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SOME RESULTS ON THIN SETS IN A HALF PLANE

by H. L. JACKSON

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1. Introduction.

Let C denote the complex plane, $\hat{C}$ the extended plane, and D the open right half plane. Following Brelot [3], we define $X \subset C$ to be *internally thin* at $z_0 \in C$ if it is thin there in the classical sense, [2]. If $\partial D = \{z \in C : \operatorname{Re} z = 0\}$ and $\{\infty\} = \hat{C} - C$ then the Martin boundary of D can be identified with $\partial D \cup \{\infty\}$. Let ΔD be the Martin boundary of D and $\hat{D}$ the Martin compactification of D. If H^+ is the set of all non negative, classical hyperharmonic functions on D, $X \subset D$ and φ any non negative superharmonic function on D, we define the *reduced function* (réduite) [5], p. 36, of φ on X relative to H^+ to be

$$R_\varphi^X = \inf\{h \in H^+ : h \geqslant \varphi \text{ on } X\}, \qquad \text{and} \qquad \hat{R}_\varphi^X$$

the superharmonic function which coincides q.e. with R_φ^X. If $\hat{z}_0 \in \Delta D$, and $M_{\hat{z}_0}$ is a minimal harmonic function on D whose pole is at $\hat{z}_0$ then we define $X \subset D$ to be *minimally thin* at $\hat{z}_0 \in \Delta D$ iff $M_{\hat{z}_0} \not\equiv \hat{R}^X_{M_{\hat{z}_0}}$. It is possible to formulate analogous definitions in higher dimensional Euclidean spaces and even in more general spaces, [3], [4].

In 1948, M^me^ J. Lelong [14], p. 130, introduced a definition for a subset of an open half space in n-dimensional Euclidean space $(n \geqslant 2)$ to be thin at a Martin boundary point of the half space. Naïm [15], has pointed out that the type of thinness which M^{me} Lelong introduced for a half space is in fact the minimal thinness we have just defined. M^{me} Lelong found necessary and sufficient conditions in order to ensure that a subset X of a half space must be minimally thin at the origin or at ∞. In the case where $n \geqslant 3$ she showed that internal thinness always implies minimal thinness, and that if X is contained in a Stolz domain with vertex at the origin then internal thinness and minimal thinness are equivalent for any X so restricted. She then remarked that the two types of thinness are non comparable if $n = 2$ but gave no details, [14], p. 132. In various publications ([3], [4]), Brelot noted this remark and claimed that whereas M^{me} Lelong did not prove her claim for $n = 2$, Choquet did prove here assertion in detail but did not publish the result. On the basis of this claim, Brelot ([3], [4], [6]) assumed that there could not be any axiomatic implication between the two types of thinness, but was able to prove that a statistical (i.e. almost everywhere) type of implication does exist between them in a way which he makes precise ([4], p. 10, théorème 5, and p. 14, théorème 5').

The main purpose of this paper is to prove that internal thinness at the origin always implies minimal thinness there in the case where $n = 2$ thus correcting completely a published error of M^{me} Lelong and an unpublished one of Choquet. Furthermore we shall show that unlike the case for $n \geqslant 3$, this implication continues to be strict when one is restricted to a Stolz domain with vertex at the origin. We shall also work out some of the relations between minimal thinness, finite logarithmic length, and minimal semithinness for sets restricted to a Stolz domain. Finally we shall apply our results on thin sets to a theorem of Ahlfors and Heins ([1], p. 341 theorem B) and M^{me} Lelong ([14], p. 144, théorème (1*c*)). In particular we shall point out that the P — L exceptional sets of Ahlfors and Heins are, in fact, minimally thin sets at ∞ with respect to the right half plane and then show that the « finite logarithmic length » character of their theorem B can be improved. We should expect that these results can contribute to an improvement of certain theorems of Essén [10].

2. Some Required Definitions and Theorems.

Let $B(r) = \{z \in C : |z| < r\}$ where $r > 0$, and h the extended real valued function $h(z, w) = \log \dfrac{1}{|z - w|}$ on $C \times C$. In order to ensure that h will be positive, we shall in future restrict the domain of h to $B\left(\frac{1}{2}\right) \times B\left(\frac{1}{2}\right)$, and shall temporarily be only concerned about thinness at the origin for those Borel sets $X \subset B\left(\frac{1}{2}\right)$ which are also relatively compact in $B\left(\frac{1}{2}\right)$. If $K \subset B\left(\frac{1}{2}\right)$ is compact, $\exists$ a Borel measure μ on $B\left(\frac{1}{2}\right)$ which is carried by K so that the h-potential of μ, denoted by $U^\mu = \int_K h(., w)\, d\mu_w$ is the equilibrium h-potential such that $U^\mu(z) \equiv 1$ q.e. on K. Such a measure μ shall be called the h-capacitary distribution and $\int_K d\mu$ shall be called the ordinary capacity of K, henceforth to be denoted by $c(K)$. The ordinary capacity function c extends naturally to Borel sets and even to analytic sets. It is a capacity function in the sense of Choquet [9]. If $X \subset B\left(\frac{1}{2}\right)$ and $\lambda(X)$ is the *logarithmic capacity* of X then

$$\left.\begin{aligned} c(X) &= \frac{1}{\log\left(\dfrac{1}{\lambda(X)}\right)} && \text{if} \quad \lambda(X) > 0 \\ &= 0 && \text{if} \quad \lambda(X) = 0 \end{aligned}\right\} \quad [2], \text{ p. } 321.$$

