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COMINUSCULE POINTS AND SCHUBERT VARIETIES

by William GRAHAM & Victor KREIMAN

Abstract. — We introduce the notion of a cominuscule point in a Schubert variety in a generalized flag variety for a semisimple group. We derive formulas expressing the Hilbert series and multiplicity of a Schubert variety at a cominuscule point in terms of the restrictions of classes in torus-equivariant K-theory and cohomology to that point, generalizing previously known formulas for flag varieties of cominuscule type. Thus, we can calculate Hilbert series and multiplicities in cases where these were previously unknown. The formulas for Schubert varieties are special cases of more general formulas valid at generalized cominuscule points of schemes with torus actions.

Résumé. — Nous définissons la notion de point cominuscule d'une variété de Schubert dans une variété de drapeaux généralisée pour un groupe semi-simple. Nous en déduisons des formules exprimant les séries de Hilbert et multiplicités des variétés de Schubert en des points cominuscules en termes de restrictions de classes de la K-théorie tore-équivariante et de la cohomologie en ces points, ce qui permet de généraliser des formules précédemment connues pour les variétés de drapeaux de type cominuscule. Nous pouvons ainsi calculer les séries de Hilbert et les multiplicités dans de nouveaux cas. Les formules pour les variétés de Schubert sont des cas particuliers de formules qui valent plus généralement en des points cominuscules généralisés de schémas munis d'actions de tores.

1. Introduction

Torus-equivariant K-theory and Chow groups can be used to calculate Hilbert series and multiplicities for Schubert varieties in a cominuscule flag variety. The purpose of this paper is to extend these methods to calculate Hilbert series and multiplicities at certain other points of Schubert varieties, which we call cominuscule points. Our methods apply in the more general setting of generalized cominuscule points of varieties with torus actions.

Hilbert series and multiplicity calculations on a cominuscule flag variety are possible because such a flag variety X has an action of a torus T with

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the following property. For any T-fixed point x in X, there is an element v in the Lie algebra of T such that $\alpha(v) = -1$ for any weight α of T on the T-representation $T_x X$. Note that the condition concerns the tangent space to the ambient variety X, not the Schubert variety. See [15] and [19] for a discussion and references.

The key observation of this paper is that these calculations can be carried out under a weaker tangent space condition, where the tangent space of the ambient flag variety X is replaced by the tangent space of a slice to the Schubert variety. If there exists v such that $\alpha(v) = -1$ for all weights α of the tangent space to the slice at x, we call the point x a cominuscule point of the Schubert variety. If X is a cominuscule flag variety, any torus-fixed point of a Schubert variety is a cominuscule point. However, our condition is more widely satisfied, and provides new examples where the calculation of Hilbert series and multiplicities is possible.

There is an effective test for whether a point in a Schubert variety is cominuscule, in case G is classical. The reason is that the tangent spaces of the slices can be described in terms of the tangent spaces of the Schubert varieties, and for classical groups, these have been described combinatorially. Using this, we produce examples of cominuscule points in Schubert varieties which are essentially new, in the sense that the Schubert varieties are not inverse images of Schubert varieties in cominuscule flag varieties under a projection of flag varieties.

In a sequel [16] to this paper we further explore computational and combinatorial aspects of cominuscule points. In particular, in type A, we characterize these points and give combinatorial rules for their Hilbert series and multiplicities in terms of the pipe dreams of Fomin–Kirillov [12], Bergeron–Billey [2], and Knutson–Miller [22]. We also plan to explore generalizations of the excited Young diagrams of [15, 20, 24] and [25] in this setting.

Slices to Schubert varieties were previously studied by Li and Yong [30]. They work in type A; the slices they consider are Kazhdan–Lusztig varieties, and they study these varieties by using coordinates to study their ideals. They observe that if the Kazhdan–Lusztig variety is invariant under dilation—which occurs if the Weyl group element defining the fixed point is λ-cominuscule in the sense of Peterson (cf. Section 5.2)—then their methods can be used to obtain Hilbert series and multiplicities. Our approach differs in that the slices we use are generally smaller than the Kazhdan–Lusztig varieties. Moreover, instead of using the slice itself, we (in effect) replace the slice by its tangent cone at the point, so that we can work inside inside the tangent space at the point. See Example 3.7 for some discussion and an example. Thus, we can obtain Hilbert series and multiplicity formulas in cases where the results of [30] do not apply. Moreover, our results are not limited to type A.

The contents of the paper are as follows. Section 2 contains some definitions and results about K-theory and intersection theory of schemes with torus actions. Section 3 defines the notion of a generalized cominuscule point of a scheme with a torus action. The remainder of the paper focuses on Schubert varieties. Section 4 contains some background about algebraic groups, Weyl groups, and Schubert varieties, leading to the definition of the slices to Schubert varieties used in the definition of cominuscule points. Section 5 defines the notion of cominuscule points of Schubert varieties, gives some examples, and obtains Hilbert series and multiplicity formulas in this case. Some of the results can be stated more simply in type A, because the tangent spaces to Schubert varieties are easier to describe. Section 6 contains some additional examples.

2. Notation and preliminaries

2.1. Cones

We work with schemes of finite type over an algebraically closed field $\mathbb{F}$. Given a (finite-dimensional) vector space V over $\mathbb{F}$, $S(V)$ denotes the symmetric algebra of V, that is, the ring of polynomials on V^*. Given a scheme X, $A_i(X)$ denotes the i^{th} Chow group of X, and $A^*(X)$ denotes the operational Chow ring of X (see [13]). We write $h = c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) \in A^1(\mathbb{P}(V))$. A cone C is a subscheme of V which is invariant under the action of the multiplicative group $\mathbb{G}_m$. If C is a closed cone in V of pure dimension, and k is the codimension of C in V, then $[\mathbb{P}(C)] = ah^k[\mathbb{P}(V)] \in A_*(\mathbb{P}(V))$; the degree of the cone is the integer a.

2.2. The tangent cone, multiplicities and Hilbert series

Let x be a closed point in a scheme X. Working locally, we may assume $X = \operatorname{Spec} A$. Let $\mathfrak{m}$ be the maximal ideal corresponding to x. The tangent space (by which we mean the Zariski tangent space) is $T_xX = (\mathfrak{m}/\mathfrak{m}^2)^*$. The tangent cone to X at x is defined as $C_xX = \operatorname{Spec}(\operatorname{gr}_{\mathfrak{m}} A)$, where $\operatorname{gr}_{\mathfrak{m}} A = \oplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1}$. There is a surjection $S(\mathfrak{m}/\mathfrak{m}^2) \to \operatorname{gr}_{\mathfrak{m}} A$, so C_xX is a closed cone in T_xX, and the multiplicity $\operatorname{mult}(X,x)$ is the degree of this cone.

The Hilbert function is the function $n \mapsto \dim(\mathfrak{m}^n/\mathfrak{m}^{n+1})$. For sufficiently large values of n it is a polynomial $h(X,x)(n)$ in n, called the Hilbert polynomial; this is related to the multiplicity by the equation $\operatorname{mult}(X,x) = a_d/d!$, where a_d is the leading coefficient of $h(X,x)(n)$. The Hilbert series of X at x is the power series $H(X,x) = \sum \dim(\mathfrak{m}^i/\mathfrak{m}^{i+1})t^i$. Observe that $H(X,x) = H(C_xX,x)$ and $\operatorname{mult}(X,x) = \operatorname{mult}(C_xX,x)$. If x and y are closed points in X and Y, respectively, then $C_{(x,y)}(X\times Y) \cong C_xX \times C_yY$. Hence

$$(2.1)\quad \begin{aligned} &H(X\times Y,z)\\ &\quad = H(X,x)H(Y,y),\ \operatorname{mult}(X\times Y,z) = \operatorname{mult}(X,x)\operatorname{mult}(Y,y). \end{aligned}$$

2.3. Tori, completions, and evaluation maps

Let $T \cong (\mathbb{G}_m)^n$ be a torus. The character group of T is $\widehat{T} = \operatorname{Hom}(T,\mathbb{G}_m)$. We can view $\widehat{T}$ as a subset of the dual $\mathfrak{t}^*$ of the Lie algebra of T. If we want to view $\lambda \in \widehat{T} \subset \mathfrak{t}^*$ as a homomorphism $T \to \mathbb{G}_m$, we will write it as e^λ. We write $\mathbb{F}_\lambda$ for the 1-dimensional representation of T of weight λ (that is, on which T acts by e^λ). If V is a representation of T, we denote by $\Phi(V) \subset \widehat{T}$ the set of weights of T on V. We identify $\operatorname{Hom}(\mathbb{G}_m,T)$ with the set of $v \in \mathfrak{t}$ such that $\lambda(v) \in \mathbb{Z}$ for all $\lambda \in \widehat{T}$; we will say that such a v is integral. If a nonzero integer multiple of v is integral, we say v is rational.

The representation ring $R(T)$ of T is the free $\mathbb{Z}$-module with basis e^λ, for $\lambda \in \widehat{T}$, and multiplication given by $e^\lambda e^\mu = e^{\lambda+\mu}$. Let $S(\widehat{T})$ be the symmetric algebra on $\widehat{T}$; if $\lambda_1, \ldots, \lambda_n$ is a basis for $\widehat{T} \cong \mathbb{Z}^n$, then $S(\widehat{T})$ is the polynomial ring $\mathbb{Z}[\lambda_1, \ldots, \lambda_n]$.

Let $v \in \mathfrak{t}$ be rational, and let d be a positive integer such that dv is integral. For $i \in \frac{1}{d}\mathbb{Z}$, let $R^i(T)$ be the span of e^λ with $\lambda(v) = i$. Then $R(T) = \oplus_i R^i(T)$. Let $\widehat{R}(T)$ be the completion of $R(T)$ with respect to the ideal of positive degree elements. An element of $\widehat{R}(T)$ can be written as a (possibly) infinite sum $\sum r_i$, where $r_i \in R^i(T)$, and the set of i such that $r_i \neq 0$ is bounded below. If v is integral, define a homomorphism $\operatorname{ev}_v : R(T) \to \mathbb{Z}[t,t^{-1}]$ by $\operatorname{ev}_v(e^\lambda) = t^{\lambda(v)}$. We extend this to $\operatorname{ev}_v : \widehat{R}(T) \to \mathbb{Z}[t^{-1}][[t]]$ by $\operatorname{ev}_v(\sum r_i) = \sum \operatorname{ev}_v(r_i)$. If v is rational, this construction gives a map $\operatorname{ev}_v : \widehat{R}(T) \to \mathbb{Z}[u^{-1}][[u]]$, where $u = t^{1/d}$; if v is integral, this map can be viewed as the composition of the map $\operatorname{ev}_v : \widehat{R}(T) \to \mathbb{Z}[t^{-1}][[t]]$ and the inclusion $\mathbb{Z}[t^{-1}][[t]] \to \mathbb{Z}[u^{-1}][[u]]$ which takes t to u^d. Note that if

$f, g \in R(T)$ such that g is a unit in $\widehat{R}(T)$, then $\mathrm{ev}_v(f/g)$ is the expansion of the rational function $\mathrm{ev}_v(f)/\mathrm{ev}_v(g)$ in positive powers of t (i.e., the Laurent series expansion at $t = 0$, or $u = 0$ in the rational case).

We can also define an evaluation map using $S(\widehat{T})$ in place of $R(T)$. Precisely, if $\lambda_1, \ldots, \lambda_n$ is a basis of $\widehat{T}$, then $S(\widehat{T})$ is the polynomial ring $\mathbb{Z}[\lambda_1, \ldots, \lambda_n]$. If $f(\lambda_1, \ldots, \lambda_n)$ and $g(\lambda_1, \ldots, \lambda_n)$ are in $S(\widehat{T})$, define

$$\mathrm{ev}_v\left(\frac{f}{g}\right) = \frac{f\left(\lambda_1(v), \ldots, \lambda_n(v)\right)}{g\left(\lambda_1(v), \ldots, \lambda_n(v)\right)} \in \mathbb{Q},$$

provided the denominator is nonzero.

