Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Alex Degtyarev

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Tome 72, nº 6 (2022), p. 2299-2338.

https://doi.org/10.5802/aif.3491

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
72, 6 (2022) 2299-2338

TRITANGENTS TO SMOOTH SEXTIC CURVES

by Alex DEGTYAREV (*)

Abstract. — We prove that a smooth plane sextic curve can have at most 72 tritangents, whereas a smooth real sextic may have at most 66 real tritangents.

Résumé. — On montre qu'une courbe plane lisse de degré six a au plus 72 tritangentes, alors qu'une courbe lisse réelle de degré six a au plus 66 tritangentes réelles.

1. Introduction

All algebraic varieties considered in the paper are over $\mathbb{C}$. Sumtimes, we discuss subfields $\Bbbk \subset \mathbb{C}$ (most notably, $\Bbbk = \mathbb{R}$: by definition, a *real* variety is a complex one equipped with an anti-holomorphic involution), but we *never* consider fields of positive characteristic.

1.1. Principal results

This paper concludes the study of the maximal number of straight lines in a smooth polarized $K3$-surface. The most classical case, viz. that of spatial quartics $X \subset \mathbb{P}^3$, goes as far back as to A. Clebsch [2] (the upper bound of $m(11m-24)$ lines in a smooth degree m surface $X \subset \mathbb{P}^3$, yielding at most 80 lines in a quartic) and F. Schur [27] (an example of a smooth quartic with 64 lines). The sharp upper bound of 64 lines was established in B. Segre [28]. Two recent papers aroused new interest to this classical problem: first, a minor gap in Segre's proof was discovered and corrected by S. Rams and M. Schütt [24], where the argument was also extended to all characteristics

Keywords: $K3$-surface, sextic curve, tritangent, Niemeier lattice.
Math. classification: 14J28, 14H50, 14N20, 14N25.
(*) The author was partially supported by the TÜBİTAK grant 118F413.

$p \geqslant 5$ of the ground field, and second, an alternative, purely arithmetical (or rather lattice theoretic) proof was given in [8]; this approach allowed us to obtain, in addition to mere upper bounds, a complete classification of all large (i.e., more than 52) configurations of lines, prove the uniqueness of the line maximizing quartic, and establish the sharp upper bound of 56 *real* lines in a *real* smooth quartic surface (also realized by a unique real quartic). A great deal of other works have appeared almost immediately and, for the moment, the case of spatial quartic surfaces still remains the best studied one: there are sharp upper bounds on the number of lines over algebraically closed fields of positive characteristic (see [4, 23, 24, 25]) and over $\mathbb{R}$ (see [8]), partial bounds over $\mathbb{Q}$ (see [6]), upper bounds for singular quartics, both $K3$ (see [31, 32]) and not (see [11]), explicit equations of quartics with many lines (see [8, 29, 33]), etc.

Lines in smooth polarized $K3$-surfaces $X \to \mathbb{P}^{d+1}$ of all degrees $2d \geqslant 4$, both birational and hyperelliptic (cf. [26]), were studied in [5], using an arithmetical reduction similar to [8] and an appropriate taxonomy of prospective Fano graphs. (It appears that, so far, the more conventional geometric arguments have failed to produce even reasonable bounds on the number of lines.) Among other results, found in [5] are sharp upper bounds on the number of lines, both over $\mathbb{C}$ and $\mathbb{R}$, and a complete description of all large configurations of lines, especially in the two most "classical" cases, *viz.* sextics in $\mathbb{P}^4$ and octics (most notably, triquadrics) in $\mathbb{P}^5$, which give rise to a number of interesting Fano graphs. An unexpected discovery is the fact that the configurations of lines simplify dramatically when the degree grows: asymptotically, for $2d \gg 0$, all lines are either linearly independent in $H_2(X)$ or, else, among the fiber components of a certain fixed elliptic pencil; in either case, their number does not exceed 24. (The true sharp bound oscillates between 21, 22, and 24, periodically in the degree $2d \gg 0$, whereas its real counterpart oscillates between 19, 20, and 21, with a larger period.) The other side of the coin is the fact that in the remaining "classical" case, the smallest degree $2d = 2$ (double planes), the dual adjacency graph of lines may be too large: the star of a single vertex is more complicated than the whole Fano graph of a quartic. For this reason, the case $2d = 2$ was left out as not feasible in [5]; it is treated in the present paper by means of considerably different arithmetical techniques (see Section 1.2).

In [6] it was conjectured that the maximal number of lines in a smooth 2-polarized $K3$-surface is 144, with the maximum realized by the double plane $X \to \mathbb{P}^2$ ramified over the sextic curve

$$z_0^6 + z_1^6 + z_2^6 = 10\left(z_0^3 z_1^3 + z_1^3 z_2^3 + z_2^3 z_0^3\right). \tag{1.1}$$

(This equation is borrowed from Sh. Mukai [16], as the surface in question admits a faithful action of the Mukai group M_9; explicit equations of the predicted 144 lines were found independently by D. Festi and Y. Zaytman, private communication.) The conjecture is motivated by the fact that, like Schur's quartic [27] and some line maximizing sextics in $\mathbb{P}^4$ and octics in $\mathbb{P}^5$ (see [5, 6]), this surface minimizes the discriminant of a singular $K3$-surface admitting a smooth model of a given degree. In the present paper we settle (in the affirmative) and extend the conjecture, see Theorem 1.1, Addendum 1.2, and Theorem 1.3. We state our principal results in terms of tritangents to the ramification locus $C \subset \mathbb{P}^2$ (a smooth sextic curve) rather than lines in the surface $X \to \mathbb{P}^2$, dividing the numbers by 2 (see Section 2.2 below for further details). Certainly, when speaking about tritangents, we allow the collision of some of the tangency points; in other words, a *tritangent* to a smooth sextic $C \subset \mathbb{P}^2$ is merely a line $L \subset \mathbb{P}^2$ such that the local intersection index $(L \circ C)_P$ at each intersection point $P \in L \cap C$ is even.

Theorem 1.1 (see Sections 9.1 and 9.2). — *Let $t(C)$ denote the number of tritangents to a smooth sextic $C \subset \mathbb{P}^2$. Then either*

- $t(C) = 72$, *and then C is the sextic given by* (1.1), *or*
- $t(C) = 66$, *and then C is one of the two sextics that are described in Section 9.1*(2), (3), *or*
- $t(C) \leqslant 65$.

Previously known bounds are $t(C) \leqslant 76$ in N. Elkies [9] (cf. Corollary 2.5 below) and $t(C) \leqslant 108$ given by Plücker's formulas. Note that, unlike the 28 bitangents to any smooth quartic curve (and like the case of other polarized $K3$-surfaces), counting tritangents to a sextic is not an enumerative problem: tritangents are not stable under deformation and a typical sextic has no tritangents at all.

Addendum 1.2 (see Section 9.1). — *The number $t(C)$ as in Theorem 1.1 takes all values in the set $\{0, 1, \ldots, 65, 66, 72\}$ except, possibly,* 61.

Twelve sextics (six configurations of lines) with $62 \leqslant t(C) \leqslant 65$ are described in Section 9.1(4)–(9), but we do not assert the completeness of this list. In spite of extensive, although not exhaustive, search, we could not find a sextic with 61 tritangents. There are reasons (e.g., Corollary 2.5

below or the large number of sextics with 60 tritangents) to believe that 61 is a natural threshold in the problem, but taking the classification down to 61 tritangents would require too much computing power.

As a by-product of the partial classification given by Theorem 1.1, we obtain a sharp upper bound on the number of real tritangents to a real sextic.

Theorem 1.3 (see Section 9.3). — *The number of real tritangents to a real smooth (over $\mathbb{C}$) sextic $C \subset \mathbb{P}^2$ does not exceed* 66. *Up to real projective transformation, a smooth real sextic with* 66 *real tritangents is unique, see Section 9.1*(2)*.*

Remark 1.4. — At present, I do not know what other values are taken by the number $t_{\mathbb{R}}$ of real tritangents to a real smooth sextic. In the range $61 \leqslant t_{\mathbb{R}} \leqslant 65$, *among the known examples*, there is but one other configuration, with $t_{\mathbb{R}} = 63$ tritangents, see Section 9.1(6) (and Section 9.3 for the explanation and further remarks).

1.2. Contents of the paper

As in [5, 8], the line counting problem has a simple arithmetical reduction (see Theorem 2.1): one can effectively decide whether a given graph Γ can serve as the Fano graph of a polarized $K3$-surface. The candidates Γ to be tried were constructed in [5, 8] line by line, starting from a sufficiently large and sufficiently simple graph. Unfortunately, this straightforward approach seems to diverge in the case of degree 2, and we choose another one, viz. we replant the prospective Néron–Severi lattice $NS := \mathbb{Z}\Gamma/\ker$ to an appropriate Niemeier lattice. (The idea of embedding $h^\perp \subset NS$ to a Niemeier lattice is not new, cf. Kondō [12], Nikulin [19], Nishiyama [20], etc. The novelty is the fact that, as we need to keep track of the polarization h, we have to rebuild the *hyperbolic* lattice NS to embed it to a *definite* Niemeier lattice N. As a result, instead of counting roots in NS, we work with square 4 vectors in a certain root-free sublattice $S \subset N$; unlike [6], this lattice S or even its genus is not assumed fixed. This construction is explained in Section 2.4, see Proposition 2.2.) Then, instead of dealing with abstract graphs of a priori unbounded complexity, we merely need to consider subsets $\mathfrak{L}$ of several finite sets $\mathfrak{F}(\hbar)$ *known in advance*. The precise arithmetical conditions on the subsets $\mathfrak{L}$ that may serve as Fano graphs are stated in Section 2.5 and Section 2.6.

This approach has a number of advantages. First, for most 6-polarized Niemeier lattices $N \ni \hbar$ we have an immediate bound $|\mathfrak{L}| \leqslant 130$ (often even $|\mathfrak{F}(\hbar)| \leqslant 130$) obtained as explained in Section 4. Second, the sets $\mathfrak{F}(\hbar)$ have rich intrinsic structure, splitting into orbits and combinatorial orbits (see Section 3.1), which can be used in the construction of large geometric subsets: instead of building them line-by-line from scratch, we try to patch together precomputed close to maximal intersections with the combinatorial orbits. These algorithms are described in Section 3. Finally, since we are working with known sets, all symmetry groups can be expressed in terms of permutations, which makes the computation in `GAP` [10] extremely effective.

In Sections 5–8 we treat, one by one, the 23 Niemeier lattices rationally generated by roots, outlining the details of the computation in those few cases where the a priori upper bound $|\mathfrak{L}| \leqslant 130$ fails. In Section 9, we draw a formal punch-line, collecting together our findings for individual Niemeier lattices and completing the proofs of the principal results of the paper.

1.3. Acknowledgements

I would like to express my gratitude to Noam Elkies, Dino Festi, Dmitrii Pasechnik, Ichiro Shimada, and Davide Veniani for a number of fruitful discussions concerning the subject. This paper was completed during my research stay at the *Max–Planck–Institut für Mathematik*, Bonn; I am grateful to this institution for its hospitality and financial support.

2. The reduction

The tritangent problem is reduced to an arithmetical question about the Néron–Severi lattice $NS(X)$ of a smooth 2-polarized $K3$-surface X, see Theorem 2.1. The construction of Section 2.4, combined with Proposition 2.2, replants $NS(X)$ to a Niemeier lattice. The invertibility of this construction is discussed in Section 2.6.

2.1. Lattices (see [18])

The principal goal of this section is fixing the terminology and notation. A *lattice* is a free abelian group L of finite rank equipped with a symmetric bilinear form $b\colon L\otimes L\to\mathbb{Z}$. Since b is assumed fixed (and omitted from the notation), we abbreviate $x\cdot y:=b(x,y)$ and $x^2:=b(x,x)$. A lattice L is *even* if $x^2=0 \bmod 2$ for all $x\in L$; otherwise, L is *odd*. The *determinant* $\det L\in\mathbb{Z}$ is the determinant of the Gram matrix of b in any integral basis; L is called *nondegenerate* (*unimodular*) if $\det L\neq 0$ (respectively, $\det L=\pm1$). The *inertia indices* $\sigma_\pm L$ are those of $L\otimes\mathbb{R}$. A nondegenerate lattice L is called *hyperbolic* if $\sigma_+L=1$.

The *hyperbolic plane* is the only unimodular even lattice of rank 2. Explicitly, $\mathbf{U}=\mathbb{Z}a+\mathbb{Z}b$, where $a^2=b^2=0$ and $a\cdot b=1$. One has $\sigma_+\mathbf{U}=\sigma_-\mathbf{U}=1$.

A nondegenerate lattice L admits a canonical inclusion

$$L\hookrightarrow L^\vee:=\{x\in L\otimes\mathbb{Q}\,|\,x\cdot y\in\mathbb{Z}\text{ for all }y\in L\}$$

to the dual group $L^\vee$. The finite abelian group $\mathcal{L}:=\operatorname{discr}L:=L^\vee/L$ (q_L in [18]) is called the *discriminant group* of L. Clearly, $|\mathcal{L}|=(-1)^{\sigma_-L}\det L$. This group is equipped with the nondegenerate symmetric bilinear form

$$\mathcal{L}\otimes\mathcal{L}\to\mathbb{Q}/\mathbb{Z},\quad (x \bmod L)\otimes(y \bmod L)\mapsto(x\cdot y)\bmod\mathbb{Z},$$

and, if L is even, its quadratic extension

$$\mathcal{L}\to\mathbb{Q}/2\mathbb{Z},\quad x \bmod L\mapsto x^2 \bmod 2\mathbb{Z}.$$

We denote by $\mathcal{L}_p:=\operatorname{discr}_p L:=\mathcal{L}\otimes\mathbb{Z}_p$ the p-primary components of $\operatorname{discr}L$. The 2-primary component $\mathcal{L}_2$ is called *even* if $x^2\in\mathbb{Z}$ for all order 2 elements $x\in\mathcal{L}_2$; otherwise, $\mathcal{L}_2$ is *odd*. The *determinant* $\det\mathcal{L}_p$ is the determinant of the "Gram matrix" of the quadratic form in any minimal set of generators. (This is equivalent to the alternative definition given in [18].) Unless $p=2$ and $\mathcal{L}_2$ is odd (in which case the determinant is not defined or used), we have $\det\mathcal{L}_p=u_p/|\mathcal{L}_p|$, where u_p is a well-defined element of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^2$.

The *length* $\ell(\mathcal{A})$ of a finite abelian group $\mathcal{A}$ is the minimal number of generators of $\mathcal{A}$. We abbreviate $\ell_p(\mathcal{A}):=\ell(\mathcal{A}\otimes\mathbb{Z}_p)$ for a prime p.

Given a lattice L and $q\in\mathbb{Q}$, we use the notation $L(q)$ for the same abelian group with the form $x\otimes y\mapsto q(x\cdot y)$, assuming that it is still a lattice. We abbreviate $-L:=L(-1)$, and this notation applies to discriminant forms as well. The notation nL, $n\in\mathbb{Z}_+$, is used for the orthogonal direct sum of n copies of L.

A *root* in an even lattice L is a vector of square ±2. A *root system* is a positive definite lattice generated by roots. Any root system has a unique decomposition into orthogonal direct sum of irreducible components, which are of types $\mathbf{A}_n$, $n \geqslant 1$, $\mathbf{D}_n$, $n \geqslant 4$, $\mathbf{E}_6$, $\mathbf{E}_7$, or $\mathbf{E}_8$ (see, e.g., [1]), according to their *Dynkin diagrams*.

A *Niemeier lattice* is a positive definite unimodular even lattice of rank 24. Up to isomorphism, there are 24 Niemeier lattices (see [17]): the *Leech lattice* Λ, which is root free, and 23 lattices *rationally* generated by roots. In the latter case, the isomorphism class of a lattice $N := N(D)$ is uniquely determined by that of its maximal root system D. For more details, see [3].

2.2. The covering $K3$-surface

Given a smooth sextic curve $C \subset \mathbb{P}^2$, the double covering $\varphi\colon X \to \mathbb{P}^2$ ramified over C is a $K3$-surface. The "hyperplane section" $\varphi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is a 2-*polarization* of X, i.e., a complete fixed point free degree 2 linear system; it is viewed as an element

$$h \in \operatorname{Pic} X = NS(X) \subset H_2(X;\mathbb{Z}) \cong -2\mathbf{E}_8 \oplus 3\mathbf{U}.$$

Here, the group $H_2(X;\mathbb{Z}) = H^2(X;\mathbb{Z})$ is regarded as a lattice via the intersection form; it can be characterized as the only unimodular even lattice of rank 22 and signature $\sigma_+ - \sigma_- = -16$. The *Néron–Severi lattice* $NS(X) = H^{1,1}(X) \cap H_2(X;\mathbb{Z})$ is a primitive hyperbolic sublattice; in particular, $\rho(X) := \operatorname{rk} NS(X) \leqslant 20$.