We mention here the fact, apparently not generally known, that the logarithmic capacity λ is not a capacity in the sense of Choquet.

If $X \subset B\left(\frac{1}{2}\right)$ so that the origin 0 is not an isolated point (usual topology) of $X \cup \{0\}$ then X is defined to be internally thin at 0 if $\exists$ a positive superharmonic function v

on $B\left(\frac{1}{2}\right)$ such that $v(0) < \varliminf_{\{z \to 0, z \in X\}} v(z)$. We shall now mention some further criteria, by now classical, each of which is both necessary and sufficient to ensure that a set $X \subset B\left(\frac{1}{2}\right)$ must be internally thin at 0. Let s be any real number, temporarily fixed, such that $0 < s < 1$. We define the n^{th} annular domain to be $I_n = B(s^n) - B(s^{n+1})$, $c(r) = c(X \cap B(r))$, $X_n = X \cap I_n$ and $c_n = c(X_n)$. Similarly we define

$$\lambda(r) = \lambda(X \cap B(r)) \qquad \text{and} \qquad \lambda_n = \lambda(X \cap I_n).$$

It has been shown (see [2], p. 325, and [16], p. 104) that a Borel set $X \subset B\left(\frac{1}{2}\right)$ is internally thin at 0 iff any one of the following equivalent conditions holds; (i) $\sum_{n=1}^{\infty} c(s^n) < +\infty$, (ii) $\sum_{n=1}^{\infty} nc_n < +\infty$, (iii) $\int_0^{\delta} \frac{c(r)}{r}\, dr < +\infty$ for some $\delta > 0$. For non-capacitable sets, one could replace « capacity » by « outer capacity ». The convergence of each of the series (i) and (ii) is independent of s, so that in future we shall choose $s = \frac{1}{e}$. For the metrical properties of internally thin sets the reader is referred to Brelot ([2], pp. 335-337).

The Green's function for the right half plane D and pole at $w \in D$ is the extended real valued function g defined so that $g(z, w) = \log \frac{|z + \overline{w}|}{|z - w|} = h(z, w) - h(z, -\overline{w})$ on D. If

$$D(r) = D \cap B(r), \qquad \text{and} \qquad (z, w) \in D\left(\frac{1}{2}\right) \times D\left(\frac{1}{2}\right)$$

then $|z + \overline{w}| < 1$ so that $h(z, -\overline{w}) > 0$ and therefore $g(z, w) < h(z, w)$. If $K \subset D$ is compact and $\varphi(z) \equiv 1$ on D, then $\hat{R}_{\varphi}^{K}$ is a Green potential on D whose measure ν is carried by K and whose total mass $\int_K d\nu$ is called the Green capacity of K, henceforth to be denoted by $\sigma(K)$. The set function σ can, at least, be extended to all Borel sets, and analytic sets which are relatively compact in D. It too is a capacity in the sense of Choquet. Since $g < h$ on

$D\left(\frac{1}{2}\right) \times D\left(\frac{1}{2}\right)$ it follows *directly* that if X is relatively compact in $D\left(\frac{1}{2}\right)$ then $\sigma(X) \geqslant c(X)$. If X is relatively compact in $D\left(\frac{1}{2}\right)$ we shall define $\sigma(r) = \sigma(X \cap B(r))$, and $\sigma_n = \sigma(X_n)$. Let $\mathcal{K}$ be a Stolz domain in D such that $z \in \mathcal{K}$ iff $|\text{Arg}\, z| \leqslant \theta_0 < \pi/2$. If we require X to be contained in $D\left(\frac{1}{2}\right) \cap \mathcal{K}$, then X_n is relatively compact in $D\left(\frac{1}{2}\right)$ for all n sufficiently large. M$^{\text{me}}$ Lelong has shown that X thus restricted is minimally thin at the origin iff $\sum_{n=1}^{\infty} \sigma_n < +\infty$, [14], p. 131. One can make use of her arguments to show that if $X \subset D$ is not necessarily restricted to a Stolz domain, but X_n is relatively compact in D for every n then the condition $\sum_{n=1}^{\infty} \sigma_n < +\infty$ is still sufficient to ensure that X must be minimally thin at 0. She also introduced a concept which she called *semi-thinness* and we will call *minimal semithinness.* A set X, restricted to a Stolz domain, is semi-thin according to M$^{\text{me}}$ Lelong's definition iff $\lim_{n\to\infty} (\sigma_n) = 0$. The theory of minimal semi-thinness has been developed in more general spaces by Brelot and Doob [7].