2.4. Equivariant K-theory and Chow groups for torus actions

If a torus T acts on a scheme X, we denote by $K_T(X)$ (resp. $G_T(X)$) the Grothendieck group of T-equivariant vector bundles (resp. coherent sheaves) on X. If X is nonsingular, the natural map $K_T(X) \to G_T(X)$ is an isomorphism (see [35, Corollary 7.8]). The representation ring $R(T)$ is identified with the Grothendieck group $K_T(pt)$ of a point, and $K_T(X)$ is an $R(T)$-module. Similarly, we let $A_*^T(X)$ denote the T-equivariant Chow groups of X (see [9]). By definition, $A_i^T(X) = A_{i+N}(\mathcal{U} \times^T X)$, where $\mathcal{U}$ is an open T-invariant subset in a representation $\mathcal{V}$ of T such that the codimension of $\mathcal{V} \backslash \mathcal{U}$ in $\mathcal{V}$ is greater than $\dim X - i$, and $N = \dim \mathcal{V} - \dim T$. We abuse notation and write $X_T = \mathcal{U} \times^T X$, although X_T depends on the choice of a suitable $\mathcal{U} \subset \mathcal{V}$. The ring $S(\widehat{T})$ can be identified with the operational Chow ring $A_T^*(pt)$ of a point. This acts on $A_*^T(X)$; by definition, the element $\lambda \in \widehat{T}$ acts by multiplication by the first Chern class of the line bundle $\mathcal{U} \times^T (X \times \mathbb{F}_\lambda) \to \mathcal{U} \times^T X$. If T acts freely on X, then flat pullback via the map $q : \mathcal{U} \times^T X \to X/T$ induces an isomorphism $A_i(X/T) \to A_i^T(X)$.

A T-equivariant vector bundle V on X has equivariant Chern classes $c_i^T(V)$ in the operational equivariant Chow groups $A_T^i(X)$. A representation V of T can be viewed as a T-equivariant vector bundle over a point. If $\dim V = d$, then the top equivariant Chern class of V is $c_d^T(V) = \prod_{\alpha \in \Phi(V)} \alpha \in A_*^T(pt)$. Another important class is

$$\lambda_{-1}(V^*) = \prod_{\alpha \in \Phi(V)} \left(1 - e^{-\alpha}\right) \in K_T(pt) = R(T).$$

For any point x with a trivial T-action we identify $K_T(\{x\})$ with $R(T)$ and $A_T^*(\{x\})$ with $S(\widehat{T})$. Thus, if $i_x : \{x\} \to X$ is the inclusion of a T-fixed point of a nonsingular T-variety, we have pullback maps $i_x^* : K_T(X) \to$

$K_T(\{x\}) = R(T)$ and $i_x^* : A_*^T(X) \to A_T^*(\{x\}) = S(\widehat{T})$. If $i_V : \{0\} \to V$ is the inclusion of $\{0\}$ into a representation V, then pullback induces isomorphisms $i_V^* : K_T(V) \to R(T)$ and $i_V^* : A^*(V) \to S(\widehat{T})$.

Let X be a T-scheme (i.e., a scheme with T-action), and Y a closed T-stable subscheme. We denote by $[\mathcal{O}_Y]_X$ and $[Y]_X$ the structure sheaf and equivariant fundamental classes of Y in $G_T(X)$ and $A_*^T(X)$, respectively. If it is understood that we are working in $G_T(X)$ or $A_*^T(X)$, the subscript X is frequently omitted.

Lemma 2.1. — *Suppose a torus T acts on smooth varieties M_1, M_2, and suppose Y is a closed subscheme of M_2. Let m_i be a T-fixed point of M_i $(i = 1, 2)$, and let i_{m_1}, i_{m_2}, and $i_{(m_1,m_2)}$ be the inclusions of m_1, m_2 and (m_1, m_2) into M_1, M_2 and $M_1 \times M_2$, respectively. Then*

$$i^*_{(m_1,\, m_2)}\, [\mathcal{O}_{M_1 \times Y}]_{M_1 \,\times\, M_2} = i^*_{m_2} [\mathcal{O}_Y]_{M_2}$$

and

$$i^*_{(m_1,\, m_2)}\, [M_1 \times Y]_{M_1 \,\times\, M_2} = i^*_{m_2} [Y]_{M_2}.$$

Proof. — Let $k : \{(m_1, m_2)\} \to \{m_2\}$. Under our identifications of $K_T(\{(m_1, m_2)\})$ and $K_T(\{m_2\})$ with $R(T)$, the K-theory pullback k^* is the identity. Similar remarks apply for equivariant Chow groups.

Let $\pi : M_1 \times M_2 \to M_2$ be projection on the second factor. Then

$$i^*_{(m_1,\, m_2)} \pi^* = \left(\pi \circ i_{(m_1,\, m_2)}\right)^* = (i_{m_2} \circ k)^* = k^* i^*_{m_2} = i^*_{m_2}.$$

Also, $\pi^*([\mathcal{O}_Y]_{M_2}) = [\mathcal{O}_{M_1 \times Y}]_{M_1 \times M_2}$ and $\pi^*([Y]_{M_2}) = [M_1 \times Y]_{M_1 \times M_2}$. Hence,

$$i^*_{(m_1,\, m_2)}\, [\mathcal{O}_{M_1 \times Y}]_{M_1 \,\times\, M_2} = i^*_{(m_1,\, m_2)} \pi^* \left([\mathcal{O}_Y]_{M_2}\right) = i^*_{m_2} \left([\mathcal{O}_Y]_{M_2}\right)$$

and

$$i^*_{(m_1,\, m_2)}\, [M_1 \times Y]_{M_1 \,\times\, M_2} = i^*_{(m_1,\, m_2)} \pi^* \left([Y]_{M_2}\right) = i^*_{m_2} \left([Y]_{M_2}\right),$$

as desired. □

Lemma 2.2. — *If $\mathcal{L}$ is a T-equivariant line bundle on a T-scheme X, and s is a T-invariant regular section of $\mathcal{L}$ (cf.* [13, Section 14.1]*) with zero-scheme Y, then $[Y] = c_1^T(\mathcal{L}) \cap [X]$ in $A_*^T(X)$, and $[\mathcal{O}_Y] = [\mathcal{O}_X] - [\mathcal{L}^*]$ in $G_T(X)$.*

Proof. — By definition, $A_i^T(X) = A_{i+N}(X_T)$, where X_T and N are as above. The T-equivariant line bundle $\mathcal{L}$ defines a line bundle $\mathcal{L}_T$ on X_T whose zero-scheme is $[Y_T]$. By [13, Proposition 14.1], $[Y_T] = c_1(\mathcal{L}_T) \cap [X_T]$

in $A_*(X_T)$. Since by definition the class $[Y]$ in $A_*^T(X)$ is the class $[Y_T]$ in $A_*(X_T)$, and $c_1^T(\mathcal{L})$ is $c_1(\mathcal{L}_T) \in A^1(X_T)$, we see that $[Y] = c_1^T(\mathcal{L}) \cap [X]$ in $A_*^T(X)$. For the assertion about classes in $G_T(X)$, see [17, Remark 6.3]. $\square$

Lemma 2.3. — *Let $T \cong \mathbb{G}_m$ and let λ be a generator of $\widehat{T}$. Assume T acts on $V = \mathbb{F}^{n+1}$ with all weights equal to $r\lambda$, $r \neq 0$. Let $V^0 = V \setminus \{0\}$.*

(1) *Suppose $r = -1$, so T acts freely on V^0. Under the isomorphism $A_*^T(V^0) \cong A_*(\mathbb{P}(V))$, the action of $\lambda \in A_T^1(pt)$ on $A_*^T(V^0)$ corresponds to multiplication by $-h$ on $A_*(\mathbb{P}(V))$, where*

$$h = c_1\left(\mathcal{O}_{\mathbb{P}(V)}(1)\right).$$

(2) *If $C \subset V$ is a closed cone of pure codimension k, then for some $a \in \mathbb{Z}$, $[\mathbb{P}(C)] = ah^k \cap [\mathbb{P}(V)]$ in $A_*(\mathbb{P}(V))$ and*

$$[C] = ar^k\lambda^k \cap [V] \text{ in } A_*^T(V).$$

Proof. — (1) The element $\lambda \in A_T^1(pt)$ acts on $A_*^T(V^0)$, so via the isomorphism $A_*^T(V^0) \cong A_*(\mathbb{P}(V))$, λ acts on $A_*(\mathbb{P}(V))$. The action of λ on $A_*(\mathbb{P}(V))$ is multiplication by the first Chern class of the line bundle $(V^0 \times \mathbb{F}_\lambda)/T \to V^0/T = \mathbb{P}(V)$. This line bundle is isomorphic to the tautological subbundle S of the trivial bundle $V \times \mathbb{P}(V) \to \mathbb{P}(V)$. Part (1) follows since $c_1(S) = -h$.

(2) First assume $r = -1$. We have $[\mathbb{P}(C)] = ah^k[\mathbb{P}(V)]$ for some $a \in \mathbb{Z}$. Write $C^0 = C \setminus \{0\}$. Under the isomorphism of $A_*^T(V^0)$ with $A_*(\mathbb{P}(V))$, the class $[C^0]$ corresponds to $[\mathbb{P}(C)]$, and we have proved that the action of λ corresponds to the action of $-h$. Hence, $[C^0] = (-1)^k a\lambda^k[V^0]$ in $A_*^T(V^0)$. The restriction map

$$A_{n+1-k}^T(V) \to A_{n+1-k}^T(V^0)$$

(where $n+1 = \dim V$) is an isomorphism taking $[C]$ to $[C^0]$ and $\lambda^k[V]$ to $\lambda^k[V^0]$, so $[C] = (-1)^k a\lambda^k[V]$. This proves (2) for $r = -1$.

We now consider the case of general r. Define $\phi : T \cong \mathbb{G}_m \to T_1 = \mathbb{G}_m$ by $t \mapsto t^{-r}$. Let $\lambda_1 \in \widehat{T_1}$ be the character whose image under the pullback $\phi^* : \widehat{T_1} \to \widehat{T}$ equals $-r\lambda$. If T_1 acts on V with all weights equal to λ_1, then the T-action on V is induced by the T_1-action via the map ϕ. There is a pullback map $\phi^* : A_{T_1}^*(V) \to A_T^*(V)$, defined as follows. Let $\mathcal{U}$ and $\mathcal{U}_1$ be open subsets of representations of T and T_1 respectively, as in the definition of equivariant Chow groups given in Section 2.4. The torus T acts on $\mathcal{U}_1$ via the map ϕ. Thus, there is a map

$$(\mathcal{U} \times \mathcal{U}_1) \times^T V \to \mathcal{U}_1 \times^{T_1} V,$$

and pullback along this map yields ϕ^*. By part (1), $[C]_{T_1} = (-1)^k a\lambda^k [V]_{T_1}$. Also, $\phi^*(\lambda_1) = -r\lambda$, and $\phi^*[C]_{T_1} = [C]_T$, $\phi^*[V]_{T_1} = [V]_T$, where the subscripts denote which equivariant group we are considering. Part (2) follows. □

3. Generalized cominuscule points

In this section we define the notion of a generalized cominuscule point of a scheme with a torus action. We give formulas for the Hilbert series and multiplicity at a generalized cominuscule point (see Theorem 3.9), generalizing formulas used by Ikeda–Naruse and Graham–Kreiman; see [15] for a discussion and references.

3.1. Degenerating to the tangent cone

Let V be a representation of a torus T. Let V^* denote the dual representation of T, so the symmetric algebra $S(V^*)$ is the ring of regular functions on V. Let $X = \operatorname{Spec} A$ be a T-invariant closed subscheme of V containing 0, so there is a surjection $S(V^*) \to A$. Let $\mathfrak{m} \subset A$ be the maximal ideal of 0. The tangent cone of X at 0 is $\mathcal{C} = \operatorname{Spec} B$, where $B = \operatorname{gr}_{\mathfrak{m}} A = A/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \cdots$. There is a T-equivariant surjection $S(V^*) \to B$, so $\mathcal{C}$ is a T-invariant closed subscheme of V. The next proposition is known, but for lack of a precise reference we provide a proof.

