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HOLOMORPHIC GERMS ON BANACH SPACES*

by SOO BONG CHAE

Introduction.

Let E and F be complex Banach spaces, U a non-empty open subset of E and K a compact subset of E. The concept of holomorphy type θ between E and F, and the natural locally convex topology $\mathcal{T}_{\omega,\theta}$ on the space $\mathcal{H}_\theta(\mathrm{U};\mathrm{F})$ of all holomorphic mappings of a given holomorphy type θ from U to F were considered first by L. Nachbin in his monograph [N6]. Motivated by [N6], we introduce the locally convex space $\mathcal{H}_\theta(\mathrm{K};\mathrm{F})$ of all germs of holomorphic mappings into F around K of a given holomorphy type θ and study its interplay with $\mathcal{H}_\theta(\mathrm{U};\mathrm{F})$. If E is infinite dimensional, a study of the locally convex space $\mathcal{H}_\theta(\mathrm{U};\mathrm{F})$ is by no means straightforward.

The organization of the paper is as follows : In the chapter on preliminaries, we have included statements of basic definitions and results from [N6] for convenience of reference.

In Chapter 2 the locally convex space $\mathcal{H}_\theta(\mathrm{K};\mathrm{F})$ is introduced. Let $\varepsilon > 0$ be a real number. We denote by $\mathcal{H}_{\theta\varepsilon}(\mathrm{U};\mathrm{F})$ the vector subspace of $\mathcal{H}_\theta(\mathrm{U};\mathrm{F})$ consisting of those mappings f such that

$$\|f\|_\theta = \sum_{m=0}^{\infty} \varepsilon^m \sup_{x\in \mathrm{U}} \left\| \frac{1}{m!} \hat{d}^m f(x)\right\|_\theta < \infty .$$

Then $\mathcal{H}_{\theta\varepsilon}(\mathrm{U};\mathrm{F})$ is a Banach space with respect to the norm $\|\ \|_{\theta\varepsilon}$. We define the natural locally convex topology on $\mathcal{H}_\theta(\mathrm{K};\mathrm{F})$ by considering $\mathcal{H}_\theta(\mathrm{K};\mathrm{F})$ as the inductive limit of Banach spaces $\mathcal{H}_{\theta\varepsilon}(\mathrm{U};\mathrm{F})$,

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for all real numbers $\varepsilon > 0$ and open subsets U of E containing K with respect to the natural linear mapping $\mathcal{H}_{\theta\varepsilon}(U ; F) \to \mathcal{H}_\theta(K ; F)$ assigning to each $f \in \mathcal{H}_{\theta\varepsilon}(U ; F)$ the germ $\widetilde{f} \in \mathcal{H}_\theta(K ; F)$ determined by f. This topology then is equal to the topology obtained by considering $\mathcal{H}_\theta(K ; F)$ as the inductive limit of $\mathcal{H}_\theta(U ; F)$ endowed with $\mathcal{T}_{\omega,\theta}$, for all open subsets U of E containing K with respect to the natural linear mapping $\mathcal{H}_\theta(U ; F) \to \mathcal{H}_\theta(K ; F)$.

Bounded sets, compact sets and Cauchy filters in $\mathcal{H}(K ; F)$ and $\mathcal{H}_\theta(K ; F)$ are characterized in Chapter 3 and Chapter 4 respectively. In Chapter 4 the Nachbin inequalities play an important role.

The main result is presented in Chapter 6. The following problem has been considered : When does $\mathcal{T}_{\omega,\theta} = \mathcal{T}_{\pi,\theta}$ on $\mathcal{H}_\theta(U ; F)$? The topology $\mathcal{T}_{\pi,\theta}$ is discussed in Chapter 5, which is the projective limit of the topology on $\mathcal{H}_\theta(K ; F)$, for all compact subsets K of U with respect to the linear mapping $\mathcal{H}_\theta(U ; F) \to \mathcal{H}_\theta(K ; F)$. The two topologies are identical for every open subset of E if $\dim E < \infty$. If $\dim E = \infty$, then we prove that they are equal for every open subset of E satisfying the θ-Runge property. Applying this result, we prove that $\mathcal{H}_\theta(U ; F)$ is complete for $\mathcal{T}_{\omega,\theta}$ if U satisfies the θ-Runge property. This has been done by characterizing bounded subsets, compact subsets and Cauchy filters of $\mathcal{H}_\theta(K ; F)$, and proving the completeness of $\mathcal{H}_\theta(K ; F)$.

We also have the following results in Chapter 7. If $\dim E = \infty$, then $\mathcal{H}(U ; F)$ and $\mathcal{H}(K ; F)$ are neither Montel, nor Schwartz, nor nuclear spaces. If E is reflexive and there exists a non-compact m-linear mapping from E^m to F, then neither $\mathcal{H}(U ; F)$ nor $\mathcal{H}(K ; F)$ is reflexive. In particular, if E is a Hilbert space, then $\mathcal{H}(U ; F)$ or $\mathcal{H}(K ; F)$ is reflexive if and only if $\dim E < \infty$. If E is not reflexive, then both $\mathcal{H}(U ; F)$ and $\mathcal{H}(K ; F)$ are not reflexive. If E is separable, then $\mathcal{H}(U ; F)$ is bornological if and only if every sequentially continuous semi-norm on $\mathcal{H}(U ; F)$ is continuous.

1. Preliminaries.

For the convenience of the reader, we devote this chapter to the compilation of several basic facts and definitions in [N6].

The letters E and F will denote two complex Banach spaces, U a non-empty open subset of E, K a compact subset of E. By m and n we mean natural numbers 0, 1, 2, ... The open and closed balls with center ξ and radius ρ in E are denoted by $B_\rho(\xi)$ and $\overline{B}_\rho(\xi)$, respectively. For a subset X of E, we set

$$B_\rho(X) = \bigcup_{x \in X} B_\rho(x)$$

$$\overline{B}_\rho(X) = \bigcup_{x \in X} \overline{B}_\rho(x) .$$

For each m $\mathcal{L}(^m E ; F)$ represents the Banach space of all continuous m-linear mappings of E^m into F endowed with the norm

$$\| A \| = \sup \| A(x_1 , \ldots , x_m) \|$$

where $x_1 , \ldots , x_m$ are elements in the closed unit ball of E. We denote by $\mathcal{L}_s(^m E ; F)$ the vector subspace of $\mathcal{L}(^m E ; F)$ consisting of symmetric m-linear mappings of E^m into F. $\mathcal{L}_s(^m E ; F)$ is a Banach space with respect to the induced norm by the norm of $\mathcal{L}(^m E ; F)$. We shall let $\mathcal{L}(^0 E ; F) = \mathcal{L}_s(^0 E ; F) = F$ as a Banach space. A *continuous m-homogeneous* polynomial P from E to F is a mapping $P : E \to F$ for which there is some $A \in \mathcal{L}(^m E ; F)$ such that $P(x) = Ax^m = A(x , \ldots , x)$ for every $x \in E$, where x is repeated m times, $m \neq 0$; $P(x) = Ax^0 = A$, $m = 0$. We denote by $\mathcal{P}(^m E ; F)$ the Banach space of continuous m-homogeneous polynomials from E to F endowed with the norm

$$\| P \| = \sup \| P(x) \|$$

where x are elements in the closed unit ball of E. The mapping $A \in \mathcal{L}_s(^m E ; F) \mapsto \hat{A} \in \mathcal{P}(^m E ; F)$, where $\hat{A}(x) = Ax^m$, establishes a vector space isomorphism and a homeomorphism of the first space onto the second one. A continuous polynomial P from E to F is a mapping $P : E \to F$ for which there are m and $P_k \in \mathcal{P}(^k E ; F)$, $k = 1, \ldots , m$, such that $P = P_1 + \cdots + P_m$. This representation is unique. We denote by $\mathcal{P}(E ; F)$ the vector space of all continuous polynomials from E to F.

A *power series* from E to F about $\xi \in E$ is a series in $x \in E$ of the form

$$\sum_{m=0}^{\infty} P_m(x - \xi)$$

where $P_m \in \mathcal{P}({}^mE ; F)$. The P_m are called the coefficients of the power series. The *radius of convergence* of a power series about ξ is the largest real number r, $0 \leqslant r \leqslant \infty$, such that the power series is uniformly convergent on every $B_\rho(\xi)$ for $0 \leqslant \rho < r$. The power series is said to be convergent if its radius of convergence is strictly positive.

A mapping $f : U \to F$ is said to be *holomorphic* on U if, corresponding to every $\xi \in U$, there is a convergent power series from E to F about ξ,

$$f(x) = \sum_{m=0}^{\infty} P_m(x - \xi) .$$

The sequence (P_m) is then unique at every point ξ. We refer to this convergent power series as the Taylor series of f about ξ. $\mathcal{H}(U : F)$ denotes the vector space of all holomorphic mappings from U to F. We set

$$P_m = \frac{1}{m!} \hat{d}^m f .$$

Then we have the differential mapping

$$\hat{d}^m f : x \in U \mapsto \hat{d}^m f(x) \in \mathcal{P}({}^mE ; F)$$

and the differential operator

$$\hat{d}^m : f \in \mathcal{H}(U ; F) \mapsto \hat{d}^m f \in \mathcal{H}(U ; \mathcal{P}({}^mE ; F))$$

of order m.

Cauchy inequality. Let $f \in \mathcal{H}(U ; F)$, $\rho > 0$ and $B_\rho(\xi) \subset U$. Then

$$\left\| \frac{1}{m!} \hat{d}^m f(\xi) \right\| \leqslant \frac{1}{\rho^m} \sup_{\|x-\xi\|=\rho} \| f(x) \|$$

for every m.

A *holomorphy type* θ from E to F is a sequence of Banach spaces $\mathcal{P}_\theta({}^mE ; F)$, the norm on each of them denoted by $\| \ \|_\theta$, such that the following conditions hold true :

1) Each $\mathcal{P}_\theta({}^mE ; F)$ is a vector subspace of $\mathcal{P}({}^mE ; F)$;

2) $\mathcal{P}_\theta({}^0E ; F)$ coïncides with $\mathcal{P}({}^0E ; F)$ as a normed vector space ;

3) There is a real number $\sigma \geqslant 1$ for which the following is true : Given any k, m, $k \leqslant m$, $x \in E$, and $P \in \mathcal{P}_\theta({}^mE ; F)$, we have

$$\hat{d}^k P(x) \in \mathcal{P}_\theta({}^k E ; F) \ ;$$

$$\left\| \frac{1}{k!} \hat{d}^k P(x) \right\|_\theta \leqslant \sigma^m \, \| P \|_\theta \, \| x \|^{m-k} \ .$$

(We call σ the *holomorphy constant*).

It is immediate that each inclusion mapping

$$\mathcal{P}_\theta({}^m E ; F) \hookrightarrow \mathcal{P}({}^m E ; F)$$

is continuous and $\| P \| \leqslant \sigma^m \, \| P \|_\theta \, \| x \|^m$.

A given $f \in \mathcal{H}(U ; F)$ is said to be of *holomorphy type* θ at $\xi \in U$ if

1) $\hat{d}^m f(\xi) \in \mathcal{P}_\theta({}^m E ; F)$, for every m ;

2) There are real numbers $C > 0$ and $c > 0$ such that

$$\left\| \frac{1}{m!} \hat{d}^m f(\xi) \right\|_\theta < C c^m, \text{ for every } m \ .$$

Moreover, f is said to be of holomorphy type θ on U if f is of holomorphy type θ at each point of U. We shall denote by $\mathcal{H}_\theta(U ; F)$ the vector space of all mappings of holomorphy type θ on U.