3. Some Results on Minimally Thin and Semithin Sets in a Half Plane.

We shall now prove our main theorems.

THEOREM 1. — *If* $\mathcal{K} = \{z \in D : |\text{Arg}\, z| \leqslant \theta_0 < \pi/2\}$ *and* X *is contained in* $D\left(\frac{1}{2}\right) \cap \mathcal{K}$, *then* X *is minimally thin at* 0 *if* $\sum_{n=1}^{\infty} c_n < +\infty$ *and* $\overline{\lim_{n\to\infty}}\, (nc_n) < 1$.

Proof. — Let ν_n be the mass distribution on $D\left(\frac{1}{2}\right)$ whose Green potential $\hat{R}_\varphi^{X_n} = \int_X g(\cdot, w)\, d\nu_n(w)$ coincides q.e.

with the reduced function of $\varphi(z) \equiv 1$ on X_n. Then $\hat{R}^{X_n}_{\varphi} \equiv 1$ q.e. on X_n and $\nu_n(X_n) = \int_{X_n} d\nu_n = \sigma_n$. If $\sigma_n \neq 0$, $\exists z_n \in X_n$ so that $\hat{R}^{X_n}_{\varphi}(z_n) = 1$ and therefore $1 = \int_{X_n} g(z_n, w)\, d\nu_n(w)$, or $1 = \int_{X_n} \log \frac{1}{|z_n - w|}\, d\nu_n(w) - \int_{X_n} \log \frac{1}{|z_n + \bar{w}|}\, d\nu_n(w)$ which is equivalent to

$$\text{(i)} \quad \int_{X_n} \log \frac{1}{|z_n - w|}\, d\nu_n(w) = 1 + \int_{X_n} \log \frac{1}{|z_n + \bar{w}|}\, d\nu_n(w).$$

If $z \in (I_n \cap \mathcal{K})$ and $\zeta \in I_n \cap \mathcal{K}$ then the inequalities

$$\text{(ii)} \qquad 2s^{n+1} \cos \theta_0 \leqslant |z + \zeta| < 2s^n$$

easily follow so that $2s^{n+1} \cos \theta_0 \leqslant |z_n + \bar{w}| < 2s^n$ when z_n and w are chosen as in (i).

By making use of the left inequality above it follows that $\frac{1}{|z_n + \bar{w}|} \leqslant \frac{1}{2s^{n+1} \cos \theta_0}$ which in turn implies

$$\text{(iii)} \quad \log \frac{1}{|z_n + \bar{w}|} \leqslant (n + 1) \log \left(\frac{1}{s}\right) + \log \frac{1}{2 \cos \theta_0}.$$

We agreed earlier to let $s = \frac{1}{e}$ so that (iii) becomes

$$\text{(iii')} \qquad \log \frac{1}{|z_n + \bar{w}|} \leqslant (n + 1) + \log \frac{1}{2 \cos \theta_0},$$

or $\log \frac{1}{|z_n + \bar{w}|} \leqslant n + A$ where $A = 1 + \log \frac{1}{2 \cos \theta_0}$. We note that $A \geqslant 1 - \log 2$.

If we integrate over X_n with respect to ν_n we obtain

$$\text{(iv)} \quad \int_{X_n} \log \frac{1}{|z_n + \bar{w}|}\, d\nu_n \leqslant (A + n) \int_{X_n} d\nu_n = (A + n)\, \sigma_n,$$

and combining (i) and (iv) the inequality

$$\text{(v)} \quad \int_{X_n} \log \frac{1}{|z_n - w|}\, d\nu_n \leqslant 1 + (A + n)\, \sigma_n \quad \text{results.}$$

Let $\omega_n = \frac{\nu_n}{1 + (A + n)\sigma_n}$ and note that the h-potential of

ω_n at z_n is $\int_{X_n} h(z_n, w)\, d\omega_n(w) \leqslant 1$. In fact

$$\int_{X_n} h(z, w)\, d\omega_n(w) \leqslant 1$$

q.e. on X_n which means that $\omega_n(X_n) \leqslant \mu_n(X_n)$, ([8], p. 41) where μ_n is the h-capacitary distribution on X_n. Hence the inequality

(vi) $\dfrac{\sigma_n}{1 + (A + n)\sigma_n} \leqslant c_n$ follows. Taking the inverse mapping we obtain the fundamental inequality:

(vii) $\sigma_n \leqslant \dfrac{c_n}{1 - (A + n)c_n}$ provided $0 \leqslant c_n < \dfrac{1}{A + n}$.

