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PERTURBATION OF HARMONIC STRUCTURES AND AN INDEX-ZERO THEOREM

by Bertram WALSH (1)

0. Introduction.

This paper gives a complete solution to the problem that motivates the paper [11] : given a pair $(W, \mathcal{H})$ consisting of a locally compact space and a complete presheaf of vector spaces of continuous functions on open subsets of W satisfying the assumptions of an axiomatic theory of "harmonic" functions, and supposing W compact, determine the sheaf cohomology groups $H^q(W, \mathcal{H})$, $q \geqslant 1$. The treatment here is much more general : the hypothesis placed on $\mathcal{H}$ is much less severe than that of the local validity of the axioms of Brelot [3] or even of the weaker axioms of Bauer [1], so that the present material is applicable not only to elliptic differential equations but also to some parabolic equations. However, most of the attention is given to the case in which W is compact (normal structures in the sense of [11] are not considered at all). The end result is easily stated : for compact W,

$$\dim H^0(W, \mathcal{H}) = \dim H^1(W, \mathcal{H}) < \infty,$$

and (as in [11]) $H^q(W, \mathcal{H}) = 0$ for $q \geqslant 2$. In the classical setting in which W is a compact manifold and $\mathcal{H}$ the solutions of a second-order elliptic differential equation on the manifold, the equality of the dimensions of $H^0(W, \mathcal{H})$ and $H^1(W, \mathcal{H})$ is equivalent to the equality of the dimensions of the spaces of solutions of the given equation and of its adjoint ; we give an axiomatic version of that classical theorem in 4.2.5 below. In order to establish these end results, we introduce a notion of perturbation of the given presheaf

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$\mathcal{H}$ that we believe is of independent interest. This notion is an analogue in the axiomatic setting of the replacement of a given differential operator L by an operator of the form $u \longrightarrow Lu + f{\cdot}u$, although it is much more general even in the classical cases. This perturbation theory does not depend on the compactness of W, and we intend in subsequent papers to make use of it for purposes other than that for which we introduce it here.

The ground plan of the paper is as follows. § 1 is devoted to matters which are well known for the most part, but unfortunately not well known in the generality needed here. We introduce axioms for the presheaves that we are going to study, develop some of their basic properties, and then discuss such things as the notions of specific restriction and the extension theorem of [7], the sheaves $\mathcal{R}$ and $\mathcal{Q}$ of [11], and some of the properties of the potential "kernels" of [9]. The axioms are probably not much different from those of [2], although a local Trennungsaxiom is added. It is with some trepidation that we introduce another set of axioms into a field already burdened with so many of them ; however, in § 3 below we have to construct new sheaves $\mathcal{G}$ out of the given sheaf $\mathcal{H}$, and it is a great technical convenience to work in an axiomatic framework in which the $\mathcal{G}$'s inherit the properties that $\mathcal{H}$ is known to possess. Most of § 1 is implicitly devoted to showing that the proofs of the theorems we need from [7], [9] and [11] are valid in the present axiomatic setting. Since this paper is not primarily expository, however, we have refrained from transcribing the proofs, and we simply refer the reader, whom we have armed with appropriate lemmas (minimum principles, etc.), to the theorems and proofs given in those papers ; he should be able to verify their validity in the present context with no great difficulty. § 2 consists mostly of technical preparation for later sections, although we do prove one reasonably general theorem (2.1.2). § 3 is the perturbation theory. Finally, § 4 contains the end results of the paper, theorems 4.1.4 and 4.2.5.

A word about some standard notation : if Z is a topological space, $\mathcal{C}(Z)$, $\mathcal{C}_o(Z)$, $\mathcal{C}_b(Z)$ and $\mathcal{K}(Z)$ denote its spaces of all continuous (real-valued) functions, all continuous functions vanishing at ∞, all bounded continuous functions and all continuous functions of compact support, respectively. $\mathcal{C}(Z)$ and spaces of not-necessarily-bounded continuous functions are given the topology of uniform convergence on compacta ; spaces of bounded functions are given the topology of uniform convergence. In addition to that last topological convention, however, we make a metric one : if E is a space of bounded real-valued functions on some set, E will be given the supremum norm, which will invariably be denoted by $\| \ \|_\infty$, unless very explicit mention is made to the contrary (cf. 1.7.3 and 1.7.4 below) ; if F is a normed space, the linear-transformation space $\mathcal{L}(E, F)$ will invariably be given the operator norm corresponding to the $\| \ \|_\infty$ norm on E and the norm given on F, unless explicit mention is made to the contrary. The pointwise infimum and supremum of two real-valued functions f and g will be denoted by $f \wedge g$ and $f \vee g$ respectively.

1. Axioms and other preliminaries.

1.1. Axioms. – We shall use four axioms. Rather than listing all of them at once, we shall discuss each a bit before stating the next.

AXIOM I. – W *is a connected, locally connected locally compact Hausdorff space with a countable basis.* $\mathcal{H}$ *is a complete presheaf of vector spaces of real-valued continuous functions over the base space* W.

This requires no discussion. We define regular sets in the usual way [1, p. 10], but we use $H(f, V)$ instead of H_f^V.

AXIOM II. – *To every* $x \in W$ *is assigned a connected open neighborhood* U_x *on which there exists a strictly positive section of* $\mathcal{H}$, *and a neighborhood basis* $\mathfrak{U}(x)$ *consisting of regions that are relatively compact in* U_x *and regular for* $\mathcal{H}$.

The representing measures for points in regular open sets are defined in the usual way [1, p. 12] ; the representing measure for a point x in a regular set V will be denoted by ρ_x^V.

AXIOM III. – *If* U *is an open subset of* W, *then every uniformly bounded subset of* $\mathcal{H}_U$ *is equicontinuous.*

It is evident that local uniform boundedness is sufficient for equicontinuity, and that such equicontinuity is uniform on compacta.

We define superharmonic functions in essentially the usual way, except that it is convenient to build local boundedness into the definition.

DEFINITION 1.1.1. — *Given an open set* U *in* W, *a superharmonic function on* U *is a lower-semicontinuous, locally bounded real-valued function* s *on* U, *such that for every* $x \in \mathrm{U}$ *and every* $\mathrm{V} \in \mathfrak{U}(x)$ *with* $\overline{\mathrm{V}} \subseteq \mathrm{U}$, *the inequality* $\mathrm{H}(f, \mathrm{V}) \leqslant s$ *holds throughout* V *whenever* $f \in \mathcal{C}(\partial \mathrm{V})$ *has the property that* $f \leqslant s \,|\, \partial \mathrm{V}$.

Clearly a restriction of a superharmonic function is a superharmonic function. By the definition of the representing measure and of the (upper) integral for bounded lower-semicontinuous functions, the last defining condition is exactly that $\int s \, d\rho_x^{\mathrm{V}} \leqslant s(x)$. It should be noted that the convergence axiom III is sufficiently strong that for any bounded function f on the boundary of a regular set V, the functions $x \longrightarrow \int^* f \, d\rho_x^{\mathrm{V}}$ and $x \longrightarrow \int_* f \, d\rho_x^{\mathrm{V}}$ belong to $\mathcal{H}_{\mathrm{V}}$.

AXIOM IV. — *On each of the sets* U_x *of axiom* II, *sufficiently many continuous superharmonic functions exist that they separate the points of* U_x *strongly* (i.e., *for any pair of distinct points* y *and* z *in* U_x *there exist continuous superharmonic* s *and* t *defined in* U_x, *such that* $s(y)\,t(z) \neq t(y)\,s(z)$).

1.2. Superharmonic functions.

DEFINITION 1.2.1. — *Given open* $\mathrm{U} \subseteq \mathrm{W}$ *and a basis* $\mathfrak{B}$ *for the topology of* U *consisting of open sets regular with respect to* $\mathcal{H}$, *a* $\mathfrak{B}$*-nearly superharmonic function on an open set* $\mathrm{X} \subseteq \mathrm{U}$ *is a locally bounded real-valued function* s *on* X *with the property that for every* $\mathrm{V} \in \mathfrak{B}$ *with* $\overline{\mathrm{V}} \subseteq \mathrm{X}$ *and every* $x \in \mathrm{V}$, *the inequality* $\int^* s \, d\rho_x^{\mathrm{V}} \leqslant s(x)$ *holds. If* s *is lower-semicontinuous,* s *will be called* $\mathfrak{B}$*-superharmonic.*

The considerations of [1, Ch. I, § 3] now operate to prove the Bauer minimum principle :

LEMMA 1.2.2. — *If* u *is a nonnegative* $\mathfrak{B}$*-superharmonic function on* U, *if the* $\mathfrak{B}$*-superharmonic functions on* U *strongly separate the points of* U, *and if every compact subset of* U *has a neighborhood on which there exists a strictly positive section of* $\mathcal{H}$, *then* $u^{-1}(0)$ *is not compact in* U.

PROPOSITION 1.2.3. – *Let u and* U *be as in 1.2.2, except that u is not assumed nonnegative. Suppose* Z *is a compactification of* U *such that (1) u has a lower-semicontinuous extension to* Z *and (2) there is a continuous strictly positive function on* Z *whose restriction to* U *is $\mathfrak{B}$-superharmonic. Then if u is nonnegative on* Z\U, *it is nonnegative on all of* Z.

COROLLARY 1.2.4. – *If* X $\subseteq$ W *is an open set and $\mathfrak{B}$ is a basis for the topology of* X, *such that the $\mathfrak{B}$-superharmonic functions on* X *strongly separate the points of* X, *and if there is a strictly positive section of $\mathcal{H}$ defined on a neighborhood of every compact subset of* X, *then the inequality $\int s\, d\rho_x^{\mathrm{V}} \leqslant s(x)$ holds for every $\mathfrak{B}$-superharmonic function s on* X, *regular relatively compact* V *with* $\overline{\mathrm{V}} \subseteq \mathrm{X}$, *and $x \in \mathrm{V}$.*

In particular, we can apply this corollary to those open sets X $\subseteq$ W with the property that the continuous superharmonic functions defined on X separate the points of X strongly and a strictly positive section of $\mathcal{H}$ is defined on a neighborhood of every compact subset of X : we merely take $\mathfrak{B} = \{\mathrm{V} : \mathrm{V} \in \mathfrak{U}(x)$ for some $x \in \mathrm{X}$, and $\overline{\mathrm{V}} \subseteq \mathrm{X}\}$. It is useful to make the following definition.

