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NORMAL REAL AFFINE VARIETIES WITH CIRCLE ACTIONS

by Adrien DUBOULOZ & Alvaro LIENDO (*)

Dedicated to Mikhail Zaidenberg on his 70th birthday

Abstract. — We provide a complete description of normal affine algebraic varieties over the real numbers endowed with an effective action of the real circle, that is, the real form of the complex multiplicative group whose real locus consists of the unitary circle in the real plane. Our approach builds on the geometrico-combinatorial description of normal affine varieties with effective actions of split tori in terms of proper polyhedral divisors on semiprojective varieties due to Altmann and Hausen.

Résumé. — On donne une description complète des variétés algébriques affines normales sur le corps des réels munies d'une action effective du cercle réel, c'est-à-dire la forme réelle du groupe complexe multiplicatif dont le locus réel est constitué du cercle unité dans le plan réel. Notre approche repose sur une description géométrique et combinatoire des variétés normales affines avec des actions effectives de tores en termes de diviseurs propres polyédraux sur des variétés semiprojectives dues à Altmann et Hausen.

Introduction

Normal algebraic varieties X over a field k endowed with actions of split tori $\mathbb{T} = \mathbb{G}_{m,k}^n$ are quite well understood in terms of various geometrico-combinatorial presentations. The case where $\mathbb{T}$ acts faithfully on X and $\dim(\mathbb{T}) = \dim(X)$ is known as toric varieties and was first studied by Demazure in [4]. These varieties are fully described in combinatorial terms by means of suitable collections of convex polyhedral cones in the real vector

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space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ obtained from the lattice N of 1-parameter subgroups of $\mathbb{T}$. Successive further generalizations [1, 2, 5, 9] have led to complete descriptions of normal k-varieties endowed with $\mathbb{T}$-actions in terms of certain collections of so-called *polyhedral divisors*, which are Weil divisors $\mathcal{D}$ on suitable rational quotients for the action, whose coefficients are convex rational polyhedra in the vector space $N_{\mathbb{R}}$.

For normal algebraic k-varieties endowed with actions of non-split tori, that is, algebraic groups G defined over k whose base extensions to an algebraic closure $\overline{k}$ of k are isomorphic to split tori $\mathbb{G}_{m,\overline{k}}^n$ but which are not isomorphic over k to $\mathbb{G}_{m,k}^n$, much less is known regarding the existence of geometrico-combinatorial descriptions similar to the split case. Toric varieties with respect to non-split tori have been considered by several authors, see for instance [7, 8, 17]. In another direction, the geometrico-combinatorial presentation of Altmann–Hausen was partially extended by Langlois [13] to yield a description of affine varieties X endowed with an effective action of a quasi-split torus G of dimension $\dim(X)-1$. Nevertheless, the general case remains elusive. A natural and crucial step towards a geometrico-combinatorial description of such varieties would be to extend the Altmann–Hausen presentation in terms of polyhedral divisors [1] to arbitrary normal affine k-varieties X endowed with effective actions of tori G, split or not. Since every torus G splits after base change to a finite Galois extension K/k of k, this naturally leads to seek for such an extension in the form of a geometrico-combinatorial description of affine K-varieties X with effective actions of split tori $\mathbb{T}$ which are compatible with additional Galois descent data on X and $\mathbb{T}$ for the finite Galois cover $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$.

In this article we lend support to this approach by considering a simple case of independant geometric interest for which both the combinatorics and the Galois descent machinery are reduced to their minimum: normal real affine varieties with an effective action of the unit circle

$$\mathbb{S}^1 = \mathrm{Spec}(\mathbb{R}[x,y]/(x^2+y^2-1)),$$

the only non-split real form of $\mathbb{G}_{m,\mathbb{R}}$. In this context, a descent datum on a normal complex affine variety V for the Galois cover $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ boils down to anti-regular involution σ of V called a *real structure*.

Our main results, Theorem 2.7 and Theorem 2.11, give a description of $\mathbb{S}^1$-actions in the language of [1] extended to complex affine varieties with real structures. We chose to use the Altmann–Hausen formalism since it is particularly suited for a generalization to any non-necessarily split algebraic torus over a field of characteristic zero which we will tackle in a near future. Nevertheless, it is well-known that for a 1-dimensional split torus, polyhedral divisors in the sense of [1] correspond equivalently to data consisting of certain pairs of Weil $\mathbb{Q}$-divisors, a formalism first used by Dolgachev, Pinkham and Demazure and then extended by Flenner–Zaidenberg to describe $\mathbb{G}_{m,\mathbb{C}}$-actions on normal complex affine surfaces via $\mathbb{Q}$-divisors on their quotients (see the references in [9]). Corollary 2.16 provides a description of normal real affine varieties with an effective $\mathbb{S}^1$-action in this equivalent language, which can be summarized as follows:

Theorem. — *A normal real affine variety X endowed with an effective $\mathbb{S}^1$-action is uniquely determined by the following data:*

(1) *A normal real semiprojective variety Z corresponding to a normal complex semi-projective variety with real structure (Y,τ) representing the "real Altmann–Hausen quotient" of X by $\mathbb{S}^1$ (see Definition 2.1)*

(2) *A pair (D,h) consisting of a big and semiample $\mathbb{Q}$-Cartier divisor D on Y and a τ-invariant rational function h on Y satisfying $D+\tau^*D \leqslant \operatorname{div}(h)$.*

The contents of the article is as follows. In Section 1 we recall the classical equivalence of categories between quasi-projective real varieties and quasi-projective complex varieties equipped with a real structure. We establish in Lemma 1.4 the corresponding representation of quasi-projective real varieties with circle actions under this equivalence of categories. In Section 2 we establish the main classification results extending the description by Altmann and Hausen for split torus actions to the case of circle actions on normal real affine varieties. Finally, in Sections 3 and 4 we present several instances of applications of our techniques to examples taken from algebraic and differential geometry.

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1. Basic facts on real algebraic varieties and circle actions

In what follows, we identify the field $\mathbb{R}$ of real numbers with a subfield of $\mathbb{C}$ via the standard inclusion $j^* : \mathbb{R} \hookrightarrow \mathbb{C} = \mathbb{R}[i]/(i^2+1)$ so that the usual complex conjugation $J : \mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$ coincides with the homomorphism of $\mathbb{R}$-algebra defined by $i \mapsto -i$.

The term k-variety, where $\mathrm{k} = \mathbb{R}$ or $\mathbb{C}$, will refer to a geometrically integral scheme X of finite type over k. A morphism of k-varieties is a morphism of k-schemes.

1.1. Real quasi-projective varieties as complex varieties with real structures

Let us briefly recall the classical correspondence [3] between quasi-projective real algebraic varieties and quasi-projective complex algebraic varieties equipped with a real structure.

Every complex algebraic variety $p : V \to \mathrm{Spec}(\mathbb{C})$ can be viewed as an $\mathbb{R}$-scheme $j \circ p : V \to \mathrm{Spec}(\mathbb{R})$, and a *real structure* on such a variety V is an involution $\sigma : V \to V$ of $\mathbb{R}$-schemes such that $p \circ \sigma = J \circ p$, where J denotes the complex conjugation.

Every complex variety $X_{\mathbb{C}} = X \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C})$ obtained from a real algebraic variety X by the base change $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ is canonically endowed with a real structure $\sigma_X = \mathrm{id}_X \times J$ for which the morphism $\mathrm{pr}_1 : X_{\mathbb{C}} \to X$ coincides with the quotient $X_{\mathbb{C}}/\langle \sigma_X \rangle$. Conversely, if $p : V \to \mathrm{Spec}(\mathbb{C})$ is equipped with a real structure σ and covered by σ-invariant affine open subsets (so for instance if V is quasi-projective), then the quotient $\pi : V \to V/\langle \sigma \rangle$ exists in the category of schemes and the structure morphism $p : V \to \mathrm{Spec}(\mathbb{C})$ descends to a morphism $V/\langle \sigma \rangle \to \mathrm{Spec}(\mathbb{R}) = \mathrm{Spec}(\mathbb{C})/\langle \tau \rangle$ making $V/\langle \sigma \rangle$ into a real algebraic variety X such that $V \simeq X_{\mathbb{C}}$. This correspondence extends to a well-known equivalence of categories which can be summarized as follows (see e.g. [11, Exposé VIII]).

Lemma 1.1. — *The category of quasi-projective real algebraic varieties is equivalent to the category $\mathcal{C}$ whose objects are pairs (V, σ) consisting of a quasi-projective complex algebraic variety V and a real structure $\sigma : V \to V$ and whose morphisms $(V, \sigma) \to (V', \sigma')$ are morphisms of complex algebraic varieties $f : V \to V'$ such that $\sigma' \circ f = f \circ \sigma$.*

In particular, two real structures σ and σ' on the same quasi-projective complex variety V define isomorphic real algebraic varieties $V/\langle \sigma \rangle$ and $V/\langle \sigma' \rangle$ if and only if there exists an isomorphism of complex algebraic varieties $f : V \to V$ such that $\sigma' \circ f = f \circ \sigma$.

In the sequel, we will represent a quasi-projective real variety X by a pair (V, σ) where V is a quasi-projective complex variety and σ is a real structure on V such that $V/\langle\sigma\rangle$ is isomorphic to X. Similarly, we will represent a morphism (resp. a rational map) $f : X \to X'$ between real varieties represented by pairs (V,σ) and (V',σ') respectively by a morphism (resp. a rational map) $\widetilde{f} : V \to V'$ such that $\sigma' \circ \widetilde{f} = \widetilde{f} \circ \sigma$. We sometimes abbreviate this condition by saying that $\widetilde{f}$ is a *real morphism* (resp *real rational map*).

Definition 1.2. — *A* real form *of a real algebraic variety $X = (V,\sigma)$ is a real algebraic variety $X' = (V',\sigma')$ such that V and V' are isomorphic as complex varieties. Isomorphism classes of real forms of X are classified by the Galois cohomology group $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}_{\mathbb{C}}(V))$ where the non-trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mu_2$ acts on $\mathrm{Aut}_{\mathbb{C}}(V)$ by conjugation $f \mapsto \sigma f \sigma^{-1}$.*

Recall that an algebraic variety V is said to be *semi-projective* if its coordinate ring $\Gamma(V,\mathcal{O}_V)$ is finitely generated and the canonical morphism $V \to \mathrm{Spec}(\Gamma(V,\mathcal{O}_V))$ is projective. When X is a real algebraic variety represented by a pair (V,σ), we denote by $\Gamma(\sigma)$ the unique real structure on $\mathrm{Spec}(\Gamma(V,\mathcal{O}_V))$ for which the canonical morphism given by $(V,\sigma) \to (\mathrm{Spec}(\Gamma(V,\mathcal{O}_V),\Gamma(\sigma))$ is a real morphism.

