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SOME EXAMPLES OF VECTOR FIELDS ON THE 3-SPHERE

by F. Wesley WILSON, Jr. *

1. Homotopy theory of vector fields on S^3.

The n-sphere S^n will denote the set of points with unit modulus in Euclidean $(n+1)$-space. A vector field on S^n is a cross section in the tangent bundle TS^n. Since changing the lengths of these vectors does not alter the geometry of the integral curves, only the speed with which they are transversed, we will always consider a nonsingular vector field as a C^1 cross section in the tangent $(n-1)$-sphere bundle $\tilde{T}S^n$. We shall be investigating homotopy properties of nonsingular vector fields on S^3. Nonsingular homotopies of these vector fields are just homotopies of cross sections in $\tilde{T}S^3$. By differential approximation techniques [4; page 25] it suffices to consider homotopies in which the intermediate stages may fail to be differentiable.

A parallelization of S^3 is the choice of three pointwise linearly independent vector fields on S^3, indexed so that they form a right handed 3-frame at each point. Since every parallelization can be made orthonormal by a homotopy, only orthonormal parallelizations will be considered. In this case, homotopies between parallelizations can always be

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achieved by a rigid motion of the 3-frame at each point.

Since S^3 is parallelizable, the set of homotopy classes of nonsingular vector fields on S^3 can be identified with $\pi_3(S^2)$ and the set of homotopy classes of parallelizations of S^3 can be identified with $\pi_3(SO(3))$ where $SO(3)$ denotes the group of rotations of 3-space. We can describe these identifications explicitly: let Π denote some fixed parallelization of S^3; then expressing any nonsingular vector field F in terms of its Π-coordinates yields a mapping $(F : \Pi) : S^3 \to S^2$ (homotopies of F correspond to homotopies of the mapping); if Σ is another similarly oriented parallelization of S^3, then at each point of S^3, there is a unique element of $SO(3)$ which carries Π onto Σ, and so we have a mapping $[\Sigma : \Pi] : S^3 \to (SO(3)$. (homotopies of Σ correspond to homotopies of the mapping).

Let F be a nonsingular vector field on S^3. We say that a parallelization Σ is an extension of F if the first coordinate direction of Σ coincides with F. The following is an easy consequence of elementary obstruction theory.

Proposition 1.1. — *Every nonsingular vector field on* S^3 *has an extension to a parallelization. Any two extensions of* F *are homotopic through parallelizations which extend* F.

Corollary 1.2. — *The one-to-one correspondence between homotopy classes of vector fields and homotopy classes of parallelizations determines an isomorphism between* $\pi_3(S^2)$ *and* $\pi_3(SO(3))$.

The primary (and only) obstruction to a homotopy between nonsingular vector fields F and G on S^3 is the difference element $d(F, G) \in H^3(S^3 : \pi_3(S^2))$. This difference element has the following geometric interpretation: let Π be an extension of F to a parallelization; then $d(F, G)$ is represented by $\{(G : \Pi)\} \in \pi_3(S^2)$. Identifying $\pi_3(S^2)$ with the integers via the Hopf invariant, we can interpret the difference element for a pair of vector fields as being the (integral) distance between the vector fields. Relative to a fixed vector field H, and parallelization Π which extends H

$$\begin{aligned} d(F, G) = d(F, H) + d(H, G) &= d(F, H) - d(G, H) \\ &= \{(F : \Pi)\} - \{(G : \Pi)\} \end{aligned}$$

(this distance does not depend on the choice of the reference field H). We shall investigate in what sense there is a natural choice for the reference field H.

2. Some classical examples.

Considering S^3 as the Lie group of unit quaternions, we have the right and left invariant vector fields. Since we are interested in the homotopy class, it does not matter which direction we pick at the identity. We also have the right and left invariant parallelizations as extensions of these respective fields. If R is the right invariant parallelization which is given by $[i; j; k]$ at the identity, then at any point $q \in S^3$, $R_q = [iq; jq; kq]$. Similarly, the left invariant parallelization is described by $L_q = [qi; qj; qk]$.

We have seen that relative to a fixed parallelization Π, the parallelizations are in one-to-one correspondence with the elements of $\pi_3(SO(3))$. This correspondence can be described by $g \to \Pi . g$ for $g \in \pi_3(SO(3))$. In particular, if g is a generator, then $\Pi(n) = \Pi . ng$ yields a complete set of homotopy representatives of parallelizations. In [4; page 115], an extremely useful generator for $\pi_3(SO(3))$ is described. Briefly, let S^2 denote the set of unit pure quaternions. For $\pm q \in S^3$, a rotation of S^2 is given by $g(q)(x) = qxq^{-1}$. The mapping $g : S^3 \to SO(3)$ thus defined is a 2-fold covering of $SO(3)$ which generates $\pi_3(SO(3))$. Applying g to R_q yields L_q, showing that R and L lie in adjacent homotopy classes and this is also the case for the right and left invariant vector fields. These fields are excellent choices for the reference field. There is no clear advantage in choosing either one over the other in general. In fact each has its advantages in particular situations, as we shall see.

H. Hopf [1] has described a method for differentiably fibering S^3 by S^1 so that the base space is S^2. There is such a fibering for each pair of relatively prime integers (m, n) with $m \neq 0$, $n > 0$. In the case $m = \pm 1$, $n = 1$ the structure is that of a principal fiber bundle. The fields of tangent vectors to the fibers will be denoted by $H_+ (m = +1)$ and $H_- (m = -1)$. In the other cases, these fiberings do not have the structure of fiber bundles, but are singular fiberings.

