any isolated singularities at all.

(4.3) THEOREM. – *If $\mathfrak{A}$ satisfies the Analytic Ditkin condition then Φ_g does not have any isolated singularities.*

Proof. – Suppose Φ_g has a pole at $z = 1$ with order M, say

$$\Phi_g(z) = \alpha(z - 1)^{-M} + \ldots \quad , \quad \alpha \neq 0, \tag{15}$$

near 1 (so, in particular, $1 \in Z(I) \cap \partial D$). Let $\{\tau_N\}$ be a sequence in $\mathfrak{A}^+$ such that $\tau_N(0) = 1$ and $\tau_N\, g \to 0$ in norm in $\mathfrak{A}^+$ as $N \to \infty$. Approximating each τ_N by a Taylor polynomial P_N in $\mathfrak{A}$ (via (3) and the Cesaro kernel for instance) we obtain a sequence $\{P_N\}$ satisfying

$$\text{(a)}\ \ P_N\, g \to 0 \text{ in } \mathfrak{A}^+, \qquad \text{(b)}\ \ P_N(0) \to 1, \tag{16}$$

as $N \to \infty$. With $\phi_N \in PM_{\mathfrak{A}}$ defined by $\langle \phi_N , h\rangle = \langle \phi_g , P_N h\rangle$, (16) (a) shows that $\{\phi_N\}$ converges weak* to 0. In particular, $\{\phi_N\}$ is uniformly bounded in norm in $PM_{\mathfrak{A}}$. Furthermore, as a sequence in $\mathcal{D}'(\partial D)$, $\{\phi_N\}$ converges weakly, hence strongly, to 0. But then $(\phi_N)_\pm$ also converges strongly to 0 in $\mathcal{D}'(\partial D)$ since the 'Riesz projections' are continuous on $\mathcal{D}'(\partial D)$. Thus

$$(\phi_N)_+(z) \to 0, \quad |z| < 1, \qquad (\phi_N)_-(z) \to 0, \quad |z| > 1.$$

On the other hand, defining $\psi_N \in PM_{\mathfrak{a}}$ by $<\psi_N, h> = <\phi_N, fh>$, f fixed in I, we can soon check that $\psi_N(z) = P_N(z)\, g(z)\, \psi(z)$. Hence the Carleman Transform Φ_N of ϕ_N is given by

$$P_N\, g(\psi/f) - (\phi_N)_+ \;, \quad |z| < 1 \;; \quad (\phi_N)_- \;, \quad |z| > 1.$$

Consequently $\dot{\Phi}_N(z) \to 0$ whenever $z \notin \partial D$ since $P_N g \to 0$ in $\mathfrak{A}$.

There is an alternative description of Φ_N : fix $n > 0$ and define ϕ_n by $<\phi_n, h> = <\phi_g, \chi^n h>$, $h \in \mathfrak{A}$. A calculation shows that the Carleman Transform Φ_n of ϕ_n is

$$z^n\, \Phi_g(z) + z^n \left[\sum_{1-n}^{0} \hat{\phi}_g(m)\, z^m \right].$$

Hence, for each N, there is a function Q_N analytic in **C** such that

$$\Phi_N(z) = P_N(z)\, \Phi_g(z) + Q_N(z),$$

in particular, a neighborhood $\mathfrak{N}$ of 1 in which each function $(z-1)^M \Phi_N$ is analytic. By (15) and (16) (b)

$$(z-1)^M\, P_N(z)\, \Phi_g(z) \to \alpha, \qquad z \to 1,\ N \to \infty\,;$$

thus $(z-1)^M\, \Phi_N(z) \to \alpha (\neq 0)$ as $z \to 1$, $N \to \infty$. We shall prove, however, that some subsequence of $\{(z-1)^M\, \Phi_N\}$ converge uniformly to 0 as $N \to \infty$ in some neighborhood $\mathfrak{N}$ of 1. This contradiction shows that Φ_g cannot have a pole at $z = 1$.

That some subsequence of $\{(z-1)^M\, \Phi_N\}$ converges uniformly to 0 as $N \to \infty$ follows provided $(z-1)^M\, \Phi_N$ is uniformly bounded in some neighborhood of 1 since $\Phi_N(z) \to 0$, $z \notin \partial D$. Choose a suitable contour Γ containing 1 as an interior point and a function f in I such that $f \neq 0$ on $\Gamma \cap \bar{D}$ (such a choice is always possible). Now, by (4) and the uniform boundedness of the ϕ_N's,

$$|\hat{\phi}_N(n)| \leqslant \text{const.}\ (1 + |n|^{N_{\mathfrak{a}}})\;;$$

hence

$$|\phi_N)_\pm (re^{i\theta})| \leqslant \text{const. } |1 - r|^{-(N_\alpha + 1)}, \quad r \neq 1,$$

uniformly in N. A similar calculation shows that

$$|\psi_N (re^{i\theta})| \leqslant \text{const. } |1 - r|^{-(N_\alpha + 1)}, \quad r \neq 1,$$

again uniformly in N. Thus

$$|(z - 1)^M \Phi_N(z)| \leqslant \text{const. } |1 - |z||^{-(N_\alpha + 1)}$$

whenever $z \in \Gamma$, $|z| \neq 1$. Hence (cf. proof of lemma (8.3) in [11, p. 180]) the functions $(z - 1)^M \Phi_N(z)$ are uniformly bounded in some disk centered at 1. This completes the proof of theorem (4.3).

Completion of the proof of theorem (A) : When $K = Z(I) \cap \partial D$ is at most countable there is a decreasing chain of subspaces K_α of K defined inductively as follows (cf. [17, p. 40]) : $K_0 = K$ and K_α is the derived set of K_β, $\alpha = \beta + 1$, if α is not a limit ordinal, or $K_\alpha = \bigcap_{\beta < \alpha} K_\beta$ if α is a limit ordinal. There will be a first ordinal, say γ, necessarily a non-limit ordinal, such that K_α is empty ; consequently there is a last ordinal λ such that K_λ is non-empty. Now by (4.1) the singularities of Φ_g must lie in K_0. Suppose that any singularity of Φ_g lies in each K_β, $\beta < \alpha$. Now either $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$, or $\alpha = \beta + 1$ and $K_\beta \backslash K_\alpha$ consists of isolated points. Since Φ_g cannot have isolated singularities each singularity of Φ_g must therefore lie in K_α. Hence by transfinite induction the singularities must all lie in K_λ. But K_λ is finite and a singularity in K_λ would have to be isolated. Thus Φ_g is entire which as we remarked earlier ensures :

$$g \in qH^\infty(D) \cap 1_\alpha(K) \Rightarrow g \varepsilon I,$$

the reverse inclusion to (13).

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