Proposition 3.1. — *With the assumptions and notation of the previous paragraph, we have:*

(1) *If X has pure dimension k, then so does $\mathcal{C}$, and $[\mathcal{C}] = [X]$ in $A_k^T(V)$.*
(2) *$[\mathcal{O}_{\mathcal{C}}] = [\mathcal{O}_X]$ in $G_T(V)$.*

Proof. — Let $\mathcal{A} = \cdots \oplus u^{-2}\mathfrak{m}^2 \oplus u^{-1}\mathfrak{m} \oplus A \oplus uA \oplus u^2 A \oplus \cdots$ denote the Rees algebra of A (see [11, Section 6.5]). We extend the T-action on A to an action on $\mathcal{A}$ by requiring $t \cdot u^k a = u^k(t \cdot a)$ for $k \in \mathbb{Z}$, $a \in A$. Let $\mathcal{X} = \operatorname{Spec} \mathcal{A}$. There is a T-equivariant $\mathbb{F}[u]$-algebra homomorphism $\varphi : S(V^*)[u] \to \mathcal{A}$ characterized by $\varphi(\zeta) = u^{-1}\zeta$ for $\zeta \in V^*$. The algebra $\mathcal{A}$ is spanned by elements of the form $u^{-m}\zeta_1 \cdots \zeta_n = \varphi(u^{n-m}\zeta_1 \cdots \zeta_n)$, where $\zeta_i \in V^*$ and $m \leqslant n$, so φ is surjective, and hence induces a T-equivariant closed embedding $\mathcal{X} \subset V \times \mathbb{A}^1$. Let $\mathcal{X}_c$ denote the fiber of $\mathcal{X}$ over $c \in \mathbb{A}^1$. The composition $\mathcal{X} \to V \times \mathbb{A}^1 \to V$ is T-equivariant, and it takes $\mathcal{X}_1$

(resp. $\mathcal{X}_0$) isomorphically onto X (resp. $\mathcal{C}$). The fact that if X has pure dimension k, then so does $\mathcal{C}$, is a special case of [13, Appendix B.6.6].

We have a T-equivariant open embedding $V \times \mathbb{A}^1 \subset V \times \mathbb{P}^1$ (where T acts trivially on $\mathbb{P}^1$). Let $\overline{\mathcal{X}}$ denote the closure of $\mathcal{X}$ in $V \times \mathbb{P}^1$ Then $\overline{\mathcal{X}} \cap (V \times \mathbb{A}^1)$ equals $\mathcal{X}$ as a scheme, so under the map $\pi : \overline{\mathcal{X}} \to \mathbb{P}^1$, the inverse image of $\mathbb{A}^1$ is $\mathcal{X}$. Thus, if $\overline{\mathcal{X}}_c$ denotes the fiber of $\overline{\mathcal{X}}$ over $c \in \mathbb{P}^1$, then if $c \in \mathbb{A}^1$, we have $\mathcal{X}_c = \overline{\mathcal{X}}_c$.

There are sections s_0 and s_1 of $\mathcal{O}_{\mathbb{P}^1}(1)$ whose zero-schemes are the points 0 and 1, respectively. These sections are T-invariant since T acts trivially on $\mathbb{P}^1$. The pullbacks $\pi^* s_0$ and $\pi^* s_1$ are T-invariant regular sections of $\pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ whose zero-schemes are $\mathcal{X}_0$ and $\mathcal{X}_1$, respectively. Lemma 2.2 implies that in $A_k^T(\overline{\mathcal{X}})$, $[\mathcal{X}_0]$ and $[\mathcal{X}_1]$ are each equal to $c_1^T(\pi^* \mathcal{O}_{\mathbb{P}^1}(1)) \cap [\overline{\mathcal{X}}]$, which implies $[\mathcal{X}_0] = [\mathcal{X}_1]$. Let p denote the composition $\overline{\mathcal{X}} \to V \times \mathbb{P}^1 \to V$. Then p is proper, and it takes $\mathcal{X}_0$ isomorphically onto $\mathcal{C}$, and $\mathcal{X}_1$ isomorphically onto X. Thus, $[\mathcal{C}] = p_*[\mathcal{X}_0] = p_*[\mathcal{X}_1] = [X]$, proving (1). Similarly, in $G_T(\overline{\mathcal{X}})$, we have $[\mathcal{O}_{\mathcal{X}_0}] = [\mathcal{O}_{\mathcal{X}_1}]$, since by Lemma 2.2, each is equal to $[\mathcal{O}_{\overline{\mathcal{X}}}] - [\pi^* \mathcal{O}_{\mathbb{P}^1}(-1)]$. Thus, $[\mathcal{O}_{\mathcal{C}}] = p_*[\mathcal{O}_{\mathcal{X}_0}] = p_*[\mathcal{O}_{\mathcal{X}_1}] = [\mathcal{O}_X]$, proving (2). ☐

Recall that i_V denotes the inclusion of $\{0\}$ into a representation V.

Proposition 3.2. — *Keep the notation and hypotheses of Proposition 3.1. Suppose $k : V_1 \hookrightarrow V$ is the inclusion of a T-invariant subspace such that all the weights of T on V/V_1 are nonzero, and suppose that V_1 contains $\mathcal{C}$. Let $d = \dim V$ and $d_1 = \dim V_1$.*

$$i_{V_1}^* \left([\mathcal{O}_{\mathcal{C}}]_{V_1}\right) = \frac{i_V^* \left([\mathcal{O}_{\mathcal{C}}]_V\right)}{\lambda_{-1}\left((V/V_1)^*\right)} = \frac{i_V^* \left([\mathcal{O}_X]_V\right)}{\lambda_{-1}\left((V/V_1)^*\right)}. \tag{1}$$

$$i_{V_1}^* \left([\mathcal{C}]_{V_1}\right) = \frac{i_V^*([\mathcal{C}]_V)}{c_{d-d_1}^T(V/V_1)} = \frac{i_V^*([X]_V)}{c_{d-d_1}^T(V/V_1)}. \tag{2}$$

Proof. — The self-intersection formula in equivariant K-theory (see e.g. [10, Section 3.1]) implies that $k^* k_*$ is multiplication by $\lambda_{-1}((V/V_1)^*)$, since the normal bundle of V_1 in V is the bundle $V_1 \times (V/V_1) \to V_1$, which is trivial, but not equivariantly trivial. Hence

$$\begin{aligned} i_V^* \left([\mathcal{O}_{\mathcal{C}}]_V\right) &= i_V^* k_* \left([\mathcal{O}_{\mathcal{C}}]_{V_1}\right) = i_{V_1}^* k^* k_* \left([\mathcal{O}_{\mathcal{C}}]_{V_1}\right) \\ &= i_{V_1}^* \left(\lambda_{-1}\left((V/V_1)^*\right)[\mathcal{O}_{\mathcal{C}}]_{V_1}\right) = \lambda_{-1}\left((V/V_1)^*\right) i_{V_1}^* \left([\mathcal{O}_{\mathcal{C}}]_{V_1}\right). \end{aligned}$$

The last equality holds because $i_{V_1}^*$ is an $R(T)$-module map, so $i_{V_1}^*$ commutes with multiplication by $\lambda_{-1}((V/V_1)^*)$. Dividing by $\lambda_{-1}((V/V_1)^*)$, and using the equality $[\mathcal{O}_{\mathcal{C}}]_V = [\mathcal{O}_X]_V$, proves (1). The proof of (2) is similar, using the fact that in equivariant Chow groups, the self-intersection formula implies that k^*k_* is multiplication by $c^T_{d-d_1}(V/V_1)$. □

In the above proposition, the equalities in (1) and (2) are to be interpreted as equalities in $R(T)$ and $A^T_*(pt)$, respectively. There is no need to localize (i.e. invert elements): (1) implies that $i^*[\mathcal{O}_X]$ is divisible by $\lambda_{-1}((V/V_1)^*)$, and (2) implies that $i^*[X]$ is divisible by $c^T_{d-d_1}(V/V_1)$.

3.2. Generalized cominuscule points of schemes

Definition 3.3. — *Suppose X is a closed T-invariant subscheme of a nonsingular T-variety M. A point $x \in X^T$ is said to be a generalized cominuscule point of X if there are*

(1) *Representations V' and V of T, such that all the weights of T on V are nonzero, and an isomorphism of $V' \times V$ with an open subscheme M_0 of M containing x. Let $X_0 = X \cap M_0$.*
(2) *A T-invariant subscheme $\mathcal{N}$ of V, called a slice, such that the isomorphism of (1) restricts to an isomorphism of $V' \times \mathcal{N}$ with X_0. We identify $\mathcal{N}$ with $\{0\} \times \mathcal{N}$ and via $V' \times \mathcal{N} \cong X_0$ we view $\mathcal{N}$ as a subscheme of X_0, and x as a point of $\mathcal{N}$.*
(3) *A element $v \in \mathfrak{t}$ (which can be assumed to be rational) such that for each weight α of T on the Zariski tangent space $T_x\mathcal{N}$, we have $\alpha(v) = -1$.*

Remark 3.4. — This situation differs from the situation considered in [20, Section 9], because $\mathcal{N}$ need not be embedded as a cone in V. One can find a possibly different embedding of $\mathcal{N}$ into V which factors through the inclusion $V_1 = T_x\mathcal{N} \hookrightarrow V$, such that under this embedding, $\mathcal{N}$ is a cone in V_1. See Example 3.7 below. If we let $\mathcal{N}_1$ denote the image of $\mathcal{N}$ in V under the original embedding, and $\mathcal{N}_2$ the the image under the new embedding, then we obtain classes $[\mathcal{O}_{\mathcal{N}_1}]$ and $[\mathcal{O}_{\mathcal{N}_2}]$ in $K_T(V)$, and $[\mathcal{N}_1]$ and $[\mathcal{N}_2]$ in $A^T_*(V)$. If one showed that $[\mathcal{O}_{\mathcal{N}_1}] = [\mathcal{O}_{\mathcal{N}_2}]$ and $[\mathcal{N}_1] = [\mathcal{N}_2]$, then some of our results and proofs (for example, part of the proof of Theorem 3.9) would follow by applying the results described in [20] to the classes $[\mathcal{O}_{\mathcal{N}_2}]$ and $[\mathcal{N}_2]$. The approach in this paper is somewhat different: it uses the tangent cone $\mathcal{C}$ to $\mathcal{N}$ at x, which lies in V_1 by construction, and which by Proposition 3.1 yields the same classes in K-theory or Chow groups as $\mathcal{N}$.

Remark 3.5. — We can assume that the element v is rational because if S is a finite subset of $\widehat{T}$, and $v \in \mathfrak{t}$ satisfies $\alpha(v) \in \mathbb{Q}$ for all $\alpha \in S$, then there exists a rational element v' of $\mathfrak{t}$ such that $\alpha(v') = \alpha(v)$ for all $\alpha \in S$. This can be seen as follows. Let $\{\lambda_1, \ldots, \lambda_n\}$ be a basis for $\widehat{T} \otimes \mathbb{Q}$ such that $\lambda_1, \ldots, \lambda_k$ are elements of S which form a basis for the subspace of $\widehat{T} \otimes \mathbb{Q}$ spanned by S. Let $v_1, \ldots, v_n$ denote the dual basis of $\mathrm{Hom}(\widehat{T} \otimes \mathbb{Q}, \mathbb{Q})$. Let $a_i = \lambda_i(v)$, and let $v' = a_1 v_1 + \cdots + a_k v_k \in \mathrm{Hom}(\widehat{T} \otimes \mathbb{Q}, \mathbb{Q})$. Since $\mathrm{Hom}(\widehat{T} \otimes \mathbb{Q}, \mathbb{Q})$ can be viewed as the set of rational elements of $\mathfrak{t}$, this suffices. Thus, if in (3), the element v was not rational, we could replace it by the rational element v', and the conditions of the definition would be satisfied.