If $\overline{\lim\limits_{n\to\infty}}\, nc_n = 1 - \varepsilon$ where $0 < \varepsilon \leqslant 1$ then

$$\overline{\lim_{n\to\infty}}\, (A + n)c_n = 1 - \varepsilon$$

also because $\lim\limits_{n\to\infty} \dfrac{A + n}{n} = 1$. Therefore $(A + n)c_n < 1 - \varepsilon/2$ for all n sufficiently large so that $\sigma_n < \dfrac{2}{\varepsilon} c_n$ for all n sufficiently large. If $\sum\limits_{n=1}^{\infty} c_n < +\infty$ then $\sum\limits_{n=1}^{\infty} \sigma_n < +\infty$ by the comparison test for series. Since X is minimally thin at 0 iff $\sum\limits_{n=1}^{\infty} \sigma_n < +\infty$ the theorem follows.

THEOREM 2. — *If* $X \subset D\left(\frac{1}{2}\right) \cap \mathcal{H}$ *is internally thin at* 0 *then* X *is minimally thin there. This implication is strict in general.*

Proof. — We recall from § 2 that X is internally thin at 0 iff $\sum\limits_{n=1}^{\infty} nc_n < +\infty$. If X is internally thin at 0 it follows that $\lim\limits_{n\to\infty} (nc_n) = 0$ and $\sum\limits_{n=1}^{\infty} c_n < +\infty$ so that X satisfies the conditions of theorem 1 and therefore is minimally thin at 0.

In order to see that the implication is strict we construct

$X = \bigcup_{n=1}^{\infty} X_n$ so that each X_n is a disk of radius e^{-n^2}. Therefore $c_n = \frac{1}{n^2}$ and it follows that X satisfies the conditions for theorem 1 so that X is minimally thin at 0. Nevertheless $\sum_{n=1}^{\infty} nc_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges so that X fails to be internally thin at 0. This proves theorem 2.

We shall now prove a theorem that will allow us to extend the implication of theorem 2 to the half plane itself.

Theorem 3. — *Let* $\overline{V} = \left\{ z \in C : \left| z - \frac{1}{2} \right| \leqslant \frac{1}{2} \right\}$ *and* $X \subset D\left(\frac{1}{2}\right) \cap \overline{V}$. *If* $\overline{\lim_{n \to \infty}} (nc_n) < \frac{1}{2}$ *and* $\sum_{n=1}^{\infty} c_n < +\infty$ *then* $\sum_{n=1}^{\infty} \sigma_n < +\infty$.

Proof. — Now $\overline{V} = \{(r, \theta) : r \leqslant \cos\theta\}$ and $\partial\overline{V} \cap D$ is $\{(r, \theta) : r = \cos\theta, r \neq 0\}$. Let $s = \frac{1}{e}$ and (s^n, θ_n) be that point of the first quadrant which is a member of $\partial B(s^n) \cap \partial\overline{V}$. Hence $s^n = \cos\theta_n$ and if $z \in X_n$ then

$$\mathrm{Re}(z) \geqslant s^{n+1} \cos(\theta_{n+1}) = (s^{n+1})^2.$$

Since $X \subset \overline{V}$ it is evident that X_n is relatively compact in D for every natural number n. We shall now follow a line of reasoning which constitutes a simple modification of that which was followed in theorem 1. By making use of the same notation as that employed in the above mentioned theorem, we obtain the following equality:

$$\text{(i)} \quad \int_{X_n} \log \frac{1}{|z-\zeta|} \, d\nu_n(\zeta) = 1 + \int_{X_n} \log \frac{1}{|z+\bar{\zeta}|} \, d\nu_n(\zeta) \quad \text{q.e.}$$

on X_n. where we recall that ν_n is the Green capacitary distribution on X_n and $\sigma_n = \nu_n(X_n)$.

We now obtain the following inequalities:

(ii) If $z \in X_n$ and $\zeta \in X_n$ then

$$2s^{n+1} \cos\theta_{n+1} = 2s^{2(n+1)} \leqslant |z + \bar{\zeta}|,$$

and therefore:

$$\text{(iii)} \qquad \log \frac{1}{|z+\overline{\zeta}|} \leqslant \log \left(\frac{1}{s}\right)^{2n} + \log \frac{1}{2s^2},$$

$$\text{or} \qquad \log \frac{1}{|z+\overline{\zeta}|} \leqslant 2n + (2 - \log 2) \quad \text{since} \quad e = \frac{1}{s}.$$

If we let $A = 2 - \log 2$, and integrate over X_n with respect to ν_n we obtain:

$$\text{(iv)} \int_{X_n} \log \frac{1}{|z+\overline{\zeta}|}\, d\nu_n(\zeta) \leqslant (A + 2n) \int_{X_n} d\nu_n = (A + 2n)\, \sigma_n$$

and if we combine (i) and (iv) then the inequality:

$$\text{(v)} \int_{X_n} \log \frac{1}{|z-\zeta|}\, d\nu_n(\zeta) \leqslant 1 + (A + 2n)\, \zeta_n \quad \text{results.}$$