Conversely, any 2-polarization h of a $K3$-surface X gives rise to a degree 2 map $\varphi_h\colon X \to \mathbb{P}^2$ ramified over a sextic curve $C \subset \mathbb{P}^2$ (see [21, 26]). This curve is smooth if and only if no (-2)-curve is contracted by φ_h, or, equivalently, there is no class $e \in NS(X)$ such that $e^2 = -2$ and $e \cdot h = 0$. With the ramification locus in mind, a 2-polarized $K3$-surface (X, h) with this extra property is called *smooth*.

A *line* in a 2-polarized $K3$-surface (X, h) is a smooth rational curve $L \subset X$ such that $L \cdot h = 1$. Any two distinct lines $L_1, L_2 \subset X$ either are disjoint, $L_1 \cdot L_2 = 0$, or intersect at a single point, $L_1 \cdot L_2 = 1$, or intersect at three points, $L_1 \cdot L_2 = 3$, the latter being the case if and only if L_1, L_2 are interchanged by the deck translation of the covering $\varphi_h\colon X \to \mathbb{P}^2$. Since, on the other hand, $L^2 = -2$, each line is unique in its homology class $[L] \in NS(X)$. Each 2-polarized $K3$-surface has finitely many lines (typically none). The *Fano graph* $\operatorname{Fn}(X, h)$ is the set of lines in X in which each pair of lines L_1, L_2 (regarded as vertices of the graph) is connected by an edge of multiplicity $L_1 \cdot L_2$ (i.e., no edge, simple edge, or triple edge).

Let $C \subset \mathbb{P}^2$ be a smooth sextic and $\varphi\colon X \to \mathbb{P}^2$ the covering $K3$-surface. If $L \subset \mathbb{P}^2$ is a tritangent to C, its pull-back $\varphi^{-1}(L)$ splits into two lines

L_1, L_2; they intersect at the three points of tangency of L and C (possibly, infinitely near) and are interchanged by the deck translation τ of φ. Conversely, any line in X projects to a tritangent to C. Thus, the set of tritangents to C is identified with $\operatorname{Fn} X/*$, where the free involution $*\colon \operatorname{Fn} X \to \operatorname{Fn} X$ induced by τ is intrinsic to the graph: it sends a vertex L to the only vertex connected to L by a triple edge.

2.3. The arithmetic reduction of the tritangent problem

Throughout this paper, by a 2-*polarized lattice* we mean a hyperbolic even lattice NS equipped with a distinguished class $h \in NS$, $h^2 = 2$. The *Fano graph* of a 2-polarized lattice $NS \ni h$ is the set

$$\operatorname{Fn}(NS, h) := \left\{ l \in NS \,\middle|\, l^2 = -2, l \cdot h = 1 \right\}$$

with two points (vertices) l_1, l_2 connected by an edge of multiplicity $l_1 \cdot l_2$. This graph is equipped with a natural involution

$$l \mapsto l^* := h - l;$$

the vertex l^*, called the *dual* of l, is connected to l by a triple edge.

Usually, we assume that the orthogonal complement $h^\perp \subset NS$ is root free. Under this additional assumption, for $l_1, l_2 \in \operatorname{Fn}(NS, h)$, one has

$$l_1 \cdot l_2 = 3\,(\text{iff } l_1 = l_2^*)\,, 1, 0, \text{ or } -2\,(\text{iff } l_1 = l_2)\,;$$

hence, all edges of $\operatorname{Fn}(NS, h)$ other than (l, l^*) are simple.

The following statement is well known: it follows from the global Torelli theorem for $K3$-surfaces [22], surjectivity of the period map [13], and Saint-Donat's results on projective $K3$-surfaces [26] (cf. also [8, Theorem 3.11] or [6, Theorem 7.3]).

Theorem 2.1. — *A graph Γ is the Fano graph of a smooth 2-polarized $K3$-surface if and only if $\Gamma \cong \operatorname{Fn}(NS, h)$ for some 2-polarized lattice $NS \ni h$ admitting a primitive embedding $NS \hookrightarrow -2\mathbf{E}_8 \oplus 3\mathbf{U}$ and such that $h^\perp \subset NS$ is root free.*

2.4. Embedding to a Niemeier lattice

Let $NS \ni h$ be a 2-polarized lattice. Consider the orthogonal complement $h^\perp \subset NS$. Each vector $l \in \operatorname{Fn}(NS, h)$ projects to $l' := l - \frac{1}{2}h \in (h^\perp)^\vee$, and, assuming $\operatorname{Fn}(NS, h) \neq \varnothing$, there is a unique index 2 extension

$$-S \supset h^\perp \oplus \mathbb{Z}\hbar, \quad \hbar^2 = -6, \tag{2.1}$$

containing all vectors $l' + \frac{1}{2}\hbar$, $l \in \mathrm{Fn}(NS, h)$. The lattice $S := S(NS, h)$ obtained from $-S$ by reverting the sign of the binary form is positive definite, and there is an obvious canonical bijection between $\mathrm{Fn}(NS, h)$ and the set

$$\mathfrak{L} = \mathfrak{L}(S, \hbar) := \left\{ l \in S \,\middle|\, l^2 = 4 \text{ and } l \cdot \hbar = 3 \right\};$$

the elements of $\mathfrak{L}$ are called *lines* in S. Furthermore, the sublattice $h^\perp \subset NS$ is root free if and only if so is $\hbar^\perp \subset S$; in this case, we call $S \ni \hbar$ *admissible*.

For the images $l_1, l_2 \in \mathfrak{L}$ of $l_1', l_2' \in \mathrm{Fn}(NS, h)$ one has $l_1 \cdot l_2 = 2 - l_1' \cdot l_2'$. Hence, if $S \ni \hbar$ is admissible, then, for $l_1, l_2 \in \mathfrak{L}$, one has

$$l_1 \cdot l_2 = -1 \,(\text{iff } l_1 = l_2^*)\,, 1, 2, \text{ or } 4 \,(\text{iff } l_1 = l_2)\,. \tag{2.2}$$

We will say that l_1, l_2 *intersect* (are *disjoint*) if $l_1 \cdot l_2 = 1$ (respectively, $l_1 \cdot l_2 = 2$). Accordingly, we regard $\mathfrak{L}$ as a graph, with two distinct vertices l_1, l_2 connected by a simple (triple) edge whenever $l_1 \cdot l_2 = 1$ (respectively, $l_1 \cdot l_2 = -1$.)

Proposition 2.2. — *Let $NS \ni h$ be a primitive 2-polarized sublattice of $-2\mathbf{E}_8 \oplus 3\mathbf{U}$, $\mathrm{Fn}(NS, h) \neq \varnothing$, and let $S := S(NS, h)$ be the lattice constructed as in* (2.1)*. Then*

1. *S admits a primitive embedding to a Niemeier lattice N;*
2. *S admits an embedding $S \hookrightarrow N$ to a Niemeier lattice such that the torsion of N/S is a 3-group and S is orthogonal to a root $\bar{r} \in N$.*

Proof. — Let $\rho := \mathrm{rk}\, NS$ and $\mathcal{N} := \mathrm{discr}\, NS$, so that $\ell(\mathcal{N}) \leqslant 22 - \rho$ by [18, Theorem 1.12.2]. Since $h \notin 2\, NS^\vee$ (by the assumption that $\mathrm{Fn}(NS, h) \neq \varnothing$), we have

$$\mathrm{discr}\, h^\perp = \left\langle \tfrac{1}{2} h \right\rangle \oplus \mathcal{N}, \quad \left(\tfrac{1}{2} h\right)^2 = \tfrac{3}{2} \bmod 2\mathbb{Z},$$

and the construction changes this to

$$\mathrm{discr}\, S = \left\langle \tfrac{1}{2} \hbar \right\rangle \oplus (-\mathcal{N}), \quad \left(\tfrac{1}{2} \hbar\right)^2 = \tfrac{2}{3} \bmod 2\mathbb{Z}.$$

In particular, $\ell(\mathrm{discr}\, S) \leqslant \ell(\mathcal{N}) + 1 < 24 - \rho$, and [18, Theorem 1.12.2] implies the existence of a primitive embedding $S \hookrightarrow N$. For the second statement, we compute

$$\mathcal{S} := \mathrm{discr}\,(S \oplus \mathbb{Z}\bar{r}) = \left\langle \tfrac{1}{2} \hbar \right\rangle \oplus \left\langle \tfrac{1}{2} \bar{r} \right\rangle \oplus (-\mathcal{N}), \quad \left(\tfrac{1}{2} \bar{r}\right)^2 = \tfrac{1}{2} \bmod 2\mathbb{Z}.$$

This time we have $\ell(\mathcal{S}_p) = \ell(\mathcal{N}_p) < 23 - \rho = 24 - \mathrm{rk}(S \oplus \mathbb{Z}\bar{r})$ for each prime $p > 3$, whereas $\ell(\mathcal{S}_p) = \ell(\mathcal{N}_p) + 1 \leqslant 24 - \mathrm{rk}(S \oplus \mathbb{Z}\bar{r})$ for $p = 2, 3$. Since $\mathcal{S}_2$ is odd, the possible equality does not impose any extra restriction at $p = 2$. For $p = 3$, in the case of equality, the "wrong" determinant

$\det(-\mathcal{S}_3) = -|\mathcal{S}| \bmod (\mathbb{Z}_3^\times)^2$ does inhibit the existence of a primitive embedding. However, since $\ell(\mathcal{S}_3) \geqslant 3$ in this case, we may pass to an iterated index 3 extension and reduce the length. □

2.5. Admissible sets

In the rest of the paper, we mainly use statement (2) of Proposition 2.2: it lets us avoid the Leech lattice, although at the expense of the possible imprimitivity (which makes some statements somewhat weaker and more complicated, see, e.g., Proposition 2.7 below). The idea is to construct a lattice S (or, rather, its set of lines) directly inside a Niemeier lattice. Thus, we fix a Niemeier lattice N, a square 6 vector $\hbar \in N$, and, optionally, a root $\bar{r} \in \hbar^\perp$ (which is typically omitted from the notation). Consider the set

$$\mathfrak{F} := \mathfrak{F}(\hbar) := \left\{ l \in N \;\middle|\; l^2 = 4, l \cdot \hbar = 3 \;\; (\text{and} \;\; l \cdot \bar{r} = 0) \right\}.$$

It is equipped with the involution

$$*\colon l \mapsto l^* := \hbar - l.$$

The elements of $\mathfrak{F}(\hbar)$ are called *lines*. The *span* of a subset $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ is the lattice

$$\operatorname{span} \mathfrak{L} := (\mathbb{Z}_3 \mathfrak{L} + \mathbb{Z}_3 \hbar) \cap N \subset N.$$

If $\mathfrak{L}$ is symmetric, $\mathfrak{L}^* = \mathfrak{L}$, the summation with $\mathbb{Z}_3 \hbar$ is redundant as $\hbar \in \mathbb{Z}\mathfrak{L}$. On a few occasions, we also consider the *integral* and *rational span*

$$\operatorname{span}_{\mathbb{Z}} \mathfrak{L} := (\mathbb{Z}\mathfrak{L} + \mathbb{Z}\hbar) \cap N \subset \operatorname{span} \mathfrak{L} \subset \operatorname{span}_{\mathbb{Q}} \mathfrak{L} := (\mathbb{Q}\mathfrak{L} + \mathbb{Q}\hbar) \cap N.$$

(The latter is primitive in N.) Via span, we extend to subsets $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ much of the terminology applied to lattices. Thus, we define the *rank* of $\mathfrak{L}$ as $\operatorname{rk} \mathfrak{L} := \operatorname{rk} \operatorname{span} \mathfrak{L}$, and we say that $\mathfrak{L}$ is *generated* by a subset $\mathfrak{L}' \subset \mathfrak{L}$ if $\mathfrak{L} = \mathfrak{F}(\hbar) \cap \operatorname{span} \mathfrak{L}'$.

By definition, the torsion of $N/\operatorname{span} \mathfrak{L}$ is a 3-group and $\hbar \in 3(\operatorname{span} \mathfrak{L})^\vee$. A finite index extension $S \supset \operatorname{span} \mathfrak{L}$ is called *mild* if

$$S \subset \{ v \in N \mid v \cdot \hbar = 0 \bmod 3 \}$$

(i.e., $S \subset N$ and still $\hbar \in 3S^\vee$) and S contains no roots $r \in \hbar^\perp \subset N$. Note that the latter condition is equivalent to the requirement that S itself should be root free. Indeed, since S is positive definite and $\hbar \in 3S^\vee$, we have $r \cdot \hbar = 0$ or ± 3 for any root $r \in S$, and in the latter case $\hbar \mp r$ is a root in $\hbar^\perp$.

Definition 2.3. — *A subset* $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ *is called* admissible *if*

(1) $\mathfrak{L}$ *is* symmetric (*or* $*$*-invariant*), *i.e.*, $\mathfrak{L}^* = \mathfrak{L}$, *and*
(2) *the sublattice* $\hbar^\perp \cap \operatorname{span} \mathfrak{L}$ *contains no roots.*

A subset $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ *is* complete *if* $\mathfrak{L} = \mathfrak{F}(\hbar) \cap \operatorname{span} \mathfrak{L}$. *A subset* $\mathfrak{L}$ *is* saturated *if the identity* $\mathfrak{L} = \mathfrak{F}(\hbar) \cap S$ *holds for any mild extension* $S \supset \operatorname{span} \mathfrak{L}$. *Finally, we say that* $\mathfrak{L}$ *is* $\mathbb{Q}$-complete *if* $\mathfrak{L} = \mathfrak{F}(\hbar) \cap \operatorname{span}_{\mathbb{Q}} \mathfrak{L}$.

Often, it is easier to check (2.2), which follows from (1), (2) above. Indeed, since S is definite, we have $-1 \leqslant l_1 \cdot l_2 \leqslant 4$. Thus, forbidden are $l_1 \cdot l_2 = 3$ or 0, as then $l_1 - l_2$ or $l_1 - l_2^* = l_1 + l_2 - \hbar$, respectively, would be a root in $\hbar^\perp$.

The following bound is due to N. Elkies.

Theorem 2.4 (N. Elkies [9]). — *Let* V *be a Euclidean vector space,* $\dim V = n$, *and let* $v_1, \ldots, v_N \in V$ *be a collection of unit vectors such that the products* $v_i \cdot v_j$, $i \neq j$, *take but two values* τ_1, τ_2. *Assume that* $\tau_1 + \tau_2 \leqslant 0$ *and* $1 + \tau_1 \tau_2 n > 0$. *Then*

$$N \leqslant \frac{(1-\tau_1)\,(1-\tau_2)\,n}{1+\tau_1\tau_2 n}.$$

Selecting a single vector from each pair $l, l^* \in \mathfrak{L}$ and applying Theorem 2.4 to the normalized projections to $\hbar^\perp \subset \operatorname{span} \mathfrak{L}$, we arrive at the following corollary.

Corollary 2.5 (N. Elkies [9]). — *The size of an admissible set* $\mathfrak{L}$ *is bounded* via

$$|\mathfrak{L}| \leqslant \frac{48\,(\operatorname{rk} \mathfrak{L} - 1)}{26 - \operatorname{rk} \mathfrak{L}}.$$

Since $|\mathfrak{L}|$ is even, this implies that $|\mathfrak{L}| \leqslant 152$ or 122 for $\operatorname{rk} \mathfrak{L} = 20$ or 19, respectively.

2.6. Geometric sets

According to Theorem 2.1 and Proposition 2.2, the Fano graph of any smooth 2-polarized $K3$-surface X can be represented as a complete admissible subset $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ for an appropriate pair $\hbar, \bar{r} \in N$ as in Section 2.5.

For some lattices (those with few roots), the admissibility condition is not enough to eliminate large sets of lines, and we need to use the full range of restrictions.

Recall that we start with the Néron–Severi lattice $NS(X) \ni h$, and as long as the configurations of lines are concerned, we can assume this lattice rationally generated by lines. Indeed, let $N := \mathbb{Q}\,\mathrm{Fn}(X) \cap NS(X)$; clearly, $h \in N$. We can pick a vector $\omega \in (N^\perp) \otimes \mathbb{C}$, $\omega^2 = 0$, in the same component of the positive cone as the period (class of a holomorphic 2-form) ω_X of X, and such that $\omega^\perp \cap H_2(X) = N$. (This condition merely means that ω is generic; it can be chosen arbitrary close to ω_X.) By the surjectivity of the period map [13], there is a $K3$-surface $X' \to \mathbb{P}^2$ such that $\omega_{X'} = \omega$, so that $NS(X') = N$. (The fact that $h \in N$ defines a map $X' \to \mathbb{P}^2$ with a smooth ramification locus follows from that for X, cf. [26] or Section 2.2.) Clearly, we have $\mathrm{Fn}(X') = \mathrm{Fn}(X)$, cf. Section 2.2, and the lines in X' generate (over $\mathbb{Q}$) its Néron–Severi lattice N by the construction.

