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ON A ABSTRACT STIELTJES MEASURE

by James E. HUNEYCUTT, Jr (2)

1. Introduction.

In 1955, A. Revuz [4] considered a type of Stieltjes measure defined on analogues of half-open, half-closed intervals in a partially ordered topological space. He states that these functions are finitely additive but his proof has an error. We shall furnish a new proof and extend some of his results to "measures" taking values in a topological abelian group.

2. Preliminaries.

If X is a set and $\mathfrak{S}$ is a non-void collection of subsets of X, then $\mathfrak{S}$ is called a semi-ring provided

i) $A, B \in \mathfrak{S} \Rightarrow A \cap B \in \mathfrak{S}$,

ii) $A, B \in \mathfrak{S}$, $A \subseteq B \Rightarrow \exists \, \{C_i\}_{i=0}^{n} \subseteq \mathfrak{S}$ such that

$$A = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n = B \quad \text{and} \quad C_i \backslash C_{i-1} \in \mathfrak{S} \quad \text{for} \quad 1 \leqslant i \leqslant n .$$

$\mathfrak{S}$ is a weak semi-ring provided that, in place of ii) we require

iii) $A, B \in \mathfrak{S}$, $A \subseteq B \Rightarrow \exists \, \{C_i\}_{i=1}^{n}$ such that

$$B \backslash A = \bigcup_{1}^{n} C_i \quad \text{and} \quad C_i \cap C_j = \emptyset \quad \text{if} \quad i \neq j .$$

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(1) The results presented in this paper are a part of the author's Ph.D. dissertation, written at the University of North Carolina at Chapel Hill under the direction of Professor B. J. Pettis.

DEFINITION. – *Let* $\mathfrak{S} \subseteq 2^{X}$ *and let* $\mathfrak{G}$ *be a topological abelian group. If* $\mu : \mathfrak{S} \to \mathfrak{G}$ *then*

i) μ *is 2-additive if* A, B, $A \cup B \in \mathfrak{S}$,

$$A \cap B = \Phi \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$$

ii) μ *is finitely additive if whenever* $\{A_i\}_1^n$ *is any finite, pairwise disjoint sequence in* $\mathfrak{S}$ *such that* $\bigcup_1^n A_i \in \mathfrak{S}$, *then* $\mu\left(\bigcup_1^n A_i\right) = \sum_1^n \mu(A_i)$.

iii) μ *is countably additive if whenever* $\{A_i\}_1^\infty$ *is any pairwise disjoint sequence in* $\mathfrak{S}$ *such that* $\bigcup_1^\infty A_i \in \mathfrak{S}$, *then* $\sum_{i=1}^{\infty} \mu(A_i) \to \mu\left(\bigcup_1^\infty A_i\right)$.

Von Neumann [3, p. 94] has shown that if $\mathfrak{S}$ is a semi-ring and $\mu : \mathfrak{S} \to \mathfrak{G}$ is 2-additive, then μ is finitely additive. This does not hold in general for weak semi-rings. The smallest ring $\mathcal{R}(\mathfrak{S})$ containing the semi-ring $\mathfrak{S}$ is the collection of all unions of finite pairwise disjoint sets of members of $\mathfrak{S}$; Von Neumann showed that a finitely (respectively countably) additive function an $\mathfrak{S}$ has a unique finitely (respectively countably) additive extension defined on $\mathcal{R}(\mathfrak{S})$.

The topology for the topological abelian group $\mathfrak{G}$ is determined by a family $\{\|\cdot\|_p : p \in P\}$ of semi-norms

$$(\|-g\|_p = \|g\|_p, \|g+h\|_p \leqslant \|g\|_p + \|h\|_p, \|g\|_p \geqslant 0).$$

Suppose $\mu : \mathfrak{S} \to \mathfrak{G}$; then for each p in P and each subset B of X, we define

1) $(\mu_R)_p(B) = \sup\{\|\mu(A)\|_p : A \in \mathfrak{S}, A \subseteq B\}$

2) $(\mu_D)_p(B) = \sup\{\|\Sigma_1^n \mu(A_i)\|_p\}$

3) $|\mu|_p(B) = \sup\{\Sigma_1^n \|\mu(A_i)\|_p\}$,

where the supremum in 2) and 3) is taken over all finite, pairwise disjoint sequences in $\mathfrak{S}$ whose union is a subset of B.