DEFINITION 1.2.5. – *A* B-*set is an open set* U $\subseteq$ W *such that there exists a neighborhood* Z *of* $\overline{\mathrm{U}}$ *such that the continuous superharmonic functions on* Z *strongly separate points of* Z *and a strictly positive element of $\mathcal{H}_{\mathrm{Z}}$ exists.*

It is evident that the bases $\mathfrak{U}(x)$ of axiom II are composed of B-sets.

COROLLARY 1.2.6. – *If $\mathfrak{B}$ is a basis for the topology of an open set* U $\subseteq$ W, *let $\mathfrak{B}_0$ denote the family of elements of $\mathfrak{B}$ that are* B-*sets. Then for a regular* $\mathrm{V} \in \mathfrak{B}_0$, *the inequality $\int s\, d\rho_x^{\mathrm{V}} \leqslant s(x)$ holds for each superharmonic function s on* U *and each $x \in \mathrm{V}$. In particular, every superharmonic function on* U *is $\mathfrak{B}_0$-superharmonic.*

Now let a $\mathfrak{B}$-superharmonic function s be given on an open set U. If Y $\subseteq$ U is a B-set, it is clear that $\mathfrak{B}_{\mathrm{Y}} = \{\mathrm{V} \in \mathfrak{B} : \overline{\mathrm{V}} \subseteq \mathrm{Y}\}$ is a basis for the topology of Y, obviously composed of B-sets. By 1.2.6, all superharmonic functions on Y are $\mathfrak{B}_{\mathrm{Y}}$-superharmonic ;

in particular, the family of all $\mathfrak{B}_Y$-superharmonic functions separates points of Y, which is the hypothesis required in 1.2.4. By that corollary, the inequality $\int s\, d\rho_x^V \leqslant s(x)$ holds for every regular relatively compact set $V \subseteq Y$, and thus in particular it holds for the elements of $\mathfrak{U}(x)$ for each $x \in U$. Hence

COROLLARY 1.2.7. – *For any $\mathfrak{B}$, a $\mathfrak{B}$-superharmonic function on an open set in W is superharmonic.*

The considerations of [1, Ch. II, § 1] show with virtually no modification that the following proposition holds ; we have already made the crucial observation that $x \longrightarrow \int^* f\, d\rho_x^V$ is harmonic on V for any bounded f on ∂V.

PROPOSITION 1.2.8. – *The operation of lower-semicontinuous regularization takes $\mathfrak{B}$-nearly superharmonic functions into superharmonic functions, and is additive and positively homogeneous ; if v is $\mathfrak{B}$-nearly superharmonic on $U \subseteq W$, then*

$$\hat{v}(x) = \lim_{V \ni x} \left\{ \int^* v\, d\rho_x^V : x \in V \subseteq \overline{V} \subseteq U \quad , \quad V \in \mathfrak{B} \right\}$$

and the limit is increasing for sufficiently small V at each point x.

One can attach superharmonic functions to each other at boundaries in the usual way :

PROPOSITION 1.2.9. – *Let s be a superharmonic function on an open set $U \subseteq W$, and Y an open subset of U. If t is a superharmonic function on Y with $\liminf_{Y \ni y \to x} t(y) \geqslant s(x)$ for each $x \in \partial Y$, then the function u defined as s in $U \setminus Y$ and as $s \wedge t$ in Y is superharmonic on U.*

The crucial point, as usual, is that one can restrict one's attention to "small" sets.

COROLLARY 1.2.10. – *If U is an open subset of W and $Y \subseteq U$ a regular B-set, then for any superharmonic function s defined on U, the function s_Y defined as s in $U \setminus Y$ and as $x \longrightarrow \int s\, d\rho_x^Y$ in Y is a superharmonic function.*

This last corollary shows that balayage of superharmonic functions results in new superharmonic functions, as long as the balayage takes place over sufficiently small sets. That $s_Y \leqslant s$, incidentally, is obvious.

1.3. Potentials. – As usual, a potential on an open set U is a non-negative superharmonic function p on U with the property that if $h \in \mathcal{H}_U$ and $h \leqslant p$, then $h \leqslant 0$. This definition makes sense even if U = W and W is compact, and the zero function is always a potential. Given a basis $\mathfrak{B}$ for the topology of U composed of regular B-sets, one can define a $\mathfrak{B}$-saturated family of superharmonic functions essentially as in [1, p. 53], and given a superharmonic function s on U one can define the $\mathfrak{B}$-saturated hull of s as

$$s_{\mathfrak{B}} = \{ (((s_{V_1})_{V_2}) \ldots)_{V_n} : n \in \mathrm{N} \quad , \quad V_1 , \ldots , V_n \in \mathfrak{B} \}$$

as in [1, p. 53]. One then has the expected

PROPOSITION 1.3.1. – *If* U *is an open set in W, s and f are super- and subharmonic functions on* U *respectively with* $f \leqslant s$, *and* $\mathfrak{B}$ *is a basis for the topology of* U *consisting of regular* B-*sets, then* $h = \inf s_{\mathfrak{B}}$ *is harmonic on* U *and* $f \leqslant h \leqslant s$.

Proof. – For any $V \in \mathfrak{B}$ one has $f \leqslant f_V \leqslant s_V \leqslant s$; by induction, every element of $s_{\mathfrak{B}}$ dominates f, and so does the infimum h, which is harmonic by the usual argument (valid in the presence of axiom III).

It is obvious that h is the greatest harmonic minorant of s (any other harmonic minorant is a candidate for use as f) so h is independent of the choice of $\mathfrak{B}$. Clearly $s - h$ is a potential. If p is a potential and u is superharmonic, then taking $f = -u$ and $s = p$ gives $-u \leqslant h = 0 \leqslant p$, and the corollary

COROLLARY 1.3.2. – *If u is superharmonic on* U, *p is a potential on* U, *and* $-p \leqslant u$, *then* $0 \leqslant u$. *More generally, if* $V \subseteq U$ *is an open set and v is a superharmonic function on* V *such that* $-p \leqslant v$ *and* $\liminf_{\partial V} v \geqslant 0$, *then* $v \geqslant 0$ *on* V. *If p is strictly positive on* V *and* $-p \leqslant v$ *is known to hold only outside a compact subset of* V, *it is still true that* $v \geqslant 0$.

Proof. – The first assertion is obvious. For the second, define w as $v \wedge 0$ in V and as 0 in U\V ; then w is superharmonic on U by 1.2.9 above, and $-p \leqslant w$, so $0 \leqslant w$ and $w \mid V \leqslant v$. For the third, replace p by such a large positive multiple αp that $-\alpha p \leqslant v$ holds.

Each of the various assertions of this corollary is sometimes known as the "minimum principle". A particular consequence of the corollary is the fact that if W is compact, then all superharmonic functions on W are nonnegative (and the only harmonic function the zero function) whenever a strictly positive potential exists on W.

The fact that the sum of two (or even a locally uniformly convergent, locally uniformly bounded series of) potentials is a potential, and the uniqueness, positivity and additivity of the decomposition $s = p + h$ of a superharmonic function with a subharmonic minorant into the sum of a potential and a harmonic function can be proved by standard methods in the present context. We omit the details.

The following definition simply introduces some notation we should need later anyway.

DEFINITION 1.3.3. – *The cone of continuous potentials on an open set* $U \subseteq W$ *will be denoted by* $\mathfrak{P}_U$, *and the space* $\mathfrak{P}_U - \mathfrak{P}_U$ *(a lattice under the pointwise operations) by* $\mathfrak{Q}_U$. *Similarly, the cone of bounded continuous potentials on* U *will be denoted by* $\mathfrak{P}_U^b$, *and the space it generates by* $\mathfrak{Q}_U^b$. *If* U *is relatively compact,* $\mathfrak{P}_U^c$ *will denote the subcone of* $\mathfrak{P}_U^b$ *consisting of those elements that have a continuous extension to* $\overline{U}$ *that is zero on* ∂U, *and* $\mathfrak{Q}_U^c$ *will denote the space generated by* $\mathfrak{P}_U^c$.

We shall also need the following approximation theorem.

PROPOSITION 1.3.4. – *Let* U *be an open set in* W. *The following conditions are equivalent :*

a) *The continuous nonnegative superharmonic functions on* U *strongly separate the points of* U.

b) *There exists a* $p \in \mathfrak{P}_U$ *such that* $p(x) > \int p \, d\rho_x^V$ *for every regular* B*-set* V *with* $\overline{V} \subseteq U$ *and every* $x \in V$.

c) *For every compact* $K \subseteq U$ *and every neighborhood* Y *of* K *in* U *the sublattice of* $\mathfrak{Q}_U$ *consisting of elements whose support is contained in* Y *is dense in* $\mathfrak{C}(K)$.

d) *The space* $\mathfrak{Q}_U$ *strongly separates the points of* U.

e) *There exists a strictly positive element of* $\mathfrak{P}_U$.

If in a) the continuous nonnegative superharmonic functions are replaced by $\mathfrak{P}_U^b$ *(or* U *is relatively compact and the continuous nonnegative superharmonic functions are replaced by* $\mathfrak{P}_U^c$*), and in b) through e)* $\mathfrak{P}_U$ *and* $\mathfrak{Q}_U$ *are replaced by* $\mathfrak{P}_U^b$ *and* $\mathfrak{Q}_U^b$ *(or by* $\mathfrak{P}_U^c$ *and* $\mathfrak{Q}_U^c$ *respectively), then the equivalence still holds, and all the resulting (equivalent) statements are valid whenever* U *is a* B-*set (or a regular* B-*set, respectively).*