In order to describe these fiberings explicity, we shall need to choose coordinates for S^2 and S^3:

S^2 can be parameterized by (r, θ) with $-1 \leqslant r \leqslant 1$ and $0 \leqslant \theta < 2\pi$ by means of the formulas

$$x_1 = \sqrt{1-r^2}\cos\theta \qquad x_2 = \sqrt{1-r^2}\sin\theta, \qquad x_3 = r$$

S^3 can be parameterized by (r, μ, ν) with $0 \leqslant r \leqslant 1$ and $0 \leqslant \mu, \nu < 2\pi$ by means of the formulas

$$x_1 = r\cos\mu, \qquad x_2 = r\sin\mu,$$
$$x_3 = \sqrt{1-r^2}\cos\nu, \qquad x_4 = \sqrt{1-r^2}\sin\nu,$$

In S^3, the sets T_r which are obtained by holding r constant are circles for $r = 0, 1$ and for $0 < r < 1$, T_r is a torus which separates T_0 and T_1, and which has T_0 and T_1 as its axes of symmetry.

We can now describe the $(m, n)^{\text{th}}$ Hopf fibering by defining the fiber projection $p_{m,n}: S^3 \to S^2$:

$$p_{m,n}(r, \mu, \nu) = (1 - 2r, -m\mu + n\nu).$$

The Hopf invariant of $p_{m,n}$ is mn. For $0 < r < 1$ and m, n relatively prime, $p_{m,n}^{-1}(1-2r, \theta)$ is a circle on T_r which wraps m times in the T_0 direction and n times in the T_1 direction. In particular, the tangent vector field is tangent to T_r and is positively directed along $T_1 (n > 0)$ and is either positively or negatively directed along T_0 depending on whether $m > 0$ or $m < 0$. Computing H_+ and H_- explicitly and using the identification $(a, b, c, d) \to a + bi + cj + dk$. we find.

Proposition 2.1. — H_+ *coincides with the left invariant vector field and* H_- *coincides with the right invariant vector field. Consequently,* H_+ *and* H_- *lie in adjacent homotopy classes.*

Another interesting vector field on S^3 is the normal field to the Reeb foliation [3]. This field has the properties that it is tangent to T_0 and T_1, and transverse to T_r for $0 < r < 1$. Since these vector fields can be extended to parallelizations, their directions can be reversed by a homotopy, and so we are free to always choose the field to be directed positively on T_0. Depending on the foliation it may then be either positive (R_+) or negative (R_-) on T_1.

PROPOSITION 2.2. — R_+ *is homotopic to* H_+ *and* R_- *is homotopic to* H_-.

This proposition is immediate from.

LEMMA 2.3. — *If* F *and* G *are vector fields on a parallelizable manifold* M^n *which are never negatives of one another, then* F *and* G *are homotopic.*

Proof. — With respect to any parallelization Π $(F : \Pi)$ and $(G : \Pi)$ are mappings of M^n into S^{n-1} which never have $(F : \Pi)(x)$ and $(G : \Pi)(x)$ antipodal for any $x \in M^n$. Therefore $(F : \Pi)$ and $(G : \Pi)$ are homotopic.

COROLLARY 2.4. — *The* $(m, n)^{\text{th}}$ *Hopf fibering is homotopic to* H_+ *if* $m > 0$ *and to* H_- *if* $m < 0$.

Proof. — Assume $n > 0$. For $m > 0$ compare with R_+; for $m < 0$ compare with R_-.

3. Isolated periodic solutions.

Let F be a C^1 vector field on an n-manifold M^n and let γ denote a nontrivial, isolated periodic solution of F. Then there is a tubular neighborhood B of γ which intersects no periodic solution of F other than γ. We shall assume that γ is parameterized by $t(0 \leqslant t < 1)$ and we denote the normal fiber to γ at time t by B_t.

For each fixed t, there is a diffeomorphism P of a neighborhood of $\gamma(t)$ into B_t called the Poincaré mapping. This mapping is defined by $P(x)$ is the first point where the positive trajectory from x intersects B_t. It follows from the uniqueness and the differentiability of the trajectories of F with respect to their initial conditions that P is well defined, one-to-one, and differentiable. Since an inverse for P can be defined near $\gamma(t)$ and the inverse is also differentiable, it follows that P is a diffeomorphism.

Consider B as a vector bundle with γ corresponding to the zero cross section. Then we have a vector field F^1_t defined on B_t by $F^1_t(x)$ is the vector $x - P(x)$. The aggragation of these vector fields is a C^1 vector field F^1 on B, which has γ for its singular set. Similarly for each nonzero integer k we have defined the k^{th} iterate P^k of P and the induced vector field F^k defined on some neighborhood of γ and having γ for its singular set. Thus we have the k^{th} index of γ defined for each nonzero integer k. The k^{th} *index* is just the index of F^k_t at the origin in B_t (it does not matter which t we use). We shall use the notation $i_k(\gamma, F)$ to denote the k^{th} index of the periodic solution γ of F. We can obtain other information about F near γ from the vector fields F^k, but first we need some additional topological machinery.

Let C denote a regular simple closed curve in S^3 and consider the set of embeddings $f:[0, 1) \times S^1 \to S^3$ such that $f(0 \times S^1) = C$. The images $Im(f)$ of these embeddings are called half-line bundles on C. Two half-line bundles $Im(f)$ and $Im(g)$ are called equivalent if f and g are isotopic rel $(0 \times S^1)$. A necessary and sufficient condition for equivalence is that $f\left(\frac{1}{2} \times S^1\right)$ and $g\left(\frac{1}{2} \times S^1\right)$ have the same linking with C. Given an equivalence class Δ of half-line bundles, we define the rotation number $r(\Delta)$ to be the linking number of $f\left(\frac{1}{2} \times S^1\right)$ and C for any representative $f \in \Delta$.