Remark 3.6. — If X is a closed T-invariant subscheme of a nonsingular T-variety M, $x \in X^T$, and there exists $v \in \mathfrak{t}$ such that $\alpha(v) = -1$ for all $\alpha \in \Phi(T_x M)$, then x is a generalized cominuscule point of X. Indeed, [7, Proposition A2] implies that there is a neighborhood X_0 of x in M isomorphic to $V = T_x M$, so taking $V' = 0$ and $\mathcal{N} = X_0$, the hypotheses of Definition 3.3 are satisfied. This is the situation if X is a Schubert variety in a cominuscule flag variety M.

Example 3.7. — It is natural to ask why the representation V is part of the definition of cominuscule point, since it might appear that we can simply replace V by its subspace $V_1 = T_x\mathcal{N}$. (By [7, Proposition A2], under the hypotheses of Definition 3.3, there is a T-invariant neighborhood $\mathcal{N}_0$ of x in $\mathcal{N}$ and a T-equivariant embedding of $\mathcal{N}_0$ into V_1 taking x to 0.) However, in applications, we may have a natural embedding $\mathcal{N} \subset V$ such that $\mathcal{N}_0$ does not lie in the subspace V_1 of V. Thus, the composition $\mathcal{N}_0 \to V_1 \to V$ is not the original embedding of $\mathcal{N}_0$ into V (cf. Remark 3.4). Here is an example (where $\mathcal{N} = \mathcal{N}_0$). Let $T = \mathbb{G}_m$ act on $V = \mathbb{A}^2$ so that $v = (1, 0)$ and $w = (0, 1)$ are weight vectors of weights 1 and 2, respectively. Let $\{x, y\}$ be the basis of V^* dual to the basis $\{v, w\}$, so x and y are T-weight vectors of weights -1, -2. Let $A = S(V^*) = \mathbb{F}[x, y]$. Let $\mathcal{N}$ be the subvariety of $\mathbb{A}^2$ defined by the equation $x^2 = y$, so $\mathcal{N} = \operatorname{Spec} B$ with $B = \mathbb{F}[x, y]/\langle x^2 - y\rangle$. Note that $\mathcal{N}$ is not a cone (i.e. is not dilation-invariant). Let $\overline{x}, \overline{y}$ denote the images of x, y in B. The surjection $V^* \to \mathfrak{m}/\mathfrak{m}^2 = V_1^*$ takes x to a basis element and y to 0. The dual map $V_1 \to V$ identifies V_1 with its image $\mathbb{F} \cdot v$, so $\mathcal{N}$ is not contained in V_1. However, the tangent cone $\mathcal{C}$ to $\mathcal{N}$ at 0 is contained in V_1—in fact, $\mathcal{C} = V_1$. Let $W = \mathbb{F} \cdot \overline{x}$, so $\mathfrak{m} = W \oplus \mathfrak{m}^2$ and $W \cong \mathfrak{m}/\mathfrak{m}^2$ as representations of T. Since W generates B as a ring, there is a surjection $S(W) \to B$, and this yields the embedding $\mathcal{N} \to W^* \cong V_1$

constructed by Brion. Thus, we obtain

$$\begin{array}{ccccc} \mathcal{N} & \to & V_1 & \to & V \\ (a,a^2) & \mapsto & av & \mapsto & (a,0). \end{array}$$

The composition is not the original embedding of $\mathcal{N}$ into V, because it does not take the point (a,a^2) to itself. Note that in this example, 0 is a generalized cominuscule point of $\mathcal{N}$.

Remark 3.8. — We can generalize Definition 3.3 in several ways. For example, we could replace the point 0 in V' with a T-fixed point in a smooth T-variety (with appropriate changes to the other conditions of the definition). This might be useful in considering slices with respect to actions of groups which are not unipotent. However, for our application to Schubert varieties, we do not need this generality.

The next result, Theorem 3.9, generalizes [20, Proposition 9.1] (cf. [15, Corollary 2.11]). The motivation for a statement of this form—in which the slice $\mathcal{N}$ does not explicitly appear—is that for Schubert varieties, the pullbacks $i_x^*[X]$ and $i_x^*[\mathcal{O}_X]$ are pullbacks of Schubert classes, which can be calculated.

THEOREM 3.9. — *With notation as in Definition 3.3, suppose that x is a generalized cominuscule point of X. Let $d' = \dim V'$ and $d = \dim V$. Let $i_x : \{x\} \to M$ be the inclusion, and let $[\mathcal{O}_X]$ and $[X]$ denote classes in $K_T(M)$ and $A_*^T(M)$, respectively. The Hilbert series $H(X,x)$ is given by*

$$H(X,x) = \frac{1}{(1-t)^{d'}} \operatorname{ev}_v \left(\frac{i_x^*[\mathcal{O}_X]}{\lambda_{-1}(V^*)} \right). \tag{3.1}$$

The multiplicity $\operatorname{mult}(X,x)$ *is given by*

$$\operatorname{mult}(X,x) = \operatorname{ev}_{-v} \left(\frac{i_x^*[X]}{c_d^T(V)} \right). \tag{3.2}$$

Proof. — Since the Hilbert series and tangent cone are defined locally, as are the pullbacks to the K-theory and Chow groups of $\{x\}$, we may assume that $M = M_0 = V' \times V$ and $X = X_0 = V' \times \mathcal{N}$. Then i_x is the inclusion $i_{V'\times V}$ of 0 into $V' \times V$, $[\mathcal{O}_X]$ is $[\mathcal{O}_X]_{V'\times V}$, and $[X]$ is $[X]_{V'\times V}$. By Lemma 2.1, $i^*_{V'\times V}([\mathcal{O}_X]_{V'\times V}) = i_V^*([\mathcal{O}_{\mathcal{N}}]_V)$ and $i^*_{V'\times V}([X]_{V'\times V}) = i_V^*([\mathcal{N}]_V)$.

Let $V_1 = T_x\mathcal{N} \subset V$, and let $\mathcal{C}$ denote the tangent cone to $\mathcal{N}$ at x. By (2.1),

$$H(X,x) = H(V',\{0\})\, H(\mathcal{N},x) = \frac{1}{(1-t)^{d'}} H(\mathcal{C},x), \tag{3.3}$$

since $H(V', \{0\}) = 1/(1-t)^{d'}$, and as observed in Section 2.2, $H(\mathcal{N}, x) = H(\mathcal{C}, x)$. We have

$$H(\mathcal{C}, x) = \frac{\mathrm{ev}_v\left(i_{V_1}^*([\mathcal{O}_{\mathcal{C}}]_{V_1})\right)}{(1-t)^{d_1}} = \mathrm{ev}_v\left(\frac{i_{V_1}^*([\mathcal{O}_{\mathcal{C}}]_{V_1})}{\lambda_{-1}(V_1^*)}\right), \tag{3.4}$$

where the first equality follows from [15, Proposition 2.2], and the second equality holds because each $\alpha \in \Phi(V_1)$ satisfies $\alpha(-v) = 1$. We have

$$\begin{aligned} \frac{i_{V_1}^*([\mathcal{O}_{\mathcal{C}}]_{V_1})}{\lambda_{-1}(V_1^*)} &= \frac{i_V^*([\mathcal{O}_{\mathcal{C}}]_V)}{\lambda_{-1}(V_1^*)\lambda_{-1}((V/V_1)^*)} \\ &= \frac{i_V^*([\mathcal{O}_{\mathcal{N}}]_V)}{\lambda_{-1}(V^*)} = \frac{i_{V'\times V}^*([\mathcal{O}_X]_{V'\times V})}{\lambda_{-1}(V^*)}. \end{aligned} \tag{3.5}$$

where the first equality is by Proposition 3.2 and the second by Proposition 3.1. Substituting this in the right hand side of (3.4), and then substituting the resulting formula for $H(\mathcal{C}, x)$ into (3.3), yields the formula (3.1)for the Hilbert series.

We now turn to the multiplicity formula, equation (3.2). By (2.1),

$$\mathrm{mult}(X, x) = \mathrm{mult}(V, \{0\})\,\mathrm{mult}(\mathcal{N}, x) = \mathrm{mult}(\mathcal{N}, x) = \mathrm{mult}(\mathcal{C}, x). \tag{3.6}$$

Write $a = \mathrm{mult}(\mathcal{C}, x)$. We claim that $a = \mathrm{ev}_{-v}(i_{V_1}^*([\mathcal{C}]_{V_1}))$. Assuming the claim, the remainder of the proof is similar to the K-theory case. The analogue of (3.4) is

$$\mathrm{mult}(\mathcal{C}, x) = \mathrm{ev}_{-v}\left(i_{V_1}^*([\mathcal{C}]_{V_1})\right) = \mathrm{ev}_{-v}\left(\frac{i_{V_1}^*([\mathcal{C}]_{V_1})}{c_{d_1}^T(V_1)}\right), \tag{3.7}$$

where the first equality holds by the claim, and the second holds because any $\alpha \in \Phi(V_1)$ satisfies $\alpha(-v) = 1$. The analogue of (3.5) is

$$\begin{aligned} \frac{i_{V_1}^*([\mathcal{C}]_{V_1})}{c_{d_1}^T(V_1)} &= \frac{i_{V_1}^*([\mathcal{C}]_{V_1})c_{d-d_1}^T(V/V_1)}{c_d^T(V)} \\ &= \frac{i_V^*([\mathcal{N}]_V)}{c_d^T(V)} = \frac{i_{V'\times V}^*([X]_{V'\times V})}{c_d^T(V)}. \end{aligned} \tag{3.8}$$

Substituting this in the right hand side of (3.7), and then substituting the resulting formula for $\mathrm{mult}(\mathcal{C}, x)$ into (3.6), yields the formula (3.2) for the multiplicity.

It remains to prove the claim. As indicated in [20, Section 9], using the relationships between K-theory and Chow groups, and between Hilbert series and multiplicity, the formula for the multiplicity in the claim can be deduced from the corresponding formula in K-theory (which is given in [15, Proposition 2.2]). A more direct argument, using Lemma 2.3, is as follows. The claim asserts that $\mathrm{ev}_{-v}(i_{V_1}^*([\mathcal{C}]_{V_1})) = a$, where the integer a is defined by the equation $[\mathbb{P}(\mathcal{C})] = ah^k \cap [\mathbb{P}(V_1)]$. Since $i^*_{V_1}([\mathcal{C}]_{V_1})$ is a homogeneous element of $S(\widehat{T})$ of degree k, where k is the codimension of $\mathcal{C}$ in V_1, the claim is equivalent to the assertion that $\mathrm{ev}_{-rv}(i^*_{V_1}([\mathcal{C}]_{V_1})) = ar^k$ for some $r \neq 0$. Let $T' \cong \mathbb{G}_m$ be the subtorus of T corresponding to the cocharacter rv, where r is a nonzero integer such that rv is an integral element of $\mathfrak{t}$. Let $v' \in \mathfrak{t}_1$ be the element mapping to $-rv \in \mathfrak{t}$, and define $\lambda \in \widehat{T'}$ by $\lambda(v') = 1$. Let res denote the restriction from T-equivariant Chow groups to T'-equivariant Chow groups. Restriction commutes with pullback to a point, and moreover, given $\xi \in A^*_T(pt)$, we have $\mathrm{ev}_{-rv}(\xi) = \mathrm{ev}_{v'}(\mathrm{res}(\xi))$. Therefore,

$$\mathrm{ev}_{-rv}\left(i^*_{V_1}([\mathcal{C}]_{V_1})\right) = \mathrm{ev}_{v'}\,\mathrm{res}\left((i^*_{V_1}([\mathcal{C}]_{V_1})\right) = \mathrm{ev}_{v'}\left(i^*_{V_1}([\mathcal{C}]_{V_1,T'})\right), \tag{3.9}$$

where the subscript T' indicates T'-equivariant Chow groups. Since T' acts on V_1 with all weights equal to $r\lambda$, Lemma 2.3 implies that $[\mathcal{C}]_{V_1,T'} = ar^k\lambda^k \cap [V_1]_{T'}$. Therefore, $\mathrm{ev}_{v'}(i^*_{V_1}([\mathcal{C}]_{V_1,T'}) = \mathrm{ev}_{v'}(ar^k\lambda^k) = ar^k$. The result follows. □