Proof. – a) $\Longrightarrow$ b). If V is a regular B-set, $\overline{V} \subseteq U$, then for any two continuous nonnegative superharmonic functions f and g on U and any $x \in V$ we have

$$\int \sqrt{fg}\, d\rho_x^V = \int \sqrt{f}\sqrt{g}\, d\rho_x^V \leqslant \sqrt{\left[\int f\, d\rho_x^V\right]}\sqrt{\left[\int g\, d\rho_x^V\right]} \leqslant$$
$$\leqslant (\sqrt{f}\sqrt{g})(x)$$

by the Schwarz inequality, with strict inequality holding unless f and g are proportional on the carrier of ρ_x^V. This tells us immediately that $\sqrt{fg}$ is superharmonic ; moreover, since the carrier of ρ_x^V is nonempty (V being a B-set) and since the nonnegative continuous superharmonic functions on U strongly separate points of U, we can always find f and g for which one of the inequalities above is strict. Taking such f and g, we shall have $\int \sqrt{fg}\, d\rho_y^V < \sqrt{fg}(y)$ for $y = x$ and therefore for all y in a neighborhood of x. By taking a countable family of such neighborhoods (each corresponding to some $\sqrt{f_m g_m}$, $m = 1, 2, \ldots$) whose union is V, then finding a sequence of strictly positive multipliers $\{\alpha_m\}_{m=1}^{\infty}$ such that $s = \sum_{m=1}^{\infty} \alpha_m \sqrt{f_m g_m}$ converges uniformly on compacta in U, we can construct a continuous nonnegative superharmonic function s on U such that $\int s\, d\rho_x^V < s(x)$ for all $x \in V$. Letting V run through a countable basis $\{V_n\}_{n=1}^{\infty}$ for the topology of U formed of regular B-sets, finding such an s_n for each V_n, and finding strictly positive multipliers β_n such that

$u = \sum_{n=1}^{\infty} \beta_n s_n$ converges uniformly on compacta in U, we shall have constructed a continuous nonnegative superharmonic function such that $\int u \, d\rho_x^{V_n} < u(x)$ for each V_n and each $x \in V_n$. By finding V_n with $x \in V_n \subset V$, it is easy to see that $\int u \, d\rho_x^{V} < u(x)$ for any regular B-set V with $\overline{V} \subseteq U$. If $u = p + h$ is the canonical decomposition of u into a potential on U and a harmonic function on U, it is obvious that $\int p \, d\rho_x^V < p(x)$ for each regular B-set V and $x \in V$.

b) $\Longrightarrow$ c). If x and y are distinct points of K, let V and Z be regular regions that are B-sets, neighborhoods of x and y respectively, with $\overline{V} \cap \overline{Z} = \emptyset$ and $\overline{V} \cup \overline{Z} \subseteq Y$. Clearly $p - p_V$ and $p - p_Z$ are supported by $\overline{V}$ and $\overline{Z}$ respectively, and $(p - p_V)(x) > 0$, $(p - p_Z)(y) > 0$ by choice of p. The statement c) then follows by the lattice form of the Stone-Weierstrass theorem, applied to K.

Verifying that c) $\Longrightarrow$ d) $\Longrightarrow$ e) is trivial : the first implication is immediate, and if $\mathfrak{Q}_U$ strongly separates the points of U then $\mathfrak{P}_U$ does also, and one may thus find a sequence $\{p_n\}_{n=1}^{\infty}$ in $\mathfrak{P}_U$ such that some $p_n(x) > 0$ at every point x in U. If $\{\gamma_n\}_{n=1}^{\infty}$ are strictly positive multipliers such that $\sum_{n=1}^{\infty} \gamma_n p_n$ converges uniformly on compacta, then the sum is the desired strictly positive potential. The proof that e) $\Longrightarrow$ a) is exactly the same as the "$(P_0) \Longrightarrow (P_1)$" part of [1, Satz 2.5.3, p. 63].

To prove the last assertion of the proposition, we need only observe that the proofs we just gave are valid in the situations mentioned in that assertion, provided that we take the sequences α_m, β_n and γ_n of multipliers in such a way that the respective series converge uniformly on U (or uniformly on $\overline{U}$ respectively) rather than merely uniformly on compacta in U. If U is a regular B-set, take $Z \supseteq \overline{U}$ an open set on which a) above is satisfied, and construct a potential $q \in \mathfrak{P}_Z$ for which b) above is satisfied ; $p = q - H(q \mid \partial U, U) \in \mathfrak{P}_U^c$ then clearly satisfies b) for the set U, Q.E.D.

1.4. Specific restriction ; the Hervé extension theorem. – This will be a short section, since we shall content ourselves with observing that the whole theory of specific restriction of potentials is valid in the present axiomatic framework ; all the minimum principles, etc., that Meyer's exposition of specific restriction in [9, pp. 357-363] requires have been verified above. (One must take the trivial precaution of restricting the "arbitrary" regular open sets in Meyer's proofs--usually denoted by the letter U --to be regular B-sets.) For $p \in \mathfrak{P}_{\mathrm{U}}$ and E a Borel set in U we shall denote by $\lambda_{\mathrm{E}} p$ the specific restriction of p to E ; this is slightly at variance with [11, § 2] where we took E to be a Borel set in W and let λ_{E} denote the operation of specific restriction to $\mathrm{E} \cap \mathrm{U}$, but the relation between these usages is natural enough. Meyer shows that for fixed $p \in \mathfrak{P}_{\mathrm{U}}$ and $x \in \mathrm{U}$ the set function $\mathrm{E} \longrightarrow (\lambda_{\mathrm{E}} p)(x)$ is a (countably additive) measure, and that for f a nonnegative bounded Borel function, the function $x \longrightarrow \int f\, d[\lambda . p](x)$ is an element of $\mathfrak{P}_{\mathrm{U}}$; we shall denote that potential by $\Lambda_f p$, thus extending the notation of [11] (where f was required to be a simple Borel function). Since $-\|f\|_\infty p \leqslant \Lambda_f p \leqslant \|f\|_\infty p$, this is an extension by continuity ; moreover, if $p \in \mathfrak{P}^b_{\mathrm{U}}$ or $\mathfrak{P}^c_{\mathrm{U}}$, it is clear that $\Lambda_f p \in \mathfrak{P}^b_{\mathrm{U}}$ or $\mathfrak{P}^c_{\mathrm{U}}$ for $f \geqslant 0$. We have $\Lambda_f(\Lambda_g p) = \Lambda_{fg} p$ for simple Borel functions by [11], and by taking limits we have it for all bounded Borel functions on U ; that $\Lambda_1 p = p$ and $\Lambda_{f+g}\, p = \Lambda_f p + \Lambda_g p$ is obvious. Meyer does not prove that $\lambda_{\mathrm{E}}\, p$ depends in a positively homogeneous and additive way on p, but the positive homogeneity is obvious and the proof of the additivity of λ_{E} for open E given in [7, Prop. 15.2, p. 466] does not depend on anything other than the fact that for any $p \in \mathfrak{P}_{\mathrm{U}}$, $\lambda_{\mathrm{E}}\, p$ is specifically smaller than any $(\mathrm{E} \cap \mathrm{U})$-majorant of p. Extending specific restriction and the operations Λ_f to $\mathfrak{Q}_{\mathrm{U}}$ in the unique linear way, we make $\mathfrak{Q}_{\mathrm{U}}$ into a module over the algebra $\mathfrak{B}_{\mathrm{U}}$ of bounded Borel functions on U, with the continuity relation $-\|f\|_\infty p \leqslant \Lambda_f p \leqslant \|f\|_\infty p$ for $p \in \mathfrak{P}_{\mathrm{U}}$ and the positivity relation $0 \leqslant f,\ p \in \mathfrak{P}_{\mathrm{U}} \Longrightarrow \Lambda_f p \in \mathfrak{P}_{\mathrm{U}}$ valid. It is easy to verify that if $0 \leqslant f \in \mathfrak{B}_{\mathrm{U}}$ and $p \in \mathfrak{P}^b_{\mathrm{U}}$ or $\mathfrak{P}^c_{\mathrm{U}}$, then $\Lambda_f p \in \mathfrak{P}^b_{\mathrm{U}}$ or $\mathfrak{P}^c_{\mathrm{U}}$; in consequence, $\mathfrak{Q}^b_{\mathrm{U}}$ and $\mathfrak{Q}^c_{\mathrm{U}}$ are sub-$\mathfrak{B}_{\mathrm{U}}$-modules of $\mathfrak{Q}_{\mathrm{U}}$.

Mme Hervé's proof of the extension theorem [7, Thm. 13.2, pp. 458-459] is valid in the present setting ; indeed, if the given superharmonic function ν of [7, Lemme 13.1, p. 457] is continuous, one does not need to assume that it is nonnegative or that there is a positive harmonic function defined in a neighborhood of its support. Thus we have

THEOREM 1.4.1 [Hervé]. – *Let* U *be an open set in* W *in which the equivalent conditions of 1.3.4 above hold. Then, given a continuous superharmonic function* v *on an open set* $V \subseteq U$, *with* v *supported by the compact set* $K \subseteq V$, *there is a unique* $p \in \mathfrak{P}_U$ *for which* $p - v$ *is harmonic in* V. *If the conditions of 1.3.4 above hold with* $\mathfrak{P}_U$ *replaced by* $\mathfrak{P}_U^b$ *or* $\mathfrak{P}_U^c$ *(in particular, if* U *is a* B*-set), then* $p \in \mathfrak{P}_U^b$ *or* $\mathfrak{P}_U^c$. *(Of course* p *is required to have support contained in* V.)

1.5. The sheaf $\mathcal{R}$. – We now make the definitions of [11, § 2], and verify that the considerations made in [11] are valid in the present axiomatic setting.

DEFINITION 1.5.1. – $\mathcal{R}$ *is the presheaf over* W *of vector spaces of scalar-valued functions on open subsets of* W *determined by the following condition : if* U *is an open subset of* W, $f \in \mathcal{R}_U$ *if and only if every* $x \in U$ *has a neighborhood* $V \subseteq U$ *such that* $f|V$ *is a linear combination of nonnegative continuous subharmonic functions with domain* V.

Again, there is no change in the definition if "subharmonic" is replaced by "superharmonic" in the definition above, since every point of W has a neighborhood in which some strictly positive harmonic function is defined. That $\mathcal{R}$ is a complete presheaf is again obvious. Because a uniform limit of continuous super- or subharmonic functions is a function of the same kind, the proof of [11, Prop. (2.2)] is valid with the present axioms. We recall the proposition :

PROPOSITION 1.5.2. – *If* Z *is an open subset of* W *such that* $1 \in \mathcal{H}|Z$, *then* $\mathcal{R}|Z$ *is a sheaf of algebras (under pointwise multiplication) over its scalar field. Moreover,* $\mathcal{R}|Z$ *is inverse-closed in the sense that an element of* $\mathcal{R}_U$ $(U \subseteq Z)$ *that has no zeros in* U *has an inverse (necessarily the pointwise inverse) in* $\mathcal{R}_U$.

For the next proposition of [11, § 2] one needs a slightly different proof. See [11, § 5 (B)].