By defining $\omega_n = \dfrac{\nu_n}{1 + (A + 2n)\sigma_n}$ and μ_n as the capacitary distribution of the logarithmic kernel so that $\mu_n(X_n) = c_n$ we obtain $\omega_n(X_n) \leqslant \mu_n(X_n)$ and therefore

(vi) $\dfrac{\sigma_n}{1 + (A + 2n)\sigma_n} \leqslant c_n$ follows. If we take the inverse mapping we obtain the basic inequality:

$$\text{(vii)} \ \sigma_n \leqslant \frac{c_n}{1 - (A + 2n)c_n} \quad \text{if} \quad 0 \leqslant c_n < \frac{1}{A + 2n}.$$

Now suppose that $\overline{\lim\limits_{n\to\infty}}\, nc_n = \dfrac{1}{2} - \varepsilon$ where $0 < \varepsilon \leqslant \dfrac{1}{2}$. Then $\overline{\lim\limits_{n\to\infty}}\, (A + 2n)c_n = 1 - 2\varepsilon$ so that $(A + 2n)c_n < 1 - \varepsilon$ for all n sufficiently large and therefore $\dfrac{1}{\varepsilon} > \dfrac{1}{1 - (A + 2n)c_n}$ so that $\dfrac{c_n}{\varepsilon} > \dfrac{c_n}{1 - (A + 2n)c_n} \geqslant \sigma_n$ for all n sufficiently large. If $\sum\limits_{n=1}^{\infty} c_n < +\infty$ and $\overline{\lim\limits_{n\to\infty}}\, nc_n = \dfrac{1}{2} - \varepsilon$ therefore $\sum\limits_{n=1}^{\infty} \sigma_n < +\infty$ and the theorem follows.

We shall now extend the implication in theorem 2 to the half plane itself.

THEOREM 4. — *If* $E \subset D$ *is internally thin at* 0 *with respect to* C *then* E *is minimally thin at* 0 *with respect to* D.

Proof. — Let $X = E \cap \overline{V}$ and $X' = E \cap (D - \overline{V})$ so that $E = X \cup X'$. The function $M = \operatorname{Re}\left(\frac{1}{z}\right)$ is a minimal harmonic function on D with pole at 0 and the reduced function $R_M^{D-\overline{V}} \equiv 1$ on $\overline{V}$ so that $D - \overline{V}$ is minimally thin at 0. Since $X' \subset D - \overline{V}$ it follows that X' is also minimally thin at 0. Now suppose that E is internally thin at 0. It follows that X is also internally thin at 0 and hence satisfies the conditions for theorem 3 so that $\sum_{n=1}^{\infty} \sigma_n < +\infty$. We noted at the end of § 2 that if each X_n is relatively compact in D then the condition $\sum_{n=1}^{\infty} \sigma_n < +\infty$ is a sufficient condition for minimal thinness at 0, though not necessary unless X is restricted to a Stolz domain. Since every X_n is relatively compact in D it follows that X is minimally thin at 0, and since X' is always minimally thin at 0 therefore $E = X \cup X'$ is minimally thin at 0.

We shall now consider the property of finite logarithmic length.

Let ψ be the circular projection mapping of D onto the positive real axis, that is if $z = re^{i\theta}$ then $\psi(z) = r$. If $X_n = X \cap I_n$ we shall let $X'_n = \psi(X_n)$ and $\lambda'_n = \lambda(X'_n)$. If m denotes the one dimensional Lebesgue measure on the positive real axis we shall let $m_n = m(X'_n)$, and note the inequalities $\lambda_n \geqslant \lambda_n \geqslant \frac{m_n}{4}$ ([16], p. 85, corollary 6). If X'_n is an interval then $\lambda'_n = \frac{m_n}{4}$. A set $X \subset D$ is defined to be an *r-set of finite logarithmic length* at the origin if $X' = \psi(X)$ has the property that for some $\delta > 0$ then $\int_{X' \cap (0,\delta)} \frac{dr}{r} < +\infty$, or equivalently $\sum_{n=1}^{\infty} e^n m_n < +\infty$.

THEOREM 5. — *If* $X \subset D\left(\frac{1}{2}\right) \cap \mathcal{K}$ *is minimally thin at* 0, *then* $X' = \psi(X)$ *possesses finite logarithmic length, but the converse does not hold in general.*

Proof. — We shall first prove the direct part. By making use of the right side of inequality (ii) in the proof of theorem 1 and reason as before, we obtain the inequality:

$$\text{(vi}'\text{)} \qquad \frac{\sigma_n}{1+(n-\log 2)\sigma_n} \geqslant c_n,$$

which implies that $\frac{1}{\sigma_n} + (n - \log 2) \leqslant \frac{1}{c_n}$ if we ignore all terms where σ_n or $c_n = 0$.