Remark 3.10. — If some $\alpha \in \Phi(V)$ satisfies $\alpha(v) = 0$, then $\mathrm{ev}_v(\lambda_{-1}(V^*)) = 0$ and $\mathrm{ev}_{-v}(c^T_d(V)) = 0$. Nevertheless, the evaluations in (3.1) and (3.2) can be carried out. Indeed, the proof of Theorem 3.9 shows that

$$\frac{i^*_x[\mathcal{O}_X]}{\lambda_{-1}(V^*)} = \frac{i^*_{V_1}([\mathcal{O}_{\mathcal{C}}]_{V_1})}{\lambda_{-1}(V_1^*)}, \tag{3.10}$$

since the left hand side is obtained from the right hand side by multiplying numerator and denominator by the common factor $\lambda_{-1}((V/V_1)^*)$. Note that ev_v can be applied to the right hand side of (3.10) since each $\alpha \in \Phi(V_1)$ satisfies $\alpha(v) \neq 0$ (in fact, $\alpha(v) = -1$). However, if one starts out with the formula on the left side of (3.10), to perform the evaluation ev_v, it is necessary first to cancel the common factors of $1 - e^{-\alpha}$ with $\alpha(v) = 0$ from the numerator and denominator. This might be nontrivial if one has a complicated expression for $i^*[\mathcal{O}_X]$. But in the case of Schubert varieties, in [16], we show that it is possible to explicitly perform this cancellation and apply the formula. Similar remarks apply in the Chow group situation.

Example 3.11. — Let $T = \mathbb{G}_m$. Let $e^\mu : T \to \mathbb{G}_m$ be defined by $e^\mu(t) = t$. Identify $\mathbb{Z}$ with $\widehat{T}$ by the map $n \mapsto n\mu$; then $R(T)$ is the span of $e^{n\mu}$ for $n \in \mathbb{Z}$, and $A^*_T(pt)$ is the polynomial ring $\mathbb{F}[\mu]$. Suppose T acts on $V = \mathbb{F}^3$ with all weights equal to -1 (that is, $-\mu$). Define $v \in \mathfrak{t}$ by $\mu(v) = 1$. Let x_1, x_2, x_3 be coordinates on V, let $s = x_1^2 + x_2^2 + x_3^2$, and let X denote the zero-scheme of s. The origin is a generalized cominuscule point of X, with v and V as above, and $V' = \{0\}$. Write $A = \mathbb{F}[V] = \mathbb{F}[x_1, x_2, x_3]$ and

$B = A/\langle s\rangle = \mathbb{F}[X]$. The function s is a weight vector for T with weight 2 (that is, 2μ). Thus, $s \otimes 1$ can be viewed as a T-invariant regular section of the bundle $V \times \mathbb{F}_{-2\mu} \to V$, so by Lemma 2.2, $[X] = -2\mu \cap [V]$. Hence $i_V^*[X] = -2\mu$, so by (3.2),

$$\operatorname{mult}(X,0) = \operatorname{ev}_{-v}\left(i_V^*[X]\right) = -2\mu(-v) = 2.$$

Similarly, we can compute the Hilbert series. Lemma 2.2 implies that $[\mathcal{O}_X] = (1-e^{2\mu})[\mathcal{O}_V]$. Hence $i_V^*[\mathcal{O}_X] = (1-e^{2\mu})$. Since $\lambda_{-1}(V^*) = (1-e^{\mu})^3$, we have

$$\frac{i_V^*[\mathcal{O}_X]}{\lambda_{-1}(V^*)} = \frac{1+e^{\mu}}{(1-e^{\mu})^2},$$

and thus

$$H(X,0) = \operatorname{ev}_v\left(\frac{1+e^{\mu}}{(1-e^{\mu})^2}\right) = \frac{1+t}{(1-t)^2}.$$

4. Schubert varieties and slices

4.1. Background

Let G be a semisimple algebraic group, and $B \supset T$ a Borel subgroup and maximal torus of G. Let U be the unipotent radical of B, so $B = TU$. The Lie algebra of an algebraic group will be denoted by the corresponding fraktur letter, so that the Lie algebras of these groups are (respectively) $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ and $\mathfrak{u}$. Let $B^- = TU^-$ denote the opposite Borel subgroup to B. Let Φ denote the set of roots of $\mathfrak{t}$ on $\mathfrak{g}$, and let $\Phi^+ = \Phi(\mathfrak{u}) \subset \mathfrak{t}^*$; that is, Φ^+ is the positive system of roots chosen so that the root spaces of $\mathfrak{u}$ correspond to roots in Φ^+. The Weyl group is $W = N_G(T)/T$, with longest element w_0. We will write x both for an element of W and for a representative in $N_G(T)$. If H is a subgroup of G normalized by T, and $x \in W$, we will write $H(x) = xHx^{-1}$; this is independent of the choice of a representative for x in $N_G(T)$. The Lie algebra of $H(x)$ will be denoted $\mathfrak{h}(x) = (\operatorname{Ad} x)(\mathfrak{h})$. Note that if U' is a unipotent subgroup of G normalized by T, then as a T-variety, U' is isomorphic to its Lie algebra $\mathfrak{u}'$, as can be deduced from the results in [33, Chapter 8].

Let $P \supset B$ be a standard parabolic subgroup of G. Let $P = LU_P$ be a Levi decomposition where $L \supset T$, and let $P^- = LU_P^-$ be the opposite parabolic. Write $U_L = U \cap L$ and $U_L^- = U^- \cap L$. The product map $U_P^- \times U_L^- \to U^-$ is an isomorphism (this follows from standard results about unipotent groups, cf. [33, Chapter 8]). Let W_P denote the Weyl group of L. Let Φ_L denote the set of roots of $\mathfrak{t}$ in $\mathfrak{l}$, and $\Phi_L^+ = \Phi_L \cap \Phi^+$, $\Phi_L^- = -\Phi_L^+$. Thus, $\Phi^+ = \Phi_L^+ \cup \Phi(\mathfrak{u}_P)$.

In each left coset wW_P of W_P in W there is a unique element w_m of minimal length. We write W^P for the set of minimal length coset representatives. We have $w \in W^P$ if and only if $w\Phi_L^+ \subset \Phi^+$, or equivalently, $w\alpha > 0$ for all simple roots in Φ_L^+ (cf. [32, Lemma 3.3] and the discussion preceding that lemma, as well as [23, Remark 5.13]). If $w \in W^P$, then $x \geqslant w \Leftrightarrow x_m \geqslant w$ (see [6, Lemma 2.8]). Observe that if $x, y \in W$ satisfy $xW_P = yW_P$, then $U_P(x) = U_P(y)$ and $U_P^-(x) = U_P^-(y)$. If x is minimal in its left W_P-coset, then $U^-(x) \cap U = U_P^-(x) \cap U$ (cf. [21]).

Lemma 4.1. — *Let $x \in W$.*

(1) *If $\alpha \in \Phi^+$, then $x\alpha > 0$ if and only if $xs_\alpha > x$.*
(2) *If $\alpha \in x\Phi^-$, then $s_\alpha x > x$ if and only if $\alpha \in \Phi^-$.*

Proof. — (1) follows from [3, Lemmas 8.10, 8.11]. For (2), write $\beta = -x^{-1}\alpha \in \Phi^+$. We have

$$s_\alpha x = xx^{-1}s_\alpha x = xs_{x^{-1}\alpha} = xs_{-\beta}x = xs_\beta.$$

By (1), $xs_\beta > x$ if and only if $x\beta = -\alpha \in \Phi^+$, that is, if $\alpha \in \Phi^-$. □

Definition 4.2. — *Let $w \in W$. Suppose $P = LU_P$ is the standard parabolic subgroup such that the simple roots of Φ_L^+ are those simple roots $\alpha \in \Phi^+$ with $w\alpha > 0$. We say P is the standard X^w-maximal parabolic determined by w, and we denote it by P^w. (This parabolic subgroup is considered in* [31, Definition 1.1]*.)*

Although P^w is generally not a maximal parabolic in the usual sense, the following proposition shows that it is the largest parabolic subgroup such that $w \in W^P$.

Proposition 4.3. — *Let $w \in W$. If $P = P^w$, then $w \in W^P$. Moreover, if Q is any standard parabolic subgroup, then $w \in W^Q$ if and only if $Q \subset P$.*

Proof. — Suppose that Q is a standard parabolic subgroup with Levi factor M containing T. As noted above, $w \in W^Q$ if and only if for each simple root α in Φ_M^+, we have $w\alpha > 0$. By definition, the standard maximal parabolic subgroup $P = LU$ is defined so that the simple roots in Φ_L^+ are exactly the simple roots α with $w\alpha > 0$. Hence $w \in W^P$. Moreover, $w \in W^Q$ if and only if the simple roots in Φ_M^+ are a subset of the simple roots of Φ_L^+, which occurs if and only if $Q \subset P$. □

4.2. Schubert varieties

In this section we recall some basic facts about Schubert varieties; one reference for some of these facts is [5]. We include some proofs for convenience or lack of a reference.

Let $X = G/B$ denote the flag variety. The T-fixed points in X are the points xB, where $x \in W$ (as usual we abuse notation and write x for either an element of W or a representative in $N(T)$). More generally, if $P = LU_P \supset B$ is a standard parabolic subgroup, we write X_P for the generalized flag variety G/P. The T-fixed points of X_P are of the form xP, for $x \in W$; write $i_{x,P}$ for the inclusion of xP into $X_P = G/P$.

Given $w \in W$, $X^w_{0,P} = B^- \cdot wP$ is a subvariety of X_P isomorphic to affine space, called a Schubert cell. Its closure is the Schubert variety $X^w_P = \overline{B^- \cdot wP}$. Since $xP = yP$ if and only if the cosets xW_P and yW_P are equal, X^w_P depends only on the coset of w mod W_P. Write $[\mathcal{O}_{X^w_P}]$ for the classes of the structure sheaves in $K_T(G/P)$, and $[X^w_P]$ for the fundamental classes in $A^T_*(G/P)$. If we are assuming $P = B$, then we generally omit the subscript P from the notation, e.g., we write X^w, i_x, etc.

Proposition 4.4. — *Let P be a standard parabolic subgroup of G, and let $w \in W$. Let $\pi : G/B \to G/P$ be the projection.*

1. *We have $\pi(X^w) = X^w_P$.*
2. *If $w \in W^P$, then $\pi^{-1}(X^w_P) = X^w$. Conversely, if X^w is the inverse image of a Schubert variety in G/P, then $w \in W^P$.*
3. *If X^w is the inverse image of X^w_Q under the map $G/B \to G/Q$, then $Q \subset P^w$, so $G/B \to G/Q \to G/P^w$, and the inverse image of $X^w_{P^w}$ in G/Q is X^w_Q.*

Proof. — (1) The map π is B^--equivariant, so

$$\pi\left(X^w_0\right) = \pi(B^- \cdot wB) = B^- \cdot \pi(wB) = B^- \cdot wP = X^w_{0,P}. \tag{4.1}$$

Since the map π is proper, it is a closed map. We have

$$\pi(X^w) = \pi(\overline{X^w_0}) = \overline{\pi(X^w_0)} = \overline{X^w_{0,P}} = X^w_P,$$

where the second equality holds because π is a closed map. This proves (1).