Now consider the set of immersions $f[0, 1) \times S^1 \to S^3$ such that $f|(0, 1) \times S^1$ is an embedding and $f|0 \times S^1 : S^1 \to C$ has degree q for some positive integer q. Then for each $s \in (0, 1)$, the set $\Delta_S = Im(f|[s, 1) \times S^1)$ is a half-line bundle in the above sense and $r(\Delta_S) = p$ for some integer p, which does not depend on the choice of s. Note that if T^2 is the boundary of a small tubular neighborhood of C chosen so that $T^2 \cap Im(f)$ is a simple curve, than this curve has type (p, q) in T^2. Hence p and q are relatively prime. We shall call the images of these immersions *half-line bundles* also, and we define the rotation of an equivalence class Δ of such bundles to be $r(\Delta) = p/q$. The f-images of nonzero sections over $0 \times S^1$ will be called *sections* in $Im(f)$. Every section in $Im(f)$ is a q-fold covering of C which links C p times.

Now let C be a nonzero regular section in B. The vector fields F^k define half-line bundles over C which we denote by $F^k|C$. It is clear that $F^k|C$ is altered isotopically when C is altered homotopically through nonzero sections in B. We define the *torsion* of F^k along γ, denoted by $t_k(\gamma;\, F)$ to be the integer $r(F^k|C)$ where C is a section in B which does not link γ.

We shall now give some examples of vector fields on the open solid torus which illustrate some properties of the index and the torsion. We assume that the reader is familiar with [2; pages 1-75]. Describe the solid torus by coordinates (r, θ, φ) where $0 \leqslant r < \infty$, $0 \leqslant \theta, \varphi < 2\pi$ and φ parameterizes the axis $r = 0$. Let B_φ denote the disk obtained by setting φ equal to a constant. Let n be any integer and choose a differential equation on B_0

$$\dot{r} = R(r, \theta); \qquad \dot{\theta} = \Theta(r, \theta)$$

such that $R(0, \theta) = 0$, such that this is the only singularity, and such that the index of this singularity is n. Then on the solid torus, the differential equation

$$F \quad \begin{cases} \dot{r} = R(r, \theta) \\ \dot{\theta} = \Theta(r, \theta) \\ \dot{\varphi} = 1 \end{cases}$$

has $\gamma = \{(r, \theta, \varphi) | r = 0\}$ for an isolated periodic solution. In this case, $i_k(\gamma;\, F) = n$ for every $k > 0$ and $t(\gamma;\, F^k) = 0$ for every $k > 0$. We call the equation on B_0 the *generating equation* for F. Suppose that the generating equation has been chosen so that there are finitely many elliptic domains, so that the attracted (repelled) trajectories are isolated, and so that the system is symmetric with respect to rotations of angle $2\pi/q$ where q is some positive integer, *i.e.*

$$R\left(r, \theta + \frac{2\pi k}{q}\right) = R(r, \theta)$$
$$\Theta\left(r, \theta + \frac{2\pi k}{q}\right) = \Theta(r, \theta)$$

Let E(H) denote the (finite) number of elliptic domains (hyperbolic domains, respectively). It follows from the symmetry condition that q divides E and H. Therefore since the index of $\gamma(0)$ is given by the formula $1 + \frac{1}{2}(\mathrm{E} - \mathrm{H})$, it follows that the index is of the form $1 + nq$. A trajectory of the vector field F can be classified as elliptic, hyperbolic, attracted, or repelled depending on which kind of trajectory on B_0 generated it. The union of the attracted (repelled) trajectories is an invariant 2-manifold which has a finite number of components. Each component has the property that its union with γ is a half-line bundle. The union of the elliptic (hyperbolic) trajectories is an open subset with E(H) components, each of which is an open solid torus. These components are called elliptic (hyperbolic) domains. We now construct another vector field G on the solid torus by cutting the torus at $\mathrm{B}_0 = \mathrm{B}_{2\pi}$ and identifying B_0 to $\mathrm{B}_{2\pi}$ by

$$(r, \theta, 2\pi) = \left(r, \theta + \frac{2\pi p}{q}, 0\right)$$

where p is some nonzero integer which is relatively prime to q. G is field of tangent vectors to the images of the trajectories of F under this operation. It is clear that G is as smooth as F. The invariant manifolds of G form half-line bundles over γ which have rotation number $r(\gamma; \mathrm{G}) = p/q$. Also, there are e elliptic domains and h hyperbolic domains for G where $\mathrm{E} = eq$ and $\mathrm{H} = hq$. The restrictions G_0^k of G^k to B_0 either have $\gamma(0)$ a node if $k \neq mq$ or similar to the original field if $k = mq$. Therefore,

$$i_k(\gamma; \mathrm{G}) = \begin{cases} 1 & \text{if} \quad k \neq mq \\ 1 + nq & \text{if} \quad k = mq. \end{cases}$$

From this we also see immediately that $t_k(\gamma; \mathrm{G}) = 0$ if $k \neq mq$, and $t_{mq}(\gamma; \mathrm{G})$ is the same for all $m > 0$. We shall now compute $t_q(\gamma; \mathrm{G})$.