Thus, assuming that $NS(X) \ni h$ is rationally generated by lines, we can pass to the positive definite lattice $S \ni \hbar$ as in (2.1) and embed the latter to a Niemeier lattice N, mapping $\mathrm{Fn}(X)$ bijectively onto the admissible set $\mathfrak{L} = \mathfrak{F}(\hbar) \cap S$. Most steps of this construction are invertible. However, starting from an admissible set $\mathfrak{L} \subset \mathfrak{F}(\hbar)$, we may have to take for S a mild extension of $\operatorname{span}\mathfrak{L}$ rather than $\operatorname{span}\mathfrak{L}$ itself and, still, we cannot guarantee that the lattice NS obtained from S by the backward construction admits a primitive embedding to $H_2(X) \cong -2\mathbf{E}_8 \oplus 3\mathbf{U}$. This discussion motivates the following definition.

Definition 2.6. — *An admissible set $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ is called* geometric *if $\mathfrak{L}$ is complete in some mild extension $S \supset \operatorname{span}\mathfrak{L}$ such that the lattice NS obtained from $S \ni \hbar$ by the inverse of construction* (2.1) *admits a primitive embedding to $-2\mathbf{E}_8 \oplus 3\mathbf{U}$.*

Using [18, Theorem 1.12.2], one can recast this property as follows. (For a *mild* extension $S \supset \operatorname{span}\mathfrak{L}$ there is a splitting $\operatorname{discr} S = \langle \frac{1}{2}\hbar \rangle \oplus \mathcal{T}$, and we merely restate the restrictions on $\mathcal{T} \cong -\operatorname{discr} NS$ in terms of $\operatorname{discr} S$.)

Proposition 2.7. — *For $N \ni \hbar$ as above, an admissible set $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ is geometric if and only if:*

(1) $\operatorname{rk}\mathfrak{L} \leqslant 20$ *; we denote $\delta := 22 - \operatorname{rk}\mathfrak{L} \geqslant 2$, and*

there is a mild extension $S \supset \operatorname{span}\mathfrak{L}$ in which $\mathfrak{L}$ is a complete subset and such that the discriminant $\mathcal{S} := \operatorname{discr} S$ has the following properties at each prime p:

(2) *if $p > 3$, then $\ell(\mathcal{S}_p) < \delta$ or $\ell(\mathcal{S}_p) = \delta$ and $\det \mathcal{S}_p = 3|\mathcal{S}| \bmod (\mathbb{Q}_p^\times)^2$;*
(3) *$\ell(\mathcal{S}_2) < \delta$ or $\ell(\mathcal{S}_2) = \delta$ and $\mathcal{S}_2$ is odd or $\det \mathcal{S}_2 = \pm 3|\mathcal{S}| \bmod (\mathbb{Q}_2^\times)^2$;*
(4) *$\ell(\mathcal{S}_3) \leqslant \delta$ or $\ell(\mathcal{S}_3) = \delta + 1$ and $\det \mathcal{S}_3 = |\mathcal{S}| \bmod (\mathbb{Q}_3^\times)^2$.*

Remark 2.8. — In practice, when eliminating large admissible sets, we use just a few simple consequences of Section 2.7. The main rôle is played by condition (1), see Section 3.2.1 below. Then, conditions (2) and (3) are used, as they apply directly to the original discriminant $\operatorname{discr}_p(\operatorname{span}\mathfrak{L}) = \mathcal{S}_p$, $p \neq 3$. Condition (4) is typically used when there is an obvious maximal mild extension, and we never insist that $\mathfrak{L}$ should be complete in S, thus eliminating both $\mathfrak{L}$ itself and all its oversets.

3. The approach

Throughout this section, we consider a Niemeier lattice $N := N(D)$ generated over $\mathbb{Q}$ by a fixed root system $D = \bigoplus_k D_k$, $k \in \Omega$, where D_k are the irreducible components (*aka* Dynkin diagrams) and Ω is the index set. We construct N as a subgroup of $\bigoplus_i D_i^\vee$; the vectors in

$$\operatorname{discr} D := D^\vee/D = \bigoplus_k \operatorname{discr} D_k$$

that are declared "integral" are as described in [3, Table 16.1]. (We also use the convention of [3] for the numbering of the discriminant classes of irreducible root systems.) We denote by $O := O(N)$ the full orthogonal group of N, and by $R := R(N) \subset O(N)$ its subgroup generated by reflections. Both groups preserve D; the reflection group $R(N)$ preserves each D_k and acts identically on $\operatorname{discr} D$.

3.1. Notation

We fix a square 6 vector $\hbar \in N$ and, sometimes, a root $\bar{r} \in D$ orthogonal to $\hbar$. (This root is usually omitted from the notation.) We denote by $O_\hbar(N) \subset O(N)$ and $R_\hbar(N) \subset R(N)$ the subgroups stabilizing $\hbar$ (and $\bar{r}$). Let

$$\mathfrak{F} = \mathfrak{F}(\hbar) = \mathfrak{F}(\hbar, \bar{r}) := \left\{ l \in N \,\middle|\, l^2 = 4, l \cdot \hbar = 3 \text{ (and } l \cdot \bar{r} = 0) \right\}$$

be the set of lines. This set splits into a number of $O_\hbar(N)$-orbits $\bar{\mathfrak{o}}_n$, which split further into $R_\hbar(N)$-orbits $\mathfrak{o} \subset \bar{\mathfrak{o}}_n$; the latter are called *combinatorial orbits*. It is immediate that the duality $l \mapsto l^*$ preserves orbits and combinatorial orbits; hence, we can speak about the dual orbits $\bar{\mathfrak{o}}_n^*$ and $\mathfrak{o}^*$. The number of combinatorial orbits in an orbit $\bar{\mathfrak{o}}_n$ is denoted by $m(\bar{\mathfrak{o}}_n)$. The set of all combinatorial orbits is denoted by $\mathfrak{O} := \mathfrak{O}(\hbar)$. This set inherits a natural action of the group

$$\operatorname{stab} \hbar := O_\hbar(N)/R_\hbar(N),$$

which preserves each orbit $\bar{\mathfrak{o}}_n$. (By an obvious abuse of notation, occasionally we treat $\bar{\mathfrak{o}}_n$ as a subset of $\mathfrak{O}$; likewise, subsets of $\mathfrak{O}$ are sometimes treated as sets of lines.) We denote by $\operatorname{Orb}_m(\bar{\mathfrak{o}}_n, k)$ the length m orbit of the action of $\operatorname{stab}\hbar$ on the set of unordered $*$-invariant (if so is $\bar{\mathfrak{o}}_n$) k-tuples of combinatorial orbits $\mathfrak{o} \subset \bar{\mathfrak{o}}_n$. The usage of this notation implies implicitly that such an orbit is unique.

The *support* of a vector $v \in N = N(D)$ is the subset

$$\operatorname{supp} v := \left\{ k \in \Omega \,\middle|\, v_k \neq 0 \in D_k^\vee \right\} \subset \Omega.$$

The support is invariant under reflections; hence, we can speak about the *support* $\operatorname{supp}\mathfrak{o}$ of a combinatorial orbit $\mathfrak{o}$.

The *count* and *bound* of a combinatorial orbit $\mathfrak{o}$ are defined via

$$c(\mathfrak{o}) := |\mathfrak{o}|, \qquad b(\mathfrak{o}) := \max\left\{ |\mathfrak{L} \cap \mathfrak{o}| \mid \mathfrak{L} \subset \mathfrak{F} \text{ is geometric} \right\}. \tag{3.1}$$

Clearly, c and b are constant within each orbit $\bar{\mathfrak{o}}_n$ and invariant under duality. In some cases, we replace $b(\mathfrak{o})$ by rough bounds, see Section 4.4 below for details. We extend these notions to subsets $\mathfrak{C} \subset \mathfrak{O}$ *by additivity*:

$$c(\mathfrak{C}) := \sum_{\mathfrak{o} \in \mathfrak{C}} c(\mathfrak{o}), \qquad b(\mathfrak{C}) := \sum_{\mathfrak{o} \in \mathfrak{C}} b(\mathfrak{o}).$$

Thus, we have a naïve a priori bound

$$|\mathfrak{L}| \leqslant b(\mathfrak{O}) = \sum m(\bar{\mathfrak{o}}_n) b(\mathfrak{o}), \quad \mathfrak{o} \subset \bar{\mathfrak{o}}_n. \tag{3.2}$$

Clearly, the true count $|\mathfrak{L} \cap \mathfrak{C}|$ is genuinely additive, whereas the sharp bound on $|\mathfrak{L} \cap \mathfrak{C}|$ is only subadditive; thus, our proof of Theorem 1.1 will essentially consist in reducing (3.2) down to a preset goal. To this end, we will consider the set

$$\mathcal{B} = \mathcal{B}(\mathfrak{F}) := \{ \mathfrak{L} \subset \mathfrak{F} \mid \mathfrak{L} \text{ is geometric} \} / O_\hbar(N)$$

and, for a collection of orbits $\mathfrak{C} = \bar{\mathfrak{o}}_1 \cup \ldots$ and integer $d \in \mathbb{N}$, let

$$\mathcal{B}_d(\mathfrak{C}) := \{ [\mathfrak{L}] \in \mathcal{B} \mid \mathfrak{L} \text{ is generated by } \mathfrak{L} \cap \mathfrak{C} \text{ and } |\mathfrak{L} \cap \mathfrak{C}| \geqslant b(\mathfrak{C}) - d \}.$$

Unless specified otherwise, the sets $\mathcal{B}_d(\mathfrak{C})$ (for reasonably small values of d) are computed by brute force, using patterns (see Section 3.3 below).

3.2. Idea of the proof

To prove Theorem 1.1, we consider, one by one, all 23 Niemeier lattices generated by roots. For each lattice N, we set a goal

$$|\mathfrak{L}| \geqslant M := 122 \text{ or } 132 \tag{3.3}$$

and try to find all geometric subsets $\mathfrak{L} \subset N$ satisfying this inequality. First, we list all $O(N)$-orbits of square 6 vectors $\hbar \in N$, compute the naïve bounds $b(\mathfrak{O})$ given by (3.2), and disregard those vectors for which $b(\mathfrak{O}) < M$. In the remaining cases, we list all $O_\hbar(N)$-orbits of roots $\bar{r}$ orthogonal to $\hbar$ and repeat the procedure. This leaves us with relatively few triples $\hbar, \bar{r} \in N$, which are treated on a case-by-case basis in Sections 5–refS.24A1 below.

A typical argument runs as follows. We choose a self-dual union $\mathfrak{C}$ of orbits $\bar{\mathfrak{o}}_n$ and use *patterns* (see Section 3.3 below) to compute the set $\mathcal{B}_{b(\mathfrak{O})-M}(\mathfrak{C})$. (As a modification, we take $\mathfrak{C}$ *disjoint* from its dual $\mathfrak{C}^*$ and use the obvious relation $\mathcal{B}_d(\mathfrak{C}) = \mathcal{B}_{2d}(\mathfrak{C} \cup \mathfrak{C}^*)$.) More generally, we can consider several pairwise disjoint self-dual unions of orbits $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$ and compute the sets $\mathcal{B}_{d_i}(\mathfrak{C}_i)$ for appropriately chosen integers $d_i \geqslant 0$ such that

$$d_1 + \ldots + d_m + 2(m-1) \geqslant b(\mathfrak{O}) - M.$$

As a result of this procedure, we can assert that, apart from a few explicitly listed exceptions $\mathfrak{L}_1, \ldots, \mathfrak{L}_s$ contained in the above sets $\mathcal{B}_{d_i}(\mathfrak{C}_i)$, we have $|\mathfrak{L}| < M$ for any geometric set $\mathfrak{L} \subset \mathfrak{F}$. In each case, we manage to choose the unions $\mathfrak{C}_i$ and goals d_i so that the exceptional sets $\mathfrak{L}_k$ are sufficiently large, so that they can be analysed further as explained below.

3.2.1. Maximal sets

The best case scenario is that of a *maximal* (with respect to inclusion, in the class of geometric sets) geometric set $\mathfrak{L}$. Such a set admits no geometric extensions; hence, it can be either discarded, if $|\mathfrak{L}_k| < M$, or listed as an exception in the respective statement. Besides, maximal sets can be discarded at early stages of the computation, without completing the whole pattern; however, we only use this approach in Section 8.3, where intermediate lists grow too large.

An obvious sufficient condition of maximality is given by Proposition 2.7.

Lemma 3.1. — *Any maximal geometric set is saturated. Conversely, any saturated geometric set $\mathfrak{L}$ of the maximal rank* $\operatorname{rk}\mathfrak{L} = 20$ *is maximal.*

3.2.2. Extension by a maximal orbit

If a set $\mathfrak{L} \in \mathcal{B}_d(\mathfrak{C})$ is not maximal, we try to list its geometric extensions $\mathfrak{L}' \supset \mathfrak{L}$ satisfying (3.3). Clearly, it suffices to consider $\mathfrak{C}$-*proper* extensions only, *i.e.*, those with the property that $\mathfrak{L}' \cap \mathfrak{C} = \mathfrak{L} \cap \mathfrak{C}$. Thus, we merely extend the partial pattern

$$\pi\colon \mathfrak{C} \to N, \quad \mathfrak{o} \mapsto |\mathfrak{L} \cap \mathfrak{o}|,$$

(see Section 3.3 below) used to construct $\mathfrak{L}$ by a few (usually one or at most two) extra values $\pi(\mathfrak{o})$, $\mathfrak{o} \in \mathfrak{O} \smallsetminus \mathfrak{C}$.

In many cases, a geometric set $\mathfrak{L} \in \mathcal{B}_d(\mathfrak{C})$ has the property that

$$\sum \bigl(b(\mathfrak{o}) - b'(\mathfrak{o})\bigr) \geqslant b(\mathfrak{O}) - M, \qquad \mathfrak{o} \in \mathfrak{O}_\delta := \bigl\{\mathfrak{o} \in \mathfrak{O} \bigm| |\mathfrak{L} \cap \mathfrak{o}| < b(\mathfrak{o})\bigr\},$$

where $b'(\mathfrak{o})$ is the second largest value taken by $|\mathfrak{L}' \cap \mathfrak{o}|$ with $\mathfrak{L}'$ admissible. This implies that any admissible extension $\mathfrak{L}' \supset \mathfrak{L}$ satisfying (3.3) must have maximal intersection, $|\mathfrak{L}' \cap \mathfrak{o}| = b(\mathfrak{o})$, for at least one orbit $\mathfrak{o} \in \mathfrak{O}_\delta$. Trying these orbits one by one (i.e., extending the pattern via $\pi(\mathfrak{o}) = b(\mathfrak{o})$), we obtain larger sets, which are usually maximal. (This computation uses patterns, see Section 3.3 below, and takes into account the symmetry of $\mathfrak{L}$.)

3.2.3. Other extensions

In the few remaining cases, we either analyze the lines contained in $\operatorname{span}_{\mathbb{Q}} \mathfrak{L}$ (if $\operatorname{rk} \mathfrak{L} = 20$) or obtain maximal $\mathfrak{C}$-proper extensions $\mathfrak{L}' \supset \mathfrak{L}$ by adding one or, rarely, two extra lines.