1.2. Circle actions on quasi-projective real varieties

Definition 1.3. — *The* real circle *is the only non-split real form*

$$\mathbb{S}^1 = \mathrm{Spec}(\mathbb{R}[x,y]/(x^2+y^2-1))$$

of the multiplicative group $\mathbb{G}_{m,\mathbb{R}}$. The group structure on $\mathbb{S}^1$ is given by

$$(x,y)\cdot(x',y') = (xx'-yy', xy'+yx'),$$

and the morphism of group schemes

$$\begin{aligned} \rho_0 : \mathbb{S}^1 &\longrightarrow \mathrm{SL}_{2,\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[a_{11},a_{12},a_{21},a_{22}]/(a_{11}a_{22}-a_{12}a_{21}-1)) \\ (x,y) &\longmapsto (x,y,-y,x) \end{aligned} \tag{1.1}$$

induces an isomorphism between $\mathbb{S}^1$ and the closed subgroup $\mathrm{SO}_{2,\mathbb{R}}$ defined by the equation $a_{11}-a_{22} = a_{12}+a_{21} = 0$.

It is straighforward to check that the map

$$\begin{aligned} \varphi : \mathbb{G}_{m,\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[t^{\pm 1}]) &\longrightarrow \mathbb{S}^1_{\mathbb{C}}, \\ t &\longmapsto (x,y) = ((t+t^{-1})/2, (t-t^{-1})/2i) \end{aligned} \tag{1.2}$$

is an isomorphism of complex group schemes. The pull-back of the canonical real structure $\sigma_{\mathbb{S}^1}$ on $\mathbb{S}^1_{\mathbb{C}}$ by φ is the real structure ρ on $\mathbb{G}_{m,\mathbb{C}}$ defined as the composition of the involution $t \mapsto t^{-1}$, induced by the involution $-\mathrm{id}_M : m \mapsto -m$ of the character lattice $M \simeq \mathbb{Z}$ of $\mathbb{G}_{m,\mathbb{C}}$, with the complex conjugation. We henceforth identify the group object $\mathbb{S}^1$ in the category of real algebraic varieties with the pair $(\mathbb{G}_{m,\mathbb{C}}, \rho)$.

Lemma 1.4. — *There is a one-to-one correspondence between quasi-projective real algebraic varieties endowed with an effective $\mathbb{S}^1$-action and triples (V, σ, μ) consisting of a quasi-projective real algebraic variety $X = (V, \sigma)$ and an effective $\mathbb{G}_{m,\mathbb{C}}$-action $\mu : \mathbb{G}_{m,\mathbb{C}} \times V \to V$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathbb{G}_{m,\mathbb{C}} \times V \arrow[r, "\mu"] \arrow[d, "\rho\times\sigma"'] & V \arrow[d, "\sigma"] \\ \mathbb{G}_{m,\mathbb{C}} \times V \arrow[r, "\mu"] & V. \end{tikzcd} \tag{1.3}$$

Proof. — An $\mathbb{S}^1$-action on $X = (V, \sigma)$ corresponds by definition to an effective $\mathbb{S}^1_{\mathbb{C}}$-action η on V such that $\sigma \circ \eta = \eta \circ (\sigma_{\mathbb{S}^1} \times \sigma)$, hence, composing with the real isomorphism φ of (1.2), to an effective $\mathbb{G}_{m,\mathbb{C}}$-action $\mu = \eta \circ (\varphi \times \mathrm{id}_V)$ with the announced property. □

Convention. — In the rest of the article, we will represent a quasi-projective real variety X endowed with an effective $\mathbb{S}^1$-action by one of the following equivalent data:

(1) a quasi-projective real algebraic variety X and a morphism of real algebraic varieties $\mathbb{S}^1 \times X \to X$ defining an effective $\mathbb{S}^1$-action; or
(2) a triple (V, σ, μ) consisting of a quasi-projective complex algebraic variety V endowed with a real structure σ and a morphism of complex algebraic varieties $\mu : \mathbb{G}_{m,\mathbb{C}} \times V \to V$ defining an effective $\mathbb{G}_{m,\mathbb{C}}$-action on V such that $\mu \circ (\rho \times \sigma) = \sigma \circ \mu$.

By an $\mathbb{S}^1$-*equivariant morphism* between quasi-projective real $\mathbb{S}^1$-varieties (V, σ, μ) and (V', σ', μ'), we mean a strictly $\mathbb{S}^1$-equivariant one, that is, a real morphism $f : V \to V'$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{G}_{m,\mathbb{C}} \times V \arrow[r, "\mu"] \arrow[d, "\mathrm{id}_{\mathbb{G}_{m,\mathbb{C}}}\times f"'] & V \arrow[d, "f"] \\ \mathbb{G}_{m,\mathbb{C}} \times V' \arrow[r, "\mu'"] & V'. \end{tikzcd}$$

Definition 1.5. — *A real form of a quasi-projective real $\mathbb{S}^1$-variety $X = (V, \sigma, \mu)$ is a quasi-projective real $\mathbb{S}^1$-variety $X' = (V', \sigma', \mu')$ such that V is $\mathbb{G}_{m,\mathbb{C}}$-equivariantly isomorphic to V'.*

1.3. The case of real affine varieties

Specializing further to the case where $X = (V, \sigma)$ is a real affine algebraic variety, say $V = \operatorname{Spec}(A)$ for some finitely generated integral $\mathbb{C}$-algebra A, the $\mathbb{G}_{m,\mathbb{C}}$-action μ on V is equivalently determined by its co-morphism $\mu^* : A \to A \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$. Recall that a semi-invariant regular function of weight $m \in \mathbb{Z}$ on V for the action μ is an element $f \in A$ such that $\mu^* f = f \otimes t^m$, and that A is then $\mathbb{Z}$-graded in a natural way by its sub-spaces A_m of semi-invariants of weight m for all $m \in \mathbb{Z}$. The action μ is said to be *hyperbolic* if there exists $m < 0$ and $m' > 0$ such that A_m and $A_{m'}$ are non-zero.

Lemma 1.6. — *Let $X = (V = \operatorname{Spec}(A), \sigma, \mu)$ be a real affine algebraic variety endowed with an effective $\mathbb{S}^1$-action and let $A = \bigoplus_{m \in \mathbb{Z}} A_m$ be the decomposition of A into semi-invariants sub-spaces for the $\mathbb{G}_{m,\mathbb{C}}$-action μ. Then the following hold:*

(i) *The action μ is hyperbolic and $A_m \neq 0$ for all $m \in \mathbb{Z}$*
(ii) *For all $m \in \mathbb{Z}$, $\sigma^*(A_m) = A_{-m}$,*
(iii) *The restriction of σ^* to $A_0 = A^{\mathbb{G}_{m,\mathbb{C}}}$ is the co-morphism of a real structure $\overline{\sigma}$ on the algebraic quotient $V /\!/ \mathbb{G}_{m,\mathbb{C}} = \operatorname{Spec}(A_0)$ of V.*

Proof. — The commutativity of the diagram (1.3) implies that for every semi-invariant f of weight $m \in \mathbb{Z}$, we have

$$\mu^* \sigma^*(f) = (\sigma^* \otimes \rho^*)(f \otimes t^m) = \sigma^*(f) \otimes t^{-m},$$

hence that $\sigma^*(f)$ is a semi-invariant of weight $-m$. The equality $\sigma^*(A_m) = A_{-m}$ follows from the fact that σ^* is an automorphism of A, which proves the second assertion. Since μ is non trivial, there exists a semi-invariant function f of non-zero weight m, and hence a semi-invariant function $\sigma^*(f)$ of non-zero weight $-m$. This shows that μ is hyperbolic, and the second part of the first assertion is a standard fact for such actions. Indeed, since the action is effective, the set $\{m \in \mathbb{Z} \mid A_m \neq \{0\}\}$ is not contained in any proper sublattice $d \cdot \mathbb{Z}$, $d > 1$. Hence, there exists $e < 0$ and $e' > 0$ relatively prime such that A_e and $A_{e'}$ are non-zero. Let f and g be non-zero elements in A_e and $A_{e'}$, respectively. Now, for every integer $m \in \mathbb{Z}$, there exist integers $a < 0$ and $b > 0$ such that $ae - be' = m$. Then $f^a g^b \in A_m$ is a non-zero element, as desired.

The last assertion is straightforward. □

With the notation above, it follows from Lemma 1.6 (iii) that $X//\mathbb{S}^1 = (V//\mathbb{G}_{m,\mathbb{C}}, \overline{\sigma})$ is a real affine algebraic variety and that the morphism

$$\pi : X = (V, \sigma) \to X//\mathbb{S}^1 = (V//\mathbb{G}_{m,\mathbb{C}}, \overline{\sigma})$$

induced by the inclusion $A_0 \hookrightarrow A$ is an $\mathbb{S}^1$-invariant morphism of real algebraic varieties, which is a categorical quotient in the category of real affine algebraic varieties.

2. Altmann–Hausen presentation of a circle action

The aim of this section is to establish a counter-part for real affine varieties with circle actions of the geometrico-combinatorial presentation of affine varieties with split tori actions developed by Altmann and Hausen in [1].

We first need to introduce a special kind of rational quotient for a real affine algebraic variety endowed with an effective $\mathbb{S}^1$-action which is the real counterpart of the quotient constructed in [1] for normal affine varieties with split tori actions. Keeping the same notation as in Section 1.3 above, we let $d > 0$ be minimal such that $\bigoplus_{m\in\mathbb{Z}} A_{dm}$ is generated by $A_{\pm d}$ as a graded A_0-algebra. By virtue of Lemma 1.6 (ii) and (iii), the closed subscheme W of $Y_0(V) = V//\mathbb{G}_{m,\mathbb{C}} = \mathrm{Spec}(A_0)$ with defining ideal $I = \langle A_d \cdot A_{-d}\rangle \subset A_0$ is $\overline{\sigma}$-invariant.

Definition 2.1. — *The* real AH-quotient *of $X = (V, \sigma, \mu)$ by the $\mathbb{S}^1$-action μ is the real quasi-projective variety formed by total space of the blow-up $\pi : Y(V) \to Y_0(V)$ of $Y_0(V)$ with center at W, endowed with the lift τ of the real structure $\overline{\sigma}$.*

The complex variety $Y(V)$ is semi-projective, and by virtue of [16, Theorem 1.9] (see also [15]), it is isomorphic to the irreducible component of the fiber product

$$\mathrm{Proj}\Big(\bigoplus_{m\leqslant 0} A_m\Big) \times_{Y_0(V)} \mathrm{Proj}\Big(\bigoplus_{m\geqslant 0} A_m\Big)$$

which dominates $Y_0(V)$. So $Y(V)$ coincides with the AH-quotient of V for the $\mathbb{G}_{m,\mathbb{C}}$-action μ in the sense of [1]. By construction, $\pi : (Y(V), \tau) \to (Y_0(V), \overline{\sigma})$ is a morphism of real algebraic varieties.

2.1. Proper hyperbolic segmental divisors

Recall that a Weil $\mathbb{Q}$-divisor D on a normal algebraic variety Y is called $\mathbb{Q}$-Cartier if nD is Cartier for some $n \geqslant 1$. Furthermore, D is semiample if there exits $n \geqslant 1$ such that the linear system $|nD|$ is base point free, equivalently such that the sheaf $\mathcal{O}_Y(nD)$ is invertible and globally generated. The divisor D is called big if there exists $E \in |nD|$ with affine complement for some $n \geqslant 1$. In the sequel, all our divisors will be $\mathbb{Q}$-Cartier Weil $\mathbb{Q}$-divisors. We will refer to such divisors simply as $\mathbb{Q}$-Cartier divisors.