LEMMA 3.1. — *Let* C *denote any nonzero cross-section of the solid torus and let* $\mathrm{G}^q|\mathrm{C}$ *denote the half-line bundle over* C *which is induced by* G^q. *Then*

$$r(\mathrm{G}^q|\mathrm{C}) = t_q(\gamma; \mathrm{G}) + \lambda(\mathrm{C}, \gamma) i_q(\gamma : \mathrm{G})$$

where $\lambda(\mathrm{C}, \gamma)$ *denotes the linking number of* C *and* γ.

Proof. — Let C_0 denote a section which does not link γ and let C_1 denote a curve in B_0 which links γ once. Then $t_q(\gamma : G) = r(G^q|C_0)$. Now homotopies of C through nonzero cross-sections induce equivalences of the bundle $G^q|C$; so we can assume that C is a section which lies very close to $C_0 \cup C_1$ (C is homologous to $C_0 + \lambda(C, \gamma)C_1$). A circuit of a loop close to C_1 rotates $G^q|C$ by $i_q(\gamma ; G)$, and so the formula is established.

Corollary 3.2. — *If* C *is a local, nonzero section which has homology type* (p', q'), *then* $r(G^q|C) = q' t_q(\gamma : G) + p' i_q(\gamma : G)$.

Corollary 3.3. — $t_{mq}(\gamma : G) = r(\gamma : G)[1 - i_{mq}(\gamma : G)]$.

Proof. — Let C be a local section which lies in an attracted manifold. Then C has homology type (p, q) and $r(G^q|C) = p$. By 3.2.

$$p = qt_q(\gamma : G) + pi_q(\gamma : G).$$

Remark. — With the simple geometry that we have assumed, we can compute the numbers $i_k(\gamma ; G)$ and $t_k(\gamma ; G)$ directly the values e, h, p, q. The values are

$$i_k(\gamma ; G) = \begin{cases} 1 & \text{if} \quad k \neq mq \\ 1 + \dfrac{q}{2}(e - h) & \text{if} \quad k = mq \end{cases}$$

$$t_k(\gamma ; G) = \begin{cases} 1 & \text{if} \quad k \neq mq \\ -\dfrac{p}{2}(e - h) & \text{if} \quad k = mq. \end{cases}$$

Definition. — *Let* F *be any vector field on* S^3 *and let* γ *be a periodic solution of* F. γ *has simple type if there is a neighborhood of* γ *on which* F *has the same form as one of the vector fields* G *(above) near its periodic solution.* The attracted and repelled manifolds of γ are called the *invariant manifolds.*

4. Geometrically simple vector fields.

We shall now investigate a special class of vector fields on an orientable 3-manifold M^3 whose trajectories have a sufficiently simple geometric structure that we can make some rather strong statements about their global qualitative structure. A vector field is called *geometrically simple* if its trajectories satisfy the following properties:

1. There is a finite number of periodic solutions.

2. Each periodic solution has simple type.

3. Each trajectory has its α- and ω-limit sets contained in the set of periodic solutions.

4. If a trajectory does not intersect an invariant manifold, then its α- and ω-limits are at the same periodic solution γ and the trajectory approaches γ in an elliptic domain.

5. The invariant manifolds, have degenerate intersection, i.e. they either coincide or else they are disjoint.

PROPOSITION 4.1. — *The closure of an invariant manifold of a geometrically simple vector field is the image of an immersion* $f\colon S^1 \times [0, 1] \to M^3$ *which is one-to-one on* $S^1 \times (0, 1)$.

Proof. — Let $\overline{N}$ denote the closure of an invariant manifold N. By the degenerate intersection property and the fact that every trajectory must have its α- and ω-limits at some periodic solution, it follows that there are periodic solutions γ_α, γ_ω such that every trajectory in N has its α-limit at γ_α and its ω-limit at γ_ω (it is not necessary that γ_α and γ_ω be distinct). Let $f_1 : S^1 \times \left[0, \frac{1}{3}\right] \to M^3$ be an immersion into $\overline{N}$ which describes the half-line bundle over γ_α and such that $f_1 | S^1 \times \left\{\frac{1}{3}\right\}$ is transverse to $F|N$, and let $f_3 : S^1 \times \left[\frac{2}{3}, 1\right]$ be a similar embedding at γ_ω (f_1 and f_3 can be obtained via Lyapunov theory, for example [5], [6]). Every trajectory which starts at a point $x \in N_1 = f_1\left(S^1 \times \left\{\frac{1}{3}\right\}\right)$ reaches $N_2 = f_3\left(S^1 \times \left\{\frac{2}{3}\right\}\right)$ in finite time $t(x)$ and it follows by the implicit function theorem and the differentiability of trajectories with respect to their initial conditions that $t\colon N_1 \to \mathbf{R}$ is as smooth as the vector field. Define $f_2 : S^1 \times \left[\frac{1}{3}, \frac{2}{3}\right] \to N$ by

$$x = f_1\left(\theta, \frac{1}{3}\right), \qquad f_2(\theta, t) = \psi\left(x, \left(t - \frac{1}{3}\right) 3t(x)\right)$$

where $\psi : M^3 \times \mathbf{R} \to M^3$ is the flow induced by the vector field. The mapping $f_1 \cup f_2 \cup f_3$ is the desired immersion except possibly for the lack of smoothness at $S^1 \times \left\{\frac{1}{3}\right\}$ and $S^1 \times \left\{\frac{2}{3}\right\}$. Smoothing techniques for Lyapunov functions [5] can be used to remedy this problem.