3.3. Patterns

Since we are interested in large geometric sets, we construct them orbit-by-orbit, by stacking together maximal or close to maximal intersections $\mathfrak{L} \cap \mathfrak{o}$. This process is guided by *patterns*, i.e., $*$-invariant functions

$$\pi \colon \mathfrak{O} \to \mathbb{N}, \quad \mathfrak{o} \mapsto |\mathfrak{L} \cap \mathfrak{o}|\,.$$

Having $\hbar, \bar{r} \in N$ fixed, we start with precomputing all geometric sets $\mathfrak{L} \subset \mathfrak{o}$ in each combinatorial orbit $\mathfrak{o}$. (Certainly, it suffices to consider one representative in each orbit $\bar{\mathfrak{o}}_n$; the rest is obtained by translations.) Then, in order to compute one of the sets $\mathcal{B}_d(\mathfrak{C})$ in Section 3.1, we list all $(\operatorname{stab} \hbar)$-orbits of restricted patterns $\pi \colon \mathfrak{C} \to \mathbb{N}$ satisfying the inequality $\sum \pi(\mathfrak{o}) \geqslant b(\mathfrak{C}) - d$, $\mathfrak{o} \in \mathfrak{C}$, order the orbits appropriately (typically, by the decreasing of $\pi(\mathfrak{o})$), and construct a geometric set $\mathfrak{L}$ by adding one orbit at a time, as a sequence $\varnothing = \mathfrak{L}_0 \subset \mathfrak{L}_1 \subset \mathfrak{L}_2 \subset \ldots$. At each step k and for each set $\mathfrak{L}_{k-1}$ constructed at the previous step, we proceed as follows:

(1) compute the stabilizer G of $\mathfrak{L}_{k-1}$ under the action of $R_\hbar(N)$;
(2) find the G-orbits of the geometric sets $\mathfrak{L}' \subset \mathfrak{o}_k$ of size $|\mathfrak{L}'| = \pi(\mathfrak{o}_k)$;
(3) for a representative $\mathfrak{L}'$ of each G-orbit, consider the set $\mathfrak{L}_k$ generated by the union $\mathfrak{L}_{k-1} \cup \mathfrak{L}'$; then, select those sets $\mathfrak{L}_k$ that are geometric;

(4) to reduce the overcounting, select, for the next step, those sets $\mathfrak{L}_k$ that satisfy the equality $|\mathfrak{L}_k \cap \mathfrak{o}_i| = \pi(\mathfrak{o}_i)$ for each $i \leqslant k$.

If the defect d is not too large, this procedure works reasonably fast and results in a reasonably small collection of sets that are to be analyzed further.

Remark 3.2. — Although it is not obvious a priori, it turns out that large geometric sets are often determined by their patterns uniquely up to $R_{\hbar}(N)$. Furthermore, a large set is easily reconstructed from its pattern, as the algorithm above converges very fast. For this reason, we often describe large geometric sets, especially those that are *not* $\mathbb{Q}$-complete (see Definition 2.3), by their patterns.

A pattern π taking a constant value v_n on each orbit $\bar{\mathfrak{o}}_n$ is described via

$$\pi = \langle\!\langle v_1, v_2, \ldots \rangle\!\rangle.$$

Sometimes, we use a "double value" $v_n = a|b$; this means that a cluster $\mathfrak{c}_n \subset \bar{\mathfrak{o}}_n$ is fixed (and described elsewhere) so that the restriction of π to $\bar{\mathfrak{o}}_n$ takes two values: $\pi(\mathfrak{o}) = a$ for $\mathfrak{o} \subset \mathfrak{c}_n$ and $\pi(\mathfrak{o}) = b$ for $\mathfrak{o} \subset \bar{\mathfrak{o}}_n \smallsetminus \mathfrak{c}_n$.

Remark 3.3. — In some cases, where $b(\mathfrak{O})$ exceeds the goal by just a few units, we use patterns to show directly that $\mathcal{B}_{b(\mathfrak{O})-M}(\mathfrak{O}) = \varnothing$. These cases are marked with a ✓ in the tables, and any further explanation is typically omitted (and so usually is the list of orbits).

3.4. Clusters

Sometimes, the number of combinatorial orbits in an orbit $\bar{\mathfrak{o}}$ is too large, making it difficult to compute all patterns. In these cases, we subdivide $\bar{\mathfrak{o}}$ into a number of *clusters* $\mathfrak{c}_k \subset \bar{\mathfrak{o}}$, not necessarily disjoint, and compute patterns and, then, geometric sets cluster by cluster. The subdivision is chosen so that $\operatorname{stab} \hbar$ acts transitively on the set of clusters. To reduce the overcounting, we assume that the clusters are ordered lexicographically, by the decreasing of the sequence

$$\bigl(|\mathfrak{L} \cap \mathfrak{c}_k|, \delta_0(\mathfrak{c}_k), \delta_1(\mathfrak{c}_k), \ldots\bigr),$$
$$\delta_i(\mathfrak{c}_k) := \#\bigl\{\mathfrak{o} \subset \mathfrak{c}_k \bigm| |\mathfrak{L} \cap \mathfrak{o}| = b(\mathfrak{o}) - i\bigr\}.$$

In particular, this convention implies that, when computing the set $\mathcal{B}_d(\bar{\mathfrak{o}})$, for the first cluster $\mathfrak{c}_1$ one must have $|\mathfrak{L} \cap \mathfrak{c}_1| \geqslant b(\mathfrak{c}_1) - md/n$, where n is the total number of clusters and the *multiplicity* m is the number of clusters containing any fixed orbit $\mathfrak{o} \subset \bar{\mathfrak{o}}$. More generally, extending a geometric set $\mathfrak{L}$ from $\mathfrak{c}_1, \ldots, \mathfrak{c}_k$ to the next cluster $\mathfrak{c}_{k+1}$, one must have $|\mathfrak{L} \cap \mathfrak{c}_{k+1}| \leqslant |\mathfrak{L} \cap \mathfrak{c}_k|$ and

$$|\mathfrak{L} \cap \mathfrak{c}_{k+1}| \geqslant b(\mathfrak{c}_{k+1}) - \frac{1}{n-k}\left(md - \sum_{i=1}^{k}(b(\mathfrak{c}_i) - |\mathfrak{L} \cap \mathfrak{c}_i|)\right).$$

Certainly, if the clusters are not disjoint, we also take into account the intersections $\mathfrak{c}_{k+1} \cap \mathfrak{c}_i$, $i = 1, \ldots, k$, when computing the restricted patterns $\pi\colon \mathfrak{c}_{k+1} \to \mathbb{N}$.

4. Counts and bounds

In this section, we explain the computation of the bounds $b(\mathfrak{o})$ on the number of lines within a combinatorial orbit $\mathfrak{o}$, see (3.1).

4.1. Blocks

Consider a combinatorial orbit $\mathfrak{o}$. In order to estimate the count $c(\mathfrak{o})$ and bound $b(\mathfrak{o})$, we break the root system D into *blocks*, $D = B_1 \oplus B_2 \oplus \ldots$, each block B_k consisting of whole components D_i. Then, $\hbar$ and $l \in \mathfrak{F}(\hbar) \cap \mathfrak{o}$ split into $\bigoplus_k \hbar_k$ and $\bigoplus_k l_k$, respectively, with $\hbar_k, l_k \in B_k^\vee$. We denote by $\mathfrak{o}|_k := \mathfrak{o}|_{B_k} \subset B_k^\vee$ the *restriction* of $\mathfrak{o}$ to B_k (which, in fact, is nothing but the orthogonal projection of $\mathfrak{o}$ to $B_k^\vee$). This restriction consists of a whole $R_{\hbar_k}(B_k)$-orbit of vectors; in particular, we have a well defined square $l_k^2 \in \mathbb{Q}$, product $l_k \cdot \hbar_k \in \mathbb{Q}$, and discriminant class $l_k \bmod B_k \in \operatorname{discr} B_k$. Usually, these data determine an irreducible block up to isomorphism, the reason being the following simple observation (which follows from the fact that *all* roots in N are assumed to lie in D):

- each vector $l_k \in \mathfrak{o}|_k$ is either integral, $l_k \in B_k$ (and then $l_k^2 = 0$, 2, or 4) or *shortest vector* in its discriminant class;
- each vector $\hbar_k$ is either integral, $\hbar_k \in B_k$ (and then $\hbar_k^2 = 0$, 2, 4, or 6) or shortest or *second shortest* vector in its discriminant class.

Here, *shortest* are the vectors minimizing the square within a given discriminant class, whereas *second shortest* are those of square (minimum $+\,2$). In fact, $\hbar_k$ can be a second shortest vector in at most one block B_k.

The *count* of a block B is defined in the obvious way: $c(B) = |\mathfrak{o}|_B|$. The *bound* is defined *via* $b(B) = \max|\mathfrak{B}|$, where $\mathfrak{B} \subset \mathfrak{o}|_B$ is a $*$-invariant (if $\mathfrak{o}^* = \mathfrak{o}$) subset satisfying the following condition: for $l', l'' \in \mathfrak{B}$, one has

$$l'^2 - l' \cdot l'' = 0\,(\text{iff}\ \ l' = l''), 2, 3,\ \text{or}\ 5\,(\text{iff}\ \ l' = (l'')^*)\,. \tag{4.1}$$

In other words, we bound the cardinality of subsets $\mathfrak{L} \subset \mathfrak{o}$ satisfying (2.2) and such that all lines $l \in \mathfrak{L}$ have the same fixed restriction to all other blocks $B' \neq B$.

If D is broken into two blocks, $B_1 \oplus B_2$, we obviously have

$$(4.2) \qquad c(\mathfrak{o}) = c(B_1)c(B_2), \qquad b(\mathfrak{o}) \leqslant \min\bigl\{c(B_1)b(B_2), b(B_1)c(B_2)\bigr\}.$$

By induction, for any number of blocks B_k, this implies

$$(4.3) \qquad c(\mathfrak{o}) = \prod_k c(B_k), \qquad b(\mathfrak{o}) \leqslant c(\mathfrak{o}) \min_k \frac{b(B_k)}{c(B_k)}.$$

This bound (with $B_k = D_k$ the irreducible components of D) and corresponding bound on $b(\mathfrak{O})$ given by (3.2) are always listed first in the tables below. If $b(\mathfrak{O}) \geqslant M$, we try to improve the bounds $b(\mathfrak{o})$ using one of the following arguments:

(1) Lemma 4.1 below applied to an appropriate splitting in two blocks;
(2) a computation using larger blocks, see Section 4.2 below;
(3) a brute force enumeration of admissible subsets $\mathfrak{L} \subset \mathfrak{o}$; the bounds whose sharpness is confirmed by this computation are underlined.

In the tables, we refer to this list for the reasons for the improved bounds.

4.2. Brute force *via* blocks

For some large combinatorial orbits $\mathfrak{o}$, the exact computation of $b(\mathfrak{o})$ by brute force is not feasible, and we improve the original bound given by (4.3) by using larger blocks. Typically, we consider two blocks B_1 (one of the irreducible components of D) and B_2 (the sum of all other components). Then, we compute all *admissible* (rather than just satisfying (4.1)) sets $\mathfrak{L}(l_1) \subset \mathfrak{o}$ with a fixed restriction $l_1 \in B_1^\vee$, replacing (4.2) with

$$b(\mathfrak{o}) \leqslant c(B_1) \max|\mathfrak{L}(l_1)|.$$

If this bound is still not good enough, we vary $l_1 \in B_1^\vee$ and try to construct an admissible set $\mathfrak{L} \subset \mathfrak{o}$ by packing together precomputed *large* (usually maximal or submaximal) sets $\mathfrak{L}(l_1')$, $\mathfrak{L}(l_1'')$, etc., obtaining a better bound and, if necessary, a complete list of large admissible sets in $\mathfrak{o}$.

4.3. Self-dual combinatorial orbits

Let $\mathfrak{o}$ be a self-dual combinatorial orbit, $\mathfrak{o}^* = \mathfrak{o}$, and break D into blocks B_k. Each block is also self-dual: $\bar{l}_k := \hbar_k - l_k \in \mathfrak{o}|_k$ whenever $l_k \in \mathfrak{o}|_k$.

In particular, $\bar{l}_k = l_k \bmod B_k$. Hence, we have

$$\begin{aligned} 2l_k \cdot \hbar_k &= \hbar_k^2 \quad \left(\text{since } \left(\bar{l}_k^2 = l_k^2\right)\right), \\ l_k \cdot \bar{l}_k &= l_k^2 - \delta_k \quad \text{for some } \delta_k \in \mathbb{Z}. \end{aligned}$$

The integer $\delta(B_k) := \delta_k = 2l_k^2 - l_k \cdot \hbar_k$, constant throughout the block, is called the *defect* of the block B_k; it takes values in the range $0 \leqslant \delta_k \leqslant 5$, and the defects of all blocks sum up to $5 = l^2 - l \cdot l^*$. Furthermore, for any pair of vectors $l', l'' \in \mathfrak{o}|_k$, the difference $l_k^2 - l' \cdot l''$ is an integer taking values in

$$0 \leqslant l_k^2 - l' \cdot l'' \leqslant \delta_k, \tag{4.4}$$

the two extreme values corresponding to $l'' = l'$ and $l'' = \bar{l}'$, respectively. As a consequence, we have $b(B_k) \leqslant 1$ if $\delta(B_k) = 1$ and $b(B_k) \leqslant 2$ if $\delta(B_k) = 2$; in the latter case, all maximal admissible subsets are of the form $\{l_k, \bar{l}_k\}$.

Lemma 4.1. — *Assume that a self-dual orbit $\mathfrak{o}$ is broken into two blocks, B_2 and B_3, of defects 2 and 3, respectively. Then*

$$b(\mathfrak{o}) \leqslant \max\left\{4u + \min\left\{c_3 - 2u, (c_2 - 2u)b_3\right\} \,\middle|\, u = 0, \ldots, \tfrac{1}{2}\min\left\{c_2, c_3\right\}\right\},$$

where we abbreviate $c_\delta := c(B_\delta)$ and $b_\delta := b(B_\delta)$, $\delta = 2, 3$.

Proof. — Let $\mathfrak{L} \subset \mathfrak{o}$ be an admissible set, and let $l_2 \oplus l_3 \in \mathfrak{L}$. There is a dichotomy: either $\bar{l}_2 \oplus l_3$ is in $\mathfrak{L}$ or it is not. In the former case, we have

$$\left\{l_2 \oplus l_3, \bar{l}_2 \oplus l_3, l_2 \oplus \bar{l}_3, \bar{l}_2 \oplus \bar{l}_3\right\} \subset \mathfrak{L}$$

and, by (2.2) and (4.4), no other vector $l_2 \oplus l_3'$ or $\bar{l}_2 \oplus l_3'$ with $l_3' \neq l_3, \bar{l}_3$ is in $\mathfrak{L}$. Each 4-element subset of this form consumes two vectors from $\mathfrak{o}|_3$, and all these vectors are pairwise distinct. Let $U \subset \mathfrak{o}|_2$ be the set of vectors l_2 as above, and denote $u := |U|$; clearly, $0 \leqslant 2u \leqslant \min\{c_2, c_3\}$.

Otherwise, in the obvious notation, we have

$$l_2 \oplus S(l_2) \subset \mathfrak{L}, \qquad \bar{l}_2 \oplus S\left(\bar{l}_2\right) \subset \mathfrak{L},$$

where $S(l_2) \subset \mathfrak{o}|_3$ is a certain subset and $S(\bar{l}_2) = \overline{S(l_2)}$. Since we assume that $S(l_2) \cap S(\bar{l}_2) = \varnothing$, all subsets $S(l_2)$, $l_2 \in \mathfrak{o}|_2 \smallsetminus U$, are pairwise disjoint and do not contain any of the $2u$ vectors l_3 coupled with $l_2 \in U$; hence, their total cardinality does not exceed $c_3 - 2u$. On the other hand, since $|S(l_2)| \leqslant b_3$ for each $l_2 \in \mathfrak{o}|_2$, this cardinality does not exceed $(c_2 - 2u)b_2$. Taking the minimum and maximizing over all values of u, we arrive at the bound in the statement. □

4.4. Computing counts and bounds

For "small" blocks $B_k \cong \mathbf{A}_{\leqslant 7}$, $\mathbf{D}_{\leqslant 7}$, $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, the counts $c(B_k)$ and bounds $b(B_k)$ used in (4.3) are obtained by a direct computation. For larger blocks, we use the standard combinatorial description of the **A**- and **D**-type root systems as sublattices of the odd unimodular lattice

$$\mathbf{H}_n := \bigoplus \mathbb{Z}e_i, \quad e_i^2 = 1, \quad i \in \mathcal{I} := \{1, \ldots, n\}.$$

(When working with this lattice, we let $oo := \sum_{i \in o} e_i$ for a subset $o \subset \mathcal{I}$.) Then, given a vector $\hbar_k = \sum_i \alpha_i e_i \in \mathbf{H}_n \otimes \mathbb{Q}$, we subdivide the block $B_k^\vee \subset \mathbf{H}_n \otimes \mathbb{Q}$ into "subblocks"

$$B_k(\alpha) := \left\{ \sum_i \beta_i e_i \,\middle|\, i \in \operatorname{supp}(\alpha) \right\}, \quad \operatorname{supp}(\alpha) := \{i \in \mathcal{I} \,|\, \alpha_i = \alpha\},$$

on which $\hbar_k$ is constant. We obtain counts and bounds, in the sense of (4.1), for each subblock and use an obvious analogue of (4.3) to estimate $b(B_k)$. The technical details are outlined in the next two sections.