Notation 2.2. — Given a $\mathbb{Q}$-Cartier divisor D on a normal variety Y, we denote the round-down of D by $\lfloor D \rfloor$. We identify the $\Gamma(Y, \mathcal{O}_Y)$-module $\Gamma(Y, \mathcal{O}_Y(\lfloor D \rfloor))$ with the sub-$\Gamma(Y, \mathcal{O}_Y)$-module of the field of rational functions $\mathrm{Frac}(Y)$ of Y generated by rational functions $g \in \mathrm{Frac}(Y)$ such that $\mathrm{div}(g) + \lfloor D \rfloor \geqslant 0$. Under this identification, a rational function $g \in \mathrm{Frac}(Y)$ satisfies $\mathrm{div}(g) + \lfloor D \rfloor \geqslant 0$ if and only if $\mathrm{div}(g) + D \geqslant 0$. So we can set without ambiguity $\Gamma(Y, \mathcal{O}_Y(\lfloor D \rfloor)) := \Gamma(Y, \mathcal{O}_Y(D))$ and remove the round-down brackets from the notation.

We now review a simple special case adapted to our context of the general notion of polyhedral divisor defined in [1]. Let $N \simeq \mathbb{Z}$ be a rank one lattice and let M be its dual. Let $\mathcal{J}$ be the set of all closed intervals $[a, b]$ of $N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}$ with rational bounds, where we admit singleton intervals with $a = b$ as $[a, a] = \{a\}$. The set $\mathcal{J}$ has the structure of an abelian semi-group for the Minkowski sum $[a, b] + [a', b'] = [a + a', b + b']$, with identity $[0, 0] = \{0\}$. Every element $m \in M$ determines a semi-group homomorphism

$$\mathrm{ev}_m : \mathcal{J} \to \mathbb{Q}, \quad [a, b] \mapsto \min(ma, mb) = \begin{cases} ma & \text{if } m \geqslant 0 \\ mb & \text{if } m < 0. \end{cases}$$

For a normal variety Y, we denote by $\mathcal{J} \otimes \mathrm{WDiv}(Y)$ the semi-group of formal finite sums with coefficients in $\mathcal{J}$ of prime Weil divisors on Y.

Definition 2.3. — *A* segmental divisor [15] *on a normal algebraic variety Y is an element*

$$\mathcal{D} = \sum [a_i, b_i] \otimes D_i \in \mathcal{J} \otimes \mathrm{WDiv}(Y)$$

A segmental divisor $\mathcal{D} = \sum [a_i, b_i] \otimes D_i$ is called proper *if for every $m \in \mathbb{Z}$, the Weil $\mathbb{Q}$-divisor*

$$\mathcal{D}(m) := (\mathrm{ev}_m \otimes \mathrm{id})(\mathcal{D}) = \sum \min(ma_i, mb_i) D_i$$

is a big, semiample $\mathbb{Q}$-Cartier divisor on Y.

Every Weil divisor D on Y determines a segmental divisor $\{1\} \otimes D$, in particular every non-zero rational function f on Y determines a *principal segmental divisor* $\{1\} \otimes \operatorname{div}(f)$. Note in addition that the definition of the evaluation homomorphisms ev_m guarantees that for every segmental divisor $\mathcal{D}$ and every pair of integers $m, n \in \mathbb{Z}$ the Weil $\mathbb{Q}$-divisors $\mathcal{D}(m) + \mathcal{D}(n) - \mathcal{D}(m+n)$ are all anti-effective. In particular, $\mathcal{D}(m) + \mathcal{D}(-m) \leqslant \mathcal{D}(0) = 0$ for all $m \in \mathbb{Z}$.

Definition 2.4 ([1, Definition 8.3]). — *Let $\psi : Y' \dashrightarrow Y$ be a rational map between normal algebraic varieties and let $\mathcal{D} = \sum [a_i, b_i] \otimes D_i$ be a segmental divisor on Y such that none of the supports* $\operatorname{Supp}(D_i)$ *contains the image of Y'. The* pull-back of $\mathcal{D}$ by ψ *is the segmental divisor $\psi^*\mathcal{D} := \sum [a_i, b_i] \otimes \psi^* D_i$ on Y' where for every i, $\psi^* D_i$ is the pull-back on Y' of the Weil divisor D_i on Y by ψ.*

Recall that the real structure ρ on $\mathbb{G}_{m,\mathbb{C}}$ is given as the composition of the automorphism induced by the involution $-\mathrm{id}_M$ of its character lattice $M \simeq \mathbb{Z}$ with the complex conjugation. The dual involution $-\mathrm{id}_N = (-\mathrm{id}_M)^*$ of the lattice $N \simeq \mathbb{Z}$ of 1-parameter subgroups of $\mathbb{G}_{m,\mathbb{C}}$ induces an involution of $\mathcal{J}$. Moreover, when Z is a normal real algebraic variety represented by a complex variety Y with real structure τ, the pull-back of Weil $\mathbb{Q}$-divisors on Y by the real structure τ induces an involution τ^* on $\mathbb{Q}$-divisors on Y. Putting these two involutions together, we obtain an involution

$$\begin{aligned}(-\mathrm{id}_M)^* \otimes \tau^* : \mathcal{J} \otimes \mathrm{WDiv}(Y) &\longrightarrow \mathcal{J} \otimes \mathrm{WDiv}(Y) \\ \mathcal{D} = \sum [a_i, b_i] \otimes D_i &\longmapsto \sum [-b_i, -a_i] \otimes \tau^* D_i.\end{aligned}$$

Definition 2.5. — *A* proper hyperbolic segmental pair (phs-pair) *on a normal real algebraic variety $Z = (Y, \tau)$ is a pair $(\mathcal{D}, h)$ consisting of a proper segmental divisor $\mathcal{D} = \sum [a_i, b_i] \otimes D_i$ and a τ-invariant rational function h on Y such that*

$$((-\mathrm{id}_M)^* \otimes \tau^*)(\mathcal{D}) = \mathcal{D} + \{1\} \otimes \operatorname{div}(h). \tag{2.1}$$

It will be convenient in practice to separate the homomorphism coming from the real structure on Y from that coming from the real structure on $\mathbb{G}_{m,\mathbb{C}}$. So, up to changing h for h^{-1}, we can rewrite (2.1) *equivalently as*

$$\tau^*(\mathcal{D}) = \widehat{\mathcal{D}} + \{1\} \otimes \operatorname{div}(h),$$

where $\widehat{\mathcal{D}} = ((-\mathrm{id}_M)^* \otimes \mathrm{id})(\mathcal{D}) = \sum [-b_i, -a_i] \otimes D_i$.

Lemma 2.6. — *Let $(\mathcal{D}, h)$ be a phs-pair on a normal semi-projective real algebraic variety $Z = (Y, \tau)$. Then the following hold:*

(i) *The sheaf* $\mathcal{O}_Y(\mathcal{D}(m))$ *has a nonzero global section for all* $m \in \mathbb{Z}$.

(ii) *The real structure* τ *induces an isomorphism of* $\Gamma(Y, \mathcal{O}_Y)$*-modules*

$$\begin{aligned} \tau_m^* : \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(m))) &\xrightarrow{\simeq} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(-m))), \\ g &\longmapsto h^m \cdot \tau^* g, \ \text{for all } m \in \mathbb{Z}. \end{aligned} \tag{2.2}$$

Furthermore, $\tau_0^* = \Gamma(\tau)^*$ *and* $\tau_{-m}^* \circ \tau_m^* = \mathrm{id}$.

Proof. — The first assertion, in analogy with Lemma 1.6 (i), is again a standard fact for proper segmental divisors. Since $\mathcal{D}(\pm 1)$ is big, there exist relatively prime positive integers e and e' and non-zero global sections f and g of $\mathcal{O}_Y(\mathcal{D}(e))$ and $\mathcal{O}_Y(\mathcal{D}(-e'))$, respectively. Now, for every integer $m \in \mathbb{Z}$, there exists integers $a < 0$ and $b > 0$ such that $ae - be' = m$. Then $f^a g^b$ is non-zero global section of $\mathcal{D}(ae - be') = \mathcal{D}(m)$ as desired.

For the second assertion, the fact that $\tau_0^* = \Gamma(\tau)^*$ follows from the definition of the real structure $\Gamma(\tau)$ on $\mathrm{Spec}(\Gamma(Y, \mathcal{O}_Y)$ (see Section 2). Given $m \in \mathbb{Z} \setminus \{0\}$, it follows from the definition of a phs-pair that $\tau^*(\mathcal{D})(m) = \mathcal{D}(-m) + m\,\mathrm{div}(h)$. This implies that the homomorphism τ_m^* in (2.2) is well-defined. Since τ^* is an involution, $\tau_{-m}^* \circ \tau_m^*$ is the identity of $\Gamma(Y, \mathcal{O}_Y(\mathcal{D}(m)))$, and so τ_m^* is an isomorphism with inverse τ_{-m}^*. □

2.2. Real Altmann–Hausen presentations

In this section, we state and prove our main theorem giving a geometrico-combinatorial presentation of normal real affine varieties endowed with a circle action.

Let Y be a normal semi-projective complex variety and let $\mathcal{D}$ be a proper segmental divisor on Y. Then it follows from [1, Theorem 3.1] that the $\mathbb{C}$-scheme

$$V = V(Y, \mathcal{D}) := \mathrm{Spec}\left(\bigoplus_{m \in \mathbb{Z}} \Gamma\left(Y, \mathcal{O}_Y(\mathcal{D}(m))\right)\right)$$

is a normal complex affine variety of dimension $\dim Y + 1$. Furthermore, the $\mathbb{Z}$-grading of the coordinate ring of V uniquely determines an effective $\mathbb{G}_{m,\mathbb{C}}$-action $\mu : \mathbb{G}_{m,\mathbb{C}} \times V \to V$ with algebraic quotient isomorphic to $\mathrm{Spec}(\Gamma(Y, \mathcal{O}_Y))$ and whose AH-quotient is birationally dominated by Y.

Theorem 2.7. — *Let* $Z = (Y, \tau)$ *be a normal semi-projective real algebraic variety, represented by a complex variety* Y *with real structure* τ, *and let* $(\mathcal{D}, h)$ *be a phs-pair on* Z. *Then the following hold:*

(1) *The normal complex affine variety* $V = V(Y, \mathcal{D})$ *carries a real structure* σ *such that* $\sigma \circ \mu = \mu \circ (\rho \times \sigma)$.

(2) *The triple* (V, σ, μ) *is a normal real affine algebraic variety* $X(Z, (\mathcal{D}, h))$ *of dimension* $\dim Z + 1$ *endowed with an effective* $\mathbb{S}^1$*-action with algebraic quotient* $(\mathrm{Spec}(\Gamma(Y, \mathcal{O}_Y), \Gamma(\tau))$ *and real AH-quotient birationally dominated to* Z.

(3) *Conversely, every normal real affine variety* $X = (V, \sigma, \mu)$ *endowed with an effective* $\mathbb{S}^1$*-action is equivariantly isomorphic to* $X(Z, (\mathcal{D}, h))$ *for a suitable phs-pair* $(\mathcal{D}, h)$ *on its real AH-quotient* $Z = (Y(V), \tau)$.

