C. IONESCU-TULCEA
R. MAHER

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Annales de l'institut Fourier, tome 21, nº 2 (1971), p. 35-41
<http://www.numdam.org/item?id=AIF_1971__21_2_35_0>

Ann. Inst. Fourier, Grenoble
21, 2 (1971), 35-41.

A NOTE ON ALMOST STRONG LIFTINGS (1)

By C. IONESCU TULCEA (2) and R. MAHER

1.

We denote below by X a *locally compact space* and by $\mathcal{M}(X)$ the vector space of Radon measures on X, endowed with the usual order relation. Let $\mu \neq 0$ be a positive Radon measure on X. We say that a *lifting* ρ of $M^{\infty}_{\mathbf{R}}(X, \mu)$ is *almost strong* (see [7], Chap. VIII) if there is a $\mu^{\bullet}$-negligible (that is, locally μ-negligible) set $A \subset X$ such that

$$\rho(f)|\complement A = f|\complement A$$

for all $f \in C^{b}_{\mathbf{R}}(X)$.

We say that the couple (X, μ) has the *almost strong lifting property* (*a.s. lifting property*) if there exists an almost strong lifting of $M^{\infty}_{\mathbf{R}}(X, \mu)$.

To shorten some of the statements below we also say that (X, μ) has the a.s. lifting property whenever $\mu = 0$.

The problem as to whether or not every (X, μ) (where X is a locally compact space and μ a positive Radon measure on X) has the a.s. lifting property is open (see [5] and [7], Chap. VIII). However there are many important examples of couples (X, μ) having the a.s. lifting property (see [5], [6], [7], Chap. VIII and [8]). Recently, K. Bichteler (see [1] and [2]) has noticed the interesting fact that the set of all Radon measures μ on X such that $(X, |\mu|)$ has the a.s. lifting property is a *band* of $\mathcal{M}(X)$. In this paper we present a short proof of this result by a method different from that of K. Bichteler.

(1) We use the notations and terminology introduced in [7].

(2) Supported by contract DAHCOH 68 C0005, U.S. Army Research Office, Durham, N.C.

2.

For *any* positive Radon measure μ on X we denote by $\mathcal{C}(X, \mu)$ the set of all locally countable families $(K_j)_{j\in J}$ having the following properties:

a) K_j is compact and $\mu(K_j) > 0$ for each $j \in J$.

b) $K_{j'} \cap K_{j''} = \emptyset$ if $j' \neq j''$.

c) The set $X - \bigcup_{j\in J} K_j$ is $\mu^\bullet$-negligible.

The following result will be often used below:

THEOREM 1. — *Let* μ *be a positive Radon measure on* X. 1.1) *If* (X, μ) *has the a.s. lifting property and* $K \subset X$ *is compact, then* (K, μ_K) *has the a.s. lifting property.* 1.2) *Conversely, let* $(K_j)_{j\in J} \in \mathcal{C}(X, \mu)$ *be such that, for each* $j \in J$, (K_j, μ_{K_j}) *has the a.s. lifting property. Then* (X, μ) *has the a.s. lifting property.*

Proof. — 1.1) It is enough to consider the case $\mu_K \neq 0$. Let ρ be an almost strong lifting of $M^\infty_{\mathbf{R}}(X, \mu)$ and let $A \subset X$ be a $\mu^\bullet$-negligible set such that the relations $\rho(f)|\complement A = f|\complement A$ are satisfied for all $f \in C^b_{\mathbf{R}}(X)$. Let χ be a character of $L^\infty_{\mathbf{R}}(K, \mu_K)$. For $f \in M^\infty_{\mathbf{R}}(K, \mu_K)$ define $f' : X \to \mathbf{R}$ by $f'(t) = f(t)$ if $t \in K$ and $f'(t) = 0$ if $t \notin K$. Then $f \longmapsto f'$ is a representation of $M^\infty_{\mathbf{R}}(K, \mu_K)$ into $M^\infty_{\mathbf{R}}(X, \mu)$. Define now $\rho'(f)$, for $f \in M^\infty_{\mathbf{R}}(K, \mu_K)$, by

$$\rho'(f)(t) = \begin{cases} \rho(f')(t) & \text{if} \quad t \in K \cap \rho(K) \\ \chi(\tilde{f}) & \text{if} \quad t \in K - \rho(K). \end{cases}$$

It is easy to see that ρ' is a lifting of $M^\infty_{\mathbf{R}}(K, \mu_K)$ and that $\rho'(f)(x) = f(x)$ if $f \in C^b_{\mathbf{R}}(K)$ and $t \notin K \cap (A \cup (K - \rho(K)))$. Hence ρ' is an almost strong lifting of $M^\infty_{\mathbf{R}}(K, \mu_K)$ and hence the couple (K, μ_K) has the a.s. lifting property.