PROPOSITION 1.5.3. – *If* Z *is an open subset of* W *for which* $1 \in \mathcal{H}|Z$, *then for every* $x_0 \in Z$ *and neighborhood* U *of* x_0 *there exists an* $f \in \mathcal{R}_W$ *for which* $0 \leqslant f \leqslant 1$, $f(x_0) = 1$ *and* $f = 0$ *outside a closed subset of* U.

Proof. – With no loss of generality, we can assume that U is a B-set contained in Z. Let Y be a neighborhood of x_0 with compact closure contained in U, and apply c) of 1.3.4 above with $K = \{x_0\}$ to produce an element $g \in \mathfrak{Q}_U$ with support contained in Y and a value larger than 1 at x_0. Since the nonnegative subharmonic functions on any open set in W are closed under the formation of suprema, $\mathcal{R}_U$ is a lattice under the pointwise operations, and so the function defined by $f = 0$ on W\U and $f = (g \wedge 1) \vee 0$ on U belongs to $\mathcal{R}_W$ and satisfies the specifications of the proposition, Q.E.D.

It now follows, just as in [11, § 2], that $\Gamma(Z, \mathcal{R})$ contains partitions of unity subordinate to any locally finite covering of Z, and consequently that $\mathcal{R}|Z$ is fine, whenever $1 \in \mathcal{H}|Z$. Since $\mathcal{H}|Z$ is multiplicatively equivalent to a presheaf containing 1 and satisfying axioms I-IV above whenever there is a strictly positive section of $\mathcal{H}$ defined on Z, axiom II implies that $\mathcal{R}$ is locally fine. Thus we have, as in [11],

PROPOSITION 1.5.4. – *The sheaf $\mathcal{R}$ is fine.*

1.6. The sheaf $\mathfrak{Q}$. – This sheaf can also be constructed under the present axioms ; all we need to use about potentials and specific restriction has been verified in 1.3 and 1.4 above. We did not make use of the hypothesis that domains of potentials are connected in 1.3 and 1.4, so the presheaf Q defined below can be thought of as being "indexed" by all the open subsets of W, instead of merely by the regions in W as in [11], although the difference is inessential.

DEFINITION 1.6.1. – *For open sets* U *and* V *with* $V \subseteq U$ *in* W, $r_{VU} : \mathfrak{Q}_U \longrightarrow \mathfrak{Q}_V$ *denotes the unique linear extension of the mapping* $r_{VU} : \mathfrak{P}_U \longrightarrow \mathfrak{P}_V$ *that assigns to* $p \in \mathfrak{P}_U$ *the function* $p|V - h \in \mathfrak{P}_V$, *where* h *is the greatest harmonic minorant in* V *of the superharmonic function* $p|V$. *The transitivity relation* $r_{YU} = r_{YV} \circ r_{VU}$ *for* $Y \subseteq V \subseteq U$ *is easily verified ; let* Q *denote the presheaf formed by the* $\mathfrak{Q}_U$ *and the* r_{VU}, *and let* $\mathfrak{Q}$ *denote the associated sheaf.*

The "Laplacian" $\Delta : \mathcal{R} \longrightarrow \mathfrak{Q}$ of [11, Thm. (2.11)] is defined in the present context just as it is there, and it is again routine to verify that the sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{R} \xrightarrow{\Delta} \mathcal{Q} \longrightarrow 0$$

is exact.

DEFINITION 1.6.2. – *Let g be a section of $\mathcal{R}$ on the open set* U. *At each $x \in \mathrm{U}$, $(\Delta g)(x) \in \mathcal{Q}_x$ is the element determined as follows : if* V *is a neighborhood of x on which $g|\mathrm{V} = p_1 - p_2 + h$, $p_i \in \mathfrak{P}_{\mathrm{V}}$ $(i = 1, 2)$ and $h \in \mathcal{H}_{\mathrm{V}}$, then $(\Delta g)(x)$ is the canonical image in $\mathcal{Q}_x$ of*

$$p_1 - p_2 \in \mathfrak{Q}_{\mathrm{V}} .$$

For the purposes of the present paper we shall need a more delicate module structure on $\mathcal{Q}$ than we had in [11]. As we saw in 1.4 above, each $\mathfrak{Q}_{\mathrm{U}}$ is a module over the algebra $\mathfrak{B}_{\mathrm{U}}$ of bounded Borel functions on U. The relation $\lambda_{\mathrm{E} \cap \mathrm{V}} r_{\mathrm{VU}} = r_{\mathrm{VU}} \lambda_{\mathrm{E}}$ proved in [11, Prop. (2.9)] is valid in the present setting ; applied to simple Borel functions on U it says that $\Lambda_{f|\mathrm{V}} r_{\mathrm{VU}} = r_{\mathrm{VU}} \Lambda_f$, and since for fixed $p \in \mathfrak{P}_{\mathrm{U}}$ both $\Lambda_{f|\mathrm{V}} r_{\mathrm{VU}} p$ and $r_{\mathrm{VU}} \Lambda_f p$ depend continuously on f (with the uniform norm topology for $\mathfrak{B}_{\mathrm{U}}$ and u.c.c. for $\mathfrak{Q}_{\mathrm{V}}$), this equality holds for all bounded Borel functions f. Thus Q is a presheaf of modules over the presheaf of algebras of bounded Borel functions on open subsets of W (the latter presheaf having ordinary restriction of functions as its restriction map). [4, Lect. 7, p. 35 ff.] then guarantees that the associated sheaf $\mathcal{Q}$ is a sheaf of modules over the associated sheaf $\mathcal{B}$ of algebras given by the presheaf of $\mathfrak{B}_{\mathrm{U}}$. It is an easy exercise to verify that the latter sheaf can be naturally identified with the complete presheaf of locally bounded Borel functions on open subsets of W, and so we can regard $\mathcal{Q}$ as a sheaf of modules over that (complete pre-) sheaf of algebras. We shall continue to denote the action of $f \in \mathcal{B}$ on $\mathrm{M} \in \mathcal{Q}$ by $\Lambda_f \mathrm{M}$, and we have verified

PROPOSITION 1.6.3. – *The action $(f, \mathrm{M}) \longrightarrow \Lambda_f \mathrm{M}$ makes $\mathcal{Q}$ a sheaf of modules over the sheaf of algebras $\mathcal{B}$ of germs of locally bounded Borel functions.*

We shall also need an order relation on $\mathcal{Q}$. We have a natural "positive" cone $\mathfrak{P}_{\mathrm{U}}$ in each space $\mathfrak{Q}_{\mathrm{U}}$ already, and the maps r_{VU} are positive with respect to these cones, so there is an inductive limit cone $\mathcal{P}_x$ in each stalk $\mathcal{Q}_x$.

DEFINITION 1.6.4. – *The natural order* $\leqslant$ *on* $\mathcal{D}_x$ *(for each* $x \in W$*) is the order whose cone of nonnegative elements is the inductive limit of the cones* $\mathfrak{P}_U (x \in U)$ *under the restriction maps* r_{VU} *of* Q. *In an open subset* U *of* W, *a section* $M \in \Gamma(U, \mathcal{D})$ *is defined to be nonnegative if and only if its value at each* $x \in U$ *belongs to* $\mathcal{P}_x$, *and the natural order* $\leqslant$ *on* $\Gamma(U, \mathcal{D})$ *is the order defined by the cone of nonnegative sections.*

PROPOSITION 1.6.5. – *Each* $\mathcal{P}_x$ *is a proper cone, and therefore the order relation on* $\Gamma(U, \mathcal{D})$ *is proper. A section of* $\mathcal{R}$ *on* U *is a superharmonic function if and only if its Laplacian is nonnegative in* U. If $0 \leqslant f \in \Gamma(U, \mathcal{B})$ *and* $0 \leqslant M \in \Gamma(U, \mathcal{D})$, *then* $\Lambda_f M \geqslant 0$. *The nonnegative cone in* $\Gamma(U, \mathcal{D})$ *generates* $\Gamma(U, \mathcal{D})$.

Proof. – Suppose we have an element of $\mathcal{D}_x$ that simultaneously belongs to $\mathcal{P}_x$ and $-\mathcal{P}_x$. By definition, this means that there exist neighborhoods U and V of x and potentials p_1 and p_2 on them, such that for some $Y \subseteq U \cap V$, $r_{YU} p_1 = -r_{YV} p_2$. Since both $r_{YU} p_1$ and $r_{YV} p_2$ are potentials on Y, both are therefore zero, and it follows that our element of $\mathcal{D}_x$ is zero.

For the next assertion, suppose $g \in \mathcal{R}_U$ and $\Delta g \geqslant 0$ at $x \in U$. This means that if $V \subseteq U$ is a neighborhood of x in which $g = g_1 - g_2$ with each g_i nonnegative, continuous and superharmonic ($i = 1,2$), and h_i is the greatest harmonic minorant in V of g_i ($i = 1,2$), then the element of $\mathcal{D}_x$ that is the natural image of

$$(g_1 - h_1) - (g_2 - h_2) \in \mathfrak{Q}_V$$

is also the natural image of some $p \in \mathfrak{P}_Z$, where $Z \subseteq U$. Thus for some open $Y \subseteq V \cap Z$, $r_{YV}[(g_1 - h_1) - (g_2 - h_2)] = r_{YZ} p$, and since the left side differs from g by a harmonic function in Y while the right side is superharmonic in Y, g is superharmonic in Y. If $\Delta g \geqslant 0$ throughout U, then g is superharmonic in a neighborhood of each point of U and thus superharmonic in U, by the considerations of 1.2 above (particularly 1.2.7).

The next assertion follows easily from the fact that if $0 \leqslant f \in \mathfrak{B}_V$ and $p \in \mathfrak{P}_V$, then $\Lambda_f p \in \mathfrak{P}_V$. To prove the last assertion, let $M \in \Gamma(U, \mathcal{D})$ be given. Then for each $x \in U$ there is a neighborhood V_x of x and potentials $p_{1,x}$ and $p_{2,x}$ in $\mathfrak{P}_{V_x}$ such that for each $y \in V_x$, $M(y)$ is the canonical image of $p_{1,x} - p_{2,x}$. If Z_x is a neighborhood of x which is relatively compact in V_x, then $\lambda_{Z_x} p_{1,x} \in \mathfrak{P}_{V_x}$ defines a nonnegative section of $\mathcal{D}$ in V_x, supported by $\overline{Z}_x$, which majorizes M in Z_x. Extend that section to all of U by setting it equal to zero outside V_x, and let N_x denote the extension. If $\{Z_{x_i}\}_{i\in I}$ are chosen to be a locally finite cover of U, then the locally finite sum

$$\sum_{i \in I} N_{x_i} \in \Gamma(U, \mathcal{D})$$

is clearly nonnegative and majorizes M at each point of U, Q.E.D.