Proof. — Since $(\mathcal{D}, h)$ is a phs-pair, it follows from Lemma 2.6 (ii) that there exist isomorphisms $\tau_m^* : \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(m))) \xrightarrow{\simeq} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(-m)))$ for every $m \in \mathbb{Z}$. These collect into an involution $\sigma^* = \bigoplus_{m \in \mathbb{Z}} \tau_m^*$ on the direct sum $A = \bigoplus_{m \in \mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(m)))$. The latter corresponds to a real structure σ on V such that by construction $\sigma \circ \mu = \mu \circ (\rho \times \sigma)$. It then follows from Lemma 1.6 that (V, σ, μ) represents a normal real affine variety X endowed with an effective $\mathbb{S}^1$-action. The facts that $X/\!/\mathbb{S}^1 \simeq \mathrm{Spec}(\Gamma(Y, \mathcal{O}_Y), \Gamma(\tau))$ and that the real AH-quotient of X is birationnally dominated by Z follow from the corresponding assertions for V and the construction of σ. This proves (1) and (2).

For the converse (3), let $A = \bigoplus_{m \in \mathbb{Z}} A_m$ be the decomposition of the coordinate ring of V into semi-invariants sub-spaces for the $\mathbb{G}_{m,\mathbb{C}}$-action μ and let $(Y_0 = \mathrm{Spec}(A_0), \overline{\sigma})$ and (Y, τ) be the algebraic quotient and the real AH-quotient of X respectively. By construction, Y is birational to Y_0, and we can therefore identify its field of rational functions with the field of fractions $\mathrm{Frac}(A_0)$ of A_0. By Lemma 1.6 (i), μ admits a semi-invariant regular function s of weight 1. Then [1, Theorem 3.4] guarantees the existence of a proper segmental divisor $\mathcal{D}$ on Y such that for every $m \in \mathbb{Z}$ the sub-A_0-module $s^{-m} A_m$ of $\mathrm{Frac}(A_0)$ is equal to $\Gamma(Y, \mathcal{O}_Y(\mathcal{D}(m)))$ and such that A is equal to $\bigoplus_{m \in \mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(m)) \cdot s^m$ as a graded sub-A_0-algebra of $\mathrm{Frac}(A_0)(s)$.

We will now check that $h = s\sigma^*(s)$ is a τ-invariant rational function on Y making $(\mathcal{D}, h)$ a phs-pair. Note that by construction, h is a σ-invariant rational function on V. Since by Lemma 1.6 (ii), $\sigma^*(s)$ is a semi-invariant regular function of weight -1, it follows that h is a also $\mathbb{G}_{m,\mathbb{C}}$-invariant, hence an element of $\mathrm{Frac}(A_0)$. Since σ^* coincides by definition with $\overline{\sigma}$ on A_0 and τ is lifted from $\overline{\sigma}$, we can thus view h as a τ-invariant rational function on Y. Let $m \geqslant 1$ be such that $\mathcal{D}(m)$ and $\mathcal{D}(-m)$ are Cartier and globally generated, and let $\{U_i\}_{i \in I}$ be an open cover of Y such that $\mathcal{D}(m)$ and $\mathcal{D}(-m)$ are principal in every U_i. Then, by the construction of $\mathcal{D}$ in [1, Theorem 3.4], for every $i \in I$ there exists a rational function $g \in \mathrm{Frac}(A_0)$ such that $gs^m \in A_m$ and $\mathrm{div}(g)|_{U_i} + \mathcal{D}(m)|_{U_i} = 0$. In particular, we have $\mathrm{div}(\tau^* g)|_{\tau^{-1}(U_i)} =$

$-\tau^*\mathcal{D}(m)|_{\tau^{-1}(U_i)}$. Since $\sigma^*(gs^m) = (\tau^*g)h^m s^{-m} \in A_{-m}$, it follows from the construction of $\mathcal{D}$ that $(\tau^*g)h^m \in \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(-m)))$ and hence, we have

$$\operatorname{div}(\tau^*g) + m\operatorname{div}(h) + \mathcal{D}(-m) \geqslant 0.$$

Substituting $\operatorname{div}(\tau^*g)|_{\tau^{-1}(U_i)} = -\tau^*\mathcal{D}(m)|_{\tau^{-1}(U_i)}$, we conclude that

$$(-\tau^*\mathcal{D}(m) + m\operatorname{div}(h) + \mathcal{D}(-m))|_{\tau^{-1}(U_i)} \geqslant 0.$$

Since $\{\tau^{-1}(U_i)\}_{i\in I}$ is also an open cover of Y, this inequality is independant on the open subset U_i, and we obtain

$$-\tau^*\mathcal{D}(m) + m\operatorname{div}(h) + \mathcal{D}(-m) \geqslant 0 \tag{2.3}$$

for every $m \in \mathbb{Z}$ since all the terms in (2.3) are linear on m. Taking now a rational function $g \in \operatorname{Frac}(A_0)$ such that $gs^{-m} \in A_{-m}$ and $\mathcal{D}(-m)|_{U_i} = -\operatorname{div}(g)|_{U_i}$, we find by the same argument that $-\tau^*\mathcal{D}(-m) - m\operatorname{div}(h) + \mathcal{D}(m) \geqslant 0$, hence applying the involution τ^* to this inequality that

$$-\tau^*\mathcal{D}(m) + m\operatorname{div}(h) + \mathcal{D}(-m) \leqslant 0 \tag{2.4}$$

for every $m \in \mathbb{Z}$. The inequalities (2.3) and (2.4) together yield that $\tau^*(\mathcal{D})(m) = \mathcal{D}(-m) + m\operatorname{div}(h)$ for every $m \in \mathbb{Z}$, hence that $\tau^*\mathcal{D} = \widehat{\mathcal{D}} + \{1\} \otimes \operatorname{div}(h)$. □

Remark 2.8. — Given a normal real affine variety $X = (V, \sigma, \mu)$ endowed with an effective $\mathbb{S}^1$-action, the construction of a proper segmental divisor $\mathcal{D}$ on the AH-quotient $Y(V)$ such that V is $\mathbb{G}_{m,\mathbb{C}}$-equivariantly isomorphic to $V = V(Y, \mathcal{D})$ depends on the non-canonical choice of a semi-invariant rational function s of weight 1 on V. Different choices for s lead of course to different segmental divisors $\mathcal{D}_s$ on $Y(V)$. In the proof of Theorem 2.7, we made a particular choice for s, but the proof actually shows that for every other choice of s, the rational function $h_s = s\sigma^*(s)$ on $Z = (Y(V), \tau))$ is τ-invariant and $(\mathcal{D}_s, h_s)$ is a phs-pair on Z.

Definition 2.9. — *A couple consisting of a real normal semiprojective variety $Z = (Y, \tau)$ and a phs-pair $(\mathcal{D}, h)$ on it is called* minimal *if Z is the real AH-quotient of $X(Z, (\mathcal{D}, h))$.*

It follows from the definition of the real AH-quotient of $X(Z, (\mathcal{D}, h))$ that a couple $(Z, (\mathcal{D}, h))$ is minimal if and only if Y is the AH-quotient of $V(Y, \mathcal{D})$. This definition is thus equivalent to requiring that the couple $(Y, \mathcal{D})$ is minimal in the sense of [1, Definition 8.7].

Corollary 2.10. — *Let $X = (V, \sigma, \mu)$ be a normal real affine variety endowed with an effective $\mathbb{S}^1$-action and let $\mathcal{D}$ be any proper segmental* *divisor on a complex semiprojective variety Y such that there exists a $\mathbb{G}_{m,\mathbb{C}}$-equivariant isomorphism $\phi : V \overset{\simeq}{\to} V(Y,\mathcal{D})$.*

If the real structure $\overline{\sigma}$ on the algebraic quotient $V/\!/\mathbb{G}_{m,\mathbb{C}} \simeq V(Y,\mathcal{D})/\!/\mathbb{G}_{m,\mathbb{C}}$ induced by σ lifts to a real structure τ on Y, which holds for instance if $(Y,\mathcal{D})$ is minimal, then there exists a τ-invariant rational function h on $Z=(Y,\tau)$ making $(Z,(\mathcal{D},h))$ a minimal phs-pair such that $X(Z,(\mathcal{D},h))$ is $\mathbb{S}^1$-equivariantly isomorphic to X.

Proof. — This follows from the same argument as in the proof of Theorem 2.7 (3), taking Y as in the corollary instead of the AH-quotient $Y(V)$ of V. □

Theorem 2.11. — *For $i=1,2$, let $Z_i=(Y_i,\tau_i)$ be normal real semiprojective varieties and let $X_i = X(Z_i,(\mathcal{D}_i,h_i))$ be normal real affine varieties with effective $\mathbb{S}^1$-actions determined by respective phs-pairs $(\mathcal{D}_i,h_i)$ on Z_i.*

Then X_1 and X_2 are $\mathbb{S}^1$-equivariantly isomorphic if and only if there exits a normal real semiprojective variety $Z=(Y,\tau)$, a phs-pair $(\mathcal{D},h)$ on Z, real birational morphisms $\psi_i : Z_i \to Z$, and rational functions f_i on Y_i, $i=1,2$, such that

$$\psi_i^*(\mathcal{D}) = \mathcal{D}_i + \{1\}\otimes \operatorname{div}(f_i) \quad \textit{and} \quad \psi_i^*(h) = (f_i\cdot \tau_i^* f_i)\cdot h_i.$$

Proof. — Recall that by definition, $X_i=(V_i,\sigma_i,\mu_i)$ where $V_i = V(Y_i,\mathcal{D}_i)$ and σ_i is the real structure on V_i constructed in Theorem 2.7 (1). Assume first that X_1 and X_2 are $\mathbb{S}^1$-equivariantly isomorphic, let $Z=(Y,\tau)$ be the AH-quotient of V_1 endowed with the real structure induced by σ_1. By [1, Definition 8.7], there exists a proper segmental divisor $\mathcal{D}$ on Y such that $V=V(Y,\mathcal{D})$ is $\mathbb{G}_{m,\mathbb{C}}$-equivariantly isomorphic to V_1. It then follows from Corollary 2.10 that there exists a τ-invariant function h on Y such that $(\mathcal{D},h)$ is phs-pair and such that $X(Z,(\mathcal{D},h)) = (V,\sigma,\mu)$ is $\mathbb{S}^1$-equivariantly isomorphic to X_1. By [1, Theorem 8.8], there exist birational morphisms $\psi_i : Y_i \to Y$ such that $\psi_i^*(\mathcal{D}) = \mathcal{D}_i + \{1\}\otimes\operatorname{div}(f_i)$ for some rational functions f_i on Y_i. Indeed, the existence of such birational morphisms is obtained in the notation of [1, Theorem 8.8] by taking $\mathfrak{D}' = (Z,(\mathcal{D},h))$, $\mathfrak{D} = (Z_i,(\mathcal{D}_i,h_i))$ and $\kappa = \mathrm{id}$.