θ always denotes a holomorphy type from E to F.

The *current holomorphy type* from E to F is the holomorphy type θ for which $\mathcal{P}_\theta({}^m E ; F) = \mathcal{P}({}^m E ; F)$ for every m as a normed space. Then $\mathcal{H}_\theta(U ; F) = \mathcal{H}(U ; F)$.

A semi-norm p on $\mathcal{H}_\theta(U ; F)$ is said to be *ported* by a compact subset K of U if corresponding to every real number $\varepsilon > 0$ and open subset V of U containing K there is a real number $c(\varepsilon, V) > 0$ such that

$$p(f) \leqslant c(\varepsilon , V) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in V} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta$$

for every $f \in \mathcal{H}_\theta(U ; F)$. It is equivalent to saying that a semi-norm p on $\mathcal{H}(U ; F)$ is ported by a compact subset K of U if for every open subset V of U containing K there corresponds a real number $c(V) > 0$ such that

$$p(f) \leqslant c(V) \sup_{x \in V} \| f(x) \|$$

for every $f \in \mathcal{H}(U ; F)$. The *compact-θ-ported topology* on $\mathcal{H}_\theta(U ; F)$ is defined by the family of semi-norms ported by compact subsets of U. We denote this topology by $\mathcal{T}_{\omega,\theta}$.

In the theory of holomorphy type θ other than the current one, the classical Cauchy inequalities are not valid. As a substitute for these inequalities the following inequalities are indispensible in the study of the compact-θ-ported topology $\mathcal{T}_{\omega,\theta}$ on $\mathcal{H}_\theta(U ; F)$ and the natural topology on $\mathcal{H}_\theta(K ; F)$ yet to be defined in the next chapter.

Nachbin inequalities. – Let $f \in \mathcal{H}_\theta(U ; F)$, $X \subset U$, and $B_\rho(X) \subset U$ with $\rho > 0$. Then

$$\sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in B_\rho(X)} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta \leqslant$$

$$\leqslant \sum_{m=0}^{\infty} (\sigma(\rho + \varepsilon))^m \sup_{x \in X} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta$$

where $\sigma \geqslant 1$ is the holomorphy constant.

We omit θ whenever our objects are for the current holomorphy type.

2. Topology on the spaces $\mathcal{H}_\theta(K ; F)$.

In this chapter we define the natural locally convex topology on the space of holomorphic germs of holomorphy type θ.

2.1. DEFINITION. – Let H(K ; F) be the set of all F-valued mappings which are holomorphic on some open subset of E containing K. Two mappings f_1 and f_2 in H(K ; F), defined on open subsets U_1 and U_2, respectively, are said to be equivalent modulo K if there is an open subset U of E containing K and contained in both U_1 and U_2 such that $f_1(x) = f_2(x)$ for every x in U. Each equivalence class is referred to as a *holomorphic germ* on K, or a *current holomorphic germ* on K. We denote by $\tilde{f}$ the equivalence class modulo K determined by f. The quotient space of $\mathcal{H}(K ; F)$ with respect to this relation will be denoted by $\mathcal{H}(K ; F)$. $\mathcal{H}(K ; F)$ becomes a vector space over C if we define

$$(f+g)^{\sim} = \widetilde{f} + \widetilde{g} ;$$
$$(\lambda f)^{\sim} = \lambda \widetilde{f}$$

for every f and g in $H(K ; F)$ and $\lambda \in \mathbf{C}$.

2.2. DEFINITION. -- A holomorphic germ $\widetilde{f} \in \mathcal{H}(K ; F)$ is said to be of holomorphy type θ if there is a representative of $\widetilde{f}$ which is of holomorphy type θ on some open subset of E containing K. For simplicity, we call such a germ a *θ-holomorphic germ* on K. The vector space of all θ-holomorphic germs on K will be denoted by $\mathcal{H}_\theta(K ; F)$.

2.3. DEFINITION. – The vector subspace of $\mathcal{H}(U ; F)$ consisting of all bounded holomorphic mappings on U is denoted by $\mathcal{H}^\infty(U ; F)$. The natural topology on $\mathcal{H}^\infty(U ; F)$ is defined by the norm

$$f \in \mathcal{H}^\infty(U ; F) \mapsto \sup_{x \in U} \| f(x) \| .$$

Then $\mathcal{H}^\infty(U ; F)$ is a Banach space.

2.4. DEFINITION. – Let $\varepsilon > 0$ be a real number. By $\mathcal{H}_{\theta\varepsilon}(U ; F)$ we denote the vector subspace of $\mathcal{H}_\theta(U ; F)$ consisting of all mappings f such that

$$\| f \|_\theta = \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in U} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta < \infty .$$

The natural topology on $\mathcal{H}_{\theta\varepsilon}(U ; F)$ is defined by the norm given above. Then the inclusion mapping

$$\mathcal{H}_{\theta\varepsilon}(U ; F) \hookrightarrow \mathcal{H}^\infty(U ; F)$$

is continuous, and $\mathcal{H}_{\theta\varepsilon}(U ; F)$ is a Banach space.

Let K be a fixed compact subset of E. Corresponding to every open subset U of E containing K, there exists a natural linear mapping $T_U : \mathcal{H}_\theta(U ; F) \to \mathcal{H}_\theta(K ; F)$ assigning every $f \in \mathcal{H}_\theta(U ; F)$ to its equivalence class $\widetilde{f}$ modulo K.

2.5. DEFINITION. – The natural topology on $\mathcal{H}_\theta(K ; F)$ is defined as the inductive limit of the natural topology on $\mathcal{H}_{\theta\varepsilon}(U ; F)$, for all open subsets U of E containing K and all real numbers $\varepsilon > 0$, i.e., the finest locally convex topology on $\mathcal{H}_\theta(K ; F)$ such that the natural linear mappings $T_U : \mathcal{H}_{\theta\varepsilon}(U ; F) \to \mathcal{H}_\theta(K ; F)$ are continuous for all open subsets U of E containing K and all real number $\varepsilon > 0$.

2.6. PROPOSITION. – *The natural topology on* $\mathcal{H}_\theta(K ; F)$ *is equal to the inductive limit of the compact-θ-ported topology on* $\mathcal{H}_\theta(U ; F)$, *for all open subsets* U *of* E *containing* K, *with respect to the linear mapping* T_U.

Proof. – Let $\mathcal{T}$ be the natural topology on $\mathcal{H}_\theta(K ; F)$ and $\mathcal{T}'$ the inductive limit of the topology $\mathcal{T}_{\omega,\theta}$ on $\mathcal{H}_\theta(U ; F)$, for all open subsets U of E containing K. Since the inclusion mapping

$$\mathcal{H}_{\theta\varepsilon}(U ; F) \hookrightarrow \mathcal{H}_\theta(U ; F)$$

is continuous, we have $\mathcal{T}' \subset \mathcal{T}$.

On the other hand, let p be a semi-norm on $\mathcal{H}_\theta(K ; F)$ which is continuous for $\mathcal{T}$. Then, corresponding to every real number $\varepsilon > 0$ and open subset U of E containing K, there is a real number $c(\varepsilon, U) > 0$ such that

$$p \circ T_U(f) \leqslant c(\varepsilon, U) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in U} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta$$

for every $f \in \mathcal{H}_{\theta\varepsilon}(U ; F)$. This implies that corresponding to every real number $\varepsilon > 0$ and open subset V of U (fixed) containing K, there is a real number $c(\varepsilon, V) > 0$ such that

$$p \circ T_U(f) \leqslant c(\varepsilon, V) \sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in V} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta$$

for every $f \in \mathcal{H}_\theta(U ; F)$. Therefore, $p \circ T_U$ is a continuous semi-norm on $\mathcal{H}_\theta(U ; F)$. Hence p is continuous for $\mathcal{T}'$.

2.7. PROPOSITION. – *The natural topology on* $\mathcal{H}_\theta(K ; F)$ *can be defined as the inductive limit of the topology on* $\mathcal{H}_{\theta\varepsilon}(U ; F)$, *where* U *runs through a fundamental sequence of open neighborhoods of* K *and* ε *a sequence of positive real numbers converging to* 0.

Proof. – Let (U_m) be a fundamental sequence of open neighborhoods of K and (ε_n) a sequence of positive real numbers converging to 0. For every open subset U of E containing K and real number $\varepsilon > 0$, there exist m and n such that $U_m \subset U$ and $\varepsilon_n < \varepsilon$.

Then the inclusion mappings

$$\begin{array}{ccc} \mathcal{H}_{\theta\varepsilon}(U;F) & \hookrightarrow & \mathcal{H}_{\theta\varepsilon_n}(U;F) \\ \downarrow & & \downarrow \\ \mathcal{H}_{\theta\varepsilon}(U_m;F) & \hookrightarrow & \mathcal{H}_{\theta\varepsilon_n}(U_m;F) \end{array}$$

are continuous. Therefore, the sequence of Banach spaces defines the natural topology on $\mathcal{H}_\theta(K;F)$.

2.8. PROPOSITION. – $\mathcal{H}_\theta(K;F)$ *is a bornological, barrelled and* (DF)-*space.*

Proof. – The inductive limit of bornological (respectively, barrelled) spaces is also bornological (respectively, barrelled). Therefore, $\mathcal{H}_\theta(K;F)$ is both bornological and barrelled. $\mathcal{H}_\theta(K;F)$ is also a (DF)-space as a countable inductive limit of (DF)-spaces.

3. Current holomorphic germs.

Attention is now restricted to the space $\mathcal{H}(K;F)$ of current holomorphic germs on a compact subset K. The natural locally convex topology on $\mathcal{H}(K;F)$ is described, in a simpler way, by means of the Banach spaces $\mathcal{H}^\infty(U;F)$ rather than the Banach spaces $\mathcal{H}_{\theta\varepsilon}(U;F)$. We then characterize the bounded subsets, compact subsets, and Cauchy filters in $\mathcal{H}(K;F)$ in terms of $\mathcal{H}^\infty(U;F)$ and show that $\mathcal{H}(K;F)$ is complete.

3.1. PROPOSITION. – *The natural topology on* $\mathcal{H}(K;F)$ *is equal to the inductive limit of the topology on* $\mathcal{H}^\infty(U;F)$, *for all open subsets* U *of* E *containing the compact subset* K, *with respect to the natural linear mapping* $\mathcal{H}^\infty(U;F) \to \mathcal{H}(K;F)$.

Proof. – For each open subset U of E containing K and each real number $\varepsilon > 0$, the following inclusion mappings are continuous for any holomorphy type θ :

$$\mathcal{H}_{\theta\varepsilon}(U\,;F) \hookrightarrow \mathcal{H}^{\infty}(U\,;F) \hookrightarrow \mathcal{H}(U\,;F)\ .$$

This fact and 2.6 prove the proposition.

Let G be a vector space, $(E_\lambda)_{\lambda \in I}$ a family of locally-convex spaces. Let T_λ be a linear mapping from E_λ to G for each $\lambda \in I$. Equip G with the inductive limit topology of E_λ with respect to T_λ for all $\lambda \in I$. For $T_\lambda(E_\lambda)$ is a vector subspace of G and the topology on E_λ can be transferred to $T_\lambda(E_\lambda)$ by taking as neighborhoods in $T_\lambda(E_\lambda)$ the images of neighborhoods of E_λ by T_λ. Then G is also the inductive limit of its vector subspaces $T_\lambda(E_\lambda)$ with respect to the inclusion mappings. In the sequel, we consider only one topology on $T_\lambda(E_\lambda)$, namely, the transferred one without specification.