4.5. Root systems $\mathbf{A}_n$

A block B_k of type $\mathbf{A}_n$ is $o\mathcal{I}^\perp \subset \mathbf{H}_{n+1}$:

$$\mathbf{A}_n = \left\{ \sum_i \alpha_i e_i \in \mathbf{H}_{n+1} \,\middle|\, \sum_i \alpha_i = 0 \right\}.$$

One has $\operatorname{discr} \mathbf{A}_n = \mathbb{Z}/(n+1)$, with a generator of square $n/(n+1) \bmod 2\mathbb{Z}$, and the shortest representatives of the discriminant classes are vectors of the form

$$\bar{e}_o := \frac{1}{n+1}\big(|\bar{o}|\bar{1}_o - |o|\bar{1}_{\bar{o}}\big), \qquad \bar{e}_o^2 = \frac{|o|\,|\bar{o}|}{n+1},$$

where $o \subset \mathcal{I}$ and $\bar{o}$ is the complement. We have $\bar{e}_{\bar{o}} = -\bar{e}_o$ and

$$\bar{e}_r \cdot \bar{e}_s = |r \cap s| - \frac{|r||s|}{n+1}.$$

If $|r| = |s|$, or, equivalently, e_r and e_s are in the same discriminant class, then

$$\bar{e}_r^2 - \bar{e}_r \cdot \bar{e}_s = \frac{1}{2}\,|r \vartriangle s|\,, \tag{4.5}$$

where $\vartriangle$ is the symmetric difference. Hence, in the case where l_k is a shortest vector in its (nonzero) discriminant class, the bound $b(B_k(\alpha))$ can be estimated by the following lemma, applied to $S = \operatorname{supp}(\alpha)$.

Lemma 4.2. — *Consider a finite set S, $|S| = n$, and let $\mathfrak{S}$ be a collection of subsets $s \subset S$ with the following properties*:

(1) *all subsets $s \in \mathfrak{S}$ have the same fixed cardinality m;*
(2) *if $r, s \in \mathfrak{S}$, then $|r \vartriangle s| \in \{0, 4, 6, 10\}$;*
(3) *in the case $(n, m) = (10, 5)$, if $s \in \mathfrak{S}$, then also $\bar{s} \in \mathfrak{S}$.*

Then, for small (n, m), the maximal cardinality $|\mathfrak{S}|$ is as follows:

(n,m) :	$(n,1)$	$(n,2)$	$(6,3)$	$(7,3)$	$(8,3)$	$(9,3)$	$(10,3)$	$(11,3)$	$(8,4)$	$(9,4)$	$(10,5)$
$\max\lvert\mathfrak{S}\rvert$:	1	$\lfloor n/2\rfloor$	4	7	8	12	13	17	9	12	24

More generally, for $m = 3$ one has $|\mathfrak{S}| \leqslant \lfloor n\lfloor(n-1)/2\rfloor/3\rfloor$.

Note that, if a collection $\mathfrak{S}$ is as in the lemma, then so is the collection $\{\bar{s} \mid s \in \mathfrak{S}\}$. Hence, we can always assume that $2m \leqslant n$.

Proof of Lemma 4.2. — The first two values are obvious; the others are obtained by listing all admissible collections. The general estimate for $m = 3$ follows from the observation that any two subsets in $\mathfrak{S}$ have at most one common point and, hence, each point of S is contained in at most $\lfloor(n-1)/2\rfloor$ subsets. □

There remains to consider a subblock $B_k(\alpha)$ of a block B_k containing vectors of the form $l_k = \bar{1}_r - \bar{1}_s$, where $r, s \subset \mathcal{I}$, $r \cap s = \varnothing$, and $|r| = |s| = 1$ or 2. In the latter case, one must have $l_k \cdot \hbar_k = 3$, and it follows that $|(r \cup s) \cap \operatorname{supp}(\alpha)| \leqslant 2$ for each $\alpha \in \mathbb{Q}$. The bounds are as follows:

(1) if $|(r \cup s) \cap \operatorname{supp}(\alpha)| = 1$, then, obviously, $b(B_k(\alpha)) = 1$;
(2) if $|r \cap \operatorname{supp}(\alpha)| = 2$ (or $|s \cap \operatorname{supp}(\alpha)| = 2$), distinct sets $r \cap \operatorname{supp}(\alpha)$ must be pairwise disjoint and, hence, $b(B_k(\alpha)) = \left\lfloor \frac{1}{2}|\operatorname{supp}(\alpha)|\right\rfloor$;
(3) if $|r \cap \operatorname{supp}(\alpha)| = |s \cap \operatorname{supp}(\alpha)| = 1$, then distinct sets $r \cap \operatorname{supp}(\alpha)$ must also be pairwise disjoint and, hence, $b(B_k(\alpha)) = |\operatorname{supp}(\alpha)|$.

4.6. Root systems $\mathbf{D}_n$

A block B_k of type $\mathbf{D}_n$ can be defined as the maximal even sublattice in $\mathbf{H}_n$:

$$\mathbf{D}_n = \left\{ \sum_i \alpha_i e_i \in \mathbf{H}_n \,\middle|\, \sum_i \alpha_i = 0 \bmod 2 \right\}. \tag{4.6}$$

One has $\operatorname{discr} \mathbf{D}_n = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ (if n is even) or $\mathbb{Z}/4$ (if n is odd); the shortest vectors are

$$e_i,\ i \in \mathcal{I}, \quad \text{and} \quad \bar{e}_o := \frac{1}{2}\left(\bar{1}_o - \bar{1}_{\bar{o}}\right),\ o \subset \mathcal{I}, \quad \bar{e}_o^2 = \frac{n}{4}$$

(the class $\bar{e}_o$ mod $\mathbf{D}_n$ depends on the parity of $|o|$) and we have a literal analogue of (4.5) for any pair $r, s \subset \mathcal{I}$. Thus, if $B_k \ni \bar{e}_o$, the bounds $b(B_k(\alpha))$ are estimated by Lemma 4.2 (if $\alpha \neq 0$) or Lemma 4.3 below (if $\alpha = 0$) applied to $S = \operatorname{supp}(\alpha)$.

Lemma 4.3. — *For $n \leqslant 10$, the maximal cardinality of a collection $\mathfrak{S}$ satisfying conditions* (2) *and* (3) *(if $n = 10$) of Lemma 4.2 is bounded as follows:*

$n =$	1	2	3	4	5	6	7	8	9	10
$\lvert\mathfrak{S}\rvert \leqslant$	1	1	1	2	2	4	8	10	16	32

These bounds are sharp for $n \leqslant 8$.

Proof. — If $n \leqslant 6$, the statement is easily proved by inspection, using Lemma 4.2.

Let $n = 8$. Represent a subset $s \in \mathfrak{S}$ as the root $\bar{e}_s \in \mathbf{D}_8^\vee$. Then, condition 2 implies that all subsets $s \in \mathfrak{S}$ have cardinality of the same parity and, hence, all roots are in the same discriminant class; thus, they lie in an extension $\mathbf{E}_8 \supset \mathbf{D}_8$. By Lemma 4.2 (2), the roots $\bar{e}_s$ constitute a union Γ of (affine) Dynkin diagrams other than $\widetilde{\mathbf{A}}_1$ admitting an isometry to $\mathbf{E}_8$, which gives us a bound $|\mathfrak{S}| \leqslant 12$. Furthermore, the roots $\bar{e}_s$ are distinguished by the property $\bar{e}_s \cdot 2e_1 = 1 \bmod 2$. Thus, each affine component of Γ must have even degree. The maximal graph with these properties is $2\widetilde{\mathbf{D}}_4$, resulting in the bound $|\mathfrak{S}| \leqslant 10$.

If $n = 7$, we extend the ambient set S and each subset by an extra point and argue as above, obtaining roots $\bar{e}_s \in \mathbf{E}_8$ with the property $\bar{e}_s \cdot 2e_1 = 1$. This time, the roots are linearly independent and $|\mathfrak{S}| \leqslant \operatorname{rk} \mathbf{E}_8 = 8$. This is realized by $2\mathbf{D}_4$.

In general, represent $s \in \mathfrak{S}$ by the vector $\bar{1}_s \in \mathbf{H}_n$. (If $n = 10$, select one subset s from each pair s, $\bar{s}$.) Then $\bar{1}_s^2 = n$ and the pairwise products $\bar{1}_r \cdot \bar{1}_s = n - 2|r \vartriangle s|$, $r \neq s$, take but two values $n - 8$ or $n - 12$. Since $n \leqslant 10$, Theorem 2.4 applies and bounds the number of vectors by 16. If $n = 10$, this bound is to be doubled. $\square$

The few remaining cases are listed below.

(1) If $B_k(\alpha) \ni \pm 2e_i$, $i \in \operatorname{supp}(\alpha)$, then $b(B_k(0)) = 1$.

Assume that $l_k = \sum(\pm e_i)$, $i \in o \subset \mathcal{S}$, $|o| \leqslant 4$. If $\alpha = 0$, then

(2) $|o \cap \operatorname{supp}(\alpha)| = 0$, 1, or 2 and $b(B_k(\alpha)) \leqslant 1$, 2, or $\frac{4}{3}|\operatorname{supp}(\alpha)|$, respectively,

similar to Section 4.5. (The last number is a bound on the size of a union of (affine) Dynkin diagrams other than $\widetilde{\mathbf{A}}_1$ admitting an isometry to $\mathbf{D}_{|\mathrm{supp}(\alpha)|}$.) If $\alpha \neq 0$, the numbers of signs $\pm$ within $\mathrm{supp}(\alpha)$ are also fixed, and the options are as follows:

(3) $m := |o \cap \mathrm{supp}(\alpha)| \leqslant 3$ and all signs are the same: by an analogue of (4.5), a bound on $b(B_k(\alpha))$ is given by Lemma 4.2 applied to $S = \mathrm{supp}(\alpha)$;

(4) $|o \cap \mathrm{supp}(\alpha)| = 2$ and the signs differ: $b(B_k(\alpha)) = |\mathrm{supp}(\alpha)|$ as in Section 4.5 (3).

Remark 4.4. — If $n \geqslant 5$, the group $O(\mathbf{D}_n)$ is an index 2 extension of $R(\mathbf{D}_n)$: it is generated by the reflection against the hyperplane orthogonal to any of e_i. Hence, up to $O(\mathbf{D}_n)$, we can assume that, in the expression $\hbar_k = \sum_i \alpha_i e_i$, all coefficients $\alpha_i \geqslant 0$. We always make this assumption (and adjust the results afterwards) when describing the orbits and computing counts and bounds.

5. Root systems with few components

In this section, we consider the 20 Niemeier lattices generated over $\mathbb{Q}$ by root systems with few (up to six) irreducible components. We set the goal

$$|\mathfrak{L}| \geqslant M := 122$$

and prove the following theorem.

THEOREM 5.1. — *Fix a root system D with at most six irreducible components and a configuration $(\hbar, \bar{r})$ in the Niemeier lattice $N(D)$. Then, with the exception of*

- $|\mathfrak{L}| = 144$ *and $\mathfrak{L}$ is conjugate to* $\mathfrak{M}^{\mathrm{i}}_{144} \subset N(4\mathbf{A}_5 \oplus \mathbf{D}_4)$, *see* (5.1), *or*
- $|\mathfrak{L}| = 130$ *and $\mathfrak{L}$ is conjugate to* $\mathfrak{L}^{\mathrm{i}}_{130} \subset N(6\mathbf{A}_4)$, *see* (5.2),

one has $|\mathfrak{L}| \leqslant 120$ for each geometric set $\mathfrak{L}$.

Proof. — For each configuration $(\hbar, \bar{r})$ (or just vector $\hbar$), we list all $O_\hbar(N)$-orbits $\bar{\mathfrak{o}}_n$ and indicate the number $m(\bar{\mathfrak{o}}_n)$ of combinatorial orbits $\mathfrak{o} \subset \bar{\mathfrak{o}}_n$, the count $c(\mathfrak{o})$, and the naïve bound on $|\mathfrak{L} \cap \mathfrak{o}|$ given by (4.3). Sometimes, this bound is improved by one of the arguments (1)–(3) in Section 4.1; the best bound obtained is denoted by $b(\mathfrak{o})$. The results are listed in several tables below.

Table 5.1. The lattice $N(4\mathbf{A}_5 \oplus \mathbf{D}_4)$

1:	$\llbracket\frac{3}{2}\rrbracket_3$	$\llbracket\frac{3}{2}\rrbracket_3$	$\llbracket\frac{3}{2}\rrbracket_3$	$\llbracket\frac{3}{2}\rrbracket_3$	$\{0\}_0$		456	**156**
1:	$[\frac{3}{2}]_3$	$[\frac{3}{2}]_3$	$[0]_0$	$[0]_0$	$[0]_1$	6^*	4	$\underline{2}$
2:	$[\frac{3}{2}]_3$	$[\frac{1}{2}]_5$	$[\frac{1}{2}]_5$	$[\frac{1}{2}]_5$	$[0]_0$	8	27	$\underline{9}$
3:	$\{6\}^+_*$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		912	$168 \to$ **126**
1:	$[3]_3$	$[0]_3$	$[0]_0$	$[0]_0$	$[0]_1$	3^{**}	160	$32 \to \underline{22}\,(3)$
2:	$[3]_3$	$[0]_5$	$[0]_5$	$[0]_5$	$[0]_0$	2^*	216	$36 \to 30\,(2)$
44:	$\llbracket\frac{3}{2}\rrbracket_3$	$\llbracket\frac{3}{2}\rrbracket_3$	$\{0\}_0$	$\llbracket 0\rrbracket_0$	$\llbracket 3\rrbracket_1$		528	$124 \to$ **122** ✓
47:	$\llbracket\frac{3}{2}\rrbracket_3$	$\llbracket\frac{3}{2}\rrbracket_3$	$\llbracket 2\rrbracket_\circ$	$\{0\}_0$	$\llbracket 1\rrbracket_1$		480	$124 \to$ **122** ✓

Convention 5.2. — *In the tables, the number $m(\bar{\mathfrak{o}}_n)$ is marked with a * if $\bar{\mathfrak{o}}_n$ is self-dual; it is marked with ** if also each combinatorial orbit $\mathfrak{o} \subset \bar{\mathfrak{o}}$ is self-dual. If $\bar{\mathfrak{o}}_n$ is not self-dual, then its dual $\bar{\mathfrak{o}}_n^* = \bar{\mathfrak{o}}_{n+1}$ is omitted.*

For the components $\hbar_k$ of $\hbar$ we use the notation $\llbracket \hbar_k^2 \rrbracket_d$, where d is either the discriminant class of $\hbar_k$ or, if $\hbar_k \in D_k$, the symbol 0 (if $\hbar_k = 0$), $\circ$ (if $\hbar_k^2 = 2$), $\bullet$ (if $\hbar_k^2 = 4$), or $*$ (if $\hbar_k^2 = 6$). If these data do not determine $\hbar_k$, we use a superscript:

- *$+$ or $-$ to select a second shortest vector (one of the form $\bar{e}_o + r$, where $r = e_i - e_j \in D_k$ is a root and $i, j \in o$ or $i, j \in \bar{o}$, respectively, see Sections 4.5 and 4.6) in a discriminant class $d \neq 0$;*
- *$+$, if $\hbar = \hbar_k \in \mathbf{D}_n \subset \mathbf{H}_n$ or $\mathbf{A}_{n-1} \subset \mathbf{H}_n$ is of the form $2e_1 - e_2 - e_3$ rather than $e_1 + e_2 + e_3 - e_4 - e_5 - e_6$, see Sections 4.5 and 4.6;*
- *the discriminant class of $\frac{1}{2}\hbar_k$, if $\hbar_2 \in \mathbf{D}_n \cap 2\mathbf{D}_n^\vee$.*

If D_k contains the root $\bar{r}$, this notation is changed to $\{\hbar_k^2\}_d$.

For the components l_k of a line, we use the notation $[l_k \cdot \hbar_k]_d$, where d and an occasional superscript have the same meaning as for $\hbar$.

Also shown in the tables is the naïve a priori estimate $b(\mathfrak{O})$ given by (3.2). For the vast majority of configurations we have $b(\mathfrak{O}) \leqslant M$, and these configurations are omitted. (The complete set of tables is available in [7].) The few cases where $b(\mathfrak{O}) \geqslant M$ are shown in bold, and we treat them separately below, except those marked with a ✓ (see Section 3.3). In these "trivial" cases marked with a ✓ we usually also omit the list of orbits.

5.1. The lattice $N(4\mathbf{A}_5 \oplus \mathbf{D}_4)$

There are 93 configurations to be considered, and the maximal naïve bound is $b(\mathfrak{O}) = 156$ (see Table 5.1).

