Annales de l'institut Fourier

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Annales de l'institut Fourier, tome 22, nº 2 (1972), p. 165-168
<http://www.numdam.org/item?id=AIF_1972__22_2_165_0>

Ann. Inst. Fourier, Grenoble
22, 2 (1972), 165-168.

FINE CONNECTEDNESS AND α-EXCESSIVE FUNCTIONS

by S. RAMASWAMY

In Classical Potential Theory, the non-negative superharmonic functions in any domain ω of $\mathbf{R}^n$, $n \geq 2$ have the following property: namely any such function vanishes everywhere in ω, if it vanishes at some point of ω. Our object in this paper is to find out whether this property holds for α-excessive functions of a standard process with state space $(E, \mathcal{E})$ as defined in [1]. We prove that if E is *finely connected*, then this property holds for α-excessive functions, $\alpha \geq 0$. Conversely, if for some $\alpha \geq 0$, the α-excessive functions have the above property, then we prove under some further conditions on X, to be made precise in § 3, that E is finely connected. Throughout we follow the terminology of [1].

The author is extremely thankful to Professor P. A. Meyer, who has been kind enough to go through an earlier version of the paper and to suggest alternate proofs of the theorems. The proofs given here are suggested by him, which we publish with his kind permission. The author originally proved the results without using D martingale theory.

Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ denote a standard process with state space $(E, \mathcal{E})$ where E is a locally compact (non-compact) Hausdorff topological space satisfying 2^{nd} axiom of countability and $\mathcal{E}$ is its Borel σ-algebra. Let ζ denote the life time of X. If A is any subset of E, we denote by $\complement A$, the complement of A in E. Let E_Δ be the one-point compactification of E where Δ is the point at infinity.

The fine topology on $E\Delta$ is the coarsest topology making all the α-excessive functions, $\alpha \geqslant 0$ continuous. Let us call by fine topology on E, the induced topology on E, by the fine topology of E_Δ.

For brevity of notation and language, let us say that a class $\mathcal{C}$ *of non-negative functions on* E *has the property* $(\mathfrak{X})$ *if for any* $f \in \mathcal{C}$, $f = 0$ *at some point of* E *implies that* $f \equiv 0$.

Definition. — *A subset* A *of* E *is said to be absorbing, if for every* $x \in E$, P^x *almost surely,* $X_t(\omega) \in A$ *for all* t *in* $[D_A, \zeta)$, *where*

$$D_A(w) = \text{Inf}\,\{t : X_t(w) \in A\}.$$

Roughly speaking, if A is absorbing then the process can never leave A during its life time, once it hits A.

Note that a nearly Borel absorbing set is always fine open.

For the proof of the theorem 1, we require the following theorem on supermartingales. For a proof see {[3], page 99, T 15}

Theorem on supermartingales. Let $(\Omega, \mathcal{F}, P)$ be any probability space and let $(\mathcal{F}_t)_{t \in R_+}$ be any increasing family of σ-algebras contained in $\mathcal{F}$ such that $(\mathcal{F}_t)$ is right continuous and that $\forall t \geqslant 0\, \mathcal{F}_t$ contains all P-negligible sets. Let $(\xi_t)_{t \in R_+}$ be a right continuous, non-negative supermartingale, adapted to $(\mathcal{F}_t)$ and let $S(w) = \inf\{t : \xi_t(w) = 0\}$. Then almost surely, $\xi_t(w) = 0$ for all $t \geqslant S(w)$.

Let $\mathcal{S}^\alpha$ stand for the class of all α-excessive functions, $\alpha \geqslant 0$.

Theorem 1. — *For every* $\alpha \geqslant 0$ $\mathcal{S}^\alpha$ *has the property* $(\mathfrak{X})$ *if and only if there does not exist any nearly Borel, fine closed absorbing sets except* $\varnothing$ *and* E.

Therefore, in particular, if E *is finely connected, then* $\mathcal{S}^\alpha$ *has the property* $(\mathfrak{X})$, *for every* $\alpha \geqslant 0$.