1.7. Kernels generated by potentials. – In addition to the rudiments of the specific restriction theory discussed in 1.4 above, we shall also need some of the material of [9, § 3] dealing with resolvents. Let U be an open subset of W, $p \in \mathfrak{P}^b_U$, and let K denote the "kernel" $K : f \longrightarrow \Lambda_f p$. K defines a linear, norm-continuous mapping of $\mathfrak{B}_U$ into $\mathcal{C}_b(U)$; if $p \in \mathfrak{P}^c_U$, as it will be in most applications below, K defines a linear, norm-continuous mapping of $\mathfrak{B}_U$ into $\mathcal{C}_0(U)$. K has the following properties :

LEMMA 1.7.1 *(Weak maximum principle). – Let g be a bounded continuous function on* U *and s be a nonnegative superharmonic function on* U. *If s majorizes* Kg *on the set* $S = \{x \in U : g(x) \geqslant 0\}$, *then s majorizes* Kg *on* U.

Proof. – Since Kg^+ is harmonic in U\S and Kg^- is superharmonic, $s - Kg = (s + Kg^-) - Kg^+$ is superharmonic in U\S. Clearly the negative of the potential $K|g|$ minorizes $s - Kg$ everywhere, and $s - Kg$ is nonnegative at $\partial S = \partial(U \backslash S)$, so s majorizes Kg on U\S by 1.3.2 above, Q.E.D.

LEMMA 1.7.2. – *Suppose the operator* $(I + tK)^{-1}$ *exists (as an element of* $\mathcal{L}(\mathcal{C}_b(U))$) *for some* $t \geqslant 0$. *Then*

a) $(I + tK)^{-1}K$ *is a nonnegative operator from* $\mathfrak{B}_U$ *into the bounded continuous functions on* U ;

b) *If s is a bounded continuous nonnegative superharmonic function on* U, *then* $(I + tK)^{-1} s \geqslant 0$.

Proof. – We may exclude $t = 0$ as trivial. Proving a) is equivalent to checking that the equivalent inequalities

$$[(I + tK)^{-1} Kf](x) \geqslant 0$$

where $0 \leqslant f \in \mathfrak{B}_U$

or $$(Kf)(x) \geqslant (K[(I + tK)^{-1} tKf])(x)$$

hold for all $x \in U$. By 1.7.1 above, the second of these inequalities holds on all of U if it holds on the set of points x where

$$[(I + tK)^{-1} tKf](x) \geqslant 0 ,$$

which is precisely where the first inequality holds. To prove b), observe that on the open set

$$S = \{x \in U : [s - (I + tK)^{-1} tKs](x) < 0\}$$

the function $(I + tK)^{-1} tKs = tK[s - (I + tK)^{-1} tKs]$ is subharmonic, and so $s - (I + tK)^{-1} tKs$ is superharmonic on S. At ∂S,

$$s - (I + tK)^{-1} tKs$$

takes the value zero. Moreover, it is minorized by the negative of

$$\begin{aligned}(I + tK)^{-1} tKs &= K[(I + tK)^{-1} ts] \\ &\leqslant K[|(I + tK)^{-1} ts|]\end{aligned}$$

and the right side of this inequality is a potential. By 1.3.2,

$$(I + tK)^{-1} s = s - (I + tK)^{-1} tKs \geqslant 0$$

in S, so $S = \emptyset$ and b) is proved.

COROLLARY 1.7.3. – *If there exists a continuous superharmonic u on U which is bounded and bounded away from zero, then there is a norm on $\mathcal{C}_b(U)$, equivalent to the usual uniform norm, for which $\|(I + tK)^{-1} tK\| \leqslant 1$ for all $t \geqslant 0$ for which $(I + tK)^{-1}$ is defined.*

Indeed, $\|f\| = \inf \{\alpha \geqslant 0 : |f| \leqslant \alpha u\}$ is such a norm.

LEMMA 1.7.4. – *If there exists a superharmonic function on U which is bounded and bounded away from zero, then $(I + tK)^{-1}$ exists for all $t \geqslant 0$.*

Proof. – The set of t for which it exists is the open set $R^+ \cap \rho(-K)^{-1}$ in R^+, and to prove that that set is all of R^+ it will suffice to show that it contains all its limit points. For

$$t_2 < t_1 < t_0 \,,\ t_1 \,,\ t_2 \in \rho(-K)^{-1}$$

we have the elementary relation

$$(I + t_2K)^{-1}K - (I + t_1K)^{-1}K = (t_1 - t_2)(I + t_2K)^{-1}K(I + t_1K)^{-1}K$$

and so for an appropriate norm (that of 1.7.3 above)

$$\|(I + t_2K)^{-1}K - (I + t_1K)^{-1}K\| \leqslant (t_1 - t_2)\frac{1}{t_2}\frac{1}{t_1} = \frac{1}{t_2} - \frac{1}{t_1}$$

as $t_2, t_1 \longrightarrow t_0$. The limit operator $U = \lim_{t \to t_0} (I + tK)^{-1}K$ satisfies the relations $(I + t_0K)(I - t_0U) = I = (I - t_0U)(I + t_0K)$, so $t_0 \in \rho(-K)^{-1}$, Q.E.D.

1.8. Subellipticity. – To sharpen some of the results below in the presence of the Brelot axioms, it will be useful to have the following material :

DEFINITION 1.8.1. – *A pair* $(W, \mathcal{H})$ *satisfying axioms I through IV above will be called subelliptic if every point has a neighborhood basis consisting of open regular regions* V *for which, for every* $x \in V$, *the carrier of* ρ_x^V *is all of* ∂V.

PROPOSITION 1.8.2. – *If* $U \subseteq W$ *is an open set and* $(W, \mathcal{H})$ *is subelliptic, then a nonnegative superharmonic function* u *on* U *is strictly positive throughout any component of* U *containing a point* x_0 *for which* $u(x_0) > 0$.

Proof. – Let x_1 be a boundary point in U of the component of $U_0 = \{x \in U : u(x) > 0\}$ containing x_0. Let V be a regular neighborhood of x_1 with $\overline{V} \subseteq U$, having the properties given in 1.8.1 above. Then ∂V intersects U_0, and so $u(x) \geqslant \int u \, d\rho_x^V > 0$ for all $x \in V$, so x_1 is interior to U_0 contrary to its choice, Q.E.D.

COROLLARY 1.8.3. — *If* $(\mathrm{W}, \mathcal{H})$ *is subelliptic,* V *is a regular region in* W, *and* $x \in \mathrm{V}$, *then the carrier of* ρ_x^{V} *is all of* $\partial \mathrm{V}$.

Proof. — If $0 \leqslant f \in \mathcal{C}(\partial \mathrm{V})$ and $f \neq 0$, then $x \longrightarrow \int f \, d\rho_x^{\mathrm{V}}$ is a nonnegative harmonic function in V, and since its boundary values are given by f it is not identically zero in V. Thus it is positive throughout V, and so for each $x \in \mathrm{V}$ the measure ρ_x^{V} cannot be carried by a smaller set than $\partial \mathrm{V}$.

The defining property of 1.8.1 is thus "inherited" by all regular regions in W.

2. Lifting sections of $\mathfrak{Q}$; topologizing spaces $\Gamma(\mathrm{U}, \mathfrak{Q})$.

2.1. Lifting sections of $\mathfrak{Q}$. — This is essentially a sharpening and localization of the Hervé extension theorem. The fundamental result is 2.1.1, and its fundamental cohomological consequence is 2.1.2, which is [11, Thm. (4.1)] in its most general form for compact W.

PROPOSITION 2.1.1. — *Let* U *be an open set in* W *satisfying the equivalent conditions of 1.3.4 above. If* M *is a section of* $\mathfrak{Q}$ *on* U *with compact support, then there exists a unique* $q \in \mathfrak{Q}_{\mathrm{U}}$ *with* $\Delta q = \mathrm{M}$. *If* U *satisfies the conditions of 1.3.4 with* $\mathfrak{P}_{\mathrm{U}}$ *replaced by* $\mathfrak{P}_{\mathrm{U}}^{b}$ *or* $\mathfrak{P}_{\mathrm{U}}^{c}$ *respectively, then* q *belongs to* $\mathfrak{Q}_{\mathrm{U}}^{b}$ *or* $\mathfrak{Q}_{\mathrm{U}}^{c}$ *respectively.*

Proof. — By 1.6.5 above, it will suffice to prove this under the additional assumption that $\mathrm{M} \geqslant 0$. For each $x \in \operatorname{Supp} \mathrm{M}$ we can find a region $\mathrm{V}_x \subseteq \mathrm{U}$ and a potential $p_x \in \mathfrak{P}_{\mathrm{V}_x}$ for which $\mathrm{M}(y)$ is the canonical image of p_x at every point $y \in \mathrm{V}_x$. Let Y_x be an open neighborhood of x with $\overline{\mathrm{Y}}_x$ compact in V_x for each $x \in \mathrm{U}$, and pick $x_1, \ldots, x_n$ for which the $\mathrm{Y}_i = \mathrm{Y}_{x_i} (1 \leqslant i \leqslant n)$ cover Supp M. Set $\mathrm{Z}_j = \mathrm{Y}_j \setminus \bigcup_{i=1}^{j-1} \mathrm{Y}_i (1 \leqslant j \leqslant n)$, and set $\mathrm{Z} = \bigcup_{j=1}^{n} \mathrm{Z}_j = \bigcup_{i=1}^{n} \mathrm{Y}_i$. Clearly $\lambda_{\mathrm{Z}} \mathrm{M} = \mathrm{M}$, so $\mathrm{M} = \sum_{j=1}^{n} \lambda_{\mathrm{Z}_j} \mathrm{M}$, and to prove the existence of q it will suffice to find $q_j \in \mathfrak{P}_{\mathrm{U}}$ with $\Delta q_j = \lambda_{\mathrm{Z}_j} \mathrm{M}$. Thus it will suffice to find q_j with $q_j - \lambda_{Z_j} p_j$ harmonic in V_{x_j}, where $p_j = p_{x_j}$ $(1 \leqslant j \leqslant n)$, since $\lambda_{Z_j} M$ is the canonical image of $\lambda_{Z_j} p_j$ in $V_j = V_{x_j}$ and is zero in the complement of $\overline{Z}_j$, $1 \leqslant j \leqslant n$. But $\lambda_{Z_j} p_j$ is a potential of compact support in V_j, and so by the Hervé extension theorem 1.4.1 above, there is a potential $q_j \in \mathfrak{P}_U$ with the desired property. That proves existence ; uniqueness follows from the fact that a difference of two $q's$ satisfying the specifications of the proposition would be harmonic in U but majorized in absolute value by a potential on U. The form of the Hervé extension theorem given in 1.4.1 also assures us that the functions q_j constructed above belong to $\mathfrak{P}^b_U$ or $\mathfrak{P}^c_U$ under the corresponding hypotheses.