5.1.1. Configuration 1

There are two sets $\mathfrak{L} \in \mathcal{B}_{17}(\bar{\mathfrak{o}}_2)$, defined by the patterns π such that $\pi|_{\bar{\mathfrak{o}}_2} = \text{const} = 7$ or 9. The former has rank 19, and its only nontrivial geometric $\bar{\mathfrak{o}}_2$-proper (see Section 3.2.2) extension has $\pi(\mathfrak{o}) = \pi(\mathfrak{o}^*) = 2$ for a pair of dual orbits $\mathfrak{o}, \mathfrak{o}^* \subset \bar{\mathfrak{o}}_1$ and $\pi(\mathfrak{o}') = 0$ for all other orbits $\mathfrak{o}' \subset \bar{\mathfrak{o}}_1$. The other set, which is denoted $\mathfrak{M}^{\mathrm{i}}_{144}$, is maximal, see Section 3.2.1. This set of size 144 is determined by the pattern (see Remark 3.2)

$$\pi = \langle\!\langle 0, 9, 9 \rangle\!\rangle \quad \text{(not } \mathbb{Q}\text{-complete).} \tag{5.1}$$

5.1.2. Configuration 3

It is not practical to compute the admissible sets for all orbits; thus, we argue as in Section 4.2 and only compute admissible subsets $\mathfrak{L} \subset \mathfrak{o} \subset \bar{\mathfrak{o}}_1$ of size at least 18. This suffices to show that there is a unique set $\mathfrak{L} \in \mathcal{B}_4(\bar{\mathfrak{o}}_1)$, with the pattern π taking values $(22, 22, 18)$ on $\bar{\mathfrak{o}}_1$ and identical 0 on $\bar{\mathfrak{o}}_2$. This set is maximal (see Section 3.2.1).

Table 5.2. The lattice $N(6\mathbf{A}_4)$

1:	$\llbracket\frac{6}{5}\rrbracket_3$	$\llbracket\frac{6}{5}\rrbracket_3$	$\llbracket\frac{6}{5}\rrbracket_2$	$\llbracket\frac{6}{5}\rrbracket_2$	$\llbracket\frac{6}{5}\rrbracket_3$	$\{0\}_0$		452	$156 \to \mathbf{150}$
1:	$[\frac{6}{5}]_3$	$[\frac{6}{5}]_3$	$[\frac{3}{5}]_1$	$[0]_0$	$[0]_0$	$[0]_1$	10^*	6	$\underline{2}$
2:	$[\frac{6}{5}]_3$	$[\frac{4}{5}]_2$	$[0]_0$	$[\frac{2}{5}]_4$	$[\frac{3}{5}]_4$	$[0]_0$	20^*	18	$\underline{6}$
3:	$[\frac{3}{5}]_4$	$[\frac{3}{5}]_4$	$[\frac{3}{5}]_1$	$[\frac{3}{5}]_1$	$[\frac{3}{5}]_4$	$[0]_0$	1^{**}	32	$16 \to \underline{10}\,(1)$
7:	$\llbracket\frac{4}{5}\rrbracket_4$	$\llbracket\frac{4}{5}\rrbracket_4$	$\llbracket\frac{6}{5}\rrbracket_3$	$\llbracket 2\rrbracket_\circ$	$\{0\}_0$	$\llbracket\frac{6}{5}\rrbracket_3$		476	$140 \to \mathbf{132}\,\checkmark$
21:	$\llbracket\frac{14}{5}\rrbracket_4^-$	$\llbracket\frac{4}{5}\rrbracket_4$	$\llbracket\frac{6}{5}\rrbracket_3$	$\{0\}_0$	$\llbracket 0\rrbracket_0$	$\llbracket\frac{6}{5}\rrbracket_3$		500	$132 \to \mathbf{126}\,\checkmark$
27:	$\{\frac{4}{5}\}_4$	$\llbracket\frac{4}{5}\rrbracket_4$	$\llbracket\frac{4}{5}\rrbracket_4$	$\llbracket\frac{6}{5}\rrbracket_2$	$\llbracket\frac{6}{5}\rrbracket_3$	$\llbracket\frac{6}{5}\rrbracket_2$		424	$128 \to \mathbf{122}\,\checkmark$

5.2. The lattice $N(6\mathbf{A}_4)$

There are 39 configurations to be considered, and the maximal naïve bound is $b(\mathfrak{O}) = 150$ (see Table 5.2).

5.2.1. Configuration 1

There is a unique set $\mathfrak{L}^{\mathrm{i}}_{130} \in \mathcal{B}_{28}(\mathfrak{O})$; it has 130 lines and is described by the pattern (see Remark 3.2)

$$\pi = \langle\!\langle 0, 6, 10 \rangle\!\rangle \quad \text{(not } \mathbb{Q}\text{-complete).} \tag{5.2}$$

This case completes the technical details of the proof of Theorem 5.1. □

6. The lattice $N(8\mathbf{A}_3)$

Starting from this section, we relax the goal to

$$|\mathfrak{L}| \geqslant M := 132.$$

The result of this section is the following theorem.

THEOREM 6.1. — *Let $\mathfrak{L} \subset N(8\mathbf{A}_3)$ be a geometric set. Then, unless $|\mathfrak{L}| = 132$ and $\mathfrak{L}$ is conjugate to $\mathfrak{S}^{\mathrm{i}}_{132}$, see* (6.1)*, or $\mathfrak{S}^{\mathrm{ii}}_{132}$, see* (6.2)*, one has $|\mathfrak{L}| \leqslant 130$.*

Proof. — We proceed as in Section 5, listing pairs $\hbar, \bar{r} \in N(8\mathbf{A}_3)$ and respective orbits (and following the notation of Section 5).

6.1. Configuration 1

The only maximal geometric set, denoted $\mathfrak{S}^{\mathrm{i}}_{132}$, does have 132 lines. It is characterized by the constant pattern (see Remark 3.2)

$$\pi = \langle\!\langle 12, 12 \rangle\!\rangle. \tag{6.1}$$

6.2. Configuration 4

There are 162 sets $\mathfrak{L} \in \mathcal{B}_{10}(\bar{\mathfrak{o}}_4)$. Most are maximal; one of them, denoted $\mathfrak{S}^{\mathrm{ii}}_{132}$, has 132 lines and is determined by the pattern

$$\pi = \langle\!\langle 0, 3, 3, 4, 4, 0, 0 \rangle\!\rangle. \tag{6.2}$$

Eleven sets are of rank 19; extending these sets by a maximal orbit (see Section 3.2.2), we arrive at the bound $|\mathfrak{L}| \leqslant 112$.

On the other hand, there is a unique set $\mathfrak{L} \in \mathcal{B}_6(\bar{\mathfrak{o}}_1 \cup \bar{\mathfrak{o}}_2 \cup \bar{\mathfrak{o}}^*_2 \cup \bar{\mathfrak{o}}_6 \cup \bar{\mathfrak{o}}^*_6)$. One has $|\mathfrak{L}| = 92$, and this set is maximal, see Section 3.2.1. □

Table 6.1. The lattice $N(8\mathbf{A}_3)$

1:	$\{4\}^+_\bullet$	$\llbracket 2\rrbracket_\circ$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$		728	$200 \to$	**132** ✓
1:	$[2]_2$	$[1]_2$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	3^{**}	72	$24 \to$	$\underline{12}$ (3)
2:	$[2]_2$	$[1]_3$	$[0]_3$	$[0]_0$	$[0]_3$	$[0]_0$	$[0]_0$	$[0]_1$	8^*	64	$16 \to$	12 (2)
4:	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket 1\rrbracket_2$	$\llbracket 1\rrbracket_2$	$\{0\}_0$	$\llbracket\frac{3}{4}\rrbracket_1$	$\llbracket 1\rrbracket_2$		448	**160**	
1:	$[\frac{3}{4}]_3$	$[\frac{3}{4}]_3$	$[0]_0$	$[1]_2$	$[0]_0$	$[0]_3$	$[0]_0$	$[\frac{1}{2}]_1$	12^*	4	$\underline{2}$	
2:	$[\frac{1}{2}]_2$	$[\frac{1}{2}]_2$	$[0]_0$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_0$	6	9	$\underline{3}$	
4:	$[\frac{1}{2}]_2$	$[\frac{3}{4}]_3$	$[\frac{3}{4}]_3$	$[0]_0$	$[\frac{1}{2}]_3$	$[0]_0$	$[0]_0$	$[\frac{1}{2}]_1$	12	12	$\underline{4}$	
6:	$[\frac{3}{4}]_3$	$[\frac{3}{4}]_3$	$[\frac{3}{4}]_3$	$[0]_0$	$[0]_0$	$[0]^+_2$	$[\frac{3}{4}]_1$	$[0]_0$	2	1	$\underline{1}$	
5:	$\{\frac{3}{4}\}_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket 1\rrbracket_2$	$\llbracket 1\rrbracket_2$	$\llbracket 0\rrbracket_0$	$\llbracket\frac{3}{4}\rrbracket_1$	$\llbracket 1\rrbracket_2$		420	**136**	✓
7:	$\{3\}^+_2$	$\llbracket 1\rrbracket_2$	$\llbracket 1\rrbracket_2$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 1\rrbracket_2$	$\llbracket 0\rrbracket_0$		600	$172 \to$	**136** ✓
10:	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket 2\rrbracket_\circ$	$\{0\}_0$	$\llbracket 1\rrbracket_2$	$\llbracket\frac{3}{4}\rrbracket_1$	$\llbracket 0\rrbracket_0$		472	$152 \to$	**148** ✓
14:	$\{\frac{11}{4}\}^-_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\llbracket 1\rrbracket_2$	$\llbracket\frac{3}{4}\rrbracket_1$	$\llbracket 0\rrbracket_0$		548	$152 \to$	**140** ✓
15:	$\llbracket\frac{11}{4}\rrbracket^-_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\{0\}_0$	$\llbracket 0\rrbracket_0$	$\llbracket 1\rrbracket_2$	$\llbracket\frac{3}{4}\rrbracket_1$	$\llbracket 0\rrbracket_0$		496	$148 \to$	**138** ✓
18:	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket 0\rrbracket_0$	$\llbracket 0\rrbracket_0$	$\{3\}^+_2$	$\llbracket\frac{3}{4}\rrbracket_1$	$\llbracket 0\rrbracket_0$		600	$152 \to$	**146** ✓
19:	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\{0\}_0$	$\llbracket 0\rrbracket_0$	$\llbracket 3\rrbracket^+_2$	$\llbracket\frac{3}{4}\rrbracket_1$	$\llbracket 0\rrbracket_0$		520	$144 \to$	**140** ✓
21:	$\llbracket 1\rrbracket_2$	$\llbracket 1\rrbracket_2$	$\llbracket 1\rrbracket_2$	$\llbracket 2\rrbracket_\circ$	$\{0\}_0$	$\llbracket 0\rrbracket_0$	$\llbracket 1\rrbracket_2$	$\llbracket 0\rrbracket_0$		472	$164 \to$	**140** ✓
27:	$\{\frac{3}{4}\}_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$	$\llbracket\frac{3}{4}\rrbracket_3$		420	**140**	✓

7. The lattice $N(12\mathbf{A}_2)$

The ultimate result of this section is the following theorem.

Theorem 7.1. — *Let $\mathfrak{L} \subset N(12\mathbf{A}_2)$ be a geometric set. Then, unless*

- $|\mathfrak{L}| = 144$ *and $\mathfrak{L}$ is conjugate to $\mathfrak{M}^{\mathrm{ii}}_{144}$, see* (7.5), *or*
- $|\mathfrak{L}| = 132$ *and $\mathfrak{L}$ is conjugate to $\mathfrak{S}^{\mathrm{iii}}_{132}$, see* (7.8),

one has $|\mathfrak{L}| \leqslant 130$.

In the course of the proof of this theorem we also discover and describe (by means of their patterns, see Remark 3.2) several geometric sets $\mathfrak{L}$ of size $|\mathfrak{L}| \geqslant 124$.

Proof of Theorem 7.1. — We proceed as in Section 5, analyzing pairs $(\hbar, \bar{r})$ one by one. (The notation in the table is explained in Section 5.) There are 9 configurations to be considered, and the maximal naïve bound is $b(\mathfrak{O}) = 190$ (see Table 7.1).

Table 7.1. The lattice $N(12\mathbf{A}_2)$

1:	$\{\frac{8}{3}\}_2^-$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		572	196 $\to$	**190**
1:	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	30^*	18	$\underline{6}$	
2:	$[\frac{4}{3}]_1$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	1^{**}	32	16 $\to$	$\underline{10}$ (1)
2:	$[\![\frac{8}{3}]\!]_2^-$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$\{0\}_0$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		492	176 $\to$	**170**
1:	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	20^*	18	$\underline{6}$	
2:	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	10^*	6	$\underline{2}$	
3:	$[\frac{4}{3}]_1$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	1^{**}	32	16 $\to$	$\underline{10}$ (1)
4:	$[2]_\circ$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	5	4	$\underline{2}$	
3:	$[\![\frac{8}{3}]\!]_2^-$	$\{\frac{2}{3}\}_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		464	**160**	
1:	$[2]_\circ$	$[0]_0$	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	4	4	$\underline{2}$	
3:	$[\frac{4}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	12	18	$\underline{6}$	
4:	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_1$	$\{0\}_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_1$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_1$		444	186 $\to$	**162**
1:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{1}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_0$	36^*	6	$\underline{2}$	
2:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_1$	$[0]_0$	$[0]_0$	$[\frac{2}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	18^*	2	$\underline{1}$	
3:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[\frac{1}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_1$	24^*	8	4 $\to$	$\underline{3}$ (3)
5:	$\{\frac{2}{3}\}_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_1$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_1$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_1$		416	160 $\to$	**144** ✓
6:	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$[\![2]\!]_\circ$	$\{0\}_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		516	**180**	
1:	$[2]_\circ$	$[1]_\circ^+$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	12^*	1	$\underline{1}$	
2:	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_2$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_0$	8^*	9	$\underline{3}$	
3:	$[1]_2$	$[1]_2$	$[1]_2$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	16^*	27	$\underline{9}$	
7:	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![2]\!]_\circ$	$[\![\frac{2}{3}]\!]_2$	$\{0\}_0$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		468	**164**	
1:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	12^*	9	$\underline{3}$	
2:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[1]_1$	$[0]_0$	$[0]_2$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	6^*	4	$\underline{2}$	
3:	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[1]_1$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	24^*	12	$\underline{4}$	
4:	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[1]_2$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	8^*	3	$\underline{1}$	
5:	$[1]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[2]_\circ$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_0$	6	2	$\underline{1}$	
8:	$\{\frac{2}{3}\}_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![\frac{2}{3}]\!]_2$	$[\![2]\!]_\circ$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![\frac{2}{3}]\!]_2$	$[\![0]\!]_0$	$[\![0]\!]_0$	$[\![0]\!]_0$		440	**150**	
1:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[\frac{2}{3}]_2$	$[0]_0$	$[0]_0$	$[\frac{\bar{1}}{3}]_2$	$[0]_0$	$[0]_0$	$[0]_0$	5	2	$\underline{1}$	
3:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[0]_0$	$[1]_2$	$[0]_0$	$[0]_0$	$[0]_2$	$[0]_0$	$[0]_0$	$[0]_0$	$[0]_2$	10	9	$\underline{3}$	
5:	$[\frac{2}{3}]_2$	$[\frac{2}{3}]_2$	$[\frac{1}{3}]_1$	$[0]_0$	$[1]_1$	$[0]_0$	$[0]_2$	$[0]_0$	$[\frac{1}{3}]_1$	$[0]_0$	$[0]_0$	$[0]_0$	10	12	$\underline{4}$	

7.1. Configuration 1

We subdivide the orbit $\bar{\mathfrak{o}}_1$ into five pairwise disjoint clusters $\mathfrak{c}_1, \ldots, \mathfrak{c}_5$ constituting $\mathrm{Orb}_5(\bar{\mathfrak{o}}_1, 6)$. Explicitly,

$$\mathfrak{c}_k := \left\{ l \in \bar{\mathfrak{o}}_1 \,\middle|\, l_k \cdot \hbar_k = \tfrac{1}{3} \right\}, \quad k \in \mathcal{K} := \left\{ k \in \Omega \,\middle|\, \hbar_k = \tfrac{2}{3} \right\}. \tag{7.1}$$

Then, arguing as in Section 3.4, we compute $\mathcal{B}_{58}(\bar{\mathfrak{o}}_1) = \varnothing$.