Let X be a topological space and $\mathfrak{S}$ a weak-semi-ring of subsets of X and let $\mu : \mathfrak{S} \to \mathfrak{G}$ be finitely additive.

DEFINITION. – μ *is* μ_R*-regular on* $\mathfrak{S}$ *provided that for all* $p \in P$, $A \in \mathfrak{S}$, *and* $\varepsilon > 0$, *there exist* C *countably* compact, o *open*, $A', A'' \in \mathfrak{S}$ *such that* $A' \subseteq C \subseteq A \subseteq o \subseteq A''$ *and* $(\mu_R)_p(A'' \setminus A') < \varepsilon$.

Similar definitions are made for μ_D- and $|\mu|$-regularity. In a previous paper [1], we have shown that a μ_D-regular, finitely additive function on a weak semi-ring is countably additive. We note that $|\mu|$-regularity $\Rightarrow$ μ_D-regularity $\Rightarrow$ μ_R-regularity and that, for a ring of sets, μ_D-regularity is the same as μ_R-regularity.

3. The main theorems.

Revuz considered the problem of obtaining countable additivity from finite additivity and derived a suitable regularity condition to obtain countable additivity for non-negative real valued functions ([4], p. 208). The work of this paper generalizes the regularity condition of Revuz so that countable additivity may be obtained from finite additivity in the case of a function with values in a topological abelian group. We also show that an argument of Revuz concerning finite additivity is wrong (Example 3.1) and we give an alternate argument (Theorem 3.2).

Let X be a non-void set and $\leqslant$ a binary relation on X. We shall say that $(X, \leqslant)$ is a *conditional lower semilattice* provided that

i) $\leqslant$ is reflexive, transitive, and antisymmetric.

ii) If x and y are members of X and there is some member z in X such that $z \leqslant x$ and $z \leqslant y$, then there is a largest (relative to $\leqslant$) such member of X ; we shall denote such a member of X by "inf xy".

We now form our "intervals" in this set. For any x in X, $C_-(x)$ will denote the set of all members y of X such that $y \leqslant x$, and $C_+(x)$ will denote the set of all members y of X such that $x \leqslant y$. For each positive integer n and each $x, u_1, u_2, \ldots, u_n$ in X, let

$$S(x; u_1, u_2, \ldots, u_n) = C_-(x) \setminus \bigcup_1^n C_-(u_i)$$

$$= \{y \in X : y \leqslant x \quad \text{but} \quad y \nleqslant u_i \quad \text{for any } i = 1, 2, \ldots, n\}.$$

Revuz ([4], p. 195) has shown that each non-empty set of the form above has a unique representation in which each $u_i \leqslant x$ but $u_i \nleqslant u_j$ for $i \neq j$. This form will be called the *canonical form*. In particular when $(X, \leqslant)$ is the real line with the usual ordering, the S's are simply intervals of the form $(a, b]$.

Let $\mathfrak{S}$ denote the collection of all sets of the form $S(x; u_1, \dots u_n)$. We note that $\Phi \in \mathfrak{S}$ since for any x in X, $S(x; x) = \Phi$.

In the case of the real line with the usual ordering,

$$\{(a, b] : -\infty < a \leqslant b < \infty\}$$

forms a semi-ring. Revuz has shown that $\mathfrak{S}$ is a weak semi-ring ([4], p. 199) ; we shall show that $\mathfrak{S}$ is actually a semi-ring.

LEMMA. – *If* $S_1 = S(x; v_1, v_2, \dots, v_n)$ *and*

$$S_2 = S(x; v_2, \dots, v_n, \textit{ then}$$

i) $S_1 \subseteq S_2$ *and in particular,* $S_1 = S_2 \cap \widetilde{C - (v_1)}$.

ii) $S_2 \backslash S_1 = S(\inf xv_1 ; v_2, v_3, \dots, v_n)$ *or* Φ *if* $\inf xv_1$ *does not exist.*