PROPOSITION 4.2. — *If* V *is an elliptic domain of a periodic solution of* F, *then the saturation* $\tilde{V} = \{p \mid \varphi(p, t) \in V$ *for some* $t \in \mathbf{R}\}$ *of* V *is open. If* V_1 *and* V_2 *are distinct elliptic domains, the* $\tilde{V}_1$ *and* $\tilde{V}_2$ *are disjoint.*

Proof. — $\tilde{V}$ is open since V is open. By the structure near a periodic solution, V_1 and V_2 can be chosen so that whenever a trajectory enters V_i positively (negatively) it stays in V_i for all future positive (negative) time. But with such choices, $\tilde{V}_1$ and $\tilde{V}_2$ must be disjoint.

PROPOSITION 4.3. — *Let* F *be a geometrically simple vector field and let* γ *be a periodic solution of* F. *Then every hyperbolic domain of* γ, *which intersects no invariant manifolds, corresponds to an elliptic domain of some periodic solution.*

Proof. — Let U denote a hyperbolic domain of γ, let $p \in U$, and let $\gamma_\alpha, \gamma_\omega$ denote the respective α- and ω-limit sets of p. Since p does not lie on an invariant manifold, $\gamma_\alpha = \gamma_\omega$. Let V denote the elliptic domain of γ_α which corresponds to the trajectory through p. Since U is open and connected and since U intersects no invariant manifolds, it follows that U is contained in the saturation of V.

PROPOSITION 4.4 — *The saturation of an elliptic domain is homeomorphic to an open solid torus.*

Proof. — Let V denote an elliptic domain of a periodic solution γ. If we can construct a cross section to the induced flow on $\tilde{V}$ which is contained in V, then the cross section will have the form $S^1 \times \mathbf{R}$. We can then obtain a homeomorphism $h: S^1 \times \mathbf{R}^2 \to \tilde{V}$ by

$$h(\theta, s, t) = \psi(g(\theta, s), t)$$

where $g: S^1 \times \mathbf{R} \to \tilde{V}$ describes the cross-section and $\psi(p, t)$ describes the induced flow. Thus it remains to construct a cross-section to the flow on $\tilde{V}$. Let us suppose $r(\gamma; F) = p/q$. Then there is a neighborhood U of γ such that $F|U$ coincides with one of the examples from § 3. Let (U', γ') be the q-fold covering space of (U, γ) and let F' denote the induced vector field, V' a related elliptic domain. Then it suffices to construct a cross-section to the flow induced on V' by F'. But F' can be described in coordinates by

$$\dot{r} = R(r, \theta); \qquad \dot{\theta} = \Theta(r, \theta); \qquad \dot{\varphi} = 1.$$

Thus it suffices to find a cross-section to the generating equation on the elliptic domain $V^1 = V^1 \cap B_0$. But for this, we can take any trajectory of a transverse vector field. There is no difficulty in choosing the transverse vector field so that the section is contained in V ([cf. [7]).

THEOREM 4.5 (STRUCTURE THEOREM). — *Let* F *be a geometrically simple vector field on* M^3. *Then* F *has a finite set of periodic solutions* $\gamma_1, \ldots, \gamma_k$ *and a finite set of invariant manifolds* $N_1, \ldots, N_l$. *Each invariant manifold* N_i *is diffeomorphic to* $S^1 \times (0, 1)$ *and contains two periodic solutions in its closure (not necessarily distinct) one attracting the trajectories of* N_i, *the other repelling them. The union of* N_i *and either of these periodic solutions forms a half-line bundle. The complement of the set* $S = \left(\bigcup_{j=1}^{k} \gamma_j\right)\left(\bigcup_{j=1}^{l} N_l\right)$ *is the union of disjoint open tori, and each such torus is the saturation of an elliptic domain of some periodic solution.*

Proof. — The properties of the N_i follow from 4.1. Let U be a component of $M^3 - S$. Then any $p \in U$ lies in the saturation of some elliptic domain (cf. 4.3) and by 4.2 U is an elliptic domain. It follows from 4.4 that U is homeomorphic to an open solid torus. Finiteness follows since there are only finitely many periodic solutions and each of these has only finitely many elliptic domains.

PROPOSITION 4.6. — *Suppose that* F *is a geometrically simple vector field having periodic solutions* γ_0 *and* γ_1 *which bound an invariant manifold* N. *Suppose that* $r(\gamma_0; F) = p/q$ *and* $r(\gamma_1; F) = p'/q'$ $(q, q' > 0$, p *or* p' *may be zero). Then*

$$\lambda(\gamma_0; \gamma_1) = \frac{p}{q'} = \frac{p'}{q}$$
$$r(\gamma_0; F).r(\gamma_1; F) = \lambda(\gamma_0; \gamma_1)^2.$$

If F *has no other periodic solutions, then*

$$t_q(\gamma_0; F) = \lambda(\gamma_0; \gamma_1)i_{q'}(\gamma_1; F) - \lambda(\gamma_0; \gamma_1)$$
$$t_{q'}(\gamma_1; F) = \lambda(\gamma_0; \gamma_1)i_q(\gamma_1; F) - \lambda(\gamma_0; \gamma_1).$$

Proof. — Let γ denote a section in the half-line bundle $N \cup \gamma_0$. Then $\lambda(\gamma; \gamma_0) = p$ and γ covers γ_0 q times. Similarly, $\lambda(\gamma; \gamma_1) = p'$ and γ covers γ_1 q' times. Translating γ by the flow to get it close to γ_0 or γ_1 as is desired, we conclude

$$q\lambda(\gamma_0; \gamma_1) = \lambda(\gamma; \gamma_1) = p'$$
$$q'\lambda(\gamma_0; \gamma_1) = \lambda(\gamma; \gamma_0) = p.$$

If F has no other periodic solutions, then the roles of e and h are interchanged for γ_0 and γ_1. Thus, using the formulas from § 3, we have

$$i_{q'}(\gamma_1; F) = 1 + \frac{q'}{2}(h - e)$$

and

$$t_q(\gamma_0; F) = -\frac{p}{2}(e - h) = \frac{p}{2} \cdot \frac{q'}{2}[i_{q'}(\gamma_i; F) - 1]$$
$$= \lambda(\gamma_0; \gamma_1)i_{q'}(\gamma_1; F) - \lambda(\gamma_0; \gamma_1).$$

5. Some examples of geometrically simple vector fields on S^3.

In this section, we shall show that there are geometrically simple vector fields on S^3 and we shall determine the properties of those vector fields which have only two periodic solutions. Preparatory for this goal, we must study the geometric properties of the closure of an invariant manifold.