THEOREM 2.1.2. – *If* W *is compact and* $\mathfrak{P}_W$ *contains a strictly positive element, then* $\Gamma(W, \mathcal{H}) = 0$ *and* $H^1(W, \mathcal{H}) = 0$; *moreover,* $\Delta : \mathfrak{Q}_W \longrightarrow \Gamma(W, \mathcal{S})$ *is onto, so* $\Gamma(W, \mathcal{R}) = \mathfrak{Q}_W$.

Proof. – We prove the second assertion first : taking U = W in 2.1.1 shows us that Δ sends $\mathfrak{Q}_W$ onto $\Gamma(W, \mathcal{S})$, so *a fortiori* Δ sends $\Gamma(W, \mathcal{R})$ onto $\Gamma(W, \mathcal{S})$. The minimum principle implies $\Gamma(W, \mathcal{H}) = 0$ and the cohomology exact sequence

$$0 \longrightarrow \Gamma(W, \mathcal{H}) \longrightarrow \Gamma(W, \mathcal{R}) \xrightarrow{\Delta} \Gamma(W, \mathcal{S}) \xrightarrow{\delta} H^1(W, \mathcal{H}) \longrightarrow 0$$

shows that Δ is 1-1 and onto from $\Gamma(W, \mathcal{R})$ to $\Gamma(W, \mathcal{S})$. It follows that $\mathfrak{Q}_W$ must be all of $\Gamma(W, \mathcal{R})$, Q.E.D.

2.2. Topologies for the spaces $\Gamma(U, \mathcal{S})$. – Our program is to topologize the spaces $\mathfrak{Q}^c_U$ and then use 2.1.1 to induce topologies on the spaces $\Gamma(U, \mathcal{S})$.

DEFINITION 2.2.1. – *The seminorm* $|||\cdot|||$ *on* $\mathfrak{Q}^c_U$ *is defined by* $|||q||| = \inf\{\|p^+\|_\infty + \|p^-\|_\infty : p^+, p^- \in \mathfrak{P}^c_U, q = p^+ - p^-\}$.

PPROPOSITION 2.2.2. – $\|q\|_\infty \leqslant |||q|||$, *so* $|||\cdot|||$ *is a norm ; under this norm,* $\mathfrak{Q}^c_U$ *is a Banach space. The specific restriction operators are continuous in this norm ; more generally, if* $f \in \mathfrak{B}_U$ *then* $|||\Lambda_f q||| \leqslant 4 \cdot \|f\|_\infty \cdot |||q|||$. *If* $\{E_i\}_{i=1}^\infty$ *is a sequence of disjoint Borel* *subsets of* U *and* $q \in \mathfrak{Q}^c_{\mathrm{U}}$, *then the series* $\sum_{i=1}^{\infty} \lambda_{\mathrm{E}_i} q$ *converges to* $\lambda_{\mathrm{E}}\, q$, *where* $\mathrm{E} = \bigcup_{i=1}^{\infty} \mathrm{E}_i$, *in the* $|||\cdot|||$*-topology. If* $\mathrm{V} \subseteq \mathrm{U}$ *is a regular* B*-set, then* $r_{\mathrm{VU}} : \mathfrak{Q}^c_{\mathrm{U}} \longrightarrow \mathfrak{Q}^c_{\mathrm{V}}$ *is continuous in the norm topologies.*

Proof. – The first assertion is obvious. To prove the second, it suffices to show that if $\{q_n\}_{n=1}^{\infty}$ is a sequence in $\mathfrak{Q}^c_{\mathrm{U}}$ with

$$\sum_{n=1}^{\infty} |||q_n||| < \infty ,$$

then $q = \lim_{k\to\infty} \sum_{n=1}^{k} q_n$ exists in the $|||\cdot|||$-topology. For each q_n choose p_n^+ and p_n^- in $\mathfrak{P}^c_{\mathrm{U}}$ with $\|p_n^+\|_\infty + \|p_n^-\|_\infty \leqslant 2 \cdot |||q_n|||$. Then $p_n^+ = \sum_{n=1}^{\infty} p_n^+$ and $p^- = \sum_{n=1}^{\infty} p_n^-$ are well-defined elements of $\mathfrak{P}^c_{\mathrm{U}}$. Set $q = p^+ - p^-$. Since

$$\left|\left|\left| p^+ - \sum_{n=1}^{k} p_n^+ \right|\right|\right| \leqslant \left\| \sum_{n>k} p_n^+ \right\|_\infty \leqslant 2 \sum_{n>k} |||q_n|||$$

and similarly for p^-, we have

$$\left|\left|\left| q - \sum_{n=1}^{k} q_n \right|\right|\right| \leqslant 4 \sum_{n>k} |||q_n|||$$

and that approaches zero as $k \longrightarrow \infty$.

The fact that $\|\Lambda_f p\|_\infty \leqslant \|f\|_\infty \cdot \|p\|_\infty$ if $0 \leqslant f \in \mathfrak{B}_{\mathrm{U}}$ and $p \in \mathfrak{P}^c_{\mathrm{U}}$ leads immediately to the norm inequality for $\Lambda_f q$ in general. For $p \in \mathfrak{P}^c_{\mathrm{U}}$ we have $\lambda_{\mathrm{E}} p = \sum_{i=1}^{\infty} \lambda_{\mathrm{E}_i} p$ pointwise on U, and since everything in sight is zero on $\partial\mathrm{U}$ the Dini theorem assures us that the convergence is uniform on $\overline{\mathrm{U}}$. Thus

$$\left|\left|\left| \lambda_{\mathrm{E}} p - \sum_{i=1}^{k} \lambda_{\mathrm{E}_i} p \right|\right|\right| \leqslant \left\| \sum_{i>k} \lambda_{\mathrm{E}_i} p \right\|_\infty \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.$$

Finally, if V is a regular B-set then $r_{\mathrm{VU}}\, q = q\,|\mathrm{V} - \mathrm{H}(q\,|\,\partial\mathrm{V}, \mathrm{V})$, so r_{VU} takes its values in $\mathfrak{Q}^c_{\mathrm{U}}$, and since it is pointwise decreasing when applied to potentials, it is certainly continuous in the $|||\cdot|||$-topologies, Q.E.D.

COROLLARY 2.2.3. – *The cone* $\mathfrak{P}_U^c$ *is normal in* $\mathfrak{Q}_U^c$ *in the sense of [10, p. 215].*

Indeed, $|||p||| = \|p\|_\infty$ if $p \in \mathfrak{P}_U^c$, and $\|\cdot\|_\infty$ is clearly monotone on $\mathfrak{P}_U^c$, so [10, Thm. 3.1 (e), p. 215] applies.

Remark. – The same sort of definition can be used to topologize $\mathfrak{Q}_U^b$ as a Banach space. That space does not appear to be as useful as $\mathfrak{Q}_U^c$, however, because it is not clear that series of the form $\sum_{i=1}^{\infty} \lambda_{E_i} q$ are norm-convergent.

DEFINITION 2.2.4. – *Let* V *be an open set in* W *satisfying the equivalent conditions of 1.3.4 above for* $\mathfrak{P}_V^c$. *For any Borel set* E *that is relatively compact in* V, *the lifting operator*

$$S_{(E,V)} : \Gamma(V, \mathfrak{L}) \longrightarrow \mathfrak{Q}_V^c$$

is the linear transformation that assigns to each $M \in \Gamma(V, \mathfrak{L})$ *the unique* $q \in \mathfrak{Q}_V^c$ *for which* $\Delta q = \lambda_E M$. *If* U *is an open set in* W,V *is an open set in* U *satisfying the equivalent conditions of 1.3.4 above for* $\mathfrak{P}_V^c$, *and* A *is a relatively compact Borel subset of* V, *the seminorm* $\|\cdot\|_{(A,V)}$ *on* $\Gamma(U, \mathfrak{L})$ *is defined by* $\|M\|_{(A,V)} = |||S_{(A,V)}[M|V]|||$.

PROPOSITION 2.2.5. – *Let* U *be an open set in* W, *and let* $\{(A_i, V_i)\}_{i\in I}$ *be a family of sets with the following properties :*

a) *Each* V_i *is an open subset of* U *for which the equivalent conditions of 1.3.4 above hold for* $\mathfrak{P}_{V_i}^c$;

b) *Each* A_i *is a Borel subset of* $V_i (i \in I)$, *and* $\overline{A}_i$ *is a compact subset of* V_i ;

c) $\{A_i\}_{i\in I}$ *is a disjoint covering of* U, *and there exists a locally finite family of open* $Z_i \supseteq \overline{A}_i (i \in I)$.

Then $\{\|\cdot\|_{(A_i,V_i)}\}_{i\in I}$ *generates a separated, complete metrizable locally convex topology on* $\Gamma(U, \mathfrak{L})$. *If* U *is compact, that topology is normable.*

Proof. – If $0 \neq M \in \Gamma(U, \mathfrak{L})$, then since $M = \sum_{i\in I} \lambda_{A_i} M$ some $\lambda_{A_i} M$ must be nonzero; thus $S_{(A_i,V_i)}M$ is nonzero, and so $\|M\|_{(A_i,V_i)} \neq 0$.