(2) First suppose that $w \in W^P$. Since $X^w_P = \pi(X^w)$, we have $\pi^{-1}(X^w_P) \supseteq X^w$; we must prove the reverse inclusion. It suffices to show that if $X^y \subset \pi^{-1}(X^w_P)$, then $y \geqslant w$. The hypothesis $X^y \subset \pi^{-1}(X^w_P)$ implies that yP is in X^w_P. Every T-fixed point in X^w_P is the image of a T-fixed point in X^w, so $yP = zP$ for some $z \geqslant w$. Since w is minimal, $z_m \geqslant w$ (see Section 4.1); since $yW_P = zW_P = z_mW_P$, we have $y \geqslant z_m$, so $y \geqslant w$. Hence

$\pi^{-1}(X^w_P) = X^w$, as desired. For the converse, suppose that $X^w = \pi^{-1}(X^y_P)$. We must show that w is minimal. Since $X^y_P = X^{y_m}_P$, we may assume $y = y_m$. Then by what we have already proved, $\pi^{-1}(X^y_P) = X^y$, so $X^y = X^w$. Hence $y = w$, so w is minimal, as desired.

(3) Let $P = P^w$. If X^w is the inverse image of X^w_Q, then by (2), $w \in W^Q$, so Proposition 4.3 implies that $Q \subset P$. Consider the projections

$$G/B \xrightarrow{f} G/Q \xrightarrow{g} G/P.$$

Since $g^{-1}(X^w_P)$ is closed, U^--stable, and irreducible, it is a Schubert variety X^u_Q for some $u \in W^Q$. By (2), we have $f^{-1}(X^u_Q) = X^u$; but $f^{-1}(X^u_Q) = f^{-1}(g^{-1}(X^w_P)) = X^w$. Hence $u = w$, so $g^{-1}(X^w_P) = X^w_Q$, as desired. □

4.3. Slices to Schubert varieties

Let $x \in W$. Let P be a standard parabolic subgroup, and let $U^-_P(x)$ be as in Section 4.1. The map $U^-_P(x) \to C_{x,P} = U^-_P(x) \cdot xP$ embeds $U^-_P(x)$ as an open subvariety of G/P. This embedding is T-equivariant, where T acts by conjugation on $U^-_P(x)$, and by left multiplication on $C_{x,P}$. We refer to $C_{x,P}$ as an open cell. If $Q = MU_Q \subset P = LU_P$ are standard parabolic subgroups, then

$$C_{x,\,Q} \cong U^-_Q(x) \cong U^-_P(x) \times \left(U^-_Q \cap L\right)(x) \cong C_{x,\,P} \times \left(U^-_Q \cap L\right)(x), \tag{4.2}$$

as follows from the Lie algebra decomposition $\mathfrak{u}^-_Q = \mathfrak{u}^-_P \oplus (\mathfrak{u}^-_Q \cap \mathfrak{l})$. If $w \in W^P$, then under the identification (4.2), we have

$$X^w_Q \cap C_{x,\,Q} = (X^w_P \cap C_{x,\,P}) \times \left(U^-_Q \cap L\right)(x). \tag{4.3}$$

Indeed, the projection $\pi : G/Q \to G/P$ takes $C_{x,\,Q}$ to $C_{x,\,P}$. Let $\rho := \pi|_{C_{x,\,Q}}$. If Z is any subscheme of $C_{x,P}$, then under the identification given by (4.2), $\rho^{-1}(Z) = Z \times (U^-_Q \cap L)(x)$. Equation (4.3) follows from this, since $\rho^{-1}(X^w_P \cap C_{x,\,P}) = X^w_Q \cap C_{x,\,Q}$ for $w \in W^P$.

Definition 4.5. — *Let $x \geqslant w$ be elements of W and let $P = P^w$. Define*

$$\mathcal{N}^w_x = \left[\left(U^-_P(x) \cap U\right) \cdot xP\right] \cap X^w_P \subset C_{x,\,P} \cap X^w_P.$$

More generally, if Q is a standard parabolic subgroup contained in P (so $w \in W^Q$), define

$$\mathcal{N}^w_{x,\,Q} = \left[\left(U^-_P(x) \cap U\right) \cdot xQ\right] \cap X^w_Q \subset C_{x,\,Q} \cap X^w_Q.$$

Note that $\mathcal{N}^w_{x,Q}$ is T-stable since it is the intersection of two T-stable subvarieties. Under the identification (4.3), we have

$$\mathcal{N}^w_{x,Q} = \mathcal{N}^w_x \times \{1\}. \tag{4.4}$$

The reason is that under the identifications (4.2) and (4.3), we have

$$\begin{aligned}\mathcal{N}^w_{x,Q} &= \left[\left(U^-_P(x) \cap U\right) \times \{1\}\right] \cap \left[(X^w_P \cap C_{x,P}) \times \left(U^-_Q \cap L\right)(x)\right] \\ &= \left[\left(U^-_P(x) \cap U\right) \cap (X^w_P \cap C_{x,P})\right] \times \{1\} = \mathcal{N}^w_x \times \{1\}.\end{aligned}$$

Hence we can view $\mathcal{N}^w_x$ as a subvariety of any generalized flag variety G/Q with $Q \subset P^w$. In particular, we can view $\mathcal{N}^w_x$ as a subvariety of G/B. Note also that if $xW_P = yW_P$, then since $xP = yP$ in G/P, we have $\mathcal{N}^w_x = \mathcal{N}^w_y$.

Lemma 4.6. — *Let H be a linear algebraic group, and X a scheme. Let H act on the product $H \times X$ by left multiplication on the first factor. Any H-invariant closed subscheme Z of $H \times X$ is of the form $Z = H \times Y$, where Y is the closed subscheme $(\{e\} \times X) \cap Z$ of $\{e\} \times X$ (which we identify with X).*

Results of this form are known (cf. the proof of [8, Proposition 1.3.5]), so we omit the proof. The lemma implies that the action map gives an isomorphism $H \times Y \to Z$; we will refer to Y as a slice to Z. The next proposition is analogous to [8, Proposition 1.3.5] (which concerns Kazhdan–Lusztig varieties), and has a similar proof.

Proposition 4.7. — *Let $x \geqslant w$ be in W. Let $P = P^w = LU_P$. The action map gives a T-equivariant isomorphism*

$$\left(U^-_P(x) \cap U^-\right) \times \mathcal{N}^w_x \to X^w_P \cap C_{x,P}. \tag{4.5}$$

Proof. — By the remarks preceding the proposition, the map (4.5) is T-equivariant. We must prove that this map is an isomorphism. We can decompose $\mathfrak{u}^-_P(x) = (\mathfrak{u}^-_P(x) \cap \mathfrak{u}^-) \oplus (\mathfrak{u}^-_P(x) \cap \mathfrak{u})$, and hence

$$\left(U^-_P(x) \cap U^-\right) \times \left(U^-_P(x) \cap U\right) \cong U^-_P(x) \cong C_{x,P}. \tag{4.6}$$

Since $C_{x,P}$ (resp. X^w_P) is stable under the left action of $U^-_P(x)$ (resp. U^-), $X^w_P \cap C_{x,P}$ is stable under the left action of $U^-_P(x) \cap U^-$. Applying Lemma 4.6, with $H = U^-_P(x) \cap U^-$, $X = (U^-_P(x) \cap U) \cdot xP$, and $Z = X^w_P \cap C_{x,P}$, we see that under the identification (4.6), the embedding $X^w_P \cap C_{x,P} \subset C_{x,P}$ corresponds to the embedding

$$\left(U^-_P(x) \cap U^-\right) \times \mathcal{N}^w_x \subset \left(U^-_P(x) \cap U^-\right) \times \left(U^-_P(x) \cap U\right).$$

This proves the result. □

The previous proposition shows shows that $\mathcal{N}_x^w$ is a slice to X_P^w, where $P = P^w$. In fact, $\mathcal{N}_x^w$ plays the role of a slice at xQ to X_Q^w for any $Q \subset P$, since by (4.3),

$$X_Q^w \cap C_{x,Q} \cong \left(U_P^-(x) \cap U^-\right) \times \mathcal{N}_x^w \times \left(U_Q^- \cap L\right)(x). \tag{4.7}$$

Proposition 4.8. — *Suppose $x \geqslant w$. Let $P = P^w = LU_P$. Then*

$$\dim \mathcal{N}_x^w = \ell(x) - \ell(w) - \left|x\Phi_L^- \cap \Phi^+\right|.$$

In particular, if $x \in W^P$, then $\dim \mathcal{N}_x^w = \ell(x) - \ell(w)$.

Proof. — Equation (4.5) implies that

$$\dim \mathcal{N}_x^w = \dim X_P^w - \dim \left(U_P^-(x) \cap U^-\right).$$

It is well known that the codimension of X_P^w in X_P is $\ell(w)$, so the dimension of X^w is $\dim X_P - \ell(w) = \dim U_P^-(x) - \ell(w)$. Hence

$$\begin{aligned}\dim \mathcal{N}_x^w &= \dim U_P^-(x) - \dim \left(U_P^-(x) \cap U^-\right) - \ell(w)\\ &= \dim \left(U_P^-(x) \cap U\right) - \ell(w)\end{aligned}$$

But

$$\dim \left(U_P^-(x) \cap U\right) = \left|x\Phi^- \cap \Phi^+\right| - \left|x\Phi_L^- \cap \Phi^+\right| = \ell(x) - \left|x\Phi_L^- \cap \Phi^+\right|.$$

The result follows. □

Remark 4.9. — Given $w \in W^P$, and $x \geqslant w$, there is a Kazhdan–Lusztig variety $\mathcal{N}_{x,P}^{w,KL}$ at xP, defined by taking the intersection of X_P^w with an opposite Schubert cell. (Kazhdan and Lusztig defined this in the case where $P = B$, but following [21, Section 7.3], we use the term Kazhdan–Lusztig variety in this more general context.) If $P = P^w$, then $\mathcal{N}_{x,P}^w = \mathcal{N}_x^w$ equals the Kazhdan–Lusztig variety $\mathcal{N}_{x,P}^{w,KL}$. Since $\mathcal{N}_{x,P}^w$ is the intersection of X_P^w with part of an opposite Schubert cell, it can be smaller than $\mathcal{N}_{x,P}^{w,KL}$.

The discussion above shows that slices to Schubert varieties in generalized flag varieties are also slices in the full flag variety. Along the same lines, the next proposition shows that computations of pullbacks and Hilbert series for generalized flag varieties can be carried out using the full flag variety.

Proposition 4.10. — *Let P be a standard parabolic subgroup, and suppose that $w \in W^P$ and $x \geqslant w$.*

(1) *We have*

$$i_{x,P}^* \left[\mathcal{O}_{X_P^w}\right] = i_x^* \left[\mathcal{O}_{X^w}\right] \quad \text{and} \quad i_{x,P}^* \left[X_P^w\right] = i_x^* \left[X^w\right]. \tag{4.8}$$

(2) *Let* $d = \dim G/B - \dim G/P = |\Phi_L^+|$. *Then*

$$H\left(X_P^w, xP\right) = (1-t)^d H\left(X^w, xB\right)$$

and

$$\operatorname{mult}\left(X_P^w, xP\right) = \operatorname{mult}\left(X^w, xB\right).$$

Proof. — (1) The proof of this equation for K-theory is in [15, Section 2.2]; the proof for Chow groups is almost the same.

(2) If $w \in W^P$ and $x \geqslant w$, then (4.3) implies

$$X^w \cap C_x \cong \left(X_P^w \cap C_{x,\,P}\right) \times U_L^-(x).$$

The group $U_L^-(x)$ is isomorphic to affine space of dimension d, so $H(U_L^-(x), e) = \frac{1}{(1-t)^d}$, and $\operatorname{mult}(U_L^-(x), e) = 1$. The result now follows immediately from (2.1). ☐

To simplify the notation, we will write simply $H(X^w, x)$ for $H(X^w, xB)$ and $\operatorname{mult}(X^w, x)$ for $\operatorname{mult}(X^w, xB)$.