LEMMA 5.1. — *Let* F *be a vector field on* S^3 *which has just two periodic solutions* γ_0 *and* γ_1. *Suppose that* γ_0 *is asymptotically stable with domain of attraction* $S^3 - \gamma_1$ *and that* γ_1 *is asymptotically stable in the negative sense with domain of (negative) attraction* $S^3 - \gamma_0$. *Then* γ_0 *and* γ_1 *are unknotted and* $\lambda(\gamma_0, \gamma_1) = \pm 1$.

Proof. — Let M denote an unknotted circle which lies in a tubular neighborhood of γ_1 and such that $\lambda(M, \gamma_1) = 1$. Let M_t denote the translate of M by the flow for time t. Then $\lambda(M_t, \gamma_1) = 1$ for all t, and for sufficiently large t', $M_{t'}$ lies in a tubular neighborhood U of γ_0. Let $\pi : U \to \gamma_0$ denote the projection of the tubular neighborhood U, and let q denote the degree of $\pi|M_{t'}$. Then

$$1 = \lambda(M_{t'}, \gamma_1) = q\lambda(\gamma_0, \gamma_1)$$

from which it follows that $q = \lambda(\gamma_0, \gamma_1) = \pm 1$. Now if γ_0 were knotted, then $M_{t'}$ would be knotted. But this cannot be since M is unknotted. Thus γ_0 is unknotted. By symmetry, γ_1 is also unknotted.

THEOREM 5.2. — *Let* F *denote a geometrically simple vector field on* S^3 *which has just two periodic solutions* γ_0 *and* γ_1. *Then* γ_0 *and* γ_1 *are unknotted and* $\lambda(\gamma_0, \gamma_1) = \pm 1$.

Proof. — There is no loss of generality in assuming that every elliptic domain of γ_0 corresponds to a hyperbolic domain of γ_1, and conversely, since if W is the closure of an elliptic domain of γ_0 which does not contain γ_1, then $S^3 - W$ is diffeomorphic to $S^3 - \gamma_0$, and the resulting diffeomorphism induces a geometrically simple vector field on S^3 with essentially the same structure except for the omission of W. We shall build a new vector field G on S^3 which has γ_0, γ_1 for its only minimal sets and which has the elliptic domains and invariant manifolds as invariant sets, but such that the hypotheses of 5.1 are satisfied. This is sufficient to prove the theorem. Since every invariant manifold of F has γ_0 and γ_1 in its closure, we define G on the invariant manifolds so that every trajectory runs from γ_1 to γ_0. Since every elliptic domain of F is a solid torus which contains γ_0, γ_1 and two invariant manifolds in its boundary, we can extend G to each elliptic domain of F so that every trajectory runs from γ_1 to γ_0 there also.

THEOREM 5.3. — *If* F *is a geometrically simple vector fields on* S^3 *which has just two periodic solutions*, γ_0, γ_1, *then*

$$r(\gamma_0\,; F) = \frac{p}{q} \neq 0$$
$$r(\gamma_1\,; F) = \pm \frac{q}{p} \neq 0.$$

Proof. — There is no loss of generality in assuming $\gamma_0 = T_0$ and $\gamma_1 = T_1$. Let $\overline{N}$ denote the closure of an invariant manifold which contains γ_0 and γ_1, and let C denote a section in N. Then C covers $\gamma_0\, q \geqslant 1$ times and $\lambda(C, \gamma_0) = p$.

By the relationship of T_0 and T_1, it follows that $\lambda(C, \gamma_1) = q$ and C covers $\gamma_0 |p|$ times, i.e. $|p| \neq 0$. The theorem follows.

Examples which have two periodic solutions. — We will now describe some examples of geometrically simple vector fields on S^3 which have just two periodic solutions. As a consequence of 4.2, 4.5, 5.2 and 5.3, it follows that these every such vector field corresponds to one of these examples.

Take $\gamma_0 = T_0$ and $\gamma_1 = T_1$. We choose integers $p \neq 0$, $q > 0$, $e > 0$, $h > 0$ such that $(p, q) = 1$ and $e + h = 0$ mod 2. Consider the arcs

$$A_k = \left\{ (r, \theta, \varphi) | 0 \leqslant r \leqslant 1, \theta = \frac{2\pi k}{q(e+h)}, \varphi = \frac{2\pi k}{p(e+h)} \right\}$$

for $k = 1, 2, \ldots, e + h$. There is one point x_r^k in $A_k \cap T_r$ for each r, $0 < r < 1$. Let C_r^k denote the cannonical curve of type (p, q) in T_r which passes through x_r^k. Then