1.2) It is enough to consider the case $\mu \neq 0$. For each $j \in J$ let ρ_j be an almost strong lifting of $M^\infty_{\mathbf{R}}(K_j, \mu_{K_j})$ and $A_j \subset K$ a $\mu^\bullet_K$-negligible set such that $\rho_j(f)|\complement A_j = f|\complement A_j$ for $f \in C^b_{\mathbf{R}}(K_j)$. Let χ be a character of $L^\infty_{\mathbf{R}}(X, \mu)$. If $f \in M^\infty_{\mathbf{R}}(X, \mu)$, then $f|K_j \in M^\infty_{\mathbf{R}}(K_j, \mu_{K_j})$ for each $j \in J$ and hence we may define

$$\rho(f)(t) = \begin{cases} \rho_j(f \mid K_j)(t) & \text{if} \quad t \in K_j \\ \chi(\tilde{f}) & \text{if} \quad t \in X - \bigcup_{j \in J} K_j. \end{cases}$$

It is easy to see that ρ is a lifting of $M^{\infty}_{\mathbf{R}}(X, \mu)$ and that $\rho(f)(x) = f(x)$ if $f \in C^b_{\mathbf{R}}(X)$ and $t \notin \left(\bigcup_{j \in J} A_j\right) \cup \left(X - \bigcup_{j \in J} K_j\right)$. Hence ρ is an almost strong lifting of $M^{\infty}_{\mathbf{R}}(X, \mu)$ and hence the couple (X, μ) has the a.s. lifting property.

Remarks. — Theorem 1 is similar to Proposition 2, [7], Chap. VIII (in fact it can be easily deduced from this proposition).

3.

If μ and ν are two positive Radon measures on X we write $\mu \prec \nu$ if μ is absolutely continuous with respect to ν (that is, if $\mu = \varphi . \nu$ with $\varphi : X \to R_+$, locally ν-integrable). We say that μ and ν are equivalent if $\mu \prec \nu$ and $\nu \prec \mu$. If μ and ν are equivalent, then (X, μ) has the a.s. lifting property if and only if (X, ν) has the a.s. lifting property.

Notice that if $\mu \prec \nu$ *then there is* $(K_j)_{j \in J} \in \mathcal{C}(X, \mu)$ *such that, for each* $j \in J$, μ_{K_j} *and* ν_{K_j} *are equivalent.*

In fact if $\mu \prec \nu$ then $\mu = \varphi . \nu$ with $\varphi : X \to \mathbf{R}_+$ locally ν-integrable. Let $A = \{x \mid \varphi(x) > 0\}$ and consider a partition of A consisting of a $\mu^{\bullet}$-negligible set N and a locally countable family of compact sets $(K_j)_{j \in L}$ such that $\varphi \mid K_j$ is continuous for each $j \in L$ (see Corollary 1, Chap. IV, § 5 [3]). If $J = \{j \in L \mid \mu(K_j) > 0\}$, then $(K_j)_{j \in J} \in \mathcal{C}(X, \mu)$. Since for each $j \in J$, $\mu_{K_j} = (\varphi \mid K_j) . \nu_{K_j}$ and since

$$0 < \inf_{x \in K_j} (\varphi \mid K_j)(x) \leqslant \sup_{x \in K_j} (\varphi \mid K_j)(x) < +\infty,$$

we deduce that μ_{K_j} and ν_{K_j} are equivalent.

THEOREM 2. — *Let* μ *and* ν *be two positive Radon measures on* X. *If* (X, ν) *has the a.s. lifting property and* $\mu \prec \nu$ *then* (X, μ) *has the a.s. lifting property* (3).

(3) See [1].

Proof. — We have noticed above that there is

$$(K_j)_{j\in J} \in \mathcal{C}(X, \mu)$$

such that, for each $j \in J$, μ_{K_j} and ν_{K_j} are equivalent. By Theorem 1, for each $j \in J$, (K_j, ν_{K_j}) has the a.s. lifting property, whence (K_j, μ_{K_j}) has the a.s. lifting property. Using again Theorem 1 we deduce that (X, μ) has the a.s. lifting property.