7.2. Configuration 2

There are four sets $\mathfrak{L} \in \mathcal{B}_{30}(\bar{\mathfrak{o}}_1)$. One, denoted $\mathfrak{L}^{\mathrm{ii}}_{130}$, is maximal and has 130 lines; it is characterized by the pattern

$$\pi := \langle\!\langle 6, 0, 10, 0, 0 \rangle\!\rangle \quad \text{(not } \mathbb{Q}\text{-complete)}. \tag{7.2}$$

The three other sets are of rank 19; extending them by an extra orbit (see Section 3.2.3), we arrive at a number of sets of size $|\mathfrak{L}| \leqslant 118$ and one, up to $O_\hbar(N)$, maximal set $\mathfrak{L}^{\mathrm{iii}}_{124}$ of size 124. The latter is characterized by any of the five patterns

$$\pi_{\mathfrak{c}} = \langle\!\langle 5 | 4, 2, 8, 0, 0 \rangle\!\rangle, \qquad \mathfrak{c} := \mathfrak{c}_1 \in \mathrm{Orb}_5(\bar{\mathfrak{o}}_1, 16); \tag{7.3}$$

explicitly, $\mathfrak{c} = \{ l \in \bar{\mathfrak{o}}_1 \mid l_k \cdot \hbar_k \neq \frac{1}{3} \}$ for some $k \in \mathcal{K}$ (see Section 7.1).

On the other hand, there are 13 sets $\mathfrak{L} \in \mathcal{B}_6(\bar{\mathfrak{o}}_2 \cup \bar{\mathfrak{o}}_3 \cup \bar{\mathfrak{o}}_4 \cup \bar{\mathfrak{o}}_4^*)$, all saturated and with $|\mathfrak{L}| \leqslant 94$. One set is of rank 19; extending it by an extra orbit (see Section 3.2.3), we obtain a number of sets with at most 92 lines.

7.3. Configuration 3

There is a single set $\mathfrak{L} \in \mathcal{B}_{14}(\bar{\mathfrak{o}}_3)$; it is maximal and $|\mathfrak{L}| = 120$.

7.4. Configuration 4

There are 733 sets $\mathfrak{L} \in \mathcal{B}_{14}(\bar{\mathfrak{o}}_1)$; they are all saturated and $|\mathfrak{L}| \leqslant 126$. Extending the 32 sets of rank 19 by a maximal orbit (see Section 3.2.2), we arrive at $|\mathfrak{L}| \leqslant 112$. The only, up to $O_\hbar(N)$, set $\mathfrak{L}^{\mathrm{iv}}_{126}$ with 126 lines constitutes $\mathcal{B}_0(\bar{\mathfrak{o}}_1)$; it is characterized by any of the four patterns

$$\pi_{\mathfrak{c}} = \langle\!\langle 2, 0, 3 | 2 \rangle\!\rangle, \qquad \mathfrak{c} := \mathfrak{c}_3 \in \mathrm{Orb}_4(\bar{\mathfrak{o}}_3, 6). \tag{7.4}$$

On the other hand, there are 105 sets $\mathfrak{L} \in \mathcal{B}_{14}(\bar{\mathfrak{o}}_2 \cup \bar{\mathfrak{o}}_3)$, which are all of rank 18 or 19. Extending them by a maximal orbit (see Section 3.2.2), we arrive at $|\mathfrak{L}| \leqslant 118$.

7.5. Configuration 6

There are 16 sets $\mathfrak{L} \in \mathcal{B}_{48}(\bar{\mathfrak{o}}_3)$. One of them, denoted $\mathfrak{M}^{\mathrm{ii}}_{144}$, is maximal and contains 144 lines. It is determined by the pattern

$$\pi = \langle\!\langle 0, 0, 9 \rangle\!\rangle \quad \text{(not } \mathbb{Q}\text{-complete)}. \tag{7.5}$$

Extending the remaining 15 sets by one or two extra orbits (see Section 3.2.3), we obtain, among others, a set with 124 lines and one with 128 lines. The latter, denoted by $\mathfrak{L}^{\mathrm{v}}_{128}$, is characterized by the pattern

$$\pi = \langle\!\langle 0, 2, 7 \rangle\!\rangle. \tag{7.6}$$

The 124-element set $\mathfrak{L}^{\mathrm{vi}}_{124}$ is characterized by any of the six patterns

$$\pi_{\mathfrak{c}} = \langle\!\langle 0, 3|2, 7|6 \rangle\!\rangle, \tag{7.7}$$

where $\mathfrak{c} := \mathfrak{c}_3 \in \mathrm{Orb}_6(\bar{\mathfrak{o}}_3, 8)$ and $\mathfrak{c}_2 \in \mathrm{Orb}_1(\bar{\mathfrak{o}}_2, 4, \operatorname{stab}\mathfrak{c})$ is determined by $\mathfrak{c}$.

7.6. Configuration 7

There are 244 sets $\mathfrak{L} \in \mathcal{B}_{22}(\bar{\mathfrak{o}}_3)$, all saturated. One of these sets, denoted $\mathfrak{S}^{\mathrm{iii}}_{132}$, has 132 lines; it is determined by the pattern

$$\pi = \langle\!\langle 3, 0, 4, 0, 0, 0 \rangle\!\rangle. \tag{7.8}$$

For the other sets, one has $|\mathfrak{L}| \leqslant 116$. Twenty sets are of rank 19; extending them by a maximal orbit (see Section 3.2.2), we obtain at most 120 lines.

On the other hand, there are nine sets $\mathfrak{L} \in \mathcal{B}_8(\bar{\mathfrak{o}}_1 \cup \bar{\mathfrak{o}}_2 \cup \bar{\mathfrak{o}}_4 \cup \bar{\mathfrak{o}}_5 \cup \bar{\mathfrak{o}}_5^*)$, which are all saturated and have $|\mathfrak{L}| \leqslant 104$. Extensions of the two sets of rank 19 by an extra orbit (see Section 3.2.3) have at most 108 lines.

7.7. Configuration 8

There are 94 sets $\mathfrak{L} \in \mathcal{B}_9(\bar{\mathfrak{o}}_5)$. Most are maximal, and $|\mathfrak{L}| \leqslant 116$. The extensions of the two sets of rank 19 by a maximal orbit (see 3.2.2) are maximal sets with at most 110 lines. $\square$

8. The lattice $N(24\mathbf{A}_1)$

The results of this section are summarized by the following theorem.

Theorem 8.1. — *Let $\mathfrak{L} \subset N(24\mathbf{A}_1)$ be a geometric set. Then, unless*

- $|\mathfrak{L}| = 144$ *and* $\mathfrak{L}$ *is conjugate to* $\mathfrak{M}^{\mathrm{iii}}_{144}$, *see* (8.9), *or*
- $|\mathfrak{L}| = 132$ *and* $\mathfrak{L}$ *is conjugate to one of the sets* $\mathfrak{S}^{\mathrm{iv}}_{132}$, $\mathfrak{S}^{\mathrm{v}}_{132}$, $\mathfrak{S}^{\mathrm{vi}}_{132}$, *or* $\mathfrak{S}^{\mathrm{vii}}_{132}$, *see* (8.1), (8.2), (8.10), *or* (8.11), *respectively,*

one has $|\mathfrak{L}| \leqslant 130$.

Proof. — We proceed as in the previous sections. Each component $v_k \in D^\vee_k$, $k \in \Omega = [1, \ldots, 24]$, of a vector $v \in N$ is a multiple of the generator $r_k \in D_k$. To save space, we use the notation

$$\cdot\,(\text{if } v_k = 0)\,, \qquad -\!\left(\text{if } v_k = \pm\tfrac{1}{2} r_k\right),$$
$$\circ\,(\text{if } v_k = \pm r_k)\,, \quad \bullet\,(\text{the position of } \bar{r})\,.$$

The signs always agree, so that $\hbar_k \cdot l_k \geqslant 0$ for any line $l \in \mathfrak{F}(\hbar)$ and $k \in \Omega$.

There are three configurations (see Table 8.1). Fix a basis $\{r_k\}$, $k \in \Omega$,

Table 8.1. The lattice $N(24\mathbf{A}_1)$

1:	○ ○ ○ ● ·		512	256 → **160** ✓
1:	— — — · — — · · · — · · · · · · · · — — · · · ·	16**	32	16 → <u>10</u> (3)
2:	— — — — — ○ ● — · · — · — · · · · · · · · · · ·		464	240 → **184**
1:	— — — — · — · · · · · · · · — · · — · · · · · —	56*	8	4 → <u>3</u> (3)
2:	○ · · · · ○ · · · · · · · · · · · · · · · · · ·	8	1	<u>1</u>
3:	— — — — — — — — ● — · · · — · · · · · · — · · —		440	**220**
1:	— — — — — · · — · · — · — · · · · · · · · · · ·	110*	4	<u>2</u>

for $24\mathbf{A}_1$ consisting of roots. The kernel

$$N \bmod 24\mathbf{A}_1 \subset \operatorname{discr} 24\mathbf{A}_1 \cong (\mathbb{Z}/2)^{24}$$

of the extension is the Golay code $\mathcal{C}_{24}$ (see [3]). The map supp identifies codewords with subsets of Ω; then, $\mathcal{C}_{24}$ is invariant under complement and, in addition to $\varnothing$ and Ω, it consists of 759 octads, 759 complements thereof, and 2576 dodecads.

To simplify the notation, we identify the basis vectors r_k (assumed fixed) with their indices $k \in \Omega$. For a subset $\mathcal{S} \subset \Omega$, we let $\bar{1}_{\mathcal{S}} := \sum r$, $r \in \mathcal{S}$, and abbreviate $[\mathcal{S}] := \frac{1}{2}\bar{1}_{\mathcal{S}} \in N$ if $\mathcal{S} \in \mathcal{C}_{24}$ is a codeword.

8.1. Configuration 1

We have $|\operatorname{stab} \hbar| = 5760$ and $\hbar$ is the sum of three roots. Using patterns (see Section 3.3), we compute $\mathcal{B}_{38}(\bar{\mathfrak{o}}_1) = \varnothing$, arriving at $|\mathfrak{L}| \leqslant 120$.

8.2. Configuration 2

We have $|\operatorname{stab} \hbar| = 1344$ and $\hbar = [\mathcal{O}] + r_\hbar$, where $\mathcal{O} \in \mathcal{C}_{24}$ is an octad and $r_\hbar \notin \mathcal{O}$. Let $\mathcal{K} := \Omega \smallsetminus (\mathcal{O} \cup \{r_\hbar, \bar{r}\})$ and break $\bar{\mathfrak{o}}_1$ into eight clusters

$$\mathfrak{c}_o := \big\{ \mathfrak{o} \subset \bar{\mathfrak{o}}_1 \bigm| (\mathcal{K} \cap \operatorname{supp} \mathfrak{o}) \subset o \big\},$$
$$o \in \mathcal{C}_{24},\ |o| = 8,\ o \cap \mathcal{O} = \varnothing,\ \bar{r} \notin o,\ r_\hbar \in o.$$

They constitute the orbit $\operatorname{Orb}_8(\bar{\mathfrak{o}}_1, 14)$. Each combinatorial orbit $\mathfrak{o} \subset \bar{\mathfrak{o}}_1$ belongs to two clusters, and each pair of clusters intersects in a single pair of dual orbits.

The set $\mathcal{B}_{52}(\bar{\mathfrak{o}}_1)$ is computed cluster by cluster, as explained in Section 3.4. We arrive at a number of sets $\mathfrak{L}$ of size $|\mathfrak{L}| \leqslant 120$ and a few those with $124 \leqslant |\mathfrak{L}| \leqslant 132$. All sets are maximal. The large sets found can be described as

$$\mathfrak{L} = \bar{\mathfrak{o}}_1 \cap \operatorname{span}\big(\bar{r}, \bar{h}, \bar{u}_s, v\big)^\perp,$$

where

- $\bar{r}$ and $\bar{h} := [\mathcal{O}] - 2r_\hbar = \hbar - 3r_\hbar$ generate the subspace $\bar{\mathfrak{o}}_1{}^\perp \subset N$,
- $\bar{u}_s := \bar{1}_\mathcal{K} - 2s$ for a certain fixed point $s \in \mathcal{O}$,

and the extra vector v is specified below, using the ad hoc notation

- $\bar{v}_o := [o \smallsetminus \mathcal{O}] - [o \cap \mathcal{O}]$ for a codeword $o \in \mathcal{C}_{24}$.

Then, the large sets are as follows:

$$\mathfrak{S}^{\mathrm{iv}}_{132} : v = k, \qquad k \in \mathcal{K}; \tag{8.1}$$
$$\mathfrak{S}^{\mathrm{v}}_{132} : v = \bar{v}_o, \qquad \bar{r} \in o,\ r \in o,\ t \in o, \quad |o \cap \mathcal{O}| = 2,\ |o| = 8; \tag{8.2}$$
$$\mathfrak{L}^{\mathrm{vii}}_{126} : v = \bar{v}_o, \qquad \bar{r} \notin o,\ r \in o,\ t \in o, \quad |o \cap \mathcal{O}| = 2,\ |o| = 8; \tag{8.3}$$
$$\mathfrak{L}^{\mathrm{viii}}_{126} : v = \bar{v}_o + r_\hbar, \qquad \bar{r} \in o,\ r \notin o,\ t \in o, \quad |o \cap \mathcal{O}| = 4,\ |o| = 8; \tag{8.4}$$
$$\mathfrak{L}^{\mathrm{ix}}_{126} : v = [o], \qquad \bar{r} \in o,\ r \notin o,\ t \notin o, \quad |o \cap \mathcal{O}| = 0,\ |o| = 8; \tag{8.5}$$
$$\mathfrak{L}^{\mathrm{x}}_{126} : v = \bar{v}_o, \qquad \bar{r} \notin o,\ r \in o,\ t \in o, \quad |o \cap \mathcal{O}| = 2,\ |o| = 12; \tag{8.6}$$
$$\mathfrak{L}^{\mathrm{xi}}_{124} : v = \bar{v}_o, \qquad \bar{r} \in o,\ r \in o,\ t \in o, \quad |o \cap \mathcal{O}| = 2,\ |o| = 12; \tag{8.7}$$
$$\mathfrak{L}^{\mathrm{xii}}_{124} : v = [o] - r_\hbar, \qquad \bar{r} \in o,\ r \in o,\ t \notin o, \quad |o \cap \mathcal{O}| = 2,\ |o| = 8. \tag{8.8}$$

In each case, it is straightforward that the set of data required for the description is unique up to $O_\hbar(N)$.

8.3. Configuration 3

We have $|\operatorname{stab}\hbar| = 7920$ and $\hbar = [\mathcal{O}]$, where $\mathcal{O} \in \mathcal{C}_{24}$ ia a dodecad. Let $\mathcal{K} := \Omega \smallsetminus (\mathcal{O} \cup \bar{r})$. Each support $o := \operatorname{supp}\mathfrak{o}$, $\mathfrak{o} \in \bar{\mathfrak{o}}_1$, is an octad, so that $|o \cap \mathcal{O}| = 6$ and $|o \cap \mathcal{K}| = 2$; conversely, each 2-element set $s \subset \mathcal{K}$ extends to a unique pair of such octads, representing a pair of dual orbits $\mathfrak{o}, \mathfrak{o}^* \subset \bar{\mathfrak{o}}_1$. We break $\bar{\mathfrak{o}}_1$ into eleven clusters

$$\mathfrak{c}_k := \left\{\mathfrak{o} \subset \bar{\mathfrak{o}}_1 \;\middle|\; \operatorname{supp}\mathfrak{o} \ni k\right\}, \quad k \in \mathcal{K}.$$

Each orbit belongs to two clusters, and each pair of clusters intersects in a single pair of dual orbits. We compute the set $\mathcal{B}_{88}(\bar{\mathfrak{o}}_1)$ cluster by cluster, as explained in Section 3.4. Note that the first cluster $\mathfrak{c}$ has $|\mathfrak{L} \cap \mathfrak{c}| \geqslant 24$ (and, hence, $\delta_0(\mathfrak{c}) \geqslant 4$) and, in case of equality, also $|\mathfrak{L} \cap \mathfrak{c}_k| = 24$ for each $k \in \mathcal{K}$. In this latter case, we reduce overcounting by using the following observations:

- if $\delta_0(\mathfrak{c}) = 6$, there is exactly one other cluster $\mathfrak{c}'$ with $\delta_0(\mathfrak{c}') = 6$, so that $\mathfrak{L} \cap \mathfrak{o} = \mathfrak{L} \cap \mathfrak{o}^* = \varnothing$ for the two orbits $\mathfrak{o}, \mathfrak{o}^* \subset \mathfrak{c} \cap \mathfrak{c}'$;
- if $\delta_0(\mathfrak{c}) = 4$, then $\delta_0(\mathfrak{c}_k) = 4$ for each $k \in \mathcal{K}$ (thus, no preferred order), and we can choose $\mathfrak{c}'$ so that $|\mathfrak{L} \cap \mathfrak{o}| = |\mathfrak{L} \cap \mathfrak{o}^*| = 1$ for $\mathfrak{o}, \mathfrak{o}^* \subset \mathfrak{c} \cap \mathfrak{c}'$.