Proof :

$$\text{i) } S_1 = C_-(x) \backslash \bigcup_1^n C_-(v_i) = C_-(x) \cap \left(\bigcap_1^n \widetilde{C_-(v_i)}\right)$$

$$= C_-(x) \cap \left(\bigcap_2^n \widetilde{C_-(v_i)}\right) \cap \widetilde{C_-(v_1)}$$

$$= S(x; v_2, \dots, v_n) \cap \widetilde{C_-(v_1)} = S_2 \cap \widetilde{C_-(v_1)}$$

$$\text{ii) } S_2 \backslash S_1 = S_2 \backslash [S_2 \cap \widetilde{C_-(v_1)}] = S_2 \backslash (\widetilde{C_-(v_1)}) = S_2 \cap C_-(v_1)$$

$$= (C_-(x) \backslash \bigcup_2^n C_-(v_i)) \cap C_-(v_1)$$

$$= (C_-(x) \cap C_-(v_1)) \backslash \bigcup_2^n C_-(v_i)$$

$$= \Phi \text{ or } S(\inf xv_1 ; v_2, v_3, \dots, v_n). \ \square$$

We shall use the preceding lemma to prove

THEOREM 3.1. – $\mathfrak{S}$ *is a semi-ring.*

Proof. – By a result of Revuz $\mathfrak{S}$ is closed under finite intersections. We must show that if S and S* are in $\mathfrak{S}$ with $S \subseteq S^*$, then there is a finite sequence $S_1, S_2, \ldots, S_m$ in $\mathfrak{S}$ with

$$S = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_{m+1} = S^* \quad \text{and} \quad S_i \backslash S_{i-1} \in \mathfrak{S}$$

for $1 \leqslant i \leqslant m + 1$. Suppose $S = S(x\,; v_1, \ldots, v_m)$ and

$$S^* = S(x^*\,; u_1, \ldots, u_n)$$

with $S \subseteq S^*$ and S^* is in canonical form ($u_i \leqslant x^*$ for $1 \leqslant i \leqslant n$, but $u_i \leqslant u_j$ if $i \neq j$). If $y \in S$, then $y \in S^*$ so y not $\leqslant u_i$ for any

$$i = 1, 2, \ldots, n\ ;$$

thus S can be put into the (not necessarily canonical) form

$$S = S(x\,; v_1, v_2, \ldots, v_m, u_1, \ldots, u_n)\,.$$

Now let

$$\begin{aligned} S_0 &= S = S(x\,; v_1, v_2, \ldots, v_m, u_1, \ldots, u_n) \\ S_1 &= S(x\,; v_2, \ldots, v_m, u_1, \ldots, u_n) \\ &\vdots \\ S_i &= S(x\,; v_{i+1}, \ldots, v_m, u_1, \ldots, u_n) \\ &\vdots \\ S_m &= S(x\,; u_1, \ldots, u_n)\,. \end{aligned}$$

By Lemma 6.2, we have that $S_i \backslash S_{i-1} \in \mathfrak{S}$ for $1 \leqslant i \leqslant m$; and we also have $S = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_m$ so we need only show that if $S^* = S_{m+1}$, then $S_m \subseteq S_{m+1}$ and $S_{m+1} \backslash S_m \in \mathfrak{S}$. If $y \in S_m$, then $y \leqslant x$; since $S \subseteq S^*$ then $x \in S^*$ and $x \leqslant x^*$; thus $y \leqslant x^*$. By definition of S_m, if $y \in S_m$ then y not $\leqslant u_i$ for

$$1 \leqslant i \leqslant n \quad \text{and so} \quad S_m \subseteq S_{m+1} = S^*\,.$$

We also note that $S_m = \{y \in X : y \leqslant x$ but y not $\leqslant u_i$ for $1 \leqslant i \leqslant n$ and $S_{m+1} = \{y \in X : y < x^*$ but y not $\leqslant u_i$ for $1 \leqslant i \leqslant n\}$.

Thus, $S_{m+1} \backslash S_m = \{y \in X : y \leqslant x^*$ but y not $\leqslant x$ and y not $\leqslant u_i$ for $1 \leqslant i \leqslant n\}$

$$= S(x^*\,; x, u_1, \ldots, u_n) \in \mathfrak{S}\,.$$

and $\mathfrak{S}$ is a semi-ring. □

We recall from Chapter II, that one property that a semi-ring has but a weak semi-ring lacks is that a two-additive function is necessarily finitely additive. In a previous paper [2], we generated such functions on $\{(a, b] : -\infty \leqslant a \leqslant b \leqslant \infty\}$ from abelian group valued functions on the reals. We perform a similar feat in our more abstract setting. Suppose F is a function on X with values in an abelian group $\mathfrak{J}$; we define μ from $\mathfrak{S}$ to $\mathfrak{J}$ as follows :