Furthermore, the real structures τ_i on Y_i and τ on Y are all lifts of the real structure $\overline{\sigma}_1$ on the algebraic quotient $V_1/\!/\mathbb{G}_{m,\mathbb{C}}$, which is birational to Y_i and Y since the $\mathbb{G}_{m,\mathbb{C}}$-actions considered are all hyperbolic by Lemma 1.6 (i). Hence, we have $\tau\circ\psi_i = \psi_i\circ\tau_i$ which shows that ψ_i is a real birational morphism. By [1, Proposition 8.6], the isomorphism between V

and V_i is given by the collection of isomorphisms

$$\Psi_i^*:\Gamma(Y,\mathcal{O}_Y(\mathcal{D}(m)))\xrightarrow{\simeq}\Gamma(Y_i,\mathcal{O}_{Y_i}(\mathcal{D}_i(m))),\quad g\longmapsto f_i^{-m}\cdot\psi_i^*(g). \tag{2.5}$$

Let s and s_i be the regular functions on V and V_i corresponding respectively to 1 in degree 1 in the grading of their coordinate rings by the subspaces $\Gamma(Y,\mathcal{O}_Y(\mathcal{D}(m)))$ and $\Gamma(Y,\mathcal{O}_Y(\mathcal{D}_i(m)))$. By construction, we have $h=s\sigma^*(s)$ and $h_i=s_i\sigma_i^*(s_i)$. Since on the other hand $\Psi_i^*(s)=f_is_i$, we have

$$\begin{aligned}\psi_i^*(h)=\psi_i^*(s\sigma^*(s))&=\Psi_i^*(s)\cdot\Psi_i^*\circ\sigma^*(s)=\Psi_i^*(s)\cdot\sigma_i^*\circ\Psi_i^*(s)\\&=f_is_i\cdot\sigma_i^*(f_i)\sigma_i^*(s_i)=f_i\cdot\sigma_i^*(f_i)\cdot h_i=(f_i\cdot\tau_i^*f_i)\cdot h_i.\end{aligned}$$

We now prove the converse statement. Let $i=1$ or $i=2$. By [1, Theorem 8.8], V is isomorphic to V_i and the isomorphism is given by (2.5). The real structures σ on V and σ_i on V_i are given as in the proof of Theorem 2.7(1) via the collection of isomorphisms (2.2). To conclude that X is $\mathbb{S}^1$-equivariantly isomorphic to X_i we only need to check that $\sigma_i^*\circ\Psi_i^*=\Psi_i^*\circ\sigma^*$. But for every $g\in\Gamma(Y,\mathcal{O}_Y(\mathcal{D}(m)))$, we have

$$\begin{aligned}\Psi_i^*\circ\sigma^*(g)&=\Psi_i^*(h^m\tau^*(g))=f_i^{-m}\cdot\psi_i^*(h^m)\cdot\psi_i^*\tau^*(g)\\&=f_i^{-m}\cdot f_i^m\cdot\tau_i^*(f_i^m)\cdot h_i^m\cdot\psi_i^*\tau^*(g)\\&=\tau_i^*(f_i^m)\cdot h_i^m\cdot\tau_i^*\psi_i^*(g)=\sigma_i^*(f_i^m\psi_i^*(g))=\sigma_i^*\circ\Psi_i^*(g),\end{aligned}$$

which concludes the proof. □

Corollary 2.12. — *Let $(Z_i,(\mathcal{D}_i,h_i))$ be minimal couples on real normal semiprojective varieties $Z_i=(Y_i,\tau_i)$, $i=1,2$, determining normal real affine varieties $X_i=X(Z_i,(\mathcal{D}_i,h_i))$ with effective $\mathbb{S}^1$-actions. Then X_1 and X_2 are $\mathbb{S}^1$-equivariantly isomorphic if and only if there exists a real isomorphism $\psi:Z_1\to Z_2$ and a rational function f_1 on Y_1 such that*

$$\psi^*(\mathcal{D}_2)=\mathcal{D}_1+\{1\}\otimes\operatorname{div}(f_1)\quad\textit{and}\quad\psi^*(h_2)=(f_1\cdot\tau_1^*f_1)\cdot h_1.$$

Proof. — This follows directly from [1, Theorem 8.8] which asserts that ψ_1 and ψ_2 in Theorem 2.11 are both isomorphisms. □

Corollary 2.13. — *Let $(Z,(\mathcal{D},h))$ be a minimal couple on a real normal semiprojective variety $Z=(Y,\tau)$ determining a normal real affine variety $X=X(Z,(\mathcal{D},h))$ with an effective $\mathbb{S}^1$-action. Then every real form of the $\mathbb{S}^1$-variety X is $\mathbb{S}^1$-equivariantly isomorphic to $X(Z',(\mathcal{D},h'))$ for some real form $Z'=(Y,\tau')$ of Z and a τ'-invariant rational function h' on Y making $(\mathcal{D},h')$ a phs-pair on Z'.*

Proof. — Recall that by definition $X = (V, \sigma, \mu)$ where $V = V(\mathcal{D}, h')$ is endowed with the $\mathbb{G}_{m,\mathbb{C}}$-action μ given by the $\mathbb{Z}$-grading of its coordinate ring, and σ is the real structure on V constructed in the proof of Theorem 2.7(1). On the other hand, by Theorem 2.7(3), every real form $X_1 = (V_1, \mu_1, \sigma_1)$ of X is $\mathbb{S}^1$ equivariantly isomorphic to $X(Z_1, (\mathcal{D}_1, h_1))$ for a suitable phs-pair $(\mathcal{D}_1, h_1)$ on its real AH-quotient $Z_1 = (Y_1, \tau_1)$. Since by definition V_1 is $\mathbb{G}_{m,\mathbb{C}}$-equivariantly isomorphic to V and the couple $(Y, \mathcal{D})$ is minimal, it follows from [1, Theorem 8.8] that there exist an isomorphism of complex varieties $\psi : Y \to Y_1$ such that $\mathcal{D} = \psi^*\mathcal{D}_1 + \mathrm{div}(f)$ for some rational function f on Y. Letting $\tau' = \psi^*(\tau_1)$, the rational function $h' = f(\psi^{-1}\tau_1\psi(f))^{-1}\psi^*h_1$ on Y is τ'-invariant, and $(\mathcal{D}, h')$ is a phs pair on $Z' = (Y, \tau')$ such that X_1 is $\mathbb{S}^1$-equivariantly isomorphic to $X(Z', (\mathcal{D}, h'))$. □

Remark 2.14. — A direct adaptation of [1, Section 8] in our context provides a notion of *morphism of phs-pairs* for which the appropriate extension of Theorem 2.11 yields that the assignment $(Z, (\mathcal{D}, h)) \mapsto X(Z, (\mathcal{D}, h))$ is a faithful covariant functor from the category of phs-pairs on normal real semiprojective varieties to the category of normal real affine varieties with effective $\mathbb{S}^1$-actions.

2.3. Real DPD presentations

It is well-known that proper segmental divisors can be described by a simpler datum consisting of a suitable pair of $\mathbb{Q}$-divisors. Indeed, given a proper segmental divisor $\mathcal{D}$ on a normal complex algebraic variety Y, we let $D_+ = \mathcal{D}(1)$ and $D_- = \mathcal{D}(-1)$. The identity

$$\mathcal{D} = \{1\} \otimes D_+ + [0,1] \otimes (-D_+ - D_-) \tag{2.6}$$

implies that $\mathcal{D}$ is equivalently fully determined by a pair of big and semi-ample $\mathbb{Q}$-Cartier divisors D_+ and D_- on Y satisfying $D_+ + D_- \leqslant 0$ (see Section 2.1). Furthermore, if $(\mathcal{D}, h)$ is a phs-pair for a given additional real structure τ on Y, then the identity $\tau^*\mathcal{D} = \widehat{\mathcal{D}} + \{1\} \otimes \mathrm{div}(h)$ is equivalent to $D_- = \tau^*D_+ - \mathrm{div}(h)$. Summing up, a phs-pair $(\mathcal{D}, h)$ on a normal real algebraic variety $Z = (Y, \tau)$ is equivalently fully determined via (2.6) by a pair (D, h) satisfying $D + \tau^*D \leqslant \mathrm{div}(h)$. The original data is recovered from (2.6) by setting $D_+ = D$ and $D_- = \tau^*D - \mathrm{div}(h)$. By analogy with the terminology introduced by Flenner and Zaidenberg [9] for the description of normal complex affine surfaces with $\mathbb{G}_{m,\mathbb{C}}$-actions, we set the following definition:

Definition 2.15. — *A* real DPD pair *on a normal real algebraic variety $Z=(Y,\tau)$ is a pair (D,h) consisting of a big and semiample $\mathbb{Q}$-Cartier divisor D and a τ-invariant rational function h on Y satisfying $D+\tau^*D\leqslant\operatorname{div}(h)$.*

Here, DPD stands for Dolgachev, Pinkham and Demazure, respectively who where the first to describe split $\mathbb{G}_{m,\mathbb{C}}$-actions via $\mathbb{Q}$-divisors on their quotients, see the references in [9]. The following corollary is a straightforward reformulation of Theorem 2.7 and Corollary 2.12 in terms of DPD-pairs:

Corollary 2.16. — *A normal real affine variety X with an effective $\mathbb{S}^1$-action is determined by the following data:*

(1) *A real normal semiprojective variety $Z=(Y,\tau)$ representing the real AH-quotient of X,*
(2) *A pair (D,h) consisting of a big and semiample $\mathbb{Q}$-Cartier divisor D and a τ-invariant rational function h on Y satisfying $D+\tau^*D\leqslant\operatorname{div}(h)$.*

Furthermore, for a fixed $Z=(Y,\tau)$, two pairs (D_1,h_1) and (D_2,h_2) determine $\mathbb{S}^1$-equivariantly isomorphic affine varieties if and only if there exists a real automorphism ψ of Z and a rational function f on Y such that

$$\psi^*D_2=D_1+\operatorname{div}(f)\quad\text{and}\quad\psi^*(h_2)=(f\cdot\tau^*f)\cdot h_1.$$

Remark 2.17. — Given a real algebraic variety $Z=(Y,\tau)$, the group $K(Y)^*$ of invertible rational functions on Y has the structure of Galois module under the action of τ. In terms of this structure, the condition $\psi^*(h_2)=(f\cdot\tau^*f)\cdot h_1$ in Corollary 2.12 and Corollary 2.16 means that h_1 and $\psi^*(h_2)$ have the same class in the Galois cohomology group

$$H^2(\operatorname{Gal}(\mathbb{C}/\mathbb{R}),K(Y)^*)=(K(Y)^*)^{\tau^*}/\operatorname{Im}(\operatorname{id}\times\tau^*).$$

3. Low dimensional examples

In this section we consider real affine curves and surfaces with $\mathbb{S}^1$-actions. In the surface case, rephrasing Corollary 2.16, we obtain in particular a real counterpart of Flenner–Zaidenberg DPD-presentation of normal complex affine surfaces with $\mathbb{G}_{m,\mathbb{C}}$-actions [9]. We illustrate the methods to explicitly find phs-pairs and DPD-pairs corresponding to given $\mathbb{S}^1$-actions on various examples.

3.1. Real affine curves with $\mathbb{S}^1$-actions

Let us first explain how to re-derive the following classical characterization of real affine curves with a effective $\mathbb{S}^1$-actions:

Proposition 3.1. — *Up to equivariant isomorphism there exists precisely two normal real affine curves with an effective $\mathbb{S}^1$-action:*

(a) *The circle* $\mathbb{S}^1 = \mathrm{Spec}(\mathbb{R}[x,y,]/(x^2+y^2-1)$ *acting on itself by translations*

(b) *The curve* $C = \mathrm{Spec}(\mathbb{R}[u,v]/(u^2+v^2+1)) \subset \mathbb{A}^2_{\mathbb{R}}$ *on which* $\mathbb{S}^1$ *acts by restriction of the representation* $\rho_0 : \mathbb{S}^1 \to \mathrm{SO}_{2,\mathbb{R}}$ *defined in* (1.1).