Now define

$$\left.\begin{aligned} \gamma(r) &= \frac{1}{\log\left(\frac{1}{r}\right)} && \text{if} \quad 0 < r < 1 \\ &= 0 && \text{if} \quad r = 0 \end{aligned}\right\}.$$

Then γ is strictly monotone increasing on $[0,1)$. Since $\frac{m_n}{4} \leqslant \lambda_n' \leqslant \lambda_n$ therefore $\gamma\left(\frac{m_n}{4}\right) \leqslant \gamma(\lambda_n') \leqslant \gamma(\lambda_n) = c_n$, and it follows that $\frac{1}{c_n} \leqslant \frac{1}{\gamma\left(\frac{m_n}{4}\right)} = \log\left(\frac{4}{m_n}\right)$. Combining this last inequality with the one above it follows that

$$\frac{1}{\sigma_n} \leqslant \log\left(\frac{4}{m_n}\right) + \log 2 - n = \log\left(\frac{8}{m_n}\right) - n$$

which implies that

$$\sigma_n \geqslant \frac{1}{\log\left(\frac{8}{m_n}\right) - n} = \frac{1}{\log\left(\frac{8}{m_n e^n}\right)} = \gamma\left(\frac{m_n e^n}{8}\right)$$

for all natural numbers n. It follows that minimal thinness at 0 implies that $\sum_{n=1}^{\infty} \gamma\left(\frac{m_n e^n}{8}\right) < +\infty$ and since $\gamma(r) > r$ where r is sufficiently small it follows that $\sum_{n=1}^{\infty} \frac{m_n e^n}{8} < +\infty$. The direct part follows.

For the converse part let $X_n' = \left(\frac{1}{e^n} - \frac{4}{n^2 e^n}, \frac{1}{e^n}\right)$ for each n. Then $e^n m_n = \frac{4}{n^2}$ so that X' has finite logarithmic length, but $\frac{m_n}{4} = \lambda_n = \frac{1}{n^2 e^n}$ so that $c_n = \frac{1}{\log\left(\frac{1}{\lambda_n}\right)} = \frac{1}{n + 2 \log n}$.

Hence $\sum_{n=1}^{\infty} c_n$ diverges and since $c_n \leqslant \sigma_n$ it follows that X′ fails to be minimally thin at 0. This proves theorem 5.

Remark. — Since $\gamma(r) > r^{\frac{1}{p}}$ for all natural numbers p it follows that if $X \subset D\left(\frac{1}{2}\right) \cap \mathcal{K}$ is minimally thin at 0, therefore $\sum_{n=1}^{\infty} (m_n e^n)^{\frac{1}{p}} < +\infty$.

We shall now find a necessary and sufficient condition for a set $X \subset D\left(\frac{1}{2}\right) \cap \mathcal{K}$ to be minimally semithin at the origin.

THEOREM 6. — *If* $X \subset D\left(\frac{1}{2}\right) \cap \mathcal{K}$ *then* X *is minimally semithin at* 0 *iff* $\lim_{n\to\infty} (\lambda_n e^n) = 0$.

Proof. — For the necessity part we use inequality (vi)′ in the proof of theorem 5 to obtain

$$\frac{1}{\sigma_n} + n - \log 2 \leqslant \frac{1}{c_n} = \log\left(\frac{1}{\lambda_n}\right),$$

or

$$\frac{1}{\sigma_n} \leqslant \log\left(\frac{1}{\lambda_n}\right) + \log 2 - n.$$

It follows that $\frac{1}{\sigma_n} \leqslant \log\left(\frac{2}{\lambda_n e^n}\right)$ and therefore

$$\sigma_n \geqslant \frac{1}{\log\left(\frac{2}{\lambda_n e^n}\right)} = \gamma\left(\frac{\lambda_n e^n}{2}\right).$$

Since $\gamma(r) > r$ it follows that minimal semithinness implies that $\lim_{n\to\infty} (\lambda_n e^n) = 0$.

For the sufficiency part we make use of inequality (vi) in the proof of theorem 1 to obtain

$$\frac{1}{\sigma_n} + (A + n) \geqslant \frac{1}{c_n} = \log\left(\frac{1}{\lambda_n}\right).$$

Therefore $\frac{1}{\sigma_n} + A \geqslant \log\left(\frac{1}{\lambda_n e^n}\right)$ so that if $\lim\limits_{n\to\infty} (\lambda_n e^n) = 0$ then $\lim\limits_{n\to\infty} \log\left(\frac{1}{\lambda_n e^n}\right) = +\infty$ and hence $\lim\limits_{n\to\infty}\left(\frac{1}{\sigma_n}\right) = +\infty$ or $\lim\limits_{n\to\infty} \sigma_n = 0$. This proves the sufficiency and the theorem.