4.4. Tangent spaces to slices

We now describe the set of weights $\Phi(T_{xB}\mathcal{N}_x^w)$ (for $x \geqslant w$) in terms of $\Phi(T_{xB}X^w)$. The sets $\Phi(T_{xB}X^w)$ have been described in classical types (see [29], Lakshmibai [26, 27, 28], [5, Chapter 5]). Thus, in classical types, we can describe $\Phi(T_{xB}\mathcal{N}_x^w)$, which is what we need to determine whether x is cominuscule in X^w. Further combinatorial refinements appear in [16].

Observe that

$$\Phi\left(T_{xB}X\right) = x\Phi^- \supset \Phi\left(T_{xB}X^w\right) \supset \Phi\left(T_{xB}\mathcal{N}_x^w\right).$$

Proposition 4.11. — *Let* $w \in W$, *and let* $P = P^w$. *If* $x \geqslant w$, *then*

$$\Phi\left(T_{xB}\mathcal{N}_x^w\right) = \Phi\left(T_{xB}X^w\right) \setminus \left(\left(x\Phi^- \cap \Phi^-\right) \sqcup \left(x\Phi_L^- \cap \Phi^+\right)\right). \tag{4.9}$$

Hence, if $x \in W^P$,

$$\Phi\left(T_{xB}\mathcal{N}_x^w\right) = \Phi\left(T_{xB}X^w\right) \setminus \left(x\Phi^- \cap \Phi^-\right). \tag{4.10}$$

Proof. — By Proposition 4.7, we have

$$\Phi\left(T_{xP}X_P^w\right) = \Phi\left(T_{xB}\mathcal{N}_x^w\right) \sqcup \left(x\Phi\left(\mathfrak{u}_P^-\right) \cap \Phi^-\right).$$

If $\pi : G/B \to G/P$ denotes the projection, then

$$\Phi(T_{xB}X^w) = \Phi(T_{xP}X_P^w) \sqcup \Phi(T_{xB}(\pi^{-1}(xP)) \tag{4.11}$$

$$= \Phi\left(T_{xP}X_P^w\right) \sqcup x\Phi_L^-. \tag{4.12}$$

Therefore,

$$\Phi\left(T_{xB}X^{w}\right)=\Phi\left(T_{xB}\mathcal{N}_{x}^{w}\right)\sqcup\left(x\Phi\left(\mathfrak{u}_{P}^{-}\right)\cap\Phi^{-}\right)\sqcup x\Phi_{L}^{-}. \tag{4.13}$$

Since $\Phi^{-}=\Phi(\mathfrak{u}_{P}^{-})\sqcup\Phi_{L}^{-}$, we can rewrite (4.13) as

$$\Phi\left(T_{xB}X^{w}\right)=\Phi\left(T_{xB}\mathcal{N}_{x}^{w}\right)\sqcup\left(x\Phi^{-}\cap\Phi^{-}\right)\sqcup\left(x\Phi_{L}^{-}\cap\Phi^{+}\right).$$

This implies (4.9). If $x\in W^{P}$, then $x\Phi_{L}^{-}\subset\Phi^{-}$, so (4.10) follows. $\square$

Suppose $x\geqslant w$. In type A, a result of Lakshmibai and Seshadri ([29]) implies that

$$\Phi\left(T_{xB}X^{w}\right)=\left\{\alpha\in x\Phi^{-}\middle|s_{\alpha}x\geqslant w\right\}. \tag{4.14}$$

This set contains the elements $\alpha\in\Phi^{-}$ such that $s_{\alpha}x>x$, which by Lemma 4.1, equals $x\Phi^{-}\cap\Phi^{-}$. Combining this with Proposition 4.11 yields the following.

Proposition 4.12. — *Suppose G is of type A. Let $w\in W$, and let $P=P^{w}$. If $x\geqslant w$, then*

$$\begin{aligned}\Phi\left(T_{xB}\mathcal{N}_{x}^{w}\right)&=\left\{\alpha\in x\Phi^{-}\setminus x\Phi_{L}^{-}\mid x>s_{\alpha}x\geqslant w\right\}\\&=\left\{\alpha\in x\Phi^{-}\setminus\left(x\Phi_{L}^{-}\cap\Phi^{+}\right)\middle|x>s_{\alpha}x\geqslant w\right\}\\&=\left\{\alpha\in x\Phi(\mathfrak{u}_{P}^{-})\middle|x>s_{\alpha}x\geqslant w\right\}.\end{aligned}$$

Hence, if $x\in W^{P}$,

$$\Phi\left(T_{xB}\mathcal{N}_{x}^{w}\right)=\left\{\alpha\in x\Phi^{-}\middle|x>s_{\alpha}x>w\right\}.$$

Note that the first equality of the proposition holds because if $\alpha\in x\Phi_{L}^{-}\cap\Phi^{-}$, then $s_{\alpha}x>x$.

Remark 4.13. — For G of arbitrary type, a result of Carrell and Peterson implies the right hand sides of the equations in Proposition 4.12 describe the sets of weights to tangent spaces of T-stable curves in $\mathcal{N}_{x}^{w}$ through xB.

5. Cominuscule points of Schubert varieties

The definition of generalized cominuscule points allows for arbitrary slices $\mathcal{N}$. For Schubert varieties, we define cominuscule points using the particular slices $\mathcal{N}_{x}^{w}$. Since by (4.7), $\mathcal{N}_{x}^{w}$ serves as a slice at xQ to X_{Q}^{w} for any standard parabolic subgroup Q such that $w\in W^{Q}$, this definition can be used to calculate Hilbert series and multiplicities in X_{Q} for any such Q. Since these Hilbert series and multiplicities can be determined from the corresponding X^{w} in the full flag variety X (Proposition 4.10), we will restrict our attention to the case $Q=B$.

5.1. Cominuscule points

In this section we define the notion of a cominuscule point in a Schubert variety. We provide some examples of cominuscule points, and in type A, we give conditions which guarantee that a cominuscule point in a Schubert variety is also a cominuscule point in a smaller Schubert variety.

Definition 5.1. — *Suppose $x \geqslant w$ are elements of W. We will say that x is cominuscule in X^w if there exists $v \in \mathfrak{t}$ (which can be assumed to be rational) such that for all α in $\Phi(T_{xB}\mathcal{N}_x^w)$, we have $\alpha(v) = -1$.*

Note that we can assume that v is rational by the discussion in Remark 3.5.

Remark 5.2. — Let $P = P^w$, and suppose $x \geqslant w$. If $xW_P = yW_P$, then since $\mathcal{N}_x^w = \mathcal{N}_y^w$, we see that if x is cominuscule in X^w then so is y for any $y \in xW_P$. Moreover, in this case, if $Q \subset P$ is a standard parabolic subgroup of G (so $w \in W^Q$), then xQ is a generalized cominuscule point of X_Q^w.

Example 5.3. — For any $w \in W$, the element w is cominuscule in X^w. Indeed, the slice $\mathcal{N}_w^w$ is a single point, so the cominuscule condition is trivially satisfied. For a similar reason, any x is cominuscule in $X^1 = X$. Indeed, in this case, since $w = 1$, we have $P^w = G$, so $U_{P^w}^- = \{1\}$. Thus for any $x \in W$, $\mathcal{N}_x^1$ is a point, so the cominuscule condition is trivially satisfied.

We now show that Schubert varieties in cominuscule flag varieties give rise to points which are cominuscule in our sense. We begin by recalling the definition of cominuscule flag varieties. Suppose that $P = LU$ is a maximal standard parabolic subgroup; then there is a unique simple root β which is not in Φ_L. The parabolic subgroup P and the corresponding flag variety G/P are said to be of cominuscule type if the simple root β appears with coefficient equal to 1 when the highest root is written as a sum of simple roots. In this case, it is known that for any $x \in W$ there exists an element $v \in \mathfrak{t}$ such that for any $\alpha \in \Phi(T_{xP}(G/P))$, $\alpha(v) = -1$ (one reference is [15, Proposition 2.9]).

Proposition 5.4. — *Suppose P is a standard parabolic subgroup of cominuscule type. If $x \in W^P$, then for any $w \leqslant x$, x is cominuscule in X^w.*

Proof. — By Remark 5.2, we may assume $w \in W^P$. The tangent space $T_{xP}(\mathcal{N}_{x,P}^w)$ is a subspace of $T_{xP}(G/P)$, so the cominuscule condition on the tangent space of the normal slice holds because it holds for $T_{xP}(G/P)$. □

The next result gives conditions under which an element x which is cominuscule in X^w is also cominuscule in a smaller Schubert variety X^v. We do not know if the result is valid in types other than type A, since the descriptions of tangent spaces are more complicated in other types.

Corollary 5.5. — *Let G be of type A. Suppose that $w \leqslant v \leqslant x$ are elements of W such that $P^w \subset P^v$ (equivalently, $v \in W^{P^w}$). If x is cominuscule in X^w, then x is cominuscule in X^v.*

Proof. — Since $P^w \subset P^v$, the reverse inclusion $\Phi(\mathfrak{u}_{P^w}^-) \supset \Phi(\mathfrak{u}_{P^v}^-)$ holds. Since also $w \leqslant v$, Proposition 4.12 implies $\Phi(T_{xB}\mathcal{N}_x^v) \subset \Phi(T_{xB}\mathcal{N}_x^w)$, implying the result. □

Without the condition $P^w \subset P^v$, the assumption that x is cominuscule in X^w does not imply that x is cominuscule in X^v. Indeed, if this were true, then since any T-fixed point is cominuscule in $X^1 = X$ by Example 5.3, any T-fixed point would be cominuscule in any Schubert variety, which is not true (see Example 6.1).

5.2. Cominuscule elements of Weyl groups

Definition 5.6. — *The element $x \in W$ is cominuscule if and only if there exists $v \in \mathfrak{t}$ such that for all $\alpha \in x\Phi^- \cap \Phi^+ = I(x^{-1})$, we have $\alpha(v) = -1$.*

This notion is due to Peterson, with different terminology: the element x is cominuscule if for some $\lambda \in \mathfrak{t}^*$, x is λ-cominuscule (in Peterson's sense) for the dual root system. See [34, Proposition 5.1].

It follows from the equality $I(x) = -x^{-1}I(x^{-1})$ that x is cominuscule if and only if x^{-1} is. In type A_{n-1}, the Weyl group is the symmetric group S_n, and the cominuscule elements are exactly the 321-avoiding permutations (see [21, p. 25]).

The next proposition shows that cominuscule elements provide cominuscule points, although the examples of Section 6 show that not all cominuscule points arise this way.

Proposition 5.7. — *If x is a cominuscule element of W, then x is cominuscule in any Schubert variety X^w containing xB (equivalently, such that $x \geqslant w$).*

Proof. — If x is cominuscule, then there exists $v \in \mathfrak{t}$ such that for all $\alpha \in I(x^{-1})$, we have $\alpha(v) = -1$. By Proposition 4.11, for any $w \leqslant x$, we have $\Phi(T_{xB}\mathcal{N}_x^w) \subset I(x^{-1})$, so by definition, x is cominuscule in X^w. □

Remark 5.8. — Knutson observed that for a λ-cominuscule element x of W, the torus action on the cell $X_x^0 = U \cdot xB$ contains the natural dilation action (see [21, p. 25]), and therefore the ideal of $X^w \cap X_x^0$ in X_x^0 is homogeneous with respect to the standard dilation action. It was noted in [30] this condition implies that the Kazhdan–Lusztig ideal can be used to compute Hilbert series and multiplicities. In the context of this paper, Proposition 5.7 is an almost immediate consequence of Knutson's observation.

Remark 5.9. — The converse of Proposition 5.7 is false. For example, in type A_2, the long element w_0 of W is cominuscule in any X^w, but in 1-line notation, $w_0 = (3, 2, 1)$, which is not cominuscule in W.

5.3. Hilbert series and multiplicity formulas at cominuscule points

The following theorem is a straightforward consequence of the corresponding result for generalized cominuscule points (Theorem 3.9).