Proof. — Since the complex punctured affine line $\mathbb{A}^1_* = \mathrm{Spec}(\mathbb{C}[z^{\pm 1}])$ is the only normal complex affine curve admitting an effective hyperbolic $\mathbb{G}_{m,\mathbb{C}}$-action, namely the one μ by translations $t \cdot z = tz$, a normal real affine curve X endowed with an effective $\mathbb{S}^1$-action is represented by a triple $(V = \mathbb{A}^1_*, \sigma, \mu)$. Its real AH-quotient is thus isomorphic to $\mathrm{Spec}(\mathbb{C})$, endowed with the complex conjugation, an a phs-pair $(\mathcal{D}, h)$ on it consists of the trivial divisor and a non-zero real number $h \in \mathbb{R}^*$. By Theorem 2.11 and Remark 2.17, two real numbers h and h' determine $\mathbb{S}^1$-equivariantly isomorphic curves if and only if they have the same class in $H^2(\mathbb{Z}_2, \mathbb{C}^*) \simeq \mathbb{R}/\mathbb{R}_+$, that is, if and only if they have the same sign. We thus have two cases:

- $h = 1$. The corresponding real structure σ on $\mathbb{A}^1_*$ as constructed in Theorem 2.7 is given by the composition of the involution $z \mapsto z^{-1}$ with the complex conjugation. The invariants are then generated by $x = \frac{1}{2}(z + z^{-1})$ and $y = \frac{1}{2i}(z - z^{-1})$, and we conclude that $X = \mathrm{Spec}(\mathbb{C}[z^{\pm 1}]^{\sigma^*}) \simeq \mathbb{S}^1$ on which $\mathbb{S}^1$ acts by translations.
- $h = -1$. The corresponding real structure σ is given by the composition of the involution $z \mapsto -z^{-1}$ with the complex conjugation The invariants are generated by $u = \frac{1}{2}(z - z^{-1})$ and $v = \frac{1}{2i}(z + z^{-1})$, and the corresponding real affine curve $X = \mathrm{Spec}(\mathbb{C}[z^{\pm 1}]^{\sigma^*})$ is isomorphic to C with the announced $\mathbb{S}^1$-action.

□

3.2. Real DPD-presentation of affine surfaces with $\mathbb{S}^1$-actions

Given a normal real affine surface endowed with an effective $\mathbb{S}^1$-action $X = (V, \sigma, \mu)$, the AH-quotient $Y(V)$ of V coincides with its algebraic quotient $Y_0(V) = V/\!/\mathbb{G}_{m,\mathbb{C}}$, which is a normal, hence smooth complex affine curve. The pair $(Y_0(V), \overline{\sigma})$ is thus a smooth real affine curve. Corollary 2.16 can be rephrased in the form of the following real counterpart of Flenner–Zaidenberg DPD-presentation of normal complex affine surfaces with $\mathbb{G}_{m,\mathbb{C}}$-actions [9].

Proposition 3.2. — *A normal real affine surface X with an effective $\mathbb{S}^1$-action is determined by a smooth real affine curve $C = (Y, \tau)$ and a pair (D, h) consisting of a Weil $\mathbb{Q}$-divisor D and a τ-invariant rational function h on Y such that $D + \tau^* D \leqslant \mathrm{div}(h)$.*

Example 3.3. — Given a non-constant polynomial $P \in \mathbb{R}[w]$, we let $X(P)$ be the normal real affine surface in $\mathbb{A}^2_{\mathbb{R}} \times \mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[x, y][w])$ defined by the equation $x^2 + y^2 - P(w) = 0$. The action of $\mathbb{S}^1$ on $\mathbb{A}^2_{\mathbb{R}} \times \mathbb{A}^1_{\mathbb{R}}$ defined by the direct sum of the representation $\rho_0 : \mathbb{S}^1 \to \mathrm{SO}_{2,\mathbb{R}}$ of (1.1) on the first factor with the trivial representation on the second factor restricts to an effective $\mathbb{S}^1$-action on $X(P)$. We will show that a DPD-presentation for $X(P)$ is $(D, h) = (0, P(w))$ on the curve $C = \mathrm{Spec}(\mathbb{R}[w]) = (\mathrm{Spec}(\mathbb{C}[w]), \tau)$, where τ is the complex conjugation.

Indeed, by making the complex coordinate change $(u, v) = (x+iy, x-iy)$, we see that $X(P)$ endowed with its $\mathbb{S}^1$-action is represented by the triple $(V(P), \sigma, \mu)$ where $V(P)$ is the normal complex surface with equation $uv - P(w) = 0$ in $\mathbb{A}^3_{\mathbb{C}}$, σ is the real structure defined as the composition of the involution $(u, v) \mapsto (v, u)$ with the complex conjugation and μ is the effective $\mathbb{G}_{m,\mathbb{C}}$-action induced by the linear action $t \cdot (u, v, w) = (t^{-1}u, tv, w)$ on $\mathbb{A}^3_{\mathbb{C}}$. The quotient $V(P)/\!/\mathbb{G}_{m,\mathbb{C}}$ is isomorphic to $\mathrm{Spec}(\mathbb{C}[w])$ on which σ induces the complex conjugation τ. Choosing v as a semi-invariant function of weight 1 on $V(P)$ as in the proof of Theorem 2.7, we deduce from the identification:

$$\Gamma(V(P), \mathcal{O}_{V(P)}) \simeq \bigoplus_{n<0} \mathbb{C}[w] \cdot P(w)^{-n} v^n \oplus \mathbb{C}[w] \oplus \bigoplus_{n>0} \mathbb{C}[w] \cdot v^n \subset \mathbb{C}(w)(v)$$

that $V(P)$ is $\mathbb{G}_{m,\mathbb{C}}$-equivariantly isomorphic to $V(\mathrm{Spec}(\mathbb{C}[w]), \mathcal{D})$, where $\mathcal{D}$ is determined by $\mathcal{D}(n) = \mathrm{div}(1) = 0$ and $\mathcal{D}(-n) = \mathrm{div}(P(w)^{-n})$, for all $n > 0$. We obtain from (2.6) that

$$\mathcal{D} = \sum [0, p_i] \otimes \{a_i\} + \sum [0, q_i] \otimes (\{b_i\} + \{\overline{b}_i\}),$$

where a_i, b_i and $\overline{b}_i$ are the real and complex roots of P respectively, and p_i, q_i are their respective multiplicities. Furthermore, $h = v\sigma^* v = vu = P(w)$, and so, the DPD-presentation of $X(P)$ is $(\mathcal{D}(1), P(w)) = (0, P(w))$ as claimed.

Note that in Example 3.3 above, the special case where $P(w) = w$ corresponds to a surface $X(P)$ equivariantly isomorphic to the affine plane $\mathbb{A}^2_{\mathbb{R}}$ endowed with the effective $\mathbb{S}^1$-action defined by the representation $\rho_0 : \mathbb{S}^1 \to \mathrm{SO}_{2,\mathbb{R}}$. The following is a counter-part for $\mathbb{S}^1$-actions of Gutwirth's linearisation theorem [12] for $\mathbb{G}_m$-actions on the plane:

Proposition 3.4. — *Every effective $\mathbb{S}^1$-action on $\mathbb{A}^2_{\mathbb{R}}$ is conjugate by an automorphism of $\mathbb{A}^2_{\mathbb{R}}$ to that defined by the representation $\rho_0 : \mathbb{S}^1 \to \mathrm{SO}_{2,\mathbb{R}}$.*

Proof. — Indeed, let $(V = \mathbb{A}^2_{\mathbb{C}}, \sigma_{\mathbb{A}^2_{\mathbb{R}}}, \mu)$ be a triple representing the given $\mathbb{S}^1$-action on $X = \mathbb{A}^2_{\mathbb{R}}$. Since by virtue of Lemma 1.6, the $\mathbb{G}_{m,\mathbb{C}}$-action μ is hyperbolic, it follows from Gutwirth's theorem [12] that μ is conjugate by an automorphism φ of $\mathbb{A}^2_{\mathbb{C}}$ to a linear action ν of the form $t \cdot (u, v) = (t^{-p}u, t^q v)$ for some relatively prime positive integers p and q. It follows that X endowed with its $\mathbb{S}^1$-action is also represented by the triple (V, σ, ν) where $\sigma = \varphi^* \sigma_{\mathbb{A}^2_{\mathbb{R}}} = \varphi^{-1} \sigma_{\mathbb{A}^2_{\mathbb{R}}} \varphi$ is the pull-back of $\sigma_{\mathbb{A}^2_{\mathbb{R}}}$ by φ.

The algebraic quotient $V/\!/\mathbb{G}_{m,\mathbb{C}}$ is isomorphic to $\mathrm{Spec}(\mathbb{C}[z]) \simeq \mathbb{A}^1_{\mathbb{C}}$, where $z = u^q v^p$. Letting a and b be positive integers such that $-ap + bq = 1$, $s = u^a v^b$ is a semi-invariant regular function of weight 1 on V which determines a $\mathbb{G}_{m,\mathbb{C}}$-equivariant isomorphism between V and $V(\mathbb{A}^1_{\mathbb{C}}, \mathcal{D}')$ for the segmental divisor $\mathcal{D}' = [-a/q, b/p] \otimes \{0\}$. Since $C = (V/\!/\mathbb{G}_{m,\mathbb{C}}, \overline{\sigma})$ is a real form of $\mathbb{A}^1_{\mathbb{R}}$, hence is isomorphic to the trivial one, there exists an automorphism $\psi : z \mapsto \alpha z + \beta$ of $V/\!/\mathbb{G}_{m,\mathbb{C}}$, where $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$, such that $\overline{\sigma} = \psi^* \sigma_{\mathbb{A}^1_{\mathbb{R}}}$. So $\overline{\sigma}$ is the composition of the automorphism $z \mapsto \overline{\alpha}\alpha^{-1} z + \alpha^{-1}(\overline{\beta} - \beta)$ of $V/\!/\mathbb{G}_{m,\mathbb{C}}$ with the complex conjugation. The condition $\overline{\sigma}^* \mathcal{D}' = \widehat{D'} + 1 \otimes \mathrm{div}(h')$ for some $\overline{\sigma}$-invariant rational function $h' \in \mathbb{C}(z)$ then reads

$$[-a/q, b/p] \otimes \{\alpha^{-1}(\overline{\beta} - \beta)/\} = [-b/p, a/q] \otimes \{0\} + 1 \otimes \mathrm{div}(h').$$