Remark. — Since $\frac{m_n}{4} \leqslant \lambda'_n \leqslant \lambda_n$ therefore $\lim\limits_{n\to\infty} (m_n e^n) = 0$, or equivalently $m_n = o\left(\frac{1}{e^n}\right)$ if X is semithin at the origin. This sharpens a result of Brelot and Doob [7], p. 406, corollaire.

Corollary. — *If* X *is contained in the positive real axis, then the properties of semithinness and of finite logarithmic length are non comparable.*

Proof of Corollary. — Suppose X is constructed so that X_n is connected for each n and $\frac{m_n}{4} = \frac{1}{ne^n}$. Then $\lambda_n = \frac{m_n}{4}$ so that $\lambda_n e^n = \frac{1}{n}$ and hence $\lim\limits_{n\to\infty} (\lambda_n e^n) = 0$ so that X is semithin at 0 but $\sum\limits_{n=1}^{\infty} e^n m_n$ diverges so that X fails to be of finite logarithmic length. We point out that if X is structured so that each X_n is connected, and if S is of finite logarithmic length then X is semithin. We shall now see that in general however, finite logarithmic length does not imply semithinness. Let us construct X so that each X_n is a Cantor set in $\left[\frac{1}{e^{n+1}}, \frac{1}{e^n}\right]$. Then $m_n = 0$ for each n so that X is of finite logarithmic length, but $\exists \alpha > 0$ such that $\lambda_n e^n > \alpha$ for all n sufficiently large, [16], p. 106-108. Hence $\varliminf\limits_{n\to\infty} \lambda_n e^n \geqslant \alpha$ and it follows that X cannot be semithin at the origin.

Remark. — The properties of internal thinness, minimal thinness, semithinness and finite logarithmic length are all preserved under the inversion mapping $f(z) = \frac{1}{z}$ so that all implications which have been proved at the origin hold equally well at ∞.

4. Applications.

In 1949, Ahlfors and Heins, [1], published the following result which we will call theorem B.

THEOREM B. — *Let* u *be a subharmonic function whose domain is the half plane* D *such that* $\overline{\lim}_{\substack{z \to z_0 \\ z_0 \in \partial D}} u(z) \leqslant 0$ *and* $\sup \left\{ \frac{u(z)}{x} : z \in D,\ x = \mathrm{Re} z \right\} = \alpha < +\infty$. *If*

$$\mathcal{K} = \{z = re^{i\theta} \in D : 0 \leqslant \theta_0 < \pi/2\},$$

$\exists$ *an r-set* X' *of finite logarithmic length such that the expression* $\lim_{r \to \infty} \frac{u(z)}{r} = \alpha \cos \theta$ *holds uniformly in* θ *where* $z \in \mathcal{K}$, *and where* r *is restricted to lie oustside of* X'.

After proving this result, they remarked that they were uncertain as to whether or not the « finite logarithmic length » character of theorem B could be improved [1], p. 345. One of the basic tools introduced by Ahlfors and Heins in order to obtain theorem B and other results was a concept, believed by them to be new, which they called a *P-L exceptional* set. A set $X \subset D$ is defined to be $P - L$ exceptional according to Ahlfors and Heins if it is open and there exists a Green potential U on D which dominates the function $\varphi(z) = \mathrm{Re} z$ everywhere on X. We shall now note the following lemma due to Brelot and published by Naïm, [15], p. 204, lemma 1. See also [14], p. 139, théorème, and [11], p. 313.

LEMMA 1. — *Let* G *be a Green space,* $\hat{z}_0$ *a minimal Martin boundary point and* $\varphi(z) = M_{\hat{z}_0}$ *a minimal harmonic function with pole at* $\hat{z}_0$. *If* $X \subset G$, *then* $\hat{R}_\varphi^X$ *is either a Green potential on* G *or the function* φ *itself.*

Remark. — From lemma 1, we obtain the following result ([11], p. 313): An open set $X \subset D$ is $P - L$ exceptional according to Ahlfors and Heins iff it is minimally thin at ∞ with respect to the half plane D.

If $\hat{z}_0 \in \Delta D$ we shall let $\mathcal{F}_{\hat{z}_0} = \{S \subset D : D - S$ is minimally thin at $\hat{z}_0\}$ be the trace on D of the filter of neighbourhoods of $\hat{z}_0$ in the space $D \cup \Delta D$ endowed with the fine topology of Naïm. For brevity we shall replace $\mathcal{F}_{\hat{z}_0}$ by $\mathcal{F}$ if $\hat{z}_0 = \infty$. We shall now mention a result of M^{me} Lelong's, ([11], p. 144, théorème 1c) which has been generalized by Naïm ([15], théorème 8′-17) and is applicable to the present discussion. Even though the result is valid in more general spaces we shall phrase it in terms of our two dimensional notation.