1) if $S = \Phi$, $\mu(S) = 0$

2) if $S \neq \Phi$ and $S = S(x ; u_1, \ldots, u_n)$ in some (not necessarily canonical) representation, define

$$(*) \qquad \mu(S) = F(x) - \Sigma_1 F(\inf xu_{i_1}) + \Sigma_2 F(\inf xu_{i_1} u_{i_2}) - \ldots$$

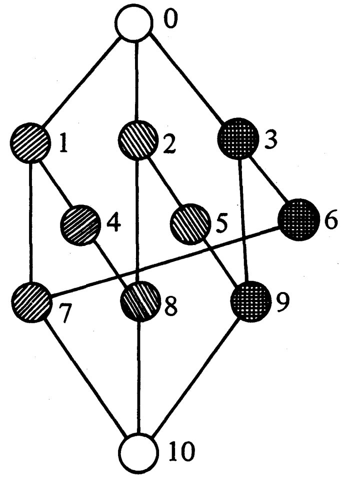
where $\Sigma_m F(\inf xu_{i_1} \ldots u_{i_m})$ represents the sum over all distinct sets of m indices $\{i_1, i_2, \ldots, i_m\}$ and $F(\inf xu_{i_1} \ldots u_{i_m}) = 0$ if that inf does not exist.

Revuz ([4], p. 197) has shown that any such real valued function μ is well-defined ; exactly the same proof carries over for the case in which μ takes values in an abelian group. We note that (*) is simply an extension of the usual modularity law :

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) .$$

Revuz attempts to show as follows that such a function μ is finitely additive. If $S = S(x ; u_1, \ldots, u_n)$ then x is called the summit of S. Revuz ([4], p. 201) defines a relation $\ll$ on arbitrary collections of pairwise disjoint members of $\mathfrak{S}$ by setting $S_1 \ll S_2$ if and only if there is some x in S_1 with $x \leqslant$ the summit of S_2. He considered $S_1, S_2, \ldots, S_n$ in $\mathfrak{S}$ with $S = \bigcup_1^n S_i \in \mathfrak{S}$ and sought to pick a minimal S_i relative to the ordering $\ll$. He asserted ([4], p. 201) that this is possible since $\ll$ is reflexive, antisymmetric, and transitive. However, the transitive property does not hold in general and a situation in which we cannot pick such a minimal element is shown in

Example 3.1. – Let $(X, \leqslant)$ be the partially ordered set determined by the following Hasse diagram where as usual, $n \leqslant m$ provided n is no higher than m and there is an ascending path from n to m. Note that $(X, \leqslant)$ is a conditional lower semilattice.

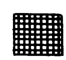


Now we pick S_0 , S_1 , S_2 , S_3 as follows :

$S_0 = \{0\} = S(0\,;1\,,2\,,3)$

$S_1 = \{1\,,4\,,7\} = S(1\,;8\,,10)$ denoted by 

$S_2 = \{2\,,5\,,8\} = S(2\,;9\,,10)$ denoted by 

$S_3 = \{3\,,6\,,9\} = S(3\,;7\,,10)$ denoted by 

$S_1 \ll S_3$ since $7 \in S_1$ and $7 \leqslant 3$ = summit of S_3 , $S_3 \ll S_2$ since $9 \in S_3$ and $9 \leqslant 2$ = summit of S_2 , and $S_2 \ll S_1$ since $8 \in S_2$ and $8 \leqslant 1$ = summit of S_1 . Thus, there is no minimal member relative to the order $\ll$.

Even though Revuz's proof is incorrect, we do get finite additivity for such a function μ. In view of Von Neumann's work and the fact that $\mathfrak{S}$ is a semi-ring, we need only prove that μ is 2-additive and a relatively trivial modification of Revuz's proof accomplishes this.

For the remainder of the chapter, we shall assume that X is a topological space and $(X, \leqslant)$ is a conditional lower semilattice and we shall be interested in the following relationships between the order and the topology :

X_a : *Each* $C_-(x)$ *is closed and the closure of each member of* $\mathfrak{S}$ *is countably compact.*

X_b : *inf is continuous from the right* in the sense that one of the following must hold for each x and y in X :

i) If $w = \inf xy$ and V is a neighborhood of w, then there exists V_x and V_y neighborhoods of x and y respectively such that if $x' \in V_x$ with $x \leqslant x'$ and $y' \in V_y$ with $y \leqslant y'$, then $\inf x'y' \in V$.

ii) If $\inf xy$ does not exist, then there exist neighborhoods V_x and V_y of x and y respectively such that if $x' \in V_x$ with $x \leqslant x'$ and $y' \in V_y$ with $y \leqslant y'$ then $\inf x'y'$ does not exist.