3.2. PROPOSITION. – *Let $\mathfrak{X}$ be a subset of $\mathcal{H}(K\,;F)$. The following are equivalent.*

a) *$\mathfrak{X}$ is bounded in $\mathcal{H}(K\,;F)$.*

b) *There exist real number $C > 0$ and $c > 0$ such that*

$$\sup_{x \in K} \left\| \frac{1}{m\,!}\, \hat{d}^m f(x) \right\| \leqslant C\, c^m$$

for every $\tilde{f} \in \mathfrak{X}$, $f \in \tilde{f}$, and m.

c) *There exists an open subset U of E containing K such that $\mathfrak{X}$ is contained and bounded in $T_U \mathcal{H}^{\infty}(U\,;F)$.*

Proof. – Since the implication c) $\Rightarrow$ a) is clear, we prove the rest.

a) $\Rightarrow$ b). Let $\mathfrak{X}$ be bounded in $\mathcal{H}(K\,;F)$. Then every continuous semi-norm on $\mathcal{H}(K\,;F)$ will be bounded on $\mathfrak{X}$. Let $\alpha = (\alpha_m)$ be a sequence of positive real numbers such that $(\alpha_m)^{1/m} \to \infty$. We define a semi-norm p_α on $\mathcal{H}(K\,;F)$ by

$$p_\alpha(\tilde{f}) = \sum_{m=0}^{\infty} \alpha_m \sup_{x \in K} \left\| \frac{1}{m\,!}\, \hat{d}^m f(x) \right\|$$

where f is a representative of $\tilde{f}$. It is well-defined since every $f \in \tilde{f}$ will take the same value on K. For every open subset U of E containing K, $p_\alpha \circ T_U$ is a continuous semi-norm on $\mathcal{H}^{\infty}(U\,;F)$ by the Cauchy inequalities. Therefore, p_α is continuous on $\mathcal{H}(\mathrm{K}\,;\mathrm{F})$. Hence p_α is bounded on $\mathfrak{X}$ for every sequence α described above. This implies that there exist real numbers $\mathrm{C} > 0$ and $c > 0$ such that

$$\sup_{x\in \mathrm{K}} \left\| \frac{1}{m\,!}\, \hat{d}^m f(x) \right\| \leqslant \mathrm{C}\, c^m$$

for every $\tilde{f} \in \mathfrak{X}$, $f \in \tilde{f}$, and m.

b) $\Rightarrow$ c). Assume that b) holds true. We choose a real number $\rho > 0$ such that $\rho c < 1$. Since K is compact, we may cover K with a finite number of open balls $\mathrm{B}_\rho(\xi_n), \ldots, \mathrm{B}_\rho(\xi_n)$ all centered in K. Let U be the union of these balls. Let $\tilde{f} \in \mathfrak{X}$ and f a representative of $\tilde{f}$. Then the Taylor series of f about ξ converges uniformly on $\mathrm{B}_\rho(\xi)$ whenever ξ is a point of K since

$$\sum_{m=0}^{\infty} \left\| \frac{1}{m\,!}\, \hat{d}^m f(\xi) \right\| \| x - \xi \|^m < \mathrm{C}/(1 - \rho c)$$

for $x \in \mathrm{B}(\xi)$. We define a mapping $g : \mathrm{U} \to \mathrm{F}$ by

$$g(x) = \sum_{m=0}^{\infty} \frac{1}{m\,!}\, \hat{d}^m f(\xi_j)\,(x - \xi_j)$$

if $x \in \mathrm{B}_\rho(\xi_j)$, for some $j = 1, \ldots, n$. Then g is holomorphic on U. We may assume that f is defined and holomorphic on some open subset V of U. Thus, $f(x) = g(x)$ for every $x \in \mathrm{V}$. Hence g is equivalent to f modulo K, i.e., $\tilde{g} = \tilde{f}$. Furthermore,

$$\sup_{x\in \mathrm{U}} \| g(x) \| < \mathrm{C}/(1 - \rho c)\ .$$

Therefore, $\mathfrak{X}$ is contained and bounded in $\mathrm{T}_\mathrm{U}\mathcal{H}^\infty(\mathrm{U}\,;\mathrm{F})$.

3.3. Corollary. – *The strong dual $\mathcal{H}'(\mathrm{K}\,;\mathrm{F})$ of $\mathcal{H}(\mathrm{K}\,;\mathrm{F})$ is a Frechet space.*

Proof. – Let (U_m) be a fundamental sequence of open neighborhoods of K. Then the natural topology on $\mathcal{H}(\mathrm{K}\,;\mathrm{F})$ is the inductive limit of the topology on $\mathcal{H}^\infty(\mathrm{U}_m\,;\mathrm{F})$, for all m by the same argument as in 2.7. Let $\mathcal{B}_m$ be the closed unit ball of $\mathcal{H}^\infty(\mathrm{U}_m : \mathrm{F})$ for every m. Set $\mathcal{Q}_m = \mathrm{T}_{\mathrm{U}_m}\mathcal{B}_m$. Then the family of semi-norms p_m on $\mathcal{H}'(\mathrm{K}\,;\mathrm{F})$, defined by

$$p_m(\mathrm{A}) = \sup_{f \in \quad m} |(\mathrm{A}, \tilde{f})| ,$$

generates the strong topology on $\mathcal{H}'(\mathrm{K} ; \mathrm{F})$. Therefore, $\mathcal{H}'(\mathrm{K} ; \mathrm{F})$ is metrizable.

Let (A_m) be a Cauchy sequence in $\mathcal{H}'(\mathrm{K} ; \mathrm{F})$. For each $\tilde{f} \in \mathcal{H}(\mathrm{K} ; \mathrm{F})$, the sequence $((\mathrm{A}_m , \tilde{f}))$ is Cauchy in $\mathbf{C}$. Let A be the pointwise limit of (A_m). Then A is continuous and linear. Thus, $\mathcal{H}'(\mathrm{K} ; \mathrm{F})$ is complete.

3.4. COROLLARY. – *The space* $\mathcal{H}(\mathrm{K} ; \mathrm{F})$ *is not metrizable.*

Proof. – Suppose that $\mathcal{H}(\mathrm{K} ; \mathrm{F})$ is metrizable. Choose f_m in $\mathcal{H}^\infty(\mathrm{U}_{m+1} ; \mathrm{F}) \backslash \mathcal{H}^\infty(\mathrm{U}_m ; \mathrm{F})$ for every m, where (U_m) is a fundamental sequence of open neighborhoods of K such that $\mathrm{U}_m \supsetneq \mathrm{U}_{m+1}$. By Mackey's countability condition (Cf. [H, 2,6]) there exists a sequence (λ_m) of positive real numbers such that the sequence $(\lambda_m \tilde{f}_m)$ is bounded in $\mathcal{H}(\mathrm{K} ; \mathrm{F})$. This is absurd because of 3.2.c.

3.5. DEFINITION. – A subset $\mathcal{X}$ of $\mathcal{H}_\theta(\mathrm{K} ; \mathrm{F})$ is said to be *relatively compact at a point* $\xi \in \mathrm{K}$ if for every m the set

$$\{\hat{d}^m f(\xi) : \tilde{f} \in \mathcal{X}, f \in \tilde{f}\}$$

is relatively compact in the Banach space $\mathcal{L}_\theta({}^m\mathrm{E} ; \mathrm{F})$.

3.6. PROPOSITION. – *Let* $\mathcal{X}$ *be a subset of* $\mathcal{H}(\mathrm{K} ; \mathrm{F})$. *The following are equivalent.*

a) $\mathcal{X}$ *is relatively compact in* $\mathcal{H}(\mathrm{K} ; \mathrm{F})$.

b) $\mathcal{X}$ *is bounded in* $\mathcal{H}(\mathrm{K} ; \mathrm{F})$ *and relatively compact at every point of* K.

c) *There exists an open subset* U *of* E *containing* K *such that* $\mathcal{X}$ *is contained and relatively compact in the Banach space* $\mathrm{T}_\mathrm{U}\mathcal{H}^\infty(\mathrm{U} ; \mathrm{F})$.

Proof. – The implication c) $\Rightarrow$ a) is clear.

a) $\Rightarrow$ b). Let $\mathcal{X}$ be relatively compact in $\mathcal{H}(\mathrm{K} ; \mathrm{F})$. Then it is bounded in $\mathcal{H}(\mathrm{K} ; \mathrm{F})$. It remains to show that $\mathcal{X}$ is relatively compact at every point of K. Let $\xi \in \mathrm{K}$. Then the mapping

$$\widetilde{f} \in \mathcal{H}(K ; F) \to \hat{d}^m f(\xi) \in \mathcal{P}(^m E ; F) ,$$

where $f \in \widetilde{f}$, is well-defined for every m. This is also linear and continuous since

$$\| \hat{d}^m f(\xi) \| \leqslant \sup_{x \in K} \| \hat{d}^m f(x) \| .$$

Therefore, $\mathfrak{X}$ is relatively compact at ξ.

b) $\Rightarrow$ c). Assume that b) holds true. Since $\mathfrak{X}$ is bounded in $\mathcal{H}(K ; F)$, by 3.2 there exist real numbers $C > 0$ and $c > 0$ such that

$$\sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\| \leqslant C c^m$$

for every $\widetilde{f} \in \mathfrak{X}$, $f \in \widetilde{f}$, and m. As in the proof b) $\Rightarrow$ c) of 3.2, choose a real number $\rho > 0$ to be $\rho c < 1$. Let U be the union of the open balls $B_\rho(\xi_1), \ldots, B_\rho(\xi_n)$ all centered in K such that $U \supset K$. Then $\mathfrak{X}$ is contained and bounded in the Banach space $T_U \mathcal{H}^\infty(U ; F)$. To prove that $\mathfrak{X}$ is relatively compact in $T_U \mathcal{H}^\infty(U ; F)$, it is sufficient to show that every sequence in $\mathfrak{X}$ admits a Cauchy subsequence in $T_U \mathcal{H}^\infty(U ; F)$. Let $\varepsilon > 0$ be given a real number. Choose an integer $N > 0$ such that

$$(*) \qquad 2C(\rho c)^{N+1}/(1 - \rho c) < \varepsilon/2 .$$

$\mathfrak{X}$ being relatively compact at each point of K, it is relatively compact at $\xi_1, \ldots, \xi_n$. Thus, corresponding to every sequence $(\widetilde{f}_m)$ in $\mathfrak{X}$, we can select a subsequence, call it again $(\widetilde{f}_m)$, with the following property : There is an integer $M > 0$ such that if p and $q \geqslant M$, then

$$(**) \qquad \left\| \frac{1}{m!} \hat{d}^m (f_p - f_q)(\xi_j) \right\| < \varepsilon/2(1 + \rho + \cdots + \rho^N)$$

for every $f_p \in \widetilde{f}_p$, $f_q \in \widetilde{f}_q$, $m = 0, \ldots, N$, and $j = 1, \ldots, n$. This can be done by a diagonal process. If $x \in B_\rho(\xi_j)$, then by the Taylor series of $f_p - f_q$ about ξ_j and the inequalities (*) and (**) we have

$$\| f_p(x) - f_q(x) \| < \varepsilon$$

for every $f_p \in \widetilde{f}_p$ and $f_q \in \widetilde{f}_q$ whenever p and $q \geqslant M$. Therefore, the subsequence $(\widetilde{f}_m)$ is a Cauchy sequence in the Banach space $T_U \mathcal{H}^\infty(U ; F)$.