THEOREM 5.10. — *Let $N = \dim G/B = |\Phi^+|$. Let $w \in W$. Suppose x is cominuscule in X^w. Let v be as in Definition 5.1. Let $P = LU$ be the standard X^w-maximal parabolic subgroup, and let $d' = N - \ell(x) + |x\Phi_L^- \cap \Phi^+|$. Then*

$$H\left(X^w, x\right) = \frac{1}{(1-t)^{d'}} \operatorname{ev}_v \left(\frac{i_x^* \left[\mathcal{O}_{X^w}\right]}{\prod_{\alpha \in x\Phi\left(\mathfrak{u}_P^-\right) \cap \Phi^+} \left(1 - e^{-\alpha}\right)} \right) \tag{5.1}$$

$$\operatorname{mult}\left(X^w, x\right) = \operatorname{ev}_{-v} \left(\frac{i_x^* \left[X^w\right]}{\prod_{\alpha \in x\Phi\left(\mathfrak{u}_P^-\right) \cap \Phi^+} \alpha} \right). \tag{5.2}$$

Note that if $x \in W^P$, then since $x\Phi_L^- \cap \Phi^+$ is empty, we have $d' = N - \ell(x)$ and $x\Phi(\mathfrak{u}_P^-) \cap \Phi^+ = x\Phi^- \cap \Phi^+ = I(x^{-1})$.

Proof. — In light of the decomposition (4.7) (with $Q = B$), this follows from Theorem 3.9, with $V' = (\mathfrak{u}_P^-(x) \cap \mathfrak{u}^-) \oplus \mathfrak{u}_L^-(x)$, $V = \mathfrak{u}_P^-(x) \cap \mathfrak{u}$, and $\mathcal{N} = \mathcal{N}_x^w$. We have

$$\begin{aligned} d' = \dim V' &= \left|x\Phi\left(\mathfrak{u}_P^-\right) \cap \Phi^-\right| + \left|x\Phi_L^-\right| \\ &= \left|x\Phi^- \cap \Phi^-\right| - \left|x\Phi_L^- \cap \Phi^-\right| + \left|x\Phi_L^-\right| \\ &= \left|x\Phi^- \cap \Phi^-\right| + \left|x\Phi_L^- \cap \Phi^+\right| = N - \ell(x) + \left|x\Phi_L^- \cap \Phi^+\right|, \end{aligned}$$

which is the formula for d' in the statement of the theorem. ☐

Remark 5.11. — As noted in Remark 3.10, the cancellations necessary to perform the evaluations are explicitly described in [16], where in fact it is shown that the formulas can be evaluated without having to find v.

As a corollary to this theorem, if x is a cominuscule element of W, we obtain formulas closely resembling the formulas for cominuscule flag varieties given in [20, Proposition 12] and [15, Theorem 2.10].

Corollary 5.12. — *Let $N = \dim G/B$. Suppose that x is a cominuscule element of W and $x \geqslant w$. Let v be an element of $\mathfrak{t}$ such that $\alpha(v) = -1$ for all $\alpha \in x\Phi^- \cap \Phi^+$. Then*

$$H\left(X^w, x\right) = \frac{\mathrm{ev}_v\left(i_x^*\left[\mathcal{O}_{X^w}\right]\right)}{(1-t)^N} \quad \textit{and} \quad \mathrm{mult}\left(X^w, x\right) = \mathrm{ev}_{-v}\left(i_x^*\left[X^w\right]\right).$$

Proof. — These formulas follow immediately from Theorem 5.10 because for all $\alpha \in x\Phi(\mathfrak{u}_P^-) \cap \Phi^+$, we have $\alpha(v) = -1$. □

6. Examples

In this section we apply the results of earlier sections to Schubert varieties in G/B, where G is of type A_5. We describe the ingredients of the computations, but omit most details. The Hilbert series and multiplicities are computed using formulas $i_x^*[X^w]$ (due to Anderson–Jantzen–Soergel and Billey ([1, 4])) and $i_x^*[\mathcal{O}_{X^w}]$ (due to Graham and Willems ([14, 36])). These formulas can be found (in a version consistent with the conventions of this paper) in [15]. We will not restate the formulas here, but note that they depend on the choice of reduced expression for x and are related to the number of subexpressions multiplying to w.

In type A_{n-1}, $G = \mathrm{SL}_n$. Let B be the Borel subgroup of upper triangular matrices in G, and T the maximal torus of diagonal matrices in G. The set of positive roots of G is $\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \leqslant i < j \leqslant n, i \neq j\}$. The simple roots are $\alpha_1, \ldots, \alpha_{n-1}$ with $\alpha_i = \epsilon_i - \epsilon_{i+1}$. The Weyl group W is the permutation group S_n, with $w \in W$ acting on $\epsilon_i - \epsilon_j$ by $w(\epsilon_i - \epsilon_j) = \epsilon_{w(i)} - \epsilon_{w(j)}$. The i^{th} simple reflection is the transposition $s_i = (i, i+1)$.

We will make use of the following descriptions of tangent spaces (for $x \geqslant w$):

$$\Phi\left(T_{xB} X^w\right) = \left\{\alpha \in x\Phi^- \middle| s_\alpha x \geqslant w\right\},$$

by (4.14), and and if $x \in W^{P^w}$,

$$\Phi\left(T_{xB} \mathcal{N}_x^w\right) = \left\{\alpha \in x\Phi^- \middle| x > s_\alpha x > w\right\},$$

by Proposition 4.12. To use these descriptions of tangent spaces, we need the following characterization of the Bruhat order in type A (see [18, Section 5.9]). If $x = (x(1), \ldots, x(n))$ and $y = (y(1), \ldots, y(n))$ are two permutations in S_n, written in 1-line notation, then $x \leqslant y$ in the Bruhat order if and only if for each $i \in \{1, \ldots, n\}$, the following holds: if $a_1, a_2, \ldots a_i$ are the numbers $x(1), x(2), \ldots, x(i)$ written in increasing order, and $b_1, b_2, \ldots b_i$ are the numbers $y(1), y(2), \ldots, y(i)$ written in increasing order, then $a_k \leqslant b_k$ for all $k \in \{1, \ldots, i\}$.

Example 6.1. — Let $w = (3,4,1,6,2,5)$ in 1-line notation. We have $\dim X = |\Phi^+| = 15$, and $\dim X^w = \dim X - \ell(w) = 15 - 6 = 9$. Let $P = P^w = LU_P$. The simple roots in Φ_L^+ are $\{\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_5 - \epsilon_6\}$. We show that for $y = (5,6,2,4,1,3) \in W^P$, yB is not a cominuscule point of X^w. We have

$$\begin{aligned}
&\Phi\left(T_{yB}\mathcal{N}_y^w\right) = \{\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_3 - \epsilon_5, \epsilon_4 - \epsilon_5, \epsilon_4 - \epsilon_6\}\\
&\Phi\left(T_{yB}X_y^w\right) =\\
&\qquad\qquad \Phi\left(T_{yB}\mathcal{N}_y^w\right) \cup \{-(\epsilon_1 - \epsilon_3), -(\epsilon_2 - \epsilon_3),\ -(\epsilon_2 - \epsilon_4), -(\epsilon_5 - \epsilon_6)\}
\end{aligned}$$

Because $\Phi(T_{yB}\mathcal{N}_x^w)$ contains the roots $\beta_1 = \epsilon_3 - \epsilon_4$, $\beta_2 = \epsilon_4 - \epsilon_5$ and $\beta_1 + \beta_2 = \epsilon_3 - \epsilon_5$, yB is not a cominuscule point of X^w, since if $\beta_1(v) = \beta_2(v) = -1$ then $(\beta_1 + \beta_2)(v) = -2$. Note that yB is a nonsingular point of X^w since $\dim T_{yB}X^w = 9 = \dim X^w$, so $\operatorname{mult}(X^w, yB) = 1$.

Example 6.2. — Let $w = (4,3,1,6,2,5)$ and let $x = (5,6,3,4,1,2)$. We have $\dim X^w = \dim X - l(w) = 15 - 7 = 8$. Let $P = P^w = LU_P$; then $\Phi_L^+ = \{\epsilon_3 - \epsilon_4, \epsilon_5 - \epsilon_6\}$. We have $x \in W^P$, and

$$\begin{aligned}
&\Phi\left(T_{xB}\mathcal{N}_x^w\right) = \{\epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_4, \epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_3 - \epsilon_6, \epsilon_4 - \epsilon_5, \epsilon_4 - \epsilon_6\}\\
&\Phi\left(T_{xB}X_x^w\right) = \Phi\left(T_{xB}\mathcal{N}_x^w\right) \cup \{-(\epsilon_1 - \epsilon_2), -(\epsilon_3 - \epsilon_4), -(\epsilon_5 - \epsilon_6)\}
\end{aligned}$$

Since $\dim T_{xB}X^w = 10 > 8 = \dim X^w$, X^w is singular at x. If $v = \frac{1}{2}\operatorname{diag}(-1,-1,-1,1,1,1)$, then $\alpha(v) = -1$ for all $\alpha \in \Phi(T_{xB}\mathcal{N}_x^w)$, so xB is a cominuscule point of X^w. The calculation of multiplicity and Hilbert series is facilitated by choosing a reduced expression for x which has few subexpressions multiplying to w. Motivated by the combinatorial results of [16] (where the calculations are carried out using pipe dreams), we choose reduced expressions for w and x given by (using the shorthand $i_1 i_2 \cdots i_k$ for $s_{i_1} s_{i_2} \cdots s_{i_k}$)

$$w = 3213254, \quad x = 432154324354.$$

There are only 3 subexpressions of x multiplying to w. One verifies that the multiplicity of X^w at x is 3. Going further, one can verify that the

Hilbert series is given by

$$H(X^w,x) = \frac{-3(t-1)^7(t+1)^4 - 2(t-1)^8(t+1)^4}{-(t-1)^{15}(t+1)^4} = \frac{3}{(t-1)^8} + \frac{2}{(t-1)^7},$$

where the middle expression is obtained from Theorem 5.10 before simplifying. We can recover the multiplicity from the Hilbert series by observing that

$$\frac{3}{(t-1)^8} + \frac{2}{(t-1)^7} = \sum_{k=0}^{\infty} \left(3\binom{k+7}{7} - 2\binom{k+6}{6} \right) t^k.$$

Thus the Hilbert polynomial

$$h(X^w,x)(k) = 3\binom{k+7}{7} - 2\binom{k+6}{6}.$$

The leading term of this polynomial is $\frac{3}{7!}k^7$, so $\mathrm{mult}(X^w,x) = 3$.

Example 6.3. — By a calculation similar to the previous example, we can verify that $x = (5,6,3,4,1,2)$ (the same x as that example) is also cominuscule in X^w for $w = (3,4,1,6,2,5)$. If we take the reduced expression for x as in the previous example, then there are again only 3 subexpressions multiplying to w, and the multiplicity of X^w at x is again 3. Note that if we took the reduced expression $x = 214354213254$, there would be 15 subexpressions of x multiplying to w, so the calculation would be more complicated.

Example 6.4. — Taking x and w as in the previous example, we can produce other Schubert varieties in which x is cominuscule using Corollary 5.5, which states that x is also cominuscule in X^v, provided that $w \leqslant v \leqslant x$ and $P^w \subset P^v$. For example, if we take $v = (3,5,1,6,2,4)$ then this holds (with $P^w = P^v$); if $v = (3,4,5,6,1,2)$ it holds (with $P^w \subsetneq P^v$).

Example 6.5. — Taking x and w as in the previous example, whenever y is in the coset xW_{P^w}, Remark 5.2 implies that y is cominuscule in X^w, and moreover that the multiplicity and Hilbert series of X^w at y are the same as those at x. Noting that $W_{P^w} = \langle s_1, s_3, s_5 \rangle$, we could take, for example, $y = (5,6,4,3,1,2)$ or $y = (6,5,4,3,2,1) = w_0$.

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