THEOREM 3. — *Let* μ and ν *be two positive Radon measures on* X *such that* (X, μ) *and* (X, ν) *have the a.s. lifting property. Then* $(X, \mu + \nu)$ *has the a.s. lifting property.*

Proof. — Let $\mu = \mu_a + \mu_s$, where μ_a is the absolutely continuous part of μ with respect to ν and μ_s the singular part of μ with respect to ν. Then

$$\mu + \nu = (\mu_a + \nu) + \mu_s.$$

Since $\mu_a + \nu \prec \nu$, the couple $(X, \mu_a + \nu)$ has the a.s. lifting property; since $\mu_s \prec \mu$, the couple (X, μ_s) has the a.s. lifting property. Moreover, there are two disjoint universally measurable parts of X, X′ and X″, the union of which is X, such that $\mu_a + \nu$ is concentrated on X′ and μ_s is concentrated on X″.

Let now $(K_j)_{j\in J} \in \mathcal{C}(X, \mu + \nu)$ such that for each $j \in J$ we have either $K_j \subset X'$ or $K_j \subset X''$ and let

$$J' = \{j \mid K_j \subset X'\} \qquad \text{and} \qquad J'' = \{j \mid K_j \subset X''\}.$$

If $j \in J'$ then $(\mu + \nu)_{K_j} = (\mu_a + \nu)_{K_j}$ so that $(K_j, (\mu + \nu)_{K_j})$ has the a.s. lifting proprety; if $j \in J''$ then $(\mu + \nu)_{K_j} = (\mu_s)_{K_j}$, so that $(K_j, (\mu + \nu)_{K_j})$ has again the a.s. lifting property. By Theorem 1, $(X, \mu + \nu)$ has the a.s. lifting property.

COROLLARY 1. — *Let* μ *and* ν *be as in the statement of Theorem* 3. *Then* $(X, \inf\{\mu, \nu\})$ *and* $(X, \sup\{\mu, \nu\})$ *have the a.s. lifting property.*

Proof. — It is enough to notice that

$$\inf\{\mu, \nu\} \prec \mu + \nu \qquad \text{and} \qquad \sup\{\mu, \nu\} \prec \mu + \nu.$$

We note before proceeding further that if $\mathcal{F}$ is a filtering set of positive Radon measures on a *compact* space X, bounded above, then there is an increasing sequence $(\mu_n)_{n\in\mathbf{N}}$ of measures belonging to $\mathcal{F}$ such that

$$\sup \mathcal{F} = \sup_{n\in\mathbf{N}} \mu_n$$

(use Theorem 4, Chap. I, [7]).

If $\lambda = \sup \mathcal{F}$ then $\lambda^\bullet(A) = 0$ if and only if $\mu_n^\bullet(A) = 0$ for every $n \in \mathbf{N}$ (use Proposition 11, Chap. V, § 1, [3]). We also notice that if $(B_n)_{n\in\mathbf{N}}$ is a sequence of parts of X such that $\mu_n^\bullet(B_n) = 0$ for every $n \in \mathbf{N}$, then

$$\lambda^\bullet (\limsup_{n\in\mathbf{N}} B_n) = 0.$$

In fact it is enough to observe that, for each $p \in \mathbf{N}$

$$\limsup_{n\in\mathbf{N}} B_n \subset \bigcup_{n=p}^{+\infty} B_n$$

and

$$\mu_p^\bullet\left(\bigcup_{n=p}^{+\infty} B_n\right) \leqslant \sum_{n=p}^{+\infty} \mu_p^\bullet(B_n) \leqslant \sum_{n=p}^{+\infty} \mu_n^\bullet(B_n) = 0.$$

THEOREM 4. — *Let $\mathcal{F}$ be a set of positive Radon measures on (the locally compact space) X, bounded above and let $\lambda = \sup \mathcal{F}$. Suppose that (X, μ) has the a.s. lifting property for every $\mu \in \mathcal{F}$. Then (X, λ) has the a.s. lifting property.*

Proof. — By Corollary 1, we may suppose that $\mathcal{F}$ is filtering. On the basis of Theorem 1 and the fact that for every compact $K \subset X$,

$$\lambda_K = \sup \{\mu_K | \mu \in \mathcal{F}\}$$

(see Proposition 5, Chap. V, § 5, [3]). It is enough to establish that (X, λ) has the a.s. lifting property *when* X *is compact.*

We may also assume $\lambda \neq 0$. Let then $(\mu_n)_{n\in\mathbf{N}}$ be an increasing sequence of strictly positive measures belonging to $\mathcal{F}$, such that $\lambda = \sup_{n\in\mathbf{N}} \mu_n$. For each $n \in \mathbf{N}$ let ρ_n be an almost strong lifting of $M_{\mathbf{R}}^\infty(X, \mu_n)$ and $A(n)$ a $\mu_n^\bullet$-negligible set such that $\rho_n(f)|\complement A(n) = f|\complement A(n)$ for all $f \in C_{\mathbf{R}}^b(X)$.