Since $\{Z_i\}_{i\in I}$ is locally finite and U is a locally compact space with a countable basis, I is finite or countably infinite. Consequently, the topology generated by the $\{\|\cdot\|_{(A_i,V_i)}\}_{i\in I}$ is metrizable. Suppose $\{M_n\}_{n=1}^{\infty}$ is a Cauchy sequence in that topology ; then for each $i \in I$ the sequence $\{S_{(A_i,V_i)}M_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathfrak{Q}^c_{V_i}$ and therefore converges to some limit $q_i \in \mathfrak{Q}^c_{V_i}$. For each $j \in I$ the canonical image of $\lambda_{A_j}[S_{(A_i,V_i)}M_n]$ in $\Gamma(V_i, \mathfrak{B})$ is $\lambda_{A_j}\lambda_{A_i}M_n = \delta_{ij}M_n$ (where $\delta_{ij} = 1$ if $i = j$ and $= 0$ otherwise), so $\lambda_{A_j}[S_{(A_i,V_i)}M_n]$ is harmonic in V_i if $i \neq j$ and therefore $\lambda_{A_j}[S_{(A_i,V_i)}M_n] = \delta_{ij}\cdot S_{(A_i,V_i)}M_n$. In the limit, it must also be true that $\lambda_{A_j}q_i = \delta_{ij}q_i$. As a first consequence of that fact, we see that the support of Δq_i, which is a section of $\mathfrak{B}$ in V_i, is contained in $\bar{A}_i$; we may extend Δq_i to U by setting it equal to zero outside $\bar{A}_i$, call the extension N_i, and define $M = \sum_{i\in I} N_i \in \Gamma(U, \mathfrak{B})$ as a locally finite sum.

We claim that the M_n converge to M. Fix an index k. It is clear that $\lambda_{A_j}N_i = \delta_{ij}\cdot N_i$, and thus $\lambda_{A_k}M = \sum_{i\in I}\lambda_{A_k}N_i = N_k$. But then $S_{(A_k,V_k)}M = q_k$ which is $\lim_{n\to\infty} S_{(A_k,V_k)}M_n$ by definition, and so $\|M - M_n\|_{(A_k,V_k)} \longrightarrow 0$. For the final assertion, we observe that if U is compact then I must be finite, Q.E.D.

THEOREM 2.2.6. – *If* U *is an open set in* W, *the topology on* $\Gamma(U, \mathfrak{B})$ *generated by all the seminorms* $\|\cdot\|_{(A,V)}$ *of 2.2.4 above is complete, metrizable, and normable if* U *is compact.*

Proof. – We have just given a construction for a complete metrizable topology in which certain of the seminorms $\|\ \|_{(A,V)}$ are continuous. It is a straightforward consequence of the "restriction" assertions of 2.2.2 above that if $\{(B_j, X_j)\}_{j\in J}$ and $\{(A_i, V_i)\}_{i\in I}$ are families satisfying the hypotheses of 2.2.5 above, such that each X_j is a regular B-set and for each $j \in J$ there exists $i \in I$ with $B_j \subseteq A_i$ and $X_j \subseteq V_i$, then the topology generated by the $\{\|\cdot\|_{(B_j,X_j)}\}_{j\in J}$ is coarser than the topology generated by the $\{\|\cdot\|_{(A_i,V_i)}\}_{i\in I}$. But two comparable complete metrizable topologies are equal by the closed graph theorem [10, Cor. 2, p. 78], so the topology is unchanged by refinement of this type. It is straightforward to verify that any two families satisfying the conditions on $\{(A_i, V_i)\}_{i \in I}$ of 2.2.5 have a common refinement $\{(B_j, X_j)\}_{j \in J}$ satisfying the conditions imposed above, and therefore any two such families $\{(A_i, V_i)\}_{i \in I}$ generate the same topology. Since any (A, V) can be made an element of such a family, that topology is precisely the topology generated by all the $\| \ \|_{(A,V)}$, and we have already seen that it is normable if U is compact, Q.E.D.

We shall simply refer to this topology as "the topology of $\Gamma(U, \mathfrak{L})$" in the remainder of the paper.

PROPOSITION 2.2.7. – *If* V *is an open subset of* U, *the restriction map from* $\Gamma(U, \mathfrak{L})$ *to* $\Gamma(V, \mathfrak{L})$ *is continuous. If* $\Gamma(U, \mathcal{B})$ *is given the topology of uniform convergence on compacta, the map*

$$(f, M) \longrightarrow \Lambda_f M \quad \textit{from} \quad \Gamma(U, \mathcal{B}) \times \Gamma(U, \mathfrak{L})$$

into $\Gamma(U, \mathfrak{L})$ *is continuous. If* $E = \bigcup_{i=1}^{\infty} E_i$ *is a disjoint union of Borel sets, then* $\lambda_E M = \sum_{i=1}^{\infty} \lambda_{E_i} M$ *for any* $M \in \Gamma(U, \mathfrak{L})$, *with convergence of the series in the topology of* $\Gamma(U, \mathfrak{L})$.

Proof. – The first statement is obvious from the way the topologies are defined. For the second statement, one must verify that $(f, M) \longrightarrow S_{(A,V)}[\Lambda_f M]$ is continuous for each pair (A, V) satisfying the specifications of 2.2.4 above. But clearly

$$S_{(A,V)}[\Lambda_f M] = \Lambda_{f \cdot \chi_A}[S_{(A,V)} M]$$

(by uniqueness of lifting), and 2.2.2 states that the right side depends continuously on $(f \cdot \chi_A, S_{(A,V)} M) \in \mathfrak{B}_V \times \mathfrak{L}^c_V$, which in turn depends continuously on (f, M). The proof of the countable additivity assertion is similar.

COROLLARY 2.2.8. – *For any compact set* $K \subseteq U$, *the mapping* $f \longrightarrow \Lambda_{f \cdot \chi_K}[M]$ *of* $\mathcal{C}(U)$ *into* $\Gamma(U, \mathfrak{L})$ *is weakly compact. In particular, if* $W = U$ *and* W *is compact, the mapping* $f \longrightarrow \Lambda_f M$ of $\mathcal{C}(W)$ *into* $\Gamma(W, \mathfrak{L})$ *is weakly compact.*

Proof. – The set function given by sending Borel sets $E \subseteq K$ to $\lambda_E M$ is a countably additive Borel vector measure on K with values in $\Gamma(U, \mathcal{D})$, and clearly $\Lambda_{f \cdot \chi_K}[M]$ is the integral of $f \cdot \chi_K$ with respect to this measure. By the Bartle-Dunford-Schwartz-Grothendieck theorem [5, p. 493], which is just as valid for Fréchet spaces as it is for Banach spaces (as one may prove, e.g., by embedding a given Fréchet space in a countable product of Banach spaces), integration of elements of $\mathcal{C}(K)$ with respect to this measure defines a weakly compact operator from $\mathcal{C}(K)$ to $\Gamma(U, \mathcal{D})$, and of course restricting elements of $\mathcal{C}(U)$ to K is a continuous linear operation.

3. Perturbation of harmonic structures.

3.1. Definition of perturbed sheaves. – Giving a formal definition for the objects of study of this section is easy. Since $\mathcal{R}$ is a subsheaf of $\mathcal{B}$, the following makes perfectly good sense.

DEFINITION 3.1.1. – *Given a fixed* $M \in \Gamma(W, \mathcal{D})$, *let* θ_M : $\mathcal{R} \longrightarrow \mathcal{D}$ *be defined by*

$$\theta_M : g \longrightarrow \Delta g + \Lambda_g M .$$

This is a homomorphism of sheaves ; let $\mathcal{G}(M)$ *denote the subsheaf* Ker θ_M *of* $\mathcal{R}$.

Since $g \longrightarrow \Delta g$ and $g \longrightarrow \Lambda_g M$ are linear, this mapping is a homomorphism of sheaves of vector spaces, and its kernel is thus a sheaf of vector spaces. $\mathcal{G}(M)$ can thus be identified with a complete presheaf of vector spaces of continuous functions, such that each $\mathcal{G}(M)_U$ is a subspace of $\mathcal{R}_U$ for open $U \subseteq W$.

The mapping θ_M can be looked at the presheaf level ; indeed, that is where we shall have to work with it in order to investigate $\mathcal{G}(M)$. The following proposition, which is nothing more than a restatement of 3.1.1 above in terms of the definitions of some of the objects involved, shows us what θ_M looks like at the presheaf level.

PROPOSITION 3.1.2. – *Given a global section* $M \in \Gamma(W, \mathcal{D})$, *let*

U *be an open set such that* $M|U$ *is the canonical image in* $\mathfrak{L}|U$ *of* $p_1 - p_2$, *where each* $p_i \in \mathfrak{P}_U (i = 1,2)$, *and let* $g \in \mathcal{R}_U$ be an element that can be written as $g = s_1 - s_2$ where each s_i is nonnegative, $\mathcal{H}$-superharmonic and bounded ($i = 1,2$). Suppose $h_i (i = 1,2)$ is the greatest $\mathcal{H}$-harmonic minorant of s_i ; then $\theta_M g$ is the canonical image of $(s_1 - h_1) - (s_2 - h_2) + \Lambda_g p_1 - \Lambda_g p_2$ throughout U. In particular, $g \in \mathfrak{G}(M)_U$ if and only if $s_1 - s_2 + \Lambda_g p_1 - \Lambda_g p_2 = g + \Lambda_g (p_1 - p_2)$ is $\mathcal{H}$-harmonic throughout U.

3.2. **Regular sets for $\mathfrak{G}(M)$.** – We now need a number of propositions of a rather technical functional-analytic nature. A fixed notation throughout these propositions will be a considerable convenience, so we shall set one up. Suppose M and N are two given global sections of $\mathfrak{L}$, with $M \geqslant N$. Let X be an open set in W in which both $M|X$ and $N|X$ are the canonical images of elements of $\mathfrak{Q}_X$ and $M - N$ is the canonical image of an element of $\mathfrak{P}_X$; in other words, suppose we can find potentials p_1, p_2 and p_3 in $\mathfrak{P}_X$ such that $M|X$ is the canonical image of $p_1 - p_2$ in $\mathfrak{L}|X$, $N|X$ is the canonical image of $p_1 - p_3$ in $\mathfrak{L}|X$, and $p_3 - p_2$ is a potential. Let $M_i \in \Gamma(X, \mathfrak{L})$ be the section whose value at each point of X is the canonical image of p_i, $1 \leqslant i \leqslant 3$; then $M|X = M_1 - M_2$, $N|X = M_1 - M_3$, and for any real numbers t_1, t_2 the section $t_1 M_1 - t_2 M_2$ is an element of $\Gamma(U, \mathfrak{L})$ whose value is the canonical image of $t_1 p_1 - t_2 p_2$.