$$N_k = \bigcup_{0<r<1} C_r^k$$

is a submanifold of S^3 whose closure can be considered as a half-line bundle with rotation $\left|\frac{p}{q}\right|$ at T_0 or with rotation $\left|\frac{q}{p}\right|$ at T_1. $S = S^3 - \bigcup_{k=1}^{e+h} \overline{N}_k$ has $(e + h)$ components, each of which is an open solid torus. By means of immersions of $S^1 \times [0, 1] \to \overline{N}_k$ and diffeomorphisms of $S^1 \times \mathbf{R}^2 \to S$ we can define a C^1 vector field F on S^3 which has $\gamma_0 = T_0$, $\gamma_1 = T_1$ periodic solutions of simple type, which has e elliptic domains at γ_0, which has h elliptic domains at γ_1, and which has each N_k as an invariant manifold. (It is also clear how to put in some elliptic domains at γ_0 which do not reach to γ_1 if we so desire). For simplicity, when we refer to these examples, we shall intend for the vector field on γ_0 to be directed with increasing φ. The vector field may be directed in either direction on γ_1. We say $\lambda(\gamma_0, \gamma_1) = +1$ if F is directed with increasing θ on γ_1.

THEOREM 5.4. — *Let* $p \neq 0$, $q > 0$, $e \geqslant 0$, $h \geqslant 0$ *be the parameters used to construct a geometrically simple vector field* F *which has two periodic solutions. Then*

i) $r(\gamma_0;\ F) = \dfrac{p}{q}$.

ii) $i_q(\gamma_0,\ F) = 1 + \dfrac{q}{2}(e - h)$.

iii) $t_q(\gamma_0;\ F) = -\dfrac{|p|}{2}(e - h)$.

iv) $r(\gamma_1;\ F) = \lambda(\gamma_0,\ \gamma_1)\dfrac{q}{p}$.

v) $i_p(\gamma_1;\ F) = 1 - \dfrac{|p|}{2}(e - h)$.

vi) $t_p(\gamma_1;\ F) = \dfrac{q}{2}(e - h)$.

Proof. — The assertions regarding γ_0 are restatements of results obtained in § 3. The assertions regarding γ_1 follow similarly after we recall that for γ_1, the roles of e and h and the roles of p and q are interchanged, and $r(\gamma_1;\ F)$ is computed by looking in the direction of F along γ_1.

Examples which have one periodic solution. We shall now describe a countably infinite set of distinct examples of geometrically simple vector fields on S^3 which have just one periodic solution. Let F be a geometrically simple vector field with just two periodic solutions, and let p, q, e, h be the parameters describing F. Observe that if p and $\lambda(\gamma_0, \gamma_1)$ have the same sign, then F could be altered homotopically, preserving each invariant manifold and elliptic domain, so that on some invariant manifold N, F is made to coincide with the field of tangents to the $(p, q)^{\text{th}}$ Hopf fibering (if $p.\lambda(\gamma_0, \gamma_1) < 0$, then this structure would force a discontinuity at γ_1). Suppose that F has been so altered. Note that there is a diffeomorphism $h : S^3 - \overline{N} \to S^3 - T_0$. Define a vector field G on S^3 by

$$G|T_0 = F|T_0$$
$$G|(S^3 - T_0) = h_*(F|S^3 - \overline{N}).$$

Then G is geometrically simple and γ_0 is the only periodic solution of G. Also, since some invariant manifolds of F remain as invariant manifolds of G near γ_0, it follows that $r(\gamma_0; G) = p|q$. Thus we obtain a different example of a geometrically simple vector field with just one periodic solution for each relatively prime pair (p, q).

6. Homotopy classification.

Let F denote a geometrically simple vector field on S^3 which has just two periodic solutions γ_0, γ_1, and let p, q, e, h be the parameters describing F.

THEOREM 6.1. — *If* $p\lambda(\gamma_0, \gamma_1) > 0$, *then* $F \simeq H_+$ *if* $p > 0$, *and* $F \simeq H_-$ *if* $p < 0$.

COROLLARY 6.2. — *The examples which we have given of geometrically simple vector fields with just one periodic solution are homotopic to* H_+ *or* H_- *according to whether* $p > 0$ *or* $p < 0$.

Proof of theorem. — Let $\{N_i\}_{i=1}^{e+h}$ denote the invariant manifolds of F and let $\{D_i\}_{i=1}^{e+h}$ denote the elliptic domains. By the construction used in the proof of 4.4, there is a cross-section M_i to the flow in D_i and we can arrange it so that M_i contains γ_0 and γ_1 in its closure and forms a half-line bundle over γ_0 and γ_1, just as the N_i do. Let $M_{i,t}$ denote the translate of M_i by the flow for time t $(-\infty < t < \infty)$. Then $M_{i,t}$ is also a half-line bundle over γ_0 and γ_1 and

$\bigcup_{-\infty<t<\infty} M_{i,t} = D_i$. Since F is transversal to $M_{i,t}$ for each i and t, there is a homotopy of F relative to $\bigcup_{i=1}^{e+h} \overline{N}_i$ which produces a vector field F' which is tangent to every $M_{i,t}$.

LEMMA 6.3. — *Let* $\{M_\alpha\}$ *be a family of embedded open cylinders which fill* $S^3 - T_0 \cup T_1$, *and such that for each* α

$$\overline{M}_\alpha = M_\alpha \cup T_0 \cup T_1$$

is a half-line bundle with rotation p/q *over* T_0 *and a half-line bundle with rotation* q/p *over* T_1. *Then there is an isotopy of* S^3 *relative to* $T_0 \cup T_1$ *which carries* $\{M_\alpha\}$ *onto* $\{N_{\theta,\varphi}\}$ *where* $N_{\theta,\varphi}$ *is the half-line bundle over* T_0 *and* T_1

$$N_{\theta,\varphi} = \bigcup_{0<r<1} C_r$$

where C_r *is the cannonical curve of type* (p, q) *in* T_r *which passes through the point* (r, θ, φ).