X_c : *If $x \in C_-(y)$ then for each neighborhood V_x of x, there exists z in $V_x \cap C_+(x) \cap C_-(y)$ such that $C_-(x) \subseteq \{C_-(z)\}^{\text{int}}$ where the interior is relative to the subspace topology of $C_-(y)$.*

The meanings of X_a and X_b are clear, but X_c may require an illustration. Let $(X, \leqslant)$ be the real line with the usual topology and the usual ordering. Let $x \leqslant Y$ and $\varepsilon > 0$. If $x = y$ then

$$(-\infty, x] = (-\infty, y]$$

and the interior of $(-\infty, x]$ relative to the subspace topology of $C_-(y)$ is $(-\infty, x]$. Thus the z whose existence is asserted in X_c is just x. Now if $x \lneqq y$, then there is some z strictly between x and y so $z \in C_+(x) \cap C_-(y)$ and $C_-(x) - (-\infty, x]_- (-\infty, z) = C_-(z)^{\text{int}}$.

In Chapter II, we have seen that an additive set function which is μ_D-regular on $\{(a, b] : a, b \in R\}$ is countably additive. Now for each $(a, b]$ with $a < b$, let c and d be numbers such that $a < c \leqslant b < d$. Then $(c, b] \subseteq [c, b] \subseteq (a, b] \subseteq (a, d) \subseteq (a, d]$ and for regularity, it is sufficient that $(a, d]\backslash(c, b]$ be "small". Now

$$(a, d]\backslash(c, b] = (a, c] \cup (b, d] = (a, b]\, \Delta\, (c, d]$$

where Δ denotes the symmetric difference. Thus, for regularity, we may require that each member $(a, b]$ be "approximated from the right" by some member $(c, d]$. To the end of generalizing this regularity for use in our more abstract setting we first of all obtain an approximation notion and then consider "approximation from the right".

Recalling the definitions of Chapter II, we define

$$(V_F)_p (S(x ; u_1, \ldots, u_m)) = \sup \left\{ \sum_1^n \| \mu(S_i) \|_p \right\}$$

$$(\mathrm{DV_F})_p\,(\mathrm{S}(x\,;u_1\,,\ldots,u_m)) = \sup\left\{ \left\| \sum_1^n \mu(\mathrm{S}_i) \right\|_p \right\}$$

where in each case the supremum is taken over all finite, pairwise-disjoint sequences of members of whose union is in $\mathrm{S}(x\,;u_1\,,\ldots,u_n)$. These will be called the variation and the Dunford variation, respectively, of F. We note that these definitions are made so that

$$|\mu|_p(s) = (\mathrm{V_F})_p(\mathrm{S}) \quad \text{and} \quad (\mu_\mathrm{D})_p(\mathrm{S}) = (\mathrm{DV_F})_p(\mathrm{S})$$

for each S in and each p in P.

For a regularity condition, we shall consider

X_d : *If* $\mathrm{S} = \mathrm{S}(x\,;u_1\,,\ldots,u_n) \in \mathfrak{S}$, $\varepsilon > 0$, *and* $p \in \mathrm{P}$, *then there exist neighborhoods* V_x *of* x *and* V_i *of* u_i $(1 < i < n)$ *such that whenever* $x' \in \mathrm{V}_x \cap \mathrm{C}_+(x)$, $u_i' \in \mathrm{V}_i \cap \mathrm{C}_+(u_i)$ $(1 \leqslant i \leqslant n)$, *then we have* $(\mu_\mathrm{D})_p(\mathrm{S}\,\Delta\,\mathrm{S}') < \varepsilon$ *where* $\mathrm{S}' = \mathrm{S}(x'\,;u_1'\,,\ldots,u_n')$.

X_d' : *Same as* X_d *but with* μ_D *replaced by* $|\mu|$.

We note that X_d' implies X_d.