With M, N, X, the p_i and the M_i as above, for each regular open B-set $U \subseteq X$ we will denote by K_i^U the "kernel" :

$$K_i^U : f \longrightarrow \Lambda_f (p_i - H(p_i | \partial U, U)) = r_{UX}[\Lambda_f p_i], \ i = 1,2,3.$$

As we observed in 1.7 above, these are continuous nonnegative linear operators on $\mathfrak{B}_U$ that take their values in $\mathcal{C}_0(U)$. Moreover, since the p_i's are continuous potentials, the functions $p_i - H(p_i|\partial U, U)$ can be made uniformly arbitrarily small on $\overline{U}$ by taking U small enough, and therefore the operator norms $\|K_i^U\|$ (relative to the supremum norm on $\mathfrak{B}_U$) can be made arbitrarily small by taking U small enough. We will let $K^U = K_1^U - K_2^U$ and $L^U = K_1^U - K_3^U$. Finally, we shall drop the superscript "U" on the K_i^U, K^U and L^U except in cases where ambiguity is possible.

Before we state the first lemma of this section, we observe that $(I + K_1^U)^{-1}$ exists for any choice of U, by 1.7.4 above ; the assumption that U is a B-set insures that there does exist a nonnegative harmonic function on U that is bounded and bounded away from zero. If U is so small that $\|K_1^U\| < 1$, then the estimate

$$\|(I + K_1^U)^{-1} - I\| \leqslant \|K_1^U\| \cdot (1 - \|K_1^U\|)^{-1}$$

is valid, and so $\|(I + K_1^U)^{-1}\|$ can be made as close to 1 as one pleases by taking U sufficiently small.

LEMMA 3.2.1. – *Suppose* $U \subseteq X$ *is a regular* B-*set so chosen that* $\|K_2^U\| < \|(I + K_1^U)^{-1}\|^{-1}$. *Then* $(I + K^U)^{-1}$ *exists, and moreover, if* u *is a nonnegative continuous bounded* $\mathcal{H}$*-superharmonic function on* U, *then* $(I + K^U)^{-1} u \geqslant 0$. *If* $\|K_3^U\| < \|(I + K_1^U)^{-1}\|^{-1}$ *also, then* $(I + L^U)^{-1} u \geqslant (I + K^U)^{-1} u$ *whenever* u *is a nonnegative continuous bounded* $\mathcal{H}$*-superharmonic function on* U. *Finally, if* U *is sufficiently small that* $|t_2| \cdot \|K_2^U\| < \|(I + t_1 K_1^U)^{-1}\|^{-1}$ *for* (t_1, t_2) *in an open neighborhood of* $[0,1] \times [0,1] \subseteq R^2$, *then for any nonnegative continuous bounded* $\mathcal{H}$*-superharmonic function* u *on* U, $(I + t_1 K_1^U - t_2 K_2^U)^{-1} u$ *is nonnegative and analytic in* (t_1, t_2), *decreasing in* t_1 *and increasing in* t_2.

Proof. – The expansion

$$(I + K_1 - K_2)^{-1} u = (I + K_1)^{-1} (I - K_2 (I + K_1)^{-1})^{-1} u$$

$$= (I + K_1)^{-1} \sum_{n=0}^{\infty} [K_2 (I + K_1)^{-1}]^n u$$

is valid under the hypotheses on K_2. By b) of 1.7.2 above,

$$(I + K_1)^{-1} u \geqslant 0$$

for any nonnegative continuous bounded $\mathcal{H}$-superharmonic function u on U ; moreover, $K_2 [(I + K_1)^{-1} u]$, being the value of K_2 on a nonnegative function, is a nonnegative superharmonic function. Therefore all the iterates $[K_2 (I + K_1)^{-1}]^n u$ are nonnegative superharmonic functions, and the sum of the (uniformly convergent) series is also such a function. Thus $(I + K)^{-1} u = (I + K_1 - K_2)^{-1} u$ is the result of applying $(I + K_1)^{-1}$ to a nonnegative bounded continuous $\mathcal{H}$-superharmonic function, and it is therefore nonnegative. The same considerations are valid with K_2 replaced by K_3, and also apply to all the operators $(I + t_1 K_1 - t_2 K_2)^{-1}$.

Now suppose S and T are linear operators on $\mathcal{C}_b(\mathrm{U})$ for which $(\mathrm{I}+\mathrm{S})^{-1}u$ and $(\mathrm{I}+\mathrm{T})^{-1}u$ are nonnegative whenever u is a nonnegative bounded continuous $\mathcal{H}$-superharmonic function on U. Then the second resolvent equation

$$(\mathrm{I}+\mathrm{S})^{-1}-(\mathrm{I}+\mathrm{T})^{-1}=(\mathrm{I}+\mathrm{T})^{-1}(\mathrm{T}-\mathrm{S})(\mathrm{I}+\mathrm{S})^{-1}$$

tells us that if $\mathrm{T}-\mathrm{S}$ sends nonnegative functions to nonnegative $\mathcal{H}$-superharmonic functions, then $(\mathrm{I}+\mathrm{S})^{-1}u \geqslant (\mathrm{I}+\mathrm{T})^{-1}u$ for any nonnegative $\mathcal{H}$-superharmonic function u. Setting $\mathrm{S}=\mathrm{L}^{\mathrm{U}}$ and $\mathrm{T}=\mathrm{K}^{\mathrm{U}}$, we have $\mathrm{T}-\mathrm{S}=\mathrm{K}_3^{\mathrm{U}}-\mathrm{K}_2^{\mathrm{U}}$, which is the kernel associated with the potential $r_{\mathrm{UX}}[p_3-p_2]$; thus $(\mathrm{I}+\mathrm{L}^{\mathrm{U}})^{-1}u \geqslant (\mathrm{I}+\mathrm{K}^{\mathrm{U}})^{-1}u$ for any nonnegative continuous bounded superharmonic u. Similarly, we see that $(\mathrm{I}+t_1\mathrm{K}_1-t_2\mathrm{K}_2)^{-1}u$ is decreasing in t_1 and increasing in t_2 for (t_1, t_2) in our neighborhood of $[0,1] \times [0,1]$. The analyticity of the $\mathcal{C}_b(\mathrm{U})$-valued function $(\mathrm{I}+t_1\mathrm{K}_1-t_2\mathrm{K}_2)^{-1}u$ is obvious, and that concludes the proof.

It may be worth pointing out here that the fact that the smallness hypothesis on U is vacuously satisfied whenever $\mathrm{K}_2=0$ (so that U can be chosen to be any regular B-set in X) is crucial at certain points below.

PROPOSITION 3.2.2. – *If* $\mathrm{U} \subseteq \mathrm{X}$ *is a regular* B-*set sufficiently small that the hypotheses of 3.2.1 above hold, then for each* $f \in \mathcal{C}(\partial \mathrm{U})$ *there exists a unique* $\mathrm{G}(f,\mathrm{U};\mathrm{M}) \in \mathcal{G}(\mathrm{M})_{\mathrm{U}}$ *possessing a continuous extension to* $\overline{\mathrm{U}}$ *that takes the boundary values f. The linear transformation* $f \longrightarrow \mathrm{G}(f,\mathrm{U};\mathrm{M})$ *of* $\mathcal{C}(\partial\mathrm{U})$ *into* $\mathcal{C}(\mathrm{U})$ *is compact, nonnegative, and differs from* $\mathrm{H}(\cdot,\mathrm{U})$ *by a compact operator from* $\mathcal{C}(\partial\mathrm{U})$ *to* $\mathcal{C}_0(\mathrm{U})$.

Proof. – Fix $f \in \mathcal{C}(\partial\mathrm{U})$ and suppose there does exist a function $g \in \mathcal{R}_{\mathrm{U}}$ satisfying the specifications given for $\mathrm{G}(f,\mathrm{U};\mathrm{M})$. As we observed in 3.1.2 above, the requirement that $\Delta g+\Lambda_g\mathrm{M}=0$ means that $g+\mathrm{K}g$ is harmonic in a neighborhood of each point of U, and thus harmonic in U ; since $g+\mathrm{K}g$ takes the same boundary values as g, we must have $g+\mathrm{K}g=\mathrm{H}(f,\mathrm{U})$. Interpreting that as an equation in $\mathcal{C}_b(\mathrm{U})$ gives $(\mathrm{I}+\mathrm{K})g=\mathrm{H}(f,\mathrm{U})$, or

$$g=(\mathrm{I}+\mathrm{K})^{-1}\mathrm{H}(f,\mathrm{U}),$$

which makes sense since the hypotheses of 3.2.1 are satisfied.

Thus g is uniquely determined by f, and moreover if $f \geq 0$ then $H(f, U)$ is a nonnegative (super-) harmonic function on U, whence $g = (I + K)^{-1}[H(f, U)] \geq 0$ by 3.2.1. On the other hand, setting $G(f, U; M) = (I + K)^{-1}[H(f, U)]$ gives for each $f \in \mathcal{C}(\partial U)$ an element of $\mathcal{R}_U$ that takes the boundary values f ; the fact that $G(f, U; M) + KG(f, U; M) = H(f, U)$ says at the sheaf level that $\Delta G(f, U; M) + \Lambda_{G(f,U;M)} M = 0$, and so $G(f, U; M)$ does belong to $\mathcal{G}(M)_U$.

To prove the compactness assertions, we observe that the relation

$$H(f, U) - G(f, U; M) = (I + K)^{-1} K[H(f, U)]$$

shows that it suffices to prove that $f \longrightarrow K[H(f, U)]$ is a compact mapping from $\mathcal{C}(\partial U)$ to $\mathcal{C}_0(U)$ in order to establish them. As A runs through the upward-directed family of compact subsets of U, $K_i(1 - \chi_A) \longrightarrow 0$ uniformly on $\bar{U}(i = 1,2)$. Therefore for $w \in \mathcal{B}_U$,

$$\|Kw - K[\chi_A \cdot w]\| \leq [\|K_1(1 - \chi_A)\| + \|K_2(1 - \chi_A)\|] \cdot \|w\| .$$

This relation shows that the mappings $f \longrightarrow K[\chi_A \cdot H(f, U)]$ converge to $f \longrightarrow K[H(f, U)]$ in the norm of $\mathcal{L}(\mathcal{C}(\partial U), \mathcal{C}_0(U))$, and since $f \longrightarrow H(f, U)|A$ is compact for each compact A, we are finished.

COROLLARY 3.2.3. – *If* U *is so small that the hypotheses of 3.2.1 regarding* $(I + t_1 K_1 - t_2 K_2)^{-1}$ *are satisfied, then for each* $0 \leq f \in \mathcal{C}(\partial U)$ *the zero-sets of the functions* $H(f, U)$ *