Since $\mathrm{div}(h')$ is an integral Weil divisor, it follows that $(-ap+bq)/pq = 1/pq$ is an integer. Thus $p = q = 1$ and we can now assume further from the very beginning that $a = 0$ and $b = 1$, so that $s = v$ and $\mathcal{D}' = [0, 1] \otimes \{0\}$. The condition $\overline{\sigma}^* \mathcal{D}' = \widehat{D'} + 1 \otimes \mathrm{div}(h')$ then implies that $\mathrm{div}(h') = \{0\} = \{\alpha^{-1}(\overline{\beta} - \beta)\}$, hence that $\beta \in \mathbb{R}$ and $h' = \gamma z$ for some γ in $\mathbb{C}^*$. The fact h' is $\overline{\sigma}$-invariant implies in turn that $\gamma = c\alpha$ for some $c \in \mathbb{R}^*$. The phs-pair $(\mathcal{D}', h')$ is thus the pull-back of the pair $(\mathcal{D}'', h'') = ([0, 1] \otimes \{\beta\}, cz)$ by the real isomorphism $\psi : (\mathbb{A}^1_{\mathbb{C}}, \overline{\sigma}) \overset{\simeq}{\to} (\mathbb{A}^1_{\mathbb{C}}, \sigma_{\mathbb{A}^1_{\mathbb{R}}})$. Since $c, \beta \in \mathbb{R}$, we see that $(\mathcal{D}'', h'')$ is in turn the pull-back of the phs-pair $(\mathcal{D}, h) = ([0, 1] \otimes \{0\}, z)$ by the real automorphism $\varphi : z \mapsto c(z - \beta)$ of $(\mathbb{A}^1_{\mathbb{C}}, \sigma_{\mathbb{A}^1_{\mathbb{R}}})$. Summing up, we conclude that X is $\mathbb{S}^1$-equivariantly isomorphic to $X(\mathrm{Spec}(\mathbb{R}[z]),$

$([0,1]\otimes\{0\},z))$ hence to $\mathbb{A}^2_{\mathbb{R}}$ endowed with the action defined by representation ρ_0. $\square$

Example 3.5 (An algebraic model of the open Moebius band). — Let $X'\simeq\mathbb{A}^1_{\mathbb{R}}\times\mathbb{S}^1$ be the real surface endowed with the free $\mathbb{S}^1$-action by translations on the second factor represented by the triple

$$(V'=\mathbb{A}^1_{\mathbb{C}}\times\mathbb{G}_{m,\mathbb{C}}=\operatorname{Spec}(\mathbb{C}[w][z^{\pm 1}]),\sigma',\mu'),$$

where σ' is the composition of the involution $(w,z)\mapsto(w,z^{-1})$ with the complex conjugation and μ' is the $\mathbb{G}_{m,\mathbb{C}}$-action by translations on the second factor. A corresponding real DPD-presentation is given by the DPD-pair $(D,h)=(\{0\},1)$ on the real affine curve $C'=(\operatorname{Spec}(\mathbb{C}[w]),\tau)$ where τ is the complex conjugation.

The involution $(w,z)\mapsto(-w,-z)$ of V' is both σ' and $\mathbb{G}_{m,\mathbb{C}}$-equivariant. The quotient of V' by this involution thus inherits a real structure and a $\mathbb{G}_{m,\mathbb{C}}$-action which correspond to a smooth real affine surface X with an effective $\mathbb{S}^1$-action. Explicitly, letting $x=wz^{-1}$ and $y=z^2$, X is represented by the triple (V,σ,μ) where $V\simeq\mathbb{A}^1_{\mathbb{C}}\times\mathbb{G}_{m,\mathbb{C}}=\operatorname{Spec}(\mathbb{C}[x][y^{\pm 1}])$, σ is the composition of involution $(x,y)\mapsto(xy,y^{-1})$ with the complex conjugation, and μ is the effective $\mathbb{G}_{m,\mathbb{C}}$-action induced by the linear action with weights $(-1,2)$. A phs-pair for X is

$$(\mathcal{D},h)=\left(\tfrac{1}{2}\otimes\{0\},x^2y\right)\ \text{on the affine curve } C=\left(\operatorname{Spec}(\mathbb{C}[x^2y]),\tau\right)\simeq\mathbb{A}^1_{\mathbb{R}},$$

where τ denotes the complex conjugation. A real DPD-presentation of X is given by the DPD-pair $(\mathcal{D}(1),h)=\left(\frac{1}{2}\cdot\{0\},x^2y\right)$. Furthermore, we have a commutative diagram of real morphisms:

$$\begin{tikzcd}
X'=(V',\sigma',\mu') \arrow[r, "{(w,z)\mapsto(x,y)=(wz^{-1},z^2)}"] \arrow[d, "p_1"'] & X=(V,\sigma,\mu) \arrow[d, "{(x,y)\mapsto x^2y}"] \\
C' \arrow[r, "{w\mapsto x^2y=w^2}"] & C.
\end{tikzcd}$$

The real locus $X'(\mathbb{R})$ of X' endowed with its natural structure of differentiable manifold is diffeomorphic to $\mathbb{R}\times S^1$, on which S^1, identified with the set of complex numbers $\exp(i\theta)$ of norm 1, acts by translations on the second factor. The real locus of X is then diffeomorphic to the open Moebius band obtained as the quotient of $\mathbb{R}\times S^1$ by the involution $(w,\exp(i\theta))\mapsto(-w,\exp(-i\theta))$, endowed with the S^1-action induced by that on $\mathbb{R}\times S^1$. We refer the reader to [6] for more examples of real DPD-presentations of algebraic models of differentiable surfaces with S^1-actions.

4. Higher dimensional examples

In this section, to continue to illustrate the methods to explicitly find phs-pairs corresponding to given $\mathbb{S}^1$-actions, we present two natural higher dimensional examples. We begin with a real form of the action of the maximal torus $\mathbb{G}_{m,\mathbb{R}}$ of $\mathrm{SL}_{2,\mathbb{R}}$ on $\mathrm{SL}_{2,\mathbb{R}}$ by multiplication, whose algebraic quotient morphism turns out to provide an algebraic model of the Hopf fibration $S^3 \to S^2$. We then consider certain families of non-trivial forms of linear $\mathbb{S}^1$-actions on $\mathbb{A}^4_{\mathbb{R}}$ constructed by Moser-Jauslin [14].

4.1. An algebraic model of the Hopf fibration $S^3 \to S^2$

Recall that the Hopf fibration $S^3 \to S^2$ realizes the real sphere S^3 as the total space of an S^1-torsor over the real sphere S^2 in the category of differentiable real manifolds. Namely, the circle S^1 identified with the set of complex numbers $z = x + iy \in \mathbb{C}^*$ of norm 1 acts by component-wise multiplication on the real sphere $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ viewed as set of pairs of complex numbers $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$ such that $|z_1|^2 + |z_2|^2 = 1$. Letting $S^2 \subset \mathbb{R}^3$ be the 2-sphere with equation $x^2 + y^2 + z^2 = 1$, the quotient map $S^3 \longrightarrow S^3/S^1 \simeq S^2$ is defined by

$$(z_1, z_2) = (x_1, y_1, x_2, y_2) \mapsto (x, y, z) = \big(|z_1|^2 - |z_2|^2, 2\,\mathrm{Re}(z_1\overline{z}_2), 2\,\mathrm{Im}(z_1\overline{z}_2)\big)$$

Putting $\mathbb{S}^3 = \mathrm{Spec}(\mathbb{R}[x_1, y_1, x_2, y_2]/(x_1^2 + y_1^2 + x_2^2 + y_2^2 - 1))$ an algebraic model of the action of S^1 on S^3 is given by the restriction to $\mathbb{S}^3$ of the $\mathbb{S}^1$-action on $\mathbb{A}^2_{\mathbb{R}} \times \mathbb{A}^2_{\mathbb{R}} = \mathrm{Spec}(\mathbb{R}[x_1, y_1][x_2, y_2])$ defined as the direct sum of two copies of the representation $\rho_0 : \mathbb{S}^1 \to \mathrm{SO}_{2,\mathbb{R}}$ in (1.1). An algebraic model of the quotient map is given by the morphism of real algebraic varieties

$$\begin{aligned} p : \mathbb{S}^3 &\longrightarrow \mathbb{S}^2 \\ (x_1, y_1, x_2, y_2) &\longmapsto (x_1^2 + y_1^2 - x_2^2 - y_2^2, 2(x_1x_2 + y_1y_2), 2(x_2y_1 - x_1y_2)), \end{aligned}$$

where $\mathbb{S}^2 = \mathrm{Spec}(\mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1))$. Recall that the divisor class group of $\mathbb{S}^2$ is trivial whereas the divisor class group of $\mathbb{S}^2_{\mathbb{C}}$ is isomorphic to $\mathbb{Z}$, generated by the class of the Cartier divisor $D = \{x + iy = 1 - z = 0\}$.

Proposition 4.1. — *The real affine threefold $\mathbb{S}^3$ endowed with the so-defined $\mathbb{S}^1$-action is $\mathbb{S}^1$-equivariantly isomorphic to $X(\mathbb{S}^2, (\{1\} \otimes D, 1 - z))$.*

Proof. — As a real threefold with $\mathbb{S}^1$-action, $\mathbb{S}^3$ can be equivalently represented by the smooth complex affine quadric $V = \{u_1v_1 + u_2v_2 = 1\}$ in $\mathbb{A}^4_{\mathbb{C}}$, equipped with the real structure σ defined as the composition of the involution $(u_1,u_2,v_1,v_2)\mapsto(v_1,v_2,u_1,u_2)$ with the complex conjugation, and endowed with the $\mathbb{G}_{m,\mathbb{C}}$-action μ defined by $t\cdot(u_1,v_1,u_2,v_2)=(tu_1,t^{-1}v_1,tu_2,t^{-1}v_2)$. The algebraic quotient $V/\!/\mathbb{G}_{m,\mathbb{C}}$ is isomorphic to the smooth affine quadric $S=\left\{uv+z^2=1\right\}$ in $\mathbb{A}^3_{\mathbb{C}}$ and the quotient morphism

$$\begin{aligned} V &\to V/\!/\mathbb{G}_{m,\mathbb{C}}=S\\ (u_1,v_1,u_2,v_2)&\mapsto(u,v,w)=(2u_1v_2,2u_2v_1,2u_1v_1-1)\end{aligned}$$

is a $\mathbb{G}_{m,\mathbb{C}}$-torsor whose class in $H^1(S,\mathcal{O}_S^*)\simeq\mathrm{Pic}(S)\simeq\mathbb{Z}$ coincides with that of the line bundle associated to the Cartier divisor $D'=\{u=1-z=0\}$ on S. It follows that S is the AH-quotient of V and that $\mathcal{D}'=\{1\}\otimes D'$ is a proper segmental divisor on S such that V is equivariantly isomorphic to $V(S,\mathcal{D}')$.

The real structure σ descends on S to the real structure $\overline{\sigma}$ defined as the composition of the involution $(u,v,w)\mapsto(v,u,w)$ with the complex conjugation. Since $\overline{\sigma}^*D'=\{v=1-z=0\}=-D'+\mathrm{div}(1-z)$ where $1-z$ is $\overline{\sigma}$-invariant, we conclude that $\mathbb{S}^3$ is equivariantly isomorphic to $X((S,\overline{\sigma}),(\{1\}\otimes D',1-z))$. The assertion then follows by noticing that the phs-pair $(\{1\}\otimes D,1-z)$ is the pull-back of $(\{1\}\otimes D',1-z)$ by the isomorphism of real algebraic surfaces

$$\varphi:(\mathbb{S}^2_{\mathbb{C}},\sigma_{\mathbb{S}^2})\xrightarrow{\simeq}(S,\overline{\sigma}),\quad (x,y,z)\longmapsto(u,v,z)=(x+iy,x-iy,z).\quad\square$$

More generally, recall that for every integer $p\geqslant 1$, the Lens space $L(p,1)$ is the quotient of $S^3=\{(z_1,z_2),\,|z_1|^2+|z_2|^2=1\}$ by the free action of the group $\mathbb{Z}_p$ defined by $(z_1,z_2)\mapsto(\zeta z_1,z_2)$, where $\zeta=\exp(2i\pi/p)$. This action is equivariant with respect to the S^1-action on S^3 by component-wise multiplication, and so, $L(p,1)$ inherits a effective action of S^1. A similar argument as in the proof of the previous proposition shows that an algebraic model of $L(p,1)$ endowed with this S^1-action is given by the real affine threefold $\mathbb{L}(p,1):=X(\mathbb{S}^2,(\{1\}\otimes(pD),(1-z)^p))$.