Proof. — For a given $\alpha \geqslant 0$, let f be an α-excessive function. Let $A = \{x \in E | f(x) = 0\}$. Then A is a nearly Borel fine closed set. Applying the above theorem on supermartingales, to the supermartingale $(e^{-\alpha t} f \circ X_t)$, we see immediately that A is absorbing too. Thus, if there does not exist any nearly Borel fine closed absorbing sets except $\varnothing$ and E, A must be either void or E. Therefore, $f > 0$ everywhere or else $f \equiv 0$.

Conversely, for some $\alpha \geqslant 0$, let $\mathcal{S}^\alpha$ have the property $(\mathfrak{L})$. Let A be any non-void fine closed nearly Borel absorbing set. Then $f = \chi_{\complement A}$ is excessive and therefore is α-excessive and $A = \{x | f = 0\}$. Therefore $f \equiv 0$ and hence $\complement A = \varnothing$ and therefore $A = E$.

If E is finely connected, then there cannot exist any nearly Borel, fine closed, absorbing set other than $\varnothing$ and E, as such a set is fine open as well. Hence for every $\alpha \geqslant 0$, $\mathcal{S}^\alpha$ has the property $(\mathfrak{L})$.

Remark. — If the life time of X is $+\infty$ and if f is an *excessive function*, $d = \inf_{x \in E} f(x)$, $d > 0$, then $Y_t = f \circ X_t - d$ is still a *non-negative* right continuous supermartingale and applying as before the supermartingale theorem, we see that $\{x | f = d\}$ is a nearly Borel, fine closed and absorbing set. Hence if E is finely connected, then $f > d$ or else $f \equiv d$. Thus, we get a minimum principle for excessive functions, in this case.

We shall now state and prove the converse under some more additional hypotheses on X. More precisely we have

THEOREM 2. — *Let us assume that there exists a* $\alpha_0 \geqslant 0$ *such that* $\mathcal{S}^{\alpha_0}$ *has the property* $(\mathfrak{L})$. *Let us assume further that*

i) X *has continuous paths in* $[0, \zeta)$.

ii) *For some* $\alpha > 0$, *all the bounded* α-*excessive functions are regular* i.e. *for any* x, P^x *almost surely* $t \longmapsto f(X_t)$ *is continuous in* $[0, \zeta)$ *for every bounded* α-*excessive* f.

iii) X *has a reference measure.*

Then, E *is finely connected.*

Remark. — The conditions (i), (ii), (iii) are fulfilled in the case of Brownian motion in $\mathbf{R}^n$ (For a proof see [2]).

Proof of theorem 2. — Let A be any proper subset of E which is fine closed and fine open at the same time. Since X has a reference measure, there exists a Borel measurable set $B \supset A$ such that $B^r = A^r$ ([1], page 202). Since $A^r = A$ and B^r is a Borel set, A is a Borel set. To prove that A must be either void or E, it is sufficient to prove that A is absorbing, in view of theorem 1.

Now $\{t\colon X_t(w) \in A\} = \{t\colon \Phi_A^\alpha \circ X_t = 1\}$. For any x, P^x almost surely, this set is a closed set in $[0, \zeta)$, as $\Phi_A^\alpha \circ X.(w)$ is continuous for P^x almost all w. Similarly, it is also an open set in $[0, \zeta)$ by considering $\complement A$ instead of A. Thus $\{t\colon X_t(w) \in A\}$ must be $[0, \zeta)$ or else void. This shows that A is absorbing.

This completes the proof of theorem 2.

BIBLIOGRAPHY

[1] Blumenthal and Getoor, Markov Processes and Potential Theory (Academic Press, 1968).
[2] P. A. Meyer, Processus de Markov, (Lecture Notes in Mathematics, nº 26 Springer-Verlag, 1967).
[3] P. A. Meyer, Probability and Potentials (Blaisdell Publishing Company, 1966).

Manuscrit reçu le 1er mars 1971,
accepté par M. Brelot.

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