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ON NONBORNOLOGICAL BARRELLED SPACES (1)

by Manuel VALDIVIA

L. Nachbin [5] and T. Shirota [6], give an answer to a problem proposed by N. Bourbaki [1] and J. Dieudonné [2], giving an example of a barrelled space, which is not bornological. Posteriorly some examples of nonbornological barrelled spaces have been given, e.g. Y. Kōmura, [4], has constructed a Montel space which is not bornological. In this paper we prove that if E is the topological product of an uncountable family of barrelled spaces, of nonzero dimension, there exists an infinite number of barrelled subspaces of E , which are not bornological. We obtain also an analogous result replacing « barrelled » by « quasi-barrelled ».

We use here nonzero vector spaces on the field K of real or complex number. The topologies on these spaces are separated.

If E is a separated locally convex space, we denote, as usual, by E' , $\sigma(E', E)$ and $\beta(E', E)$, the topological dual of E , the weak topology on E' , and the strong topology on E' , respectively. If A is a bounded, closed and absolutely convex set of E , we denote by E_A the linear hull of A equipped with the norm associated to A .

We shall need the following result of J. Dieudonné [3]:

a) Let E be a bornological space. If F is a subspace of E , of finite codimension, then F is bornological.

THEOREM 1. — *If E is the topological product of the barrelled spaces E_i , $i \in I$, where I is an uncountable set, there exists*

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an infinite family \mathcal{F} of barrelled dense subspaces of E , which are not bornological.

Proof. — Let G be the subspace of E , whose points have all components zero except a countable set. Let \mathcal{F} be the family of all the subspaces of E , such that $F \in \mathcal{F}$ if and only if $G \subset F$ and the codimension of G in F is finite and different from zero. Obviously \mathcal{F} is infinite. If $F \in \mathcal{F}$, then F is barrelled since G is barrelled. (It can be proved that G is barrelled taking any $\sigma(G', G)$ -bounded set A of $G' = E'$, and noticing that there exists a finite set $\{i_1, i_2, \dots, i_n\} \subset I$, such that $A \subset \prod_{p=1}^n E'_p$, hence A is $\sigma(E', E)$ -bounded and, therefore, equicontinuous respect to E and respect to G also.) Now we suppose that F is bornological. According to *a*) we can find a bornological space L , such that $G \subset L \subset F$, being G an hyperplane of L . In L let \mathcal{B} be the family of all the absolutely convex, closed and bounded sets. Since L is the inductive limit of $\{E_B : B \in \mathcal{B}\}$ and G is a dense hyperplane of L , there exists a $M \in \mathcal{B}$, such that $G \cap E_M$ is dense in E_M and $G \not\supset E_M$. Therefore we can find in E_M a sequence $\{x_n\}_{n=1}^\infty \subset G$, which converges to $x \notin G$. That is in contradiction with being G sequentially closed in E and also in L . Q.E.D.

THEOREM 2. — *If E is the topological product of the bornological barrelled spaces E_i , $i \in I$, where I is an uncountable set, there exists a family \mathcal{F} of barrelled dense subspaces of E , which are not bornological, so that if $F \in \mathcal{F}$, there exists a subspace H of F , of finite codimension, such that H is bornological.*

Proof. — It is enough to prove that the space G defined in the proof of Theorem 1 is bornological. Let \mathcal{M} be the family of the parts of I , which have a countable infinity of elements. For each $M \in \mathcal{M}$, we denote by $E(M)$ the subspace of E , whose points have all the components zero except at most those with indices in M . It is immediate that $E(M)$ is bornological. Since G is the inductive limit of the family of spaces $\{E(M) : M \in \mathcal{M}\}$ then G is bornological. (We can prove that G is the inductive limit of $\{E(M) : M \in \mathcal{M}\}$ of the following

way: let u be any linear form on G , such that its restriction u_M to $E(M)$ is continuous, $M \in \mathcal{M}$. Let ν_M be the continuous extension of u_M to G , such that if $x \in G$ and $x(M)$ is the projection of x on $E(M)$, then $\nu_M(x) = u_M(x(M))$. Obviously the net $\{\nu_M : M \in \mathcal{M}, \subset\}$ converges weakly to u . Furthermore, if $x \in G$, it is easy to prove that $\{\nu_M(x) : M \in \mathcal{M}\}$ is a bounded set in K , as since G is barrelled, it results that $\{\nu_M : M \in \mathcal{M}\}$ is equicontinuous set, hence u is continuous in G . Therefore, the space G and the inductive limit of $\{E(M) : M \in \mathcal{M}\}$ have the same topological dual and since, the topology of G is the Mackey one, both spaces are the same.) Q.E.D.

Note. — From the anterior proof it can be deduced that if there exists the strongly inaccessible cardinal β , then there exists a bornological space G , whose completion \hat{G} is not bornological. It is enough to carry out the topological product E of nonzero Frechet spaces, in number equal to β , and to take the subspace G formed by all points of E , whose components are nulle, except a countable set. Then G is bornological and its completion $\hat{G} = E$ is not it.

THEOREM 3. — *If E is the topological product of the quasi-barrelled spaces $E_i, i \in I$, where I is an uncountable set, there exists an infinite family of quasi-barrelled dense subspaces, which are not bornological.*

Proof. — The proof is analogous to that of Theorem 1, replacing barrelled by quasi-barrelled. (The proof of being G quasi-barrelled can be done, taking any set A of $G' = E'$, $\beta(G', G)$ -bounded, and taking into account that there exists a finite set $\{i_1, i_2, \dots, i_n\} \subset I$ so that $A \subset \prod_{p=1}^n E'_{i_p}$, hence it is easy to deduce that A is bounded for the topology $\beta(E', E)$, and since E is quasi-barrelled it results that A is equicontinuous respect to E , and also respect to G .)

Q.E.D.

THEOREM 4. — *If E is the topological product of the quasi-barrelled spaces $E_i, i \in I$, where I is an uncountable set, and there exists a $i_0 \in I$, such that E_{i_0} is not barrelled, then there*

exists a infinite family \mathcal{F} of quasi-barrelled dense subspaces of E , which are not bornological nor barrelled.

Proof. — It is enough to prove if in the Theorem 3, $F \in \mathcal{F}$, then F is not barrelled. Indeed, if F is barrelled, then its closure in E , which is equal to E , is a barrelled space. In contradiction with the fact that E_i is not barrelled. Q.E.D.

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