4.2. Linear and non-linearizable $\mathbb{S}^1$-actions on $\mathbb{A}^4_{\mathbb{R}}$

Let again $\rho_0:\mathbb{S}^1\to SO_{2,\mathbb{R}}$ be the representation defined in (1.1). For every integer $r\geqslant 0$, the morphism

$$\begin{aligned}\nu_{2,r}:\mathbb{S}^1\times\mathbb{A}^4_{\mathbb{R}}&\longrightarrow\mathbb{A}^4_{\mathbb{R}}\\ (s,(u_1,v_1,u_2,v_2))&\longmapsto(\rho_0(s)^2\cdot(u_1,v_1),\rho_0(s)^{2r+1}\cdot(u_2,v_2))\end{aligned}$$

defines an effective action of $\mathbb{S}^1$ on $\mathbb{A}^4_{\mathbb{R}} = \operatorname{Spec}(\mathbb{R}[u_1, v_1][u_2, v_2])$. With the notation of Lemma 1.4, the latter is represented by the triple $(V, \sigma, \mu_{2,r})$, where $V = \mathbb{A}^4_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[a, b][x, y])$, σ is the real structure defined as the composition of the involution $(a, b, x, y) \mapsto (b, a, y, x)$ with the complex conjugation, and $\mu_{2,r}$ is the linear hyperbolic $\mathbb{G}_{m,\mathbb{C}}$-action weights $(2, -2, 2r + 1, -2r - 1)$.

For $r = 1$, Freudenburg and Moser-Jauslin [10] constructed an $\mathbb{S}^1$-action $\nu'_{2,1}$ on $\mathbb{A}^4_{\mathbb{R}}$ which is a non-trivial form of $\nu_{2,1}$, hence in particular a non-linearizable action. The construction was generalized later on by Moser-Jauslin [14] for arbitrary $r \geqslant 2$ to yield infinite families of pairwise non-conjugate non-linearizable $\mathbb{S}^1$-actions on $\mathbb{A}^4_{\mathbb{R}}$. Our aim is to give a complementary description of these actions in terms of phs-pairs.

Given $r \geqslant 1$, we let $Q_{2,r}$ be the closed subvariety of the affine space $\mathbb{A}^4_{\mathbb{C}} = \operatorname{Spec}(\mathbb{C}[u, v, z, w])$ with equation $uv = z^{2r+1}w^2$ and we let $\pi : Y_{2,r} \to Q_{2,r}$ be the blow-up of $Q_{2,r}$ with center at the closed subscheme W with defining ideal (u, v, z^{2r+1}, w^2). We denote by E be the exceptional divisor of π, and by $D_{z,u}$, $D_{z,v}$, $D_{w,u}$ and $D_{w,v}$ the respective proper transforms in $Y_{2,r}$ of the Weil divisors $\{z = u = 0\}$, $\{z = v = 0\}$, $\{w = u = 0\}$ and $\{w = v = 0\}$ on $Q_{2,r}$. We let $\mathcal{D}_{2,r}$ be the segmental divisor on $Y_{2,r}$ defined by

$$\mathcal{D}_{2,r} = r \otimes D_{z,v} + \{1\} \otimes D_{w,v} + [2r, 2r + 1] \otimes E.$$

The main result of [14] can now be reformulated as follows:

Theorem 4.2. — *Let $r \geqslant 1$ be a fixed integer. Then for every polynomial $P \in \mathbb{R}[z]$, there exists a real structure τ_P on $Y_{2,r}$ with the following properties:*

(1) *The rational function $h_P = z^r((1 - zP^2(z))w + P^n(z)v)$ on $Y_{2,r}$ is τ_P-invariant,*
(2) *$(\mathcal{D}_{2,r}, h_P)$ is phs-pair on $Z_{2,r,P} = (Y_{2,r}, \tau_P)$,*
(3) *The affine fourfold $X(Z_{2,r,P}, (\mathcal{D}_{2,r}, h_P))$ is isomorphic to $\mathbb{A}^4_{\mathbb{R}}$.*

Furthermore, two fourfolds $X(Z_{2,r,P_i}, (\mathcal{D}_{2,r}, h_{P_i}))$, $i = 1, 2$, are $\mathbb{S}^1$-equivariantly isomorphic if and only if there exists $c \in \mathbb{R}^*$ such that $P_2(z) \equiv cP_1(c^2z)$ modulo z^r.

Proof. — Letting $n = 2r + 1$, the matrix

$$M_P = \begin{pmatrix} 1 - abP^2(ab) & a^nP^n(ab) \\ -b^nP^n(ab) & \sum_{j=0}^{2r}(abP^2(ab))^j \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[a, b])$$

defines an automorphism φ_P of V over $\operatorname{Spec}(\mathbb{C}[a, b])$ which is equivariant for the $\mathbb{G}_{m,\mathbb{C}}$-action $\mu_{2,r}$. By [14, Theorem 3.1 (i)], the composition $\sigma_P = \varphi_P \circ \sigma$ is a real structure on V such that (V, σ_P) is isomorphic to $\mathbb{A}^4_{\mathbb{R}}$. Since φ_P

is $\mathbb{G}_{m,\mathbb{C}}$-equivariant, it follows from Lemma 1.4 that the triple $(V,\sigma_P,\mu_{2,r})$ is a smooth real affine fourfold X_P endowed with an effective $\mathbb{S}^1$-action which is a real form of $X_0=(V,\sigma,\mu_{2,r})$. The main result in [14] asserts that X_{P_1} is $\mathbb{S}^1$-equivariantly isomorphic to X_{P_2} if and only if there exists $c\in\mathbb{R}^*$ such that $P_2(z)\equiv cP_1(c^2z)$ modulo z^r. So to complete the proof, it suffices to show that the AH-quotient of V is equal to $Y_{2,r}$, that V is $\mathbb{G}_{m,\mathbb{C}}$-equivariantly isomorphic to $V(Y_{2,r},\mathcal{D}_{2,r})$, and that X_P is $\mathbb{S}^1$-equivariantly isomorphic to $X(Z_{2,r,P},(\mathcal{D}_{2,r},h_P))$ for the claimed rational function h_P on $Y_{2,r}$ endowed with the real structure τ_P induced by σ_P.

First, it is clear that $Q_{2,r}$ is the algebraic quotient of V, the quotient morphism being given by $(a,b,x,y)\mapsto(u,v,z,w)=(a^ny^2,b^nx^2,ab,xy)$. The fact that $Y_{2,r}$ is the AH-quotient of V then follows from the observation that the minimal d for which the graded sub-algebra of $A=\mathbb{C}[a,b,x,y]$ consisting of semi-invariants of positive (resp. negative) weights divisible by d is generated in degree 1, is equal to $2n$, and that $A_{2n}\cdot A_{-2n}=\left(a^ny^2,b^nx^2,a^nb^n,x^2y^2\right)=\left(u,v,z^n,w^2\right)$ is precisely the defining ideal of the center of the blow-up $\pi:Y_{2,r}\to Q_{2,r}$.

The fact that $V\simeq V(Y_{2,r},\mathcal{D}_{2,r})$ can then be derived for instance from the toric downgrading method described in [1, Section 11]. In our case, V endowed with the $\mathbb{G}_{m,\mathbb{C}}$-action $\mu_{2,r}$ is the restriction to the sub-torus $\mathbb{G}_{m,\mathbb{C}}\hookrightarrow\mathbb{G}_{m,\mathbb{C}}^4$, $t\mapsto(t^2,t^{-2},t^n,t^{-n})$ of the usual structure of toric variety of $\mathbb{A}^4_{\mathbb{C}}$. This sub-torus corresponds to the injection $F:\mathbb{Z}\to\mathbb{Z}^4$, $1\mapsto(2,-2,n,-n)$ between the respective lattices of 1-parameter subgroups, and we have an exact sequence $0\to\mathbb{Z}\xrightarrow{F}\mathbb{Z}^4\xrightarrow{G}\mathbb{Z}^3\to 0$ where

$$G:\mathbb{Z}^4\to\mathbb{Z}^3\quad\text{is given by the matrix}\quad\begin{pmatrix}1&1&0&0\\0&0&1&1\\n&0&0&2\end{pmatrix}.$$

Let Σ' be the fan in $\mathbb{Z}^4\otimes_{\mathbb{Z}}\mathbb{R}$ generated by the cone $\mathrm{cone}(e_1,e_2,e_3,e_4)$ where e_i are the standard basis vectors. The coarsest fan Σ in $\mathbb{Z}^3\otimes_{\mathbb{Z}}\mathbb{R}$ generated by the image by P of Σ' is the simplicial fan generated by $f_1=(1,0,n)$, $f_2=(1,0,0)$, $f_3=(0,1,0)$, $f_4=(0,1,2)$ and the additional vector $f_5=(2,n,2n)=2f_1+nf_3=2f_2+nf_4$. This fan describes $Y_{2,r}$ as a toric threefold in which the invariant divisors corresponding the rays generated by the f_i are respectively $D(f_1)=D_{z,u}$, $D(f_2)=D_{z,v}$, $D(f_3)=D_{w,v}$, $D(f_4)=D_{w,u}$, while $D(f_5)$ is the exceptional divisor E of $\pi:Y_{2,r}\to Q_{2,r}$. A direct calculation now shows that $\mathcal{D}_{2,r}$ is equal to the proper segmental divisor supported on the union of the $D(f_i)$ whose coefficients are the images by the section $\gamma=r\,\mathrm{pr}_2+\mathrm{pr}_3:\mathbb{Z}^4\to\mathbb{Z}$ of F of the segments $\gamma\left(\mathbb{R}^4_{\geqslant 0}\cap P^{-1}(f_i)\right)$, $i=1,\ldots,5$. This implies that $V\simeq V(Y_{2,r},\mathcal{D}_{2,r})$

Now recall from the proof of Theorem 2.7 and Remark 2.8 that X_P is then $\mathbb{S}^1$-equivariantly isomorphic to $X(Z_{2,r,P}, (\mathcal{D}_{2,r}, s\sigma_P^* s))$, where s is the semi-invariant rational function of weight 1 on $\mathbb{A}^4_{\mathbb{C}}$ which provides the identification $\mathbb{C}[a,b,x,y]_m = \Gamma(Y_{2,r}, \mathcal{O}_{Y_{2,r}}(\mathcal{D}_{2,r}(m))) \cdot s^m$ for every $m \in \mathbb{Z}$. Our choice of section γ of F in the toric downgrading method used to construct $\mathcal{D}_{2,r}$ corresponds to the choice $s = b^r x$, and the fact that $h_P = (b^r x)\sigma_P^*(b^r x)$ then follows from a direct calculation. $\square$

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