3.7. COROLLARY. – *A subset* $\mathfrak{X}$ *of* $\mathcal{H}(\mathrm{K}\,;\mathrm{F})$ *is relatively compact if and only if the following are true :*

a) *There exist real numbers* $\mathrm{C} > 0$ *and* $c > 0$ *such that*

$$\sup_{x\in\mathrm{K}} \left\| \frac{1}{m\,!}\, \hat{d}^m f(x) \right\| < \mathrm{C}\, c^m$$

for every $\widetilde{f} \in \mathfrak{X}$, $f \in \widetilde{f}$, *and* m ;

b) *There exist a real number* $\rho > 0$ *and a finite number of points* $\xi_1, \ldots, \xi_n$ *in* K *such that* $\rho c < 1$, *the union of* $\mathrm{B}_\rho(\xi_1), \ldots, \mathrm{B}_\rho(\xi_n)$ *covers* K, *and* $\mathfrak{X}$ *is relatively compact at each point* $\xi_1, \ldots, \xi_n$.

Proof. – The proof of 3.6 actually shows 3.7.

3.8. PROPOSITION. – *Let* $\mathfrak{F}$ *be a bounded Cauchy filter in* $\mathcal{H}(\mathrm{K}\,;\mathrm{F})$. *Then there exists an open subset* U *of* E *containing* K *such that* $\mathfrak{F}$ *is a bounded Cauchy filter in the Banach space* $\mathrm{T_U}\mathcal{H}^\infty(\mathrm{U}\,;\mathrm{F})$.

Proof. – $\mathfrak{F}$ being bounded in $\mathcal{H}(\mathrm{K}\,;\mathrm{F})$, there exist real numbers $\mathrm{C} > 0$ and $c > 0$ such that

$$\sup_{x\in\mathrm{K}} \left\| \frac{1}{m\,!}\, \hat{d}^m f(x) \right\| \leqslant \mathrm{C}\, c^m$$

for every $\widetilde{f} \in \mathfrak{F}$, $f \in \widetilde{f}$, and m. As in the proof b) $\Rightarrow$ c) of 3.2 choose a real number $\rho > 0$ such that $\rho c < 1$. Let U be the union of the open balls $\mathrm{B}_\rho(\xi_1), \ldots, \mathrm{B}_\rho(\xi_n)$ where $\xi_1, \ldots, \xi_n$ are suitably chosen in K such that $\mathrm{U} \supset \mathrm{K}$. Then $\mathfrak{F}$ is contained and bounded in the Banach space $\mathrm{T_U}\mathcal{H}^\infty(\mathrm{U}\,;\mathrm{F})$. We claim that $\mathfrak{F}$ is a Cauchy filter in $\mathrm{T_U}\mathcal{H}^\infty(\mathrm{U}\,;\mathrm{F})$. In fact, let $\varepsilon > 0$ be a given real number. Choose an integer $\mathrm{N} > 0$ such that

$$(*) \qquad 2\mathrm{C}(\rho c)^{\mathrm{N}+1}/(1 - \rho c) < \varepsilon/2\ .$$

$\mathfrak{F}$ being a Cauchy filter in $\mathcal{H}(\mathrm{K}\,;\mathrm{F})$, corresponding to each integer $m = 0, \ldots, \mathrm{N}$, there is a set $\mathfrak{A}_m \in \mathfrak{F}$ such that

$$(**) \qquad \sup_{x\in\mathrm{K}} \left\| \frac{1}{m\,!}\, \hat{d}^m (f - g)\,(x) \right\| < \varepsilon/2(1 + \rho + \cdots + \rho^{\mathrm{N}})$$

for every $\widetilde{f}$ and $\widetilde{g}$ in $\mathfrak{A}_m$ with $f \in \widetilde{f}$ and $g \in \widetilde{g}$. $\mathfrak{A}$ denotes the intersection of $\mathfrak{A}_0, \ldots, \mathfrak{A}_\mathrm{N}$. Then $\mathfrak{A}$ belongs to the filter $\mathfrak{F}$. If $x \in \mathrm{U}$, i.e.,

$x \in B(\xi_j)$ for some j, then, by the Taylor series of $f - g$ about ξ_j and the inequalities (*) and (**), we have

$$\sup_{x \in U} \| f(x) - g(x) \| < \varepsilon$$

for $f \in \tilde{f}$ and $g \in \tilde{g}$ where $\tilde{f}$ and $\tilde{g}$ are in $\mathfrak{A}$. Therefore, $\mathfrak{F}$ is a Cauchy filter in the Banach space $T_U \mathcal{H}^\infty(U ; F)$.

3.9. Proposition. – *Every bounded subset of* $\mathcal{H}(K ; F)$ *is metrizable.*

Proof. – Let $\mathfrak{X}$ be a bounded subset of $\mathcal{H}(K ; F)$. Then there exists an open subset U of E containing K such that :

a) $\mathfrak{X}$ is contained and bounded in the Banach space $T_U \mathcal{H}^\infty(U ; F)$;

b) Every Cauchy filter in $\mathfrak{X}$ is a Cauchy filter in the space $T_U \mathcal{H}^\infty(U ; F)$ by 3.8.

On the other hand, the normed topology on $T_U \mathcal{H}^\infty(U ; F)$ is finer than the induced one by the natural topology on $\mathcal{H}(K ; F)$. Therefore, every Cauchy filter in the Banach space $T_U \mathcal{H}^\infty(U ; F)$ is also a Cauchy filter in $\mathcal{H}(K ; F)$. Therefore, the two topologies are equivalent on $\mathfrak{X}$. Hence, $\mathfrak{X}$ is metrizable.

3.10. Proposition. – *The space* $\mathcal{H}(K ; F)$ *is complete.*

Proof. – Since $\mathcal{H}(K ; F)$ is a (DF)-space by 2.8, it is sufficient to show that $\mathcal{H}(K ; F)$ is quasi-complete. (Cf. [GR, I.4]). Let $\mathfrak{X}$ be a bounded closed subset of $\mathcal{H}(K ; F)$. Then there exists an open subset U of E containing K such that a) and b) of the proof of 3.9 hold true. Since the natural linear mapping T_U is continuous, $\mathfrak{X}$ is closed in the Banach space $T_U \mathcal{H}^\infty(U ; F)$. Therefore, $\mathfrak{X}$ is complete for the normed topology of $T_U \mathcal{H}^\infty(U ; F)$. On $\mathfrak{X}$ the two topologies are equivalent, and hence $\mathfrak{X}$ is complete in $\mathcal{H}(K ; F)$.

4. θ-holomorphic germs.

In this chapter, we generalize results obtained in Chapter 3 to the space $\mathcal{H}_\theta(K ; F)$. For each result stated in terms of $\mathcal{H}^\infty(U ; F)$, the corresponding statement concerning $\mathcal{H}_{\theta\varepsilon}(\mathrm{U}\,;\mathrm{F})$ is also valid. In the theory of θ-holomorphy type other than the current one, the classical Cauchy inequalities are not true in general. Therefore, the proofs adopted in Chapter 3 can not be carried over to this chapter unless we modify them using the Nachbin inequalities.

We will omit proofs which are obvious modifications of corresponding ones in Chapter 3.

4.1. PROPOSITION. – *Let* $\mathfrak{X}$ *be a subset of* $\mathcal{H}_\theta(\mathrm{K}\,;\mathrm{F})$. *The following are equivalent.*

a) $\mathfrak{X}$ *is bounded in* $\mathcal{H}_\theta(\mathrm{K}\,;\mathrm{F})$.

b) *There exist real numbers* $\mathrm{C} > 0$ *and* $c > 0$ *such that*

$$\sup_{x\in \mathrm{K}} \left\| \frac{1}{m\,!}\hat{d}^m f(x)\right\|_\theta \leqslant \mathrm{C}\, c^m$$

for every $\tilde{f} \in \mathfrak{X}$, $f \in \tilde{f}$, *and* m.

c) *There exist an open subset* U *of* E *containing* K *and a real number* $\varepsilon > 0$ *such that* $\mathfrak{X}$ *is contained and bounded in the Banach space* $\mathrm{T_U}\mathcal{H}_{\theta\varepsilon}(\mathrm{U}\,;\mathrm{F})$.

Proof. – We will show only b) $\Rightarrow$ c). Assume that b) holds true. We choose real numbers $\rho > 0$ and $\varepsilon > 0$ such that $\sigma(\rho + \varepsilon)\, c < 1$. We cover K with a finite number of open balls $\mathrm{B}_\rho(\xi_1), \ldots, \mathrm{B}_\rho(\xi_n)$ all centered in K. Let U be the union of these balls. As in the proof b) $\Rightarrow$ c) of 3.2, for every $\tilde{f} \in \mathfrak{X}$, there exists a mapping $g \in \mathcal{H}_\theta(\mathrm{U}\,;\mathrm{F})$ such that $\tilde{f} = \tilde{g}$. Thus $\mathfrak{X}$ is contained in $\mathrm{T_U}\mathcal{H}_\theta(\mathrm{U}\,;\mathrm{F})$. By 4.1,

$$\sum_{m=0}^{\infty} \varepsilon^m \sup_{x\in \mathrm{U}} \left\| \frac{1}{m\,!}\hat{d}^m f(x)\right\|_\theta \leqslant \sum_{m=0}^{\infty} (\sigma(\rho+\varepsilon))^m \sup_{x\in \mathrm{K}} \left\| \frac{1}{m\,!}\hat{d}^m f(x)\right\|_\theta$$
$$\leqslant \mathrm{C}/(1 - \sigma(\rho + \varepsilon)\, c) < \varepsilon$$

for every $\tilde{f} \in \mathfrak{X}$, and $f \in \tilde{f}$. Therefore, $\mathfrak{X}$ is contained and bounded in the Banach space $\mathrm{T_U}\mathcal{H}_{\theta\varepsilon}(\mathrm{U}\,;\mathrm{F})$.

4.2. COROLLARY. – *The strong dual* $\mathcal{H}'_\theta(\mathrm{K}\,;\mathrm{F})$ *of* $\mathcal{H}_\theta(\mathrm{K}\,;\mathrm{F})$ *is a Frechet space.*

4.3. COROLLARY. – *The space* $\mathcal{H}_\theta(K ; F)$ *is not metrizable.*

4.4. PROPOSITION. – *Let* $\mathfrak{X}$ *be a subset of* $\mathcal{H}_\theta(K ; F)$. *The following are equivalent.*

a) $\mathfrak{X}$ *is relatively compact in* $\mathcal{H}_\theta(K ; F)$.

b) $\mathfrak{X}$ *is bounded in* $\mathcal{H}_\theta(K ; F)$ *and relatively compact at every point of* K.

c) *There exist an open subset* U *of* E *containing* K *and a real number* $\varepsilon > 0$ *such that* $\mathfrak{X}$ *is contained and relatively compact in* $T_U \mathcal{H}_{\theta\varepsilon}(U ; F)$.