In these two cases, we start with the pair $\mathfrak{c}$, $\mathfrak{c}'$ and employ the extra symmetry.

The result is one maximal set $\mathfrak{M}^{\mathrm{iii}}_{144}$ and two submaximal sets $\mathfrak{S}^{\mathrm{vi}}_{132}$, $\mathfrak{S}^{\mathrm{vii}}_{132}$. As a by-product, we have found six sets $\mathfrak{L}$ with $124 \leqslant |\mathfrak{L}| \leqslant 130$ and a number of sets of size $|\mathfrak{L}| \leqslant 120$. Most large sets can be described as

$$\mathfrak{L} = \bar{\mathfrak{o}}_1 \cap \operatorname{span}\left(\bar{r}, \bar{1}_{\mathcal{K}}, r, v\right)^{\perp},$$

where $r \in \mathcal{K}$ is a certain fixed point and the extra vector v is described below. This description depends on a codeword $o \in \mathcal{C}_{24}$ (we use the shortcut $\bar{w}_o := 3\bar{v}_o + [\mathcal{O}]$, where $\bar{v}_o$ is as in Section 8.2) and, occasionally, an extra point $s \in o \cap \mathcal{O}$ or $t \in \mathcal{K} \smallsetminus r$. Then, the large sets are as follows:

(8.9)	$\mathfrak{M}^{\mathrm{iii}}_{144} : v = t;$		
(8.10)	$\mathfrak{S}^{\mathrm{vi}}_{132} : v = [o] - s,$	$\bar{r} \notin o,\ r \notin o,$	$\lvert o \cap \mathcal{O}\rvert = 2,\ \lvert o\rvert = 8;$
(8.11)	$\mathfrak{S}^{\mathrm{vii}}_{132} : v = \bar{w}_o,$	$\bar{r} \in o,\ r \in o,$	$\lvert o \cap \mathcal{O}\rvert = 4,\ \lvert o\rvert = 8;$
(8.12)	$\mathfrak{L}^{\mathrm{xiii}}_{130} : v = [o] - s,$	$\bar{r} \in o,\ r \notin o,$	$\lvert o \cap \mathcal{O}\rvert = 2,\ \lvert o\rvert = 8;$
(8.13)	$\mathfrak{L}^{\mathrm{xiv}}_{128} : v = \bar{w}_o - 3t,$	$\bar{r} \in o,\ r \in o,\ t \in o,$	$\lvert o \cap \mathcal{O}\rvert = 4,\ \lvert o\rvert = 8;$
(8.14)	$\mathfrak{L}^{\mathrm{xv}}_{126} : v = \bar{w}_o,$	$\bar{r} \notin o,\ r \in o,$	$\lvert o \cap \mathcal{O}\rvert = 4,\ \lvert o\rvert = 8;$

(8.15) $\mathfrak{L}_{126}^{\mathrm{xvi}} : v = \bar{w}_o, \quad \bar{r} \notin o,\ r \in o, \quad |o \cap \mathcal{O}| = 4,\ |o| = 12;$

(8.16) $\mathfrak{L}_{124}^{\mathrm{xvii}} : v = \bar{w}_o, \quad \bar{r} \notin o,\ r \notin o, \quad |o \cap \mathcal{O}| = 4,\ |o| = 8.$

In (8.16) we require, in addition, that the 6-element set $(o \cap \mathcal{O}) \cup \{\bar{r}, r\}$ should not be contained in an octad. Under this extra assumption, the set of data needed for the description is unique up to $O_{\hbar}(N)$. □

Remark 8.2. — In Section 8.3, there is one more 126-element set $\mathfrak{L}$. However, since $\mathfrak{L}$ is graph isomorphic to $\mathfrak{L}_{126}^{\mathrm{xv}} \cong \mathfrak{L}_{126}^{\mathrm{xvi}}$ and, on the other hand, we do not assert the completeness in this range, we omit its description, which is more complicated.

Remark 8.3. — In the course of this computation, we have observed all even line counts $38 \leqslant |\mathfrak{L}| \leqslant 120$ realized by geometric sets of rank 20.

9. Proofs of the main results

In this concluding section, we fill in a few missing links to complete the proofs of the principal results of the paper stated in the introduction.

9.1. Proof of Theorem 1.1 and Addendum 1.2

As explained in Section 2.2, instead of counting tritangents to smooth sextics one can study (doubling the numbers) the Fano graphs of smooth 2-polarized $K3$-surfaces. By Theorem 2.1, the latter task is equivalent to the study of the Fano graphs of certain 2-polarized lattices $NS \ni h$, and Proposition 2.2 and subsequent definitions reduce it further to the study of geometric subsets $\mathfrak{L} \subset \mathfrak{F}(\hbar)$ in 6-polarized Niemeier lattices $N \ni \hbar$ *other than the Leech lattice* Λ (as we can always assume that there is a root $\bar{r} \in \hbar^{\perp}$). This is done in Theorems 5.1, 6.1, 7.1, and 8.1, and there remains to observe that all sets of size 144 are isomorphic as abstract graphs,

(1) $\mathfrak{M}_{144}^{\mathrm{i}} \cong \mathfrak{M}_{144}^{\mathrm{ii}} \cong \mathfrak{M}_{144}^{\mathrm{iii}}$, $T = [12, 6, 12]^*$,

and there are two isomorphism classes of sets of size 132:

(2) $\mathfrak{S}_{132}^{\mathrm{i}} \cong \mathfrak{S}_{132}^{\mathrm{ii}} \cong \mathfrak{S}_{132}^{\mathrm{iii}} \cong \mathfrak{S}_{132}^{\mathrm{iv}} \cong \mathfrak{S}_{132}^{\mathrm{v}} \cong \mathfrak{S}_{132}^{\mathrm{vi}}$, $T = [2, 0, 66]^*$,
(3) $\mathfrak{S}_{132}^{\mathrm{vii}}$, $T = [4, 0, 32]^*$.

The graphs are compared by means of the `GRAPE` package [14, 15, 30] in `GAP` [10]. A posteriori, the large graphs $\mathfrak{L}$ found in the paper are distinguished by their size $|\mathfrak{L}|$, discriminant form $\operatorname{discr}(\operatorname{span}_{\mathbb{Z}} \mathfrak{L})$, and, in a few cases below, the size $|\mathrm{Aut}\,\mathfrak{L}|$ of the group of abstract graph automorphisms (also computed by `GRAPE`). Instead of $\mathrm{discr}(\mathrm{span}_{\mathbb{Z}}\,\mathfrak{L})$, we give a list of representatives of the genus of the transcendental lattice $T := NS^{\perp}$ of the corresponding 2-polarized $K3$-surface, using the inline notation $[2a, b, 2c]$ for the even rank 2 form $T = \mathbb{Z}u + \mathbb{Z}v$, $u^2 = 2a$, $u \cdot v = b$, $v^2 = 2c$. The meaning of the superscript * is explained in Section 9.2 (1) below.

This observation establishes the bounds stated in Theorem 1.1, and the uniqueness is proved in Section 9.2 below. For the record, we give a similar classification for the other large geometric sets found in the course of the computation and described elsewhere in the paper:

(4) $\mathfrak{L}^{\mathrm{i}}_{130} \cong \mathfrak{L}^{\mathrm{ii}}_{130} \cong \mathfrak{L}^{\mathrm{xiii}}_{130}$, $T = [12, 3, 12]^*$;
(5) $\mathfrak{L}^{\mathrm{v}}_{128} \cong \mathfrak{L}^{\mathrm{xiv}}_{128}$, $T = [12, 2, 12]^*$;
(6) $\mathfrak{L}^{\mathrm{vii}}_{126} \cong \mathfrak{L}^{\mathrm{viii}}_{126} \cong \mathfrak{L}^{\mathrm{xv}}_{126} \cong \mathfrak{L}^{\mathrm{xvi}}_{126}$, $T = [2, 1, 72]^*$, $[6, 1, 24]$, or $[8, 1, 18]$;
(7) $\mathfrak{L}^{\mathrm{iv}}_{126} \cong \mathfrak{L}^{\mathrm{x}}_{126}$, $|\mathrm{Aut}\,\mathfrak{L}| = 144$, $T = [14, 7, 14]^*$;
(8) $\mathfrak{L}^{\mathrm{ix}}_{126}$, $|\mathrm{Aut}\,\mathfrak{L}| = 504$, $T = [14, 7, 14]^*$;
(9) $\mathfrak{L}^{\mathrm{iii}}_{124} \cong \mathfrak{L}^{\mathrm{vi}}_{124} \cong \mathfrak{L}^{\mathrm{xi}}_{124} \cong \mathfrak{L}^{\mathrm{xii}}_{124} \cong \mathfrak{L}^{\mathrm{xvii}}_{124}$, $T = [4, 0, 38]^*$ or $[6, 2, 26]$.

Besides, we have found 9 isomorphism classes of geometric sets of size 120. Note that, unlike (1)–(3), we do not assert the completeness of these lists.

The statement of Addendum 1.2 is essentially given by Remark 8.3 and the above list, as geometric sets with fewer than 38 lines are easily constructed directly, mostly within an appropriate single combinatorial orbit $\mathfrak{o}$.

9.2. Proof of Theorem 1.1 (uniqueness)

Proof. — In full agreement with Corollary 2.5, all large geometric sets listed in Section 9.1 are of the maximal rank $\mathrm{rk}\,\mathfrak{L} = 20$. Therefore, the isomorphism classes of the smooth 2-polarized $K3$-surfaces (or, equivalently, the projective equivalence classes of sextics $C \subset \mathbb{P}^2$) with the given Fano graph $\mathrm{Fn}(X, h) \cong \mathfrak{L}$ are given by the global Torelli theorem [22] (cf. also [8, Theorem 3.11]) as the classes of primitive embeddings $NS \hookrightarrow L := -2\mathbf{E}_8 \oplus 3\mathbf{U}$ up to the group $O^+_h(L)$ of auto-isometries of L preserving h and the *positive sign structure* (i.e., orientation of maximal positive definite subspaces of $L \otimes \mathbb{R}$; here, $NS \ni h$ is the 2-polarized lattice obtained from a mild extension $S \supset \mathrm{span}\,\mathfrak{L} \ni \hbar$ by the inverse construction of Section 2.4).

The classification of embeddings is done using Nikulin [18]. With an extension S (and, hence, lattice NS) fixed, the genus of the transcendental lattice $T := NS^{\perp}$ is determined by the discriminant $\mathrm{discr}\,NS \cong -\mathrm{discr}\,T$. Then, for each representative T of this genus, the isomorphism classes of the embeddings with $NS^\perp \cong T$ are in a one-to-one correspondence with the double cosets

$$O_h(NS)\backslash \operatorname{Aut}(\operatorname{discr} NS)/O^+(T).$$

There are obvious identities and inclusions

$$O_h(NS) = O_{\bar h}(S) \subset O_{\bar h}(\operatorname{span} \mathfrak{L}) \subset O_{\bar h}(\operatorname{span}_{\mathbb{Z}} \mathfrak{L}) = \operatorname{Aut} \mathfrak{L}.$$

Besides, we have the following lemma.

Lemma 9.1. — *Let $\mathfrak{L} \subset N$ be a geometric set, and assume that*

$$\operatorname{rk} \mathfrak{L} = 20, \qquad \det(\operatorname{span}_{\mathbb{Z}} \mathfrak{L}) < 1296.$$

Then the only mild extension $S \supset \operatorname{span} \mathfrak{L}$ is $S = \operatorname{span} \mathfrak{L} = \operatorname{span}_{\mathbb{Z}} \mathfrak{L}$.

Proof. — For the Néron–Severi lattice $NS(X)$ of a $K3$-surface X corresponding to S we have

$$|\det NS(X)| = \det(\operatorname{span}_{\mathbb{Z}} \mathfrak{L})/3i^2, \quad i := [S : \operatorname{span}_{\mathbb{Z}} \mathfrak{L}].$$

On the other hand, since X is smooth and of the maximal Picard rank 20, we have $|\det NS(X)| \geqslant 108$ by [6, Theorem 1.5]. This implies $i < 2$, hence $i = 1$. ☐

Lemma 9.1 applies to all geometric sets listed in Section 9.1(1)–(9) (and to the nine sets of size 120 mentioned thereafter). Then, a direct computation shows that the natural map $\operatorname{Aut} \mathfrak{L} \to \operatorname{Aut}(\operatorname{discr} NS)$ is surjective. This fact renders the other group $O^+(T)$ redundant and proves that each pair $(\mathfrak{L}, T)$ listed is realized by either

(1) a single curve $C \cong \bar{C}$ (marked with a * in the list) or
(2) a pair C, $\bar{C}$ of complex conjugate curves.

(The former is the case whenever T admits an orientation reversing autoisometry; note that we do not assert that the curve admits a real structure, although most likely it does.) In particular, each of the three configurations listed in items (1), (2), (3) is realized by a single curve, as stated in Theorem 1.1. ☐

9.3. Proof of Theorem 1.3

Let $C \subset \mathbb{P}^2$ be a real sextic. By definition, this means that C is invariant under a certain fixed *real structure* (anti-holomorphic involution) $c\colon \mathbb{P}^2 \to \mathbb{P}^2$. This involution lifts to two commuting anti-holomorphic automorphisms $c_\pm$ of the covering $K3$-surface $\varphi\colon X \to \mathbb{P}^2$, so that $c_+ \circ c_- = \tau$

is the deck translation. A priori, $c_\pm$ are either involutions or of order 4, with $c_\pm^2 = \tau$. However, if we assume that C has a real tritangent L, then at least one of the three tangency points (possibly, infinitely near) must be real; thus, the ramification locus of φ has a real point and both lifts $c_\pm$ are involutions, i.e., real structures on X. Furthermore, we can select c_+ so that both pull-backs $L_1, L_2 \subset \varphi^{-1}(L)$ are real (and then they are complex conjugate with respect to c_-); then, the pull-backs $L_1', L_2' \subset \varphi^{-1}(L')$ of any other real tritangent L' are also real, as each of L_i' intersects exactly one of L_1, L_2 at a single point, which must be real.

Thus, we reduce the problem to counting real lines in a real 2-polarized $K3$-surface (X, h). More precisely, this means that we fix a real structure $c\colon X \to X$, $c_*(h) = -h$, and count lines $L \subset X$ satisfying $c_*[L] = -[L]$. (Recall that each line is unique in its homology class and that anti-holomorphic maps reverse the orientation of complex curves.) Arguing as in [8], we can perturb the period of X to change $NS(X)$ to the sublattice rationally generated by the real lines; then, *all* lines in the new surface X are real and $\operatorname{Ker}(1-c_*) \subset T(X) = NS(X)^\perp$. Using the classification of real structures found in [18], we obtain the following statement.

Lemma 9.2 (cf. [8, Lemma 3.10]). — *A smooth 2-polarized $K3$-surface X is equilinear deformation equivalent to a real surface Y in which all lines are real if and only if the orthogonal complement $\operatorname{Fn}(X,h)^\perp$ contains $\mathbf{A}_1$ or $\mathbf{U}(2)$ as a sublattice.*

If $\operatorname{rk}\operatorname{Fn}(X,h) = 20$, the Picard rank $\operatorname{rk} NS(X) = 20$ is also maximal, the moduli space is finite, and the statement can be made more precise.

Lemma 9.3. — *Let X be a 2-polarized $K3$-surface, $\operatorname{rk}\operatorname{Fn}(X,h) = 20$. Then, the real structures $c\colon X \to X$ with respect to which all lines are real are in a one-to-one correspondence with pairs of roots $\pm r \in T(X)$. Under this correspondence, $-c_*$ is the reflection against the hyperplane $r^\perp \subset H_2(X;\mathbb{Z})$.*

There remains to examine the list found in Section 9.1 and observe that the maximum, which is 132 lines, is realized by a unique graph, viz. the one in item (2), and the corresponding transcendental lattice T contains a single pair of roots. The next *known* examples are item (6) with 126 lines and two of the nine graphs with 120 lines, but we do not assert the completeness of our lists in this range.

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Manuscrit reçu le 14 janvier 2021,
révisé le 11 avril 2021,
accepté le 6 mai 2021.

Alex DEGTYAREV
Department of Mathematics
Bilkent University
06800 Ankara (Turkey)
degt@fen.bilkent.edu.tr