LEMMA. – *Let* $\mathrm{X} = \mathrm{C}_-(y)$ *for some* y *and suppose* X_a, X_c, *and* X_d *are satisfied by* $(\mathrm{X}\,,<)$. *Then* μ *is countably additive on* $\mathfrak{S}$.

Proof. – Let $\mathfrak{U}$ be the collection of open sets of X and $\mathfrak{C}$, the collection of closed countably compact sets of X. We shall show that μ is μ_D-regular and thus μ will be countably additive.

i) "inner regularity" : Let $\mathrm{S} = \mathrm{S}(x\,;u_1\,,\ldots,u_n)$, $p \in \mathrm{P}$ and $\varepsilon > 0$; then since X satisfies X_c, for each neighborhood V_i of u_i, we can find v_i such that $v_i \in \mathrm{V}_i \cap \mathrm{C}_+(u_i)$ and $\mathrm{C}_-(u_i) \subsetneq (\mathrm{C}_-(v_i))^{\mathrm{int}}$. Now we have that $\overline{\mathrm{C}_-(x)\backslash\mathrm{C}_-(v_i)} \subsetneq \mathrm{C}_-(x)\backslash(\mathrm{C}(v_i))^{\mathrm{int}} \subsetneq \mathrm{C}_-(x)\backslash\mathrm{C}_-(u_i)$. Thus, $\mathrm{S}_1 = \mathrm{S}(x\,;v_i\,,\ldots,v_n) = \mathrm{C}_-(x)\backslash\bigcup_1^n \mathrm{C}_-(v_i) = \bigcap_1^n (\mathrm{C}_-(x)\backslash\mathrm{C}_-(v_i))$, then

$$\overline{\mathrm{S}_1} = \bigcap_1^n \overline{(\mathrm{C}_-(x)\backslash\mathrm{C}_-(v_i))} \subsetneq \bigcap_1^n \overline{(\mathrm{C}_-(x)\backslash\mathrm{C}_-(v_i)} \subsetneq \bigcap_1^n (\mathrm{C}_-(x)\backslash\mathrm{C}_-(u_i)) = \mathrm{S}\,.$$

Therefore $\mathrm{S}_1 \subsetneq \overline{\mathrm{S}_1} \subsetneq \mathrm{S}$ and $\overline{\mathrm{S}_1}$ is countably compact by X_a. Now by X_d, we may pick the neighborhoods V_i $(1 \leqslant i \leqslant n)$ such that $(\mu_\mathrm{D})_p(\mathrm{S}_1\,\Delta\,\mathrm{S}) < \varepsilon/2$. Since $\mathrm{S}_1\,\Delta\,\mathrm{S} = \mathrm{S}/\mathrm{S}_1$ whenever $\mathrm{S}_1 \subsetneq \mathrm{S}$, we have that $(\mu_\mathrm{D})_p(\mathrm{S}\backslash\mathrm{S}_1) < \varepsilon/2$.

ii) "outer regularity" : Let $S = S(x\,; u_1, \ldots, u_n)$ be in canonical form, $p \in P$ and $\varepsilon > 0$; since X satisfies X_c, for each neighborhood V_x of x, there is a z in $V_x \cap C_+(x)$ such that $C_-(x)) \subseteq (C_-(z))^{\text{int}}$. Let $S_2 = S(z\,; u_1, \ldots, u_n)$. Now $S \subseteq S_2^{\text{int}} \subseteq S_2$ since

$$S_2^{\text{int}} = (C_-(z) \setminus \bigcup_1^n C_-(u_i))^{\text{int}} = (C_-(z))^{\text{int}} \setminus \bigcup_1^n C_-(u_i) \supseteq C_-(x) \setminus \bigcup_1^n C_-(u_i) .$$

By X_d, V_x can be picked so that $(\mu_D)_p(S_2 \,\Delta\, S) < \varepsilon/2$; since

$$S_2 \,\Delta\, S = S_2 \setminus S$$

whenever $S \subseteq S_2$, we have that $(\mu_D)_p(S_2 \setminus S) < \varepsilon/2$.