Proof. – We will show only b) $\Rightarrow$ c). Assume that b) holds true. Then there exist real numbers $C > 0$ and $c > 0$ such that

$$\sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta \leqslant C c^m$$

for every $\tilde{f} \in \mathfrak{X}$, $f \in \tilde{f}$, and m. Choose real numbers $\rho > 0$ and $\varepsilon > 0$ with $\sigma(\rho + \varepsilon)\, c < 1$. Cover K with a finite number of open balls $B_\rho(\xi_j)$, $j = 1, \ldots, n$, all centered in K. Let U denote the union of these balls. Then, $\mathfrak{X}$ is contained and bounded in the Banach space $T_U \mathcal{H}_{\theta\varepsilon}(U ; F)$. To prove that $\mathfrak{X}$ is relatively compact in $T_U \mathcal{H}_{\theta\varepsilon}(U ; F)$, it is sufficient to show that every sequence in $\mathfrak{X}$ admits a Cauchy subsequence in $T_U \mathcal{H}_{\theta\varepsilon}(U ; F)$. Let $\delta > 0$ be a given real number. Choose an integer $N > 0$ such that

$$(*) \qquad 2C(\sigma(\rho + \varepsilon)\, c)^{N+1}/(1 - \sigma(\rho + \varepsilon)\, c) < \delta/2\ .$$

Since $\mathfrak{X}$ is relatively compact at every point of K, it is relatively compact at each ξ_j, $j = 1, \ldots, n$. Let $(\tilde{f}_m)$ be a sequence in $\mathfrak{X}$. Then it is possible to select a subsequence, call it again $(\tilde{f}_m)$, with the following property. There is an integer $M > 0$ such that if p and $q \geqslant M$, then

$$(**) \qquad \left\| \frac{1}{m!} \hat{d}^m (f_p - f_q)(\xi_j) \right\|_\theta < \delta/2(1 + \sigma(\rho + \varepsilon) + \cdots + (\sigma(\rho + \varepsilon))^N)$$

for every $f_p \in \tilde{f}_p$, $f_q \in \tilde{f}_q$, $m = 0, \ldots, N$, and $j = 2, \ldots, n$. If p and $q \geqslant M$, by the Nachbin inequalities, (*) and (**), we have (where $X = \{\xi_1, \ldots, \xi_n\}$) :

$$\sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in U} \left\| \frac{1}{m!} \hat{d}^m (f_p - f_q)(x) \right\|_\theta \leqslant$$

$$\leqslant \sum_{m=0}^{\infty} (\sigma(\rho + \varepsilon))^m \sup_{x \in X} \left\| \frac{1}{m!} \hat{d}^m (f_p - f_q)(x) \right\|_\theta < \delta .$$

Therefore, $(\widetilde{f}_m)$ is a Cauchy subsequence in the space $T_U \mathcal{H}_{\theta\varepsilon}(U ; F)$.

4.5. Corollary. – *A subset $\mathfrak{X}$ of $\mathcal{H}_\theta(K ; F)$ is relatively compact if and only if the following are true :*

a) *There exist real numbers $C > 0$ and $c > 0$ such that*

$$\sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta \leqslant C c^m$$

for every $\widetilde{f} \in \mathfrak{X}$, $f \in \widetilde{f}$, and m.

b) *There exist real numbers $\rho > 0$ and $\varepsilon > 0$ and a finite number of points $\xi_1, \ldots, \xi_n$ in K such that $\sigma(\rho + \varepsilon)\, c < 1$, the union of $B_\rho(\xi_1), \ldots, B_\rho(\xi_n)$ covers K, and $\mathfrak{X}$ is relatively compact at each point $\xi_1, \ldots, \xi_n$.*

4.6. Proposition. – *Let $\mathfrak{F}$ be a bounded Cauchy filter in $\mathcal{H}_\theta(K ; F)$. Then there exist a real number $\varepsilon > 0$ and an open subset U of E containing K such that $\mathfrak{F}$ is a bounded Cauchy filter in $T_U \mathcal{H}_{\theta\varepsilon}(U ; F)$.*

Proof. – Since $\mathfrak{F}$ is bounded in $\mathcal{H}_\theta(K ; F)$, there exist real numbers $C > 0$ and $c > 0$ such that

$$\sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta \leqslant C c^m$$

for every $\widetilde{f} \in \mathfrak{F}$, $f \in \widetilde{f}$, and m. Choose real numbers $\rho > 0$ and $\varepsilon > 0$ with $\sigma(\rho + \varepsilon)\, c < 1$. Let U denote the union of the open balls which we have defined before in the proof b) $\Rightarrow$ c), 4.4. Then $\mathfrak{F}$ is contained and bounded in $T_U \mathcal{H}_{\theta\varepsilon}(U ; F)$.

We now show that $\mathfrak{F}$ is a Cauchy filter in $T_U \mathcal{H}_{\theta\varepsilon}(U ; F)$. Let $\delta > 0$ be a given real number. Choose an integer $N > 0$ such that

$$(*) \qquad 2C(\sigma(\rho + \varepsilon)\, c)^{N+1} / (1 - \sigma(\rho + \varepsilon)\, c) < \delta/2 .$$

$\mathfrak{F}$ being a Cauchy filter, for each $m = 0, 1, \ldots, N$, there corresponds a set $\mathcal{A}_m \in \mathfrak{F}$ for which

$$(**) \quad \sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m (f_p - f_q)(x) \right\|_\theta$$

$$< \delta / 2(1 + \sigma(\rho + \varepsilon) + \cdots + (\sigma(\rho + \varepsilon))^N)$$

for every $\tilde{f}, \tilde{g} \in \mathcal{A}_m$, and $f \in \tilde{f}, g \in \tilde{g}$. Let $\mathcal{A}$ be the intersection of $\mathcal{A}_m$, $m = 0, 1, \ldots, N$. Then $\mathcal{A}$ belongs to the filter $\mathfrak{F}$. By the Nachbin inequalities, (*) and (**), we have

$$\sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in U} \left\| \frac{1}{m!} \hat{d}^m (f - g)(x) \right\|_\theta \leqslant$$

$$\leqslant \sum_{m=0}^{\infty} (\sigma(\rho + \varepsilon))^m \sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m (f - g)(x) \right\|_\theta < \delta ,$$

for every $\tilde{f}, \tilde{g} \in \mathcal{A}$ and $f \in \tilde{f}, g \in \tilde{g}$.

Therefore, $\mathfrak{F}$ is a Cauchy filter in $T_U \mathcal{H}_{\theta\varepsilon}(U ; F)$.

4.7. Proposition. – *Every bounded subset of $\mathcal{H}_\theta(K ; F)$ is metrizable.*

4.8. Proposition. – *The space $\mathcal{H}_\theta(K ; F)$ is complete.*

5. Topologies on the space $\mathcal{H}_\theta(U ; F)$.

If E is finite dimensional, then the natural locally convex topology on the space $\mathcal{H}(U ; F)$ is the topology $\mathcal{T}_0$ induced on it by the compact-open topology on the space $\mathcal{C}(U ; F)$ of all continuous F-valued functions on U. If E is infinite dimensional, $\mathcal{T}_0$ is not the natural topology on the space $\mathcal{H}(U ; F)$. One of many reasons is that the differential operator $\hat{d}^m$ of order m, for any $m = 1, \ldots,$ is not continuous for the topology $\mathcal{T}_0$. Nachbin [N6] has considered the topology $\mathcal{T}_\omega$ on $\mathcal{H}(U ; F)$. It is a generalization of the topology $\mathcal{T}_0$. In fact, $\mathcal{T}_0 \subset \mathcal{T}_\omega$; $\mathcal{T}_0 = \mathcal{T}_\omega$ if and only if E is finite dimensional, or $F = 0$. Let $\mathcal{T}_{0,\theta}$ denote the compact-open topology on $\mathcal{H}_\theta(U ; F)$.

We have seen that the natural topology on $\mathcal{H}_\theta(K ; F)$ is the inductive limit of the topology $\mathcal{T}_{\omega,\theta}$ on $\mathcal{H}_\theta(U ; F)$, for all open subsets U of E containing K. In this chapter and the following one, we will study the topology $\mathcal{T}_{\omega,\theta}$ on $\mathcal{H}_\theta(U ; F)$ through the natural topology on $\mathcal{H}_\theta(K ; F)$, for all compact subsets K of U.

5.1. DEFINITION. – Let U be a fixed open subset of E. The topology $\mathcal{T}_{\pi,\theta}$ on $\mathcal{H}_\theta(U ; F)$ is defined as the projective limit of the natural topology on $\mathcal{H}_\theta(K ; F)$, for all compact subsets K of U, i.e., the coarsest locally convex topology on $\mathcal{H}_\theta(U ; F)$ for which the natural linear mappings $T_K : f \in \mathcal{H}_\theta(U ; F) \mapsto \tilde{f} \in \mathcal{H}_\theta(K ; F)$ are continuous.

5.2. DEFINITION. – Corresponding to every compact subset K of U and every m we have the semi-norm p on $\mathcal{H}_\theta(U ; F)$ defined by

$$p(f) = \sup_{x \in K} \| \hat{d}^m f(x) \|_\theta$$

for $f \in \mathcal{H}_\theta(U ; F)$. The topology $\mathcal{T}_{\infty,\theta}$ on $\mathcal{H}_\theta(U ; F)$ is defined by all such semi-norms.

5.3. DEFINITION. – Corresponding to every compact subset K of U and a sequence (α_m) of positive real numbers such that $(\alpha_m)^{1/m} \to 0$ as $m \to \infty$, we have a semi-norm p on $\mathcal{H}_\theta(U ; F)$ defined by

$$p(f) = \sum_{m=0}^{\infty} \alpha_m \sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta$$

for $f \in \mathcal{H}_\theta(U ; F)$. The topology $\mathcal{T}_{\sigma,\theta}$ on $\mathcal{H}_\theta(U ; F)$ is defined by all such semi-norms.

5.4. PROPOSITION. – $\mathcal{T}_{0,\theta} \subset \mathcal{T}_{\infty,\theta} \subset \mathcal{T}_{\sigma,\theta} \subset \mathcal{T}_{\pi,\theta} \subset \mathcal{T}_{\omega,\theta}$.

Proof. – It is immediate that

$$\mathcal{T}_{0,\theta} \subset \mathcal{T}_{\infty,\theta} \subset \mathcal{T}_{\sigma,\theta} \subset \mathcal{T}_{\omega,\theta} \ ; \ \mathcal{T}_{\pi,\theta} \subset \mathcal{T}_{\omega,\theta}$$

It remains to show that $\mathcal{T}_{\sigma,\theta} \subset \mathcal{T}_{\pi,\theta}$. Let K be a compact subset of U. Then the semi-norm p described in 5.3. is defined and continuous on the Banach space $\mathcal{H}_{\theta\varepsilon}(V ; F)$, for all open subsets V of U containing K and all real numbers $\varepsilon > 0$. Let q be a semi-norm on $\mathcal{H}_\theta(K ; F)$ defined by

$$q(\widetilde{f}) = p(f)$$

for $\widetilde{f} \in \mathcal{H}_\theta(\mathrm{K}\,;\mathrm{F})$; $f \in \widetilde{f}$. Then q is well-defined and continuous on $\mathcal{H}_\theta(\mathrm{K}\,;\mathrm{F})$. It is clear that $q \circ \mathrm{T_K}(f) = p(f)$ for every $f \in \mathcal{H}_\theta(\mathrm{U}\,;\mathrm{F})$. Therefore, p is continuous on $\mathcal{H}_\theta(\mathrm{U}\,;\mathrm{F})$ for the topology $\mathcal{T}_{\pi,\theta}$, and hence, $\mathcal{T}_{\sigma,\theta} \subset \mathcal{T}_{\pi,\theta}$.