Continuing the proof of 6.1, we alter F' by the isotopy of the flow described in 6.3. By a further homotopy on each $N_{\theta,\varphi}$, we obtain the $(p, q)^{\text{th}}$ Hopf field. The theorem now follows from 2.4.

THEOREM 6.4. — If $p\lambda(\gamma_0, \gamma_1) < 0$, *then*

$$d(F, H_-) = -\, i_q(\gamma_0; F) i_p(\gamma_1; F) \quad \text{if} \quad \lambda(\gamma_0, \gamma_1) > 0.$$
$$d(F, H_+) = i_q(\gamma_0; F) i_p(\gamma_1; F) \quad \textit{if} \quad \lambda(\gamma_0, \gamma_1) < 0.$$

Proof. — We shall choose a parallelization Π which extends the $(p, q)^{\text{th}}$ Hopf fibering and compute the difference element by geometrically determining the Hopf invariant of the mapping of S^3 into S^2 induced by F and Π.

By 6.3 and the construction used in the previous proof, we can assume that F is transverse to $\{N_{\theta,\varphi}\}$ on the interior of each elliptic domain and that $N_i = N_{\theta_i,\varphi_i}$ for $i = 1, \ldots, e + h$. Let Π denote a parallelization which extends the $(p, q)^{\text{th}}$ Hopf fibering H. By 1.1, it does not matter which one we choose. On each N_i, the trajectories go either from γ_0 to γ_1 or from γ_1 to γ_0. In either case, there is no loss of generality in assuming that they are never tangent to the first coordinate of Π. Then, since F is transverse to $N_{\theta,\varphi}$ on the interior of each elliptic domain, it follows that the only places that F coincides with the first coordinate direction H of Π is on γ_0 and γ_1. On γ_0, F is positively directed and on γ_1, F is negatively directed. We can then alter Π by a homotopy relative to H so that expressed in Π-coordinates, the first coordinate of $F|T_r$ has value $1 - 2r$. Thus to compute $d(F, H)$, it suffices to consider the induced mapping $f: T_r \to S^1$. Since F is geometrically simple, sufficiently close to γ_0, F and F^q are homotopic. Since we have chosen Π to fix the magnitude of the first coordinate of F in Π-coordinates, we can use either F or F^q to determine the other two Π-coordinates of $F|T_r$.

Lemma 6.5. — *Let* Δ *denote the half-line bundle over* T_0 *induced by the second coordinate direction of the right invariant (left invariant) parallelization of* S^3. *Then*

$$\begin{aligned} r(\Delta) &= -1 \quad \textit{right invariant} \\ r(\Delta) &= +1 \quad \textit{left invariant.} \end{aligned}$$

Proof. — By 2.1, H_+ and the left invariant vector field coincide, i.e. H_+ is the (1,1)-Hopf vector field and its trajectories are the left-cosets the Lie-action of S^1 on S^3. Since these cosets carry the left invariant parallelization, $r(\Delta) = +1$ in the left invariant case. The other case is similarly verified.

Corollary 6.6. — $r(\Pi|\gamma_0) = -\lambda(\gamma_0, \gamma_1)$.

Proof. — Since Π is homotopic to one of the invariant parallelizations (depending on the sign of p) we can apply 6.5.

Returning to the proof of 6.4, we let $C_\theta(C_\varphi)$ denote a curve in T_r obtained by holding φ constant (θ constant, respectively). By the definition of torsion, $t_q(\gamma_0; F^q) = r(F|C_\varphi)$. Since $r(\Pi|C_\varphi) = -\lambda(\gamma_0, \gamma_1)$, we conclude that $f|C_\varphi$ has degree $t_q(\gamma_0; F) + \lambda(\gamma_0, \gamma_1)$. By definition, $f|C_\theta$ has degree $i_q(\gamma_0; F)$. Since these values do not depend on r, it follows that the Hopf invariant of f is

$$- i_q(\gamma_0; F)[t_q(\gamma_0; F) + \lambda(\gamma_0, \gamma_1)].$$

(cf. the description of the Hopf fiberings in § 2). By 4.7,

$$t_q(\gamma_0; F) = \lambda(\gamma_0; \gamma_1) i_p(\gamma_1; F) - \lambda(\gamma_0; \gamma_1).$$

Thus the Hopf invariant of f is

$$-\lambda(\gamma_0;\ \gamma_1)i_q(\gamma_0;\ F)i_p(\gamma_1;\ F).$$

Theorem 6.7. — *There is a geometric vector field with two periodic solutions representing every homotopy class of vector field on* S^3.

Proof. — We apply 6.4. To obtain the positive classes (above H_+), take $\lambda(\gamma_0, \gamma_1) = +1$, $p < 0$, $q > 0$ and $\frac{e-h}{2} = k$. When $q = k = 1$, then

$$i_q(\gamma_0; F) = 2; \qquad i_p(\gamma_1; F) = 1 + p.$$

Then $d(F; H_-) = -2(1+p) > 0$ and for $p = -2, -3, \ldots$ every odd class above H_+ is represented. Taking $p = -1$, $k = 2$, we find

$$i_q(\gamma_0; F) = 1 + 2q; \qquad i_p(\gamma_1; F) = -1$$

and $d(F; H_-) = 1 + 2q$ and for $q = 1, 2, \ldots,$ every even class above H_+ is represented. The negative classes (below H_-) are found by taking $\lambda(\gamma_0; \gamma_1) = -1$.

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