Now from i) and ii) we may conclude that for each S in $\mathfrak{S}$, each $p \in P$, and each $\varepsilon > 0$, there exist S_1, S_2 in $\mathfrak{S}$, $C(= S_1)$ countably compact, and $U(= S_2^{\text{int}})$ open such that $S_1 \subseteq \overline{S_1} \subseteq S \subseteq S_2^{\text{int}} \subseteq S_2$ and $(\mu_D)_p(S_2 \setminus S_1) \leqslant (\mu_D)_p(S_2 \setminus S) + (\mu_D)_p(S \setminus S_1) < \varepsilon$. Thus μ is μ_D-regular and so μ is countably additive. □

THEOREM 3.2. – *Let* X *be a topological space and* $(X, \leqslant)$ *a conditional lower semilattice. If* X_a, X_c, *and* X_d *are satisfied, then* μ *is countably additive on* $\mathfrak{S}$.

Proof. – Let $\{S_i\}_1^\infty$ be a pairwise disjoint sequence of members of $\mathfrak{S}$ such that $\bigcup_1^\infty S_i = S \in \mathfrak{S}$. Let $S = S(x\,; u_1, \ldots, u_n)$, then $S_i \subseteq S \subseteq C_-(x)$ for $1 \leqslant i < \infty$. If $X' = C_-(x)$, then $(X', \leqslant)$ satisfies X_a, X_c, and X_d. Thus, by the preceding lemma, μ is countably additive on $\mathfrak{S} \cap 2^{X'}$ and so $\sum_1^n \mu(S_i) \to \mu(S)$. Therefore, μ is countably additive on $\mathfrak{S}$. □

Since X'_d implies X_d, we obtain the following

COROLLARY 3.2.1. – *Let* X *be a topological space and* $(X, \leqslant)$ *a conditional lower semilattice. If* X_a, X_c, *and* X'_d *are satisfied, then* μ *is countably additive on* $\mathfrak{S}$.

As an additional corollary, we also obtain Revuz' original result below in Theorem 3.3 ([4], p. 208).

We shall say that a function F defined on X and taking values in the topological space Y is *continuous from the right* at x in X provided that for each neighborhood U of $F(x)$, there is a neighborhood V of x such that $F(x') \in U$ whenever $x' \in V \cap C_+(x)$.

LEMMA. – *Suppose* X *is a topological space and* $(X, \leqslant)$ *is a conditional lower semilattice. If* $(X, \leqslant)$ *satisfies* X_a *and* X_c *and if* $F : X \to R$ *is such that*

i) μ *is non-negative.*

ii) F *is continuous from the right on* X,

then if X_b *is satisfied, so is* X_d.

Proof. – Notice that under the condition that μ is non-negative we have $\mu_D(S_1 \backslash S_2) = \mu(S_1) - \mu(S_2)$ if $S_2 \subseteq S_1$. Let

$$S = S(x\,; u_1, \ldots, u_n)$$

and $\varepsilon < 0$. Now F and inf are continuous from the right on X, so for each u_i, there is a neighborhood V_i of u_i such that if

$$v_i \in V_i \cap C_+(u_i) \cap C_-(x)$$

then $0 \leqslant F(u_i) - F(v_i) \leqslant \varepsilon/2n$. Also, there is a neighborhood V of x such that if $x' \in V \cap C_+(x)$, then $0 \leqslant F(x') - F(x) < \varepsilon/2$. Now let $S^* = S(x\,; v_1, \ldots, v_n)$; then $S \backslash S^* \subseteq \bigcup_1^n S(v_i\,; u_i)$ and so

$$\mu_D(S \backslash S^*) \leqslant \sum_1^n \mu_D(S(v_i\,; u_i)) = \sum_1^n \mu(S(v_i\,; u_i)) =$$
$$= \sum_1^n F(v_i) - F(u_i) < \varepsilon/2\,.$$

Also, $S^* \backslash S = S(x'\,; x)$ and

$$\mu_D(S^* \backslash S) = \mu_D(S(x'\,; x)) = \mu(S(x'\,; x)) = F(x') - F(x) < \varepsilon/2\,.$$

Thus $\mu_D(S \,\Delta\, S^*) \leqslant \mu_D(S \backslash S^*) + \mu_D(S^* \backslash S) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ and X_d is satisfied. □

Combining the preceding lemma with Theorem 3.2, we obtain the main result of Revuz in this area.

THEOREM 3.3. – *Let* X *be a topological space and* $(X, \leqslant)$ *be a conditional lower semilattice. If* $(X, \leqslant)$ *satisfies* X_a, X_b, *and* X_c *and*

$F : X \to R$ *is such that* μ *is non-negative and* F *is continuous from the right on* X, *then* μ *is countably additive on* $\mathfrak{S}$.

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