Let $\mathfrak{U}$ be an *ultrafilter* on $\mathbf{N}$ finer than the Fréchet filter associated with $\mathbf{N}$. For every $f \in M_{\mathbf{R}}^{\infty}(X, \lambda)$ define [4]

$$\rho(f) = \lim_{n, \mathfrak{U}} \rho_n(f).$$

Then ρ is a representation of the algebra $M_{\mathbf{R}}^{\infty}(X, \lambda)$ into the algebra $B_{\mathbf{R}}^{\infty}(X)$ of all bounded functions on X to $\mathbf{R}$, such that $\rho(1) = 1$. Moreover $f \equiv g\ (\lambda)$ implies $f \equiv g\ (\mu_n)$, that is, $\rho_n(f) = \rho_n(g)$ for all $n \in \mathbf{N}$, whence $\rho(f) = \rho(g)$. Let now $f \in M_{\mathbf{R}}^{\infty}(X, \lambda)$ and for each $n \in \mathbf{N}$ let

$$B(n) = \{x \mid \rho_n(f)(x) \neq f(x)\}.$$

Clearly $\rho(f)(x) = f(x)$ for

$$x \notin \limsup_{n \in \mathbf{N}} B(n).$$

Since $\limsup_{n \in \mathbf{N}} B(n)$ is $\lambda^\bullet$-negligible, we deduce $\rho(f) \in M_{\mathbf{R}}^{\infty}(X, \lambda)$ and $\rho(f) \equiv f$. Hence ρ *is a lifting* of $M_{\mathbf{R}}^{\infty}(X, \lambda)$. In the same way we see that for every $f \in C_{\mathbf{R}}^{b}(X)$, $\rho(f)(x) = f(x)$ if $x \notin \limsup_{n \in \mathbf{N}} A(n)$. Since $\limsup_{n \in \mathbf{N}} A(n)$ is $\lambda^\bullet$-negligible we conclude that ρ is an almost strong lifting of $M_{\mathbf{R}}^{\infty}(X, \mu)$.

Hence (X, μ) has the a.s. lifting property.

Remark. — By the same method we can prove the following : *Let* $(\mu_n)_{n \in \mathbf{N}}$ *be a sequence of positive Radon measures on* X *and* λ *a positive Radon measure on* X. *Suppose that*:

i) $\mu_n \prec \mu_{n+1}$ *for all* $n \in \mathbf{N}$;

ii) $\lambda^\bullet(A) = 0$ *if and only if* $\mu_n^\bullet(A) = 0$ *for all* $n \in \mathbf{N}$. *Then* (X, λ) *has the a.s. lifting property if and only if* (X, μ_n) *has the a.s. lifting property for every* $n \in \mathbf{N}$.

We shall say that (X, μ), where $\mu \in \mathcal{M}(X)$, has the a.s. lifting property if and only if $(X, |\mu|)$ has the a.s. lifting property. Denote by $\mathbf{U}$ the set of all $\mu \in \mathcal{M}(X)$ such that (X, μ) has the a.s. lifting property. Then :

THEOREM 5 (*Bichteler*). — *The set* $\mathbf{U}$ *is a band of* $\mathcal{M}(X)$.

Proof. — The assertion follows from Theorems 2, 3 and 4.

[4] See also [4].

Let **V** be the set of all positive Radon measures μ on X such that (X, μ) has the *strong lifting property* (see Definition 1, Chap. VIII [7]). Clearly $\mathbf{V} \subset \mathbf{U}$.

COROLLARY 2. — *The set* **V** *is a cone of* $\mathcal{M}(X)$ *having the properties* :

j) *if* μ *and* ν *belong to* **V**, *then* $\sup\{\mu, \nu\} \in \mathbf{V}$;

jj) *if* $\mathcal{F} \subset \mathbf{V}$ *is bounded above, in* $\mathcal{M}(X)$, *then* $\sup \mathcal{F} \in \mathbf{V}$.

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Manuscrit reçu le 28 septembre 1970.

C. IONESCU TULCEA
Northwestern University
Department of Mathematics
Evanston, Illinois 60201 (U.S.A.).

and R. MAHER,
Loyola University
Department of Mathematics
Chicago, Illinois 60626 (U.S.A.).