5.5. REMARK. – If $\dim \mathrm{E} = \infty$, then $\mathcal{T}_0 \subsetneqq \mathcal{T}_\infty \subsetneqq \mathcal{T}_\sigma$ on $\mathcal{H}(\mathrm{U}\,;\mathrm{F})$ if $\mathrm{F} \neq 0$.

Proof. – Denote by $\mathcal{T}_n$ the topology on $\mathcal{H}(\mathrm{E})$ determined by the family of all semi-norms p of the form

$$p(f) = \sup_{x \in \mathrm{K}} \| \hat{d}^m f(x) \|$$

where K is a compact subset of E and $m = 0, 1, \ldots, n$. By the Hahn-Banach theorem, one can show that the semi-norm q_m on $\mathcal{H}(\mathrm{E})$ defined by

$$q_m(f) = \| \hat{d}^m f(0) \|$$

is continuous for the topology $\mathcal{T}_n$ if and only if $m \leqslant n$. Therefore,

$$\mathcal{T}_m \subset \mathcal{T}_{m+1} \quad \text{for every } m\,.$$

Suppose that $\mathcal{T}_\infty = \mathcal{T}_\sigma$. Let (α_m) be a sequence of positive real numbers such $(\alpha_m)^{1/m} \to 0$ as $m \to \infty$. Then the set

$$\mathcal{A} = \left\{ f \in \mathcal{H}(\mathrm{E}) \, : \, \sum_{m=0}^{\infty} \alpha_m \left\| \frac{1}{m\,!} \hat{d}^m f(0) \right\| \leqslant 1 \right\}$$

is a $\mathcal{T}_\infty$-neighborhood of 0. Therefore, there exist a compact subset K of E, a real number $r > 0$ and an integer $m > 0$ such that $\mathcal{B} \subset \mathcal{A}$, where

$$\mathcal{B} = \left\{ f \in \mathcal{H}(\mathrm{E}) \, : \, \sup_{x \in \mathrm{K}} \left\| \frac{1}{k\,!} \hat{d}^k f(x) \right\| \leqslant r,\ 0 \leqslant k \leqslant m \right\}.$$

Set $\quad \mathcal{C} = \left\{ f \in \mathcal{H}(\mathrm{E}) \, : \, \left\| \dfrac{1}{(m+1)\,!} \hat{d}^{m+1} f(0) \right\| \leqslant 1/\alpha_{m+1} \right\}.$

Then $\mathcal{B} \subset \mathcal{C}$, and hence, $\mathcal{C}$ is a $\mathcal{T}_m$-neighborhood of 0. But this is absurd since q_{m+1} is not continuous for the topology $\mathcal{T}_m$. Thus, $\mathcal{T}_\infty \neq \mathcal{T}_\sigma$.

5.6. PROPOSITION. – *The topologies $\mathcal{T}_{\sigma,\theta}$, $\mathcal{T}_{\pi,\theta}$ and $\mathcal{T}_{\omega,\theta}$ have the same family of bounded subsets and the same family of relatively compact subsets. On each bounded subset, they are equivalent.*

Proof. – See [N6, 12 and 13], or 4.1 and 4.3.

5.8. COROLLARY. – *Let $\mathfrak{X}$ be a $\mathcal{T}_{\omega,\theta}$-bounded subset of $\mathcal{H}_\theta(\mathrm{U}\,;\mathrm{F})$. Corresponding to every compact subset* K *of* U *there exist a real number* $\varepsilon > 0$ *and an open subset* V *of* U *such that* $\mathfrak{X}$ *is bounded in the Banach space* $\mathcal{H}_{\theta\varepsilon}(\mathrm{V}\,;\mathrm{F})$.

Proof. – It follows from 4.1.

5.9. PROPOSITION. – *The topology $\mathcal{T}_{\pi,\theta}$ is complete.*

Proof. – We apply the Corollary to Proposition 3 in [H, 2.11] here.

Order the family of all compact subsets of U by set inclusion. If $\mathrm{K} \subset \mathrm{J}$, then the natural inclusion

$$\mathcal{H}_\theta(\mathrm{J}\,;\mathrm{F}) \hookrightarrow \mathcal{H}_\theta(\mathrm{K}\,;\mathrm{F})$$

is continuous. Since $\mathcal{H}_\theta(\mathrm{K}\,;\mathrm{F})$ is separated for any compact subset K of U, it suffices to show that if $\tilde{f}$ belongs to every $\mathcal{H}_\theta(\mathrm{K}\,;\mathrm{F})$, for all compact subsets K of U, then f belongs to $\mathcal{H}_\theta(\mathrm{U}\,;\mathrm{F})$ for some $f \in \tilde{f}$. This is clear. Since each $\mathcal{H}_\theta(\mathrm{K}\,;\mathrm{F})$ is complete by 4.9. we conclude that $\mathcal{H}_\theta(\mathrm{U}\,;\mathrm{F})$ is complete for $\mathcal{T}_{\pi,\theta}$.

5.10. COROLLARY. – *The topologies $\mathcal{T}_{\sigma,\theta}$ and $\mathcal{T}_{\omega,0}$ are quasi-complete.*

Proof. – It is a consequence of 5.6 and 5.9.

6. Runge property.

Classically, in the complex plane **C**, a compact subset K of U is said to be U-Runge if the image of $\mathcal{H}(\mathrm{U})$ under the linear mapping $\mathrm{T_U} : \mathcal{H}(\mathrm{U}) \to \mathcal{H}(\mathrm{K})$ is dense in the space $\mathcal{H}(\mathrm{K})$. In this chapter, we extend this concept to an arbitrary Banach space E and obtain a sufficient condition for $\mathcal{T}_{\pi,\theta} = \mathcal{T}_{\omega,\theta}$.

6.1. Definition. – A compact subset K of U is said to be (θ, U)-*Runge* if for every real number $\varepsilon > 0$ and open subset V of U containing K, there exist a real number $\delta = \delta(\varepsilon, \mathrm{V})$, $0 < \delta \leqslant \varepsilon$, and an open subset $\mathrm{W} = \mathrm{W}(\mathrm{V}, \varepsilon)$ of V containing K such that given any f in $\mathcal{H}_{\theta\varepsilon}(\mathrm{V}; \mathrm{F})$ there is a sequence (f_m) in $\mathcal{H}_\theta(\mathrm{U}; \mathrm{F}) \cap \mathcal{H}_{\theta\delta}(\mathrm{W}; \mathrm{F})$ converging to f in the sense of $\mathcal{H}_{\theta\delta}(\mathrm{W}; \mathrm{F})$. U is said to satisfy *θ-Runge property* if every compact subset of U is contained in some (θ, U)-Runge compact subset of U.

6.2. Proposition. – *For the current type* θ, *a compact subset* K *of* U *is* (θ, U)-Runge *if and only if for every open subset* V *of* U *containing* K, *there exists an open subset* W *of* V *containing* K *such that given any* $f \in \mathcal{H}^\infty(\mathrm{V}; \mathrm{F})$ *there is a sequence* (f_m) *in*

$$\mathcal{H}(\mathrm{U}; \mathrm{F}) \cap \mathcal{H}^\infty(\mathrm{W}; \mathrm{F})$$

converging to f *in the sense of* $\mathcal{H}^\infty(\mathrm{W}; \mathrm{F})$.

Proof. – Let K be a (θ, U)-Runge compact subset of U and V an open subset of U containing K. Let $\nu > 0$ be a real number such that $\mathrm{B}_\nu(\mathrm{K}) \subset \mathrm{V}$. Choose real numbers $\rho > 0$ and $\varepsilon > 0$ with $\sigma(\rho + \varepsilon) < \nu$, where σ is the holomorphy constant. Set $\mathrm{V}' = \mathrm{B}_\rho(\mathrm{K})$. Then $\mathrm{V}' \subset \mathrm{V}$. By Nachbin inequalities and Cauchy inequalities we have

$$\sum_{m=0}^{\infty} \varepsilon^m \sup_{x \in \mathrm{V}'} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|$$
$$\leqslant \sum_{m=0}^{\infty} (\sigma(\rho + \varepsilon))^m \sup_{x \in \mathrm{K}} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|$$
$$\leqslant \sum_{m=0}^{\infty} \left[\frac{\sigma(\rho + \varepsilon)}{\nu} \right]^m \sup_{x \in \mathrm{B}_\nu(\mathrm{K})} \| f(x) \| < \infty$$

for every $f \in \mathcal{H}^\infty(\mathrm{V}; \mathrm{F})$, i.e., $\mathcal{H}^\infty(\mathrm{V}; \mathrm{F}) \subset \mathcal{H}_{\theta\varepsilon}(\mathrm{V}'; \mathrm{F})$. Corresponding to ε and V$'$, there exist a real number δ, $0 < \delta \leqslant \varepsilon$, and an open subset W of K such that given an $f \in \mathcal{H}_{\theta\varepsilon}(\mathrm{V}'; \mathrm{F})$, in particular, given any $f \in \mathcal{H}^\infty(\mathrm{V}; \mathrm{F})$, there is a sequence (f_m) in $\mathcal{H}(\mathrm{U}; \mathrm{F}) \cap \mathcal{H}_{\theta\delta}(\mathrm{W}; \mathrm{F})$ converging to f in the sense of $\mathcal{H}_{\theta\delta}(\mathrm{W}; \mathrm{F})$, and hence, in the sense of $\mathcal{H}^\infty(\mathrm{W}; \mathrm{F})$ since $\mathcal{H}_{\theta\delta}(\mathrm{W}; \mathrm{F}) \subset \mathcal{H}^\infty(\mathrm{W}; \mathrm{F})$ continuously.

Conversely, let V and $\varepsilon > 0$ be given as in 6.1. Corresponding to V, there exists an open subset W of V such that given any $f \in \mathcal{H}^{\infty}(V ; F)$, in particular, given any $f \in \mathcal{H}_{\theta\varepsilon}(V ; F)$, there is a sequence (f_m) in $\mathcal{H}(U ; F) \cap \mathcal{H}^{\infty}(W ; F)$ converging to f in the sense of $\mathcal{H}^{\infty}(W ; F)$. Let $\nu > 0$ be such that $B_{\nu}(K) \subset W$. Choose real numbers $\rho > 0$ and $\delta > 0$ with $\sigma(\rho + \delta) < \nu$. Then the sequence (f_m) is in $\mathcal{H}_{\theta\delta}(W' ; F)$ where $W' = B_{\rho}(K)$. Now

$$\sum_{k=0}^{\infty} \varepsilon^k \sup_{x \in W'} \left\| \frac{1}{k!} \hat{d}^k (f_m - f)(x) \right\|$$

$$\leqslant \sum_{k=0}^{\infty} (\sigma(\rho + \delta))^k \sup_{x \in K} \left\| \frac{1}{k!} \hat{d}^k (f_m - f)(x) \right\|$$

$$\leqslant \sum_{k=0}^{\infty} \left[\frac{\sigma(\rho + \delta)}{\nu} \right]^k \sup_{x \in W} \| (f_m - f)(x) \| \to 0$$

as $m \to \infty$. Therefore, K is (θ, U)-Runge.

6.3. PROPOSITION. – *Let* K *be a* (θ, U)-*Runge compact subset of* U. *Then the image of* $\mathcal{H}_{\theta}(U ; F)$ *under the linear mapping*

$$T_U : \mathcal{H}_{\theta}(U ; F) \to \mathcal{H}_{\theta}(K ; F)$$

is dense in $\mathcal{H}_{\theta}(K ; F)$.

Proof. – We show that the image of $\mathcal{H}_{\theta}(U ; F)$ in $\mathcal{H}_{\theta}(K ; F)$ is sequentially dense. Let $\tilde{f} \in \mathcal{H}_{\theta}(K ; F)$. Then there exist a real number $\varepsilon > 0$ and an open subset V of U containing K such that a representative of $\tilde{f}$, say f, comes from $\mathcal{H}_{\theta\varepsilon}(V ; F)$. Corresponding to ε and V, by 6.1. there exist a real number δ, $0 < \delta \leqslant \varepsilon$, and an open subset W of V such that for the mapping f, there is a sequence (f_m) in $\mathcal{H}_{\theta}(U ; F) \cap \mathcal{H}_{\theta\delta}(W ; F)$ converging to f in the sense of $\mathcal{H}_{\theta\delta}(W ; F)$. Therefore, the sequence $(\tilde{f}_m)$ converges to $\tilde{f}$ in $\mathcal{H}_{\theta}(K ; F)$ where $\tilde{f}_m = T_U(f_m)$ for every m. Thus the image of $\mathcal{H}_{\theta}(U ; F)$ under T_U is sequentially dense in $\mathcal{H}_{\theta}(K ; F)$.

6.5. PROPOSITION. – *If* U *satisfies the* θ-*Runge property then* $\mathcal{T}_{\omega,\theta} = \mathcal{T}_{\pi,\theta}$ *on the space* $\mathcal{H}_{\theta}(U ; F)$.

Proof. – We need to show that $\mathcal{T}_{\omega,\theta} \subset \mathcal{T}_{\pi,\theta}$. Since U satisfies θ-Runge property, the topology $\mathcal{T}_{\omega,\theta}$ on $\mathcal{H}_\theta(U ; F)$ is determined by the family of all semi-norms on $\mathcal{H}_\theta(U ; F)$ ported by (θ , U)-Runge compact subsets of U. Therefore, it is sufficient to consider only (θ , U)-Runge compact subsets of U in this proof.

Let K be a (θ , U)-Runge compact subset of U. Let $\tilde{f} \in \mathcal{H}_\theta(K ; F)$. Then there exist a real number $\varepsilon > 0$ and an open subset V of U containing K such that a representative of $\tilde{f}$, say f, comes from $\mathcal{H}_{\theta\varepsilon}(V ; F)$. Since K is (θ , U)-Runge, corresponding to ε and V, there exist a real number δ, $0 < \delta \leqslant \varepsilon$, and an open subset W of V containing K such that for the mapping f, there is a sequence (f_m) in $\mathcal{H}_\theta(U ; F) \cap \mathcal{H}_{\theta\delta}(W ; F)$ converging to f in the sense of $\mathcal{H}_{\theta\delta}(W ; F)$. Let p be an arbitrary semi-norm on $\mathcal{H}_\theta(U ; F)$ ported by K. Then there exists a real number $c(\delta , W) > 0$ such that

$$|p(f_m) - p(f_n)| \leqslant p(f_m - f_n) \leqslant$$

$$\leqslant c(\delta , W) \sum_{k=0}^{\infty} \delta^k \sup_{x \in W} \left\| \frac{1}{k!} \hat{d}^k (f_m - f_n)(x) \right\|_\theta .$$

Therefore, $\lim p(f_m)$ exists as $m \to \infty$.

Define a semi-norm q on $\mathcal{H}_\theta(K ; F)$ by

$$q(\tilde{f}) = \lim_{m \to \infty} p(f_m)$$

if $\tilde{f} \in \mathcal{H}_\theta(K ; F)$ is such that a representative of $\tilde{f}$ is the limit of a sequence (f_m) in $\mathcal{H}_\theta(U ; F) \cap \mathcal{H}_{\theta\delta}(W ; F)$ in the sense of $\mathcal{H}_{\theta\delta}(W ; F)$. It is easy to check that q is a well-defined semi-norm on $\mathcal{H}_\theta(K ; F)$.

To show that the semi-norm p is continuous on $\mathcal{H}_\theta(U ; F)$ for the topology $\mathcal{T}_{\pi,\theta}$, it is sufficient to show that q is continuous on $\mathcal{H}_\theta(K ; F)$ since

$$p(f) = q(T_K(f)), \; f \in \mathcal{H}_\theta(U ; F) .$$

By 2.8, it suffices to show that q is sequentially continuous.

Let $(\tilde{f}_m)$ be a sequence in $\mathcal{H}_\theta(K ; F)$ converging to 0. By 4.6, there exist a real number $\varepsilon > 0$ and an open subset V of U containing K such that $(\tilde{f}_m)$ is contained and converges to 0 in $T_V\mathcal{H}_{\theta\varepsilon}(V; F)$. For each m choose $f_m \in \mathcal{H}_{\theta\varepsilon}(V ; F)$ with $f_m \in \tilde{f}_m$. Then the sequence (f_m) converges to 0 in $\mathcal{H}_{\theta\varepsilon}(V ; F)$. By 6.1, corresponding to ε and

V, there exist a real number δ, $0 < \delta \leqslant \varepsilon$, and an open subset W of V containing K such that, for each m there is a sequence $(f_{m,n})$ in $\mathcal{H}_\theta(U ; F) \cap \mathcal{H}_{\theta\delta}(W ; F)$ converging to f_m in the sense of $\mathcal{H}_{\theta\delta}(W ; F)$. For each m choose an integer $m(n) > m$ satisfying the following :

$$(*) \quad \sum_{k=0}^{\infty} \delta^k \sup_{x \in W} \left\| \frac{1}{k!} \hat{d}^k (f_m - f_{m,m(n)})(x) \right\|_\theta < 1/m \; ;$$

$$(**) \quad | q(f_m) - p(f_{m,m(n)}) | < 1/m \; .$$

Since (f_m) converges to 0 in $\mathcal{H}_{\theta\delta}(W ; F)$, by (*), the sequence $(f_{m,m(n)})$ converges to 0 in $\mathcal{H}_{\theta\delta}(W ; F)$.

Therefore,

$$\lim_{m \to \infty} p(f_{m,m(n)}) = 0 \; .$$

This proves that q is sequentially continuous on the space $\mathcal{H}_\theta(K ; F)$.

The preceding proposition has the following application.

6.5. Proposition. – *If* U *satisfies the θ-Runge property then* $\mathcal{H}_\theta(U ; F)$ *is complete for* $\mathcal{T}_{\omega,\theta}$.

Proof. – Use 6.5. and 5.9.

Most of the important open subsets satisfy the θ-Runge property. All balanced open sets are in this type. We also conjecture that every open subset satisfies the θ-Runge property. It is not answered in the literature whether or not every open subset in $\mathbf{C}^n$ satisfies the θ-Runge property for the current holomorphy type θ.

7. Miscellaneous results.

In contrast with the finite dimensional theory, the spaces of holomorphic mappings and germs on infinite dimensional Banach spaces do not satisfy many nice properties in the general theory of locally convex spaces. In this chapter we examine the spaces $\mathcal{H}(U)$ and $\mathcal{H}(K)$ for the following properties : Montel ; Schwartz ; nuclear ; reflexive ; Mackey convergence. We also give a necessary and sufficient condition that the space $\mathcal{H}(U)$ be bornological. We assume the concepts of compact, nuclear and integral polynomials which can be found in [G] and [D1].

All results concerning the space $\mathcal{H}_\theta(U\,;F)$ are with respect to any locally convex topology $\mathcal{T}$, $\mathcal{T}_{\infty,\theta} \subset \mathcal{T} \subset \mathcal{T}_{\omega,\theta}$. Accordingly, on the space $\mathcal{H}_\theta(K\,;F)$ we consider the inductive limit of the topology $\mathcal{T}$ on $\mathcal{H}_\theta(U\,;F)$, for all open subsets U of E containing K.

7.1. Proposition. – *The Banach space $\mathcal{P}_\theta(^m E\,;F)$ is a closed subspace of $\mathcal{H}_\theta(U\,;F)$.*

Proof. – Let ξ be a fixed point in U. Define a linear mapping T : $\mathcal{H}_\theta(U\,;F) \to \mathcal{P}_\theta(^m E\,;F)$ by

$$T(f) = \frac{1}{m\,!}\,\hat{d}^m f(\xi)\,.$$

Then T is continuous for the topology $\mathcal{T}$. The inclusion mapping $\mathcal{P}_\theta(^m E\,;F) \hookrightarrow \mathcal{H}_\theta(U\,;F)$ is also continuous. Therefore, the composite mapping

$$\mathcal{P}_\theta(^m E\,;F) \hookrightarrow \mathcal{H}_\theta(U\,;F) \to \mathcal{P}_\theta(^m E\,;F)$$

is continuous. Since $P = \frac{1}{m\,!}\,\hat{d}^m P(\xi)$ for every P in $\mathcal{P}_\theta(^m E\,;F)$, the composite mapping is the identity mapping on $\mathcal{P}_\theta(^m E\,;F)$. Hence, $\mathcal{P}_\theta(^m E\,;F)$ is a closed subspace of $\mathcal{H}_\theta(U\,;F)$.

Let $P \in \mathcal{P}_\theta(^m E\,;F)$. Then the equivalence class $\widetilde{P}$ modulo K determined by P consists of P alone. Thus we can consider $\mathcal{P}_\theta(^m E\,;F)$ as a vector subspace of $\mathcal{H}_\theta(K\,;F)$. The inclusion mapping

$$\mathcal{P}_\theta(^m E\,;F) \hookrightarrow \mathcal{H}_\theta(K\,;F)$$

is also continuous. We also have the following proposition.

7.2. Proposition. – *The Banach space $\mathcal{P}_\theta(^m E\,;F)$ is a closed subspace of $\mathcal{H}_\theta(K\,;F)$.*

7.3. Corollary. – *The dual E′ of E is a closed subspace of $\mathcal{H}(U)$ and $\mathcal{H}(K)$.*

Proof. – $E' = \mathcal{L}(E\,;C) = \mathcal{L}_s(E\,;C) = \mathcal{P}(^1E\,;F)$. (Cf. [G]).

7.4. COROLLARY. – *The Banach space* $\mathcal{P}_c({}^m\mathrm{E}\,;\mathrm{F})$ *of all continuous compact m-homogeneous polynomials from* E *to* F *is a closed subspace of* $\mathcal{H}(\mathrm{U}\,;\mathrm{F})$ *and* $\mathcal{H}(\mathrm{K}\,;\mathrm{F})$.

Proof. – $\mathcal{L}_c({}^m\mathrm{E}\,;\mathrm{F})$ is a closed subspace of $\mathcal{L}({}^m\mathrm{E}\,;\mathrm{F})$.

7.5. PROPOSITION. – $\mathcal{H}(\mathrm{U})$ *or* $\mathcal{H}(\mathrm{K})$ *is either Montel, or Schwartz, or nuclear if and only if* $\dim \mathrm{E} < \infty$.

Proof. – If $\mathcal{H}(\mathrm{U})$ satisfies one of these properties, then its closed subspace E′ satisfies the same property. A normed space satisfies one of these properties if and only if the dimension is finite. Therefore, the necessity is proved. The sufficiency is classical. The same argument proves the proposition for $\mathcal{H}(\mathrm{K})$.

7.6. PROPOSITION. – *If* E *is not reflexive, then neither* $\mathcal{H}(\mathrm{U})$ *nor* $\mathcal{H}(\mathrm{K})$ *is reflexive. If* E *is a Hilbert space, then* $\mathcal{H}(\mathrm{U})$ *or* $\mathcal{H}(\mathrm{K})$ *is reflexive if and only if* $\dim \mathrm{E} < \infty$.

Proof. – If $\mathcal{H}(\mathrm{U})$ or $\mathcal{H}(\mathrm{K})$ is reflexive, then the closed subspace E′ is also reflexive. Therefore, E is reflexive. This proves the first part.

If E is a Hilbert space and $\mathcal{H}(\mathrm{U})$ or $\mathcal{H}(\mathrm{K})$ is reflexive, then the Banach space $\mathcal{P}_c({}^m\mathrm{E})$ is reflexive for every m. Let $\mathcal{P}_\mathrm{N}({}^m\mathrm{E})$ and $\mathcal{P}_\mathrm{I}({}^m\mathrm{E})$ be the Banach spaces of continuous nuclear and integral m-homogeneous polynomials on E respectively. By the Borel transformation

$$: \mathrm{T} \in \mathcal{P}_\mathrm{N}({}^m\mathrm{E})' \mapsto \hat{\mathrm{T}} \in \mathcal{P}({}^m\mathrm{E}')\ ;$$

$$\mathrm{T} \in \mathcal{P}_c({}^m\mathrm{E})' \mapsto \hat{\mathrm{T}} \in \mathcal{P}_\mathrm{I}({}^m\mathrm{E}')$$

defined by

$$\hat{\mathrm{T}}(\varphi) = \mathrm{T}(\varphi^m)\ ,\ \varphi \in \mathrm{E}'\ ,$$

we have

$$\mathcal{P}_\mathrm{N}({}^m\mathrm{E})' \simeq \mathcal{P}({}^m\mathrm{E}')\ ;$$

$$\mathcal{P}_c({}^m\mathrm{E})' \simeq \mathcal{P}_\mathrm{I}({}^m\mathrm{E}')\ .$$

(Cf. [G, III] and [D1, 3.2]). Now we have

$$\mathcal{P}_c({}^m E)' \simeq \mathcal{P}_I({}^m E') \hookrightarrow \mathcal{P}_N({}^m E') \ ;$$

$$\mathcal{P}_N({}^m E')' \simeq \mathcal{P}({}^m E'') \hookrightarrow \mathcal{P}_c({}^m E)'' = \mathcal{P}_c({}^m E)$$

for every m. Put $E = E''$. Then we obtain

$$\mathcal{P}({}^m E) = \mathcal{P}_c({}^m E) \ .$$

If E is infinite dimensional, the identity mapping on E is not compact, that is, there is a noncompact continuous 2-linear mapping on E since $\mathcal{L}({}^2 E) = \mathcal{L}(E \ ; E')$ and $E = E'$. Thus, neither $\mathcal{H}(U)$ nor $\mathcal{H}(K)$ is reflexive. If E is finite dimensional, then both $\mathcal{H}(U)$ and $\mathcal{H}(K)$ are Montel, and hence reflexive.

Proposition 7.6. does not seem to be true for Banach spaces. We give a possible counter-example, a Banach space E which is reflexive and infinite dimensional such that every continuous m-linear mapping on E is compact. (Compare with the preceding proof).

Let p and q be reals such that $1 \leqslant q < p < \infty \ ; (1/p) + (1/q) = 1$. Let $E = l^p$, the Banach space of complex sequences (x_m) satisfying

$$\sum_{m=0}^{\infty} (x_m)^p < \infty \ .$$

Then, E is an infinite dimensional reflexive Banach space. Furthermore, $\mathcal{L}({}^m E) = \mathcal{L}_c({}^m E)$. This fact can be shown easily by induction on m using the following : every continuous linear mapping from l^p to l^r is compact if $1 \leqslant r < p < \infty$ (Cf. [P, Theorem 1] or [R, Theorem A2]) ; E is separable ; $T \in \mathcal{L}_c({}^m E)$ if and only if $T : E \to \mathcal{L}({}^{m-1} E)$ is compact.

7.7. Proposition. – *$\mathcal{H}(U \ ; F)$ is bornological for $\mathcal{T}, \mathcal{T}_0 \subset \mathcal{T} \subsetneq \mathcal{T}_\omega$, if and only if* $\dim E < \infty$.

Proof. – $\mathcal{T}$ and $\mathcal{T}_\omega$ share the same bounded subsets.

We do not know when $\mathcal{H}(U \ ; F)$ is bornological for $\mathcal{T}_\omega$ in general. This is a problem yet to be solved. We conjecture that $\mathcal{H}(U)$ is bornological for $\mathcal{T}_\omega$ if E is separable. A necessary and sufficient condition for $(\mathcal{H}(U), \mathcal{T}_\omega)$ to be bornological is given in 7.11. Recently S. Dineen has shown that $(\mathcal{H}(l^\infty), \mathcal{T}_\omega)$ is not bornological (Cf. [D4]).

7.8. PROPOSITION. – *The space $\mathcal{H}_\theta(K;F)$ satisfies the Mackey convergence condition for the natural topology.*

Proof. – Let $(\tilde{f}_m)$ be a sequence in $\mathcal{H}_\theta(K;F)$ converging to 0. Then there exist a real number $\varepsilon > 0$ and an open subset U of E containing K such that the sequence is contained and converges to 0 in $T_U\mathcal{H}_\theta(U;F)$ by 4.6. Since every metrizable locally convex space satisfies the Mackey convergence condition, there is a sequence (λ_m) of positive real numbers converging to ∞ such that $(\lambda_m \tilde{f}_m)$ converges to 0 in $T_U\mathcal{H}_\theta(U;F)$, and hence, in $\mathcal{H}_\theta(K;F)$.

7.9. COROLLARY. – *If (f_m) is a sequence in $\mathcal{H}_\theta(U;F)$ converging to 0 for the topology $\mathcal{T}_{\omega,\theta}^{\ast}$, then for every compact subset K of U there exist a real number $\varepsilon > 0$ and an open subset V of U containing K such that*

$$\lim_{m\to\infty} \sum_{k=0}^{\infty} \varepsilon^k \lambda_m \sup_{x\in V} \left\| \frac{1}{k!}\hat{d}^k f_m(x)\right\|_\theta = 0$$

for some sequence (λ_m) of positive real numbers converging to ∞.

Proof. – Let K be a compact subset of U. Consider the sequence $(\tilde{f}_m)$ where $\tilde{f}_m = T_K(f_m)$. Then it converges to 0 in $\mathcal{H}_\theta(K;F)$. Apply 7.10 and 4.6 to obtain the desired form.

7.10. PROPOSITION. – *If E is separable, then $\mathcal{H}(U;F)$ satisfies the Mackey convergence condition.*

Proof. – Let (f_m) be a sequence in $\mathcal{H}(U;F)$ converging to 0 for $\mathcal{T}$. Since $\mathcal{T}$ and $\mathcal{T}_\omega$ have the same family of convergent sequences, it suffices to show that there is a sequence of positive real numbers λ_m, $\lambda_m \to \infty$ as $m \to \infty$, such that $\lambda_m f_m \to 0$ as $m \to \infty$ for $\mathcal{T}_\omega$.

Let $\xi \in U$. Then, by 7.9, there is an open subset V of U containing ξ such that

$$\lim_{m\to\infty} \sup_{x\in V} \|f_m(x)\| = 0.$$

Since U is separable, we can find a sequence (V_k) of open subsets of U such that :

(a) $V_k \subset V_{k+1}$ for every k ;

b) $\mathrm{U} = \bigcup_k \mathrm{V}_k$;

c) $\lim_{m \to \infty} \sup_{x \in \mathrm{V}_k} \| f_m(x) \| \to 0$ for every k.

By induction on k, we choose $m(k)$ such that $m(k) \geqslant m(k-1)$ and

$$\sup_{x \in \mathrm{V}_k} \| f_m(x) \| < 1/2^k .$$

for all $m \geqslant m(k)$. Set

$$\lambda_m = \begin{cases} 1 & \text{if } m < m(1) ; \\ \sqrt{2}^k & \text{if } m(k) \leqslant m < m(k+1) . \end{cases}$$

For each compact subset K of U, there is an integer n such that $\mathrm{K} \subset \mathrm{V}_{k+n}$ for every k. If $m \geqslant m(k+n)$,

$$\sup_{x \in \mathrm{V}_n} \| \lambda_m f_m(x) \| \leqslant \sup_{x \in \mathrm{V}_{k+n}} \| \lambda_m f_m(x) \| = \sqrt{2}^{k+n} \sup_{x \in \mathrm{V}_{k+n}} \| f_m(x) \| < 1/\sqrt{2}^{k+n}$$

for every k. This shows that $\lambda_m f_m \to 0$ as $m \to \infty$. Thus $\mathcal{H}(\mathrm{U})$ satisfies the Mackey convergence condition.

The Mackey convergence condition and the bornological property are two independent concepts.

However, one proves that every metrizable locally convex space is bornological using the Mackey convergence condition on such a space. We generalize this fact in the following way : *A locally convex separated space satisfying the Mackey convergence condition is bornological if and only if every sequentially continuous semi-norm is continuous.*

Proof. – The necessity is obvious. We show the sufficiency. Let p be a semi-norm which is bounded on bounded sets. Since the Mackey convergence condition holds true for every sequence (x_m) converging to 0, there is a sequence of real numbers $\lambda_m > 0$, $\lambda_m \to \infty$ as $m \to \infty$, such that $\lambda_m x_m \to 0$ as $m \to \infty$. Therefore, there is a real number $\mathrm{M} > 0$ such that $p(\lambda_m x_m) < \mathrm{M}$ for m. Thus p is sequentially continuous, and hence continuous.

7.11. **Proposition.** – ***Let* E *be separable. Then* $\mathcal{H}$(U ; F) *is bornological if and only if every sequentially continuous semi-norm on*** $\mathcal{H}$(U ; F) *is continuous.*

Finally we discuss the independence of the Mackey convergent condition and the bornological property.

If E is a Banach space of nonmeasurable cardinal, then it is realcompact. (Cf. [GJ, 15.24]). Then $\mathcal{C}$(E) is bornological for the compact-open topology $\mathcal{T}_0$ by Nachbin-Shirota theorem. (Cf. [N2, 29] or [N1]). Thus $\mathcal{H}$(E) is a closed subspace of a bornological space $\mathcal{C}$(E) which is not bornological for $\mathcal{T}_0$.

Let E be separable. Then $\mathcal{H}$(U) with respect to the topology $\mathcal{T}_\infty$ is a non-bornological space which satisfies the Mackey convergence condition.

The following example will show that a bornological space may not satisfy the Mackey convergence condition. Let $G = \mathbf{R}^{[0,1]}$. Then G is a bornological space if we endow G with the product topology since a nonmeasurable product of bornological spaces is also bornological by the Mackey-Ulam theorem. Let T be a one-to-one mapping from $[0,1]$ onto the set of all sequences of positive real numbers. Define a sequence (x_m) in G by

$$[x_m]_\alpha = [T(\alpha)]_m \ ,$$

where $[x]_\alpha$ is the α-th coordinate of x and $[T(\alpha)]_m$ is the m-th term of the sequence $T(\alpha)$ for $\alpha \in [0,1]$. Then it is clear that (x_m) converges to 0 in G, and there is no sequence of positive real numbers λ_m for which $(\lambda_m x_m)$ converges to 0 in G.

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