THEOREM B′. — *If* v *is a positive superharmonic function on* D *such that* $\inf\left\{\frac{v(z)}{x} : z \in D\right\} = \alpha \geqslant 0$, *then* $\lim\limits_{\mathcal{F}}\left(\frac{v(z)}{x}\right) = \alpha$.

M^{me} Lelong then observed that if v is a superharmonic function which can be decomposed into the form:

$$v(z) = \beta x + w(z)$$

where w is a non-negative superharmonic function such that $\inf\left\{\frac{w(z)}{x} : z \in D\right\} = 0$ and β is any real number, then $\lim\limits_{\mathcal{F}}\left(\frac{v(z)}{x}\right) = \beta$.

She further remarked that a subharmonic function u satisfies the Phragmén-Lindelof conditions imposed by Ahlfors and Heins in theorem B iff $(-u) = v$ is a superharmonic function which is subject to the above mentioned decomposition. When applied to subharmonic functions u of the type considered by Ahlfors and Heins, M^{me} Lelong's result can be phrased as follows:

THEOREM B″. — *If* u *is subharmonic on* D *and satisfies the restrictions imposed in Theorem* B, *then*

$$\lim_{\mathcal{F}}\left(\frac{u(z)}{x}\right) = \alpha.$$

Now suppose that z is restricted to the Stolz domain $\mathcal{K} = \{z = re^{i\theta} : |\theta| \leqslant \theta_0 < \pi/2\}$. Then it follows that

$$\left|\frac{u}{r} - \alpha \cos \theta\right| \leqslant \left|\frac{u}{x} - \alpha\right| \leqslant \frac{1}{\cos \theta_0}\left|\frac{u}{r} - \alpha \cos \theta\right|.$$

If $\mathcal{F}/\mathcal{K}$ is the trace of the filter $\mathcal{F}$ on $\mathcal{K}$ and $F \in \mathcal{F}/\mathcal{K}$

then $\left\|\frac{u}{r} - \alpha \cos \theta\right\|_F \leqslant \left\|\frac{u}{x} - \alpha\right\|_F \leqslant \frac{1}{\cos \theta_0}\left\|\frac{u}{r} - \alpha \cos \theta\right\|_F$ where $\| \ \|_F$ means the sup norm on F. Hence $\lim\limits_{\mathcal{F}/\mathcal{K}} \left\|\frac{u}{r} - \alpha \cos \theta\right\| = 0$ iff $\lim\limits_{\mathcal{F}/\mathcal{K}} \left\|\frac{u}{x} - \alpha\right\| = 0$. This last equivalence of limits does not hold of course on the half plane itself.

Remark. — (see [6], p. 44). There exists $E \subset D$ where E is minimally thin at ∞ such that the limit along $\mathcal{F}$ of a given function f on D is in fact the ordinary limit (i.e. limit in Martin topology) of f as $z \to \infty$ on $D - E$.

Lemma 2. — *The « finite logarithmic length » character of theorem B of Ahlfors and Heins can be improved.*

Proof. — For any subharmonic function u restricted as in theorem B and for any given angular domain $\mathcal{K}$ it follows that $\lim\limits_{\mathcal{F}/\mathcal{K}} \left\|\frac{u(z)}{r} - \alpha \cos \theta\right\| = 0$. There exists an exceptional set X which must be a minimally thin set at ∞ and which is restricted to $\mathcal{K}$ such that $\lim\limits_{\mathcal{F}/\mathcal{K}} = \lim\limits_{\substack{z \to \infty \\ z \in \mathcal{K} - X}}$. The projected set $X' = \psi(X)$ must also be minimally thin at ∞ with respect to D. From theorem 5 and our remark at the end of § 3, such a set must be of finite logarithmic length but not conversely. The lemma follows.

One may regard the condition of « finite logarithmic length » to be a kind of first approximation to minimal thinness. On the other hand the condition of « finite logarithmic length » does take on greater significance when theorem B is generalized to the half plane itself. This is demonstrated by theorems 2 and 5 in a paper by Hayman [12].

5. Concluding Remarks.

It would appear to the author that the result contained in theorem 4 cannot fail to exert a fundamental influence on any future work that deals with a comparison of the internal fine topology and the minimal fine topology. One would expect that some of Brelot's theorems can now be improved (eg. [4], théorème 5 and théorème 5′) from « almost everywhere » implication to « everywhere » implication at least in many instances. This can certainly be done in his « particular case » ([4], p. 10) where the boundaries under consideration are sufficiently regular and can be identified, as in the case of any ball or half space in R^n. We note in conclusion that on any closed ball in $R^n (n \geqslant 2)$, then the minimal fine (Naïm) topology is strictly finer than the internal fine (Cartan-Brelot) topology on R^n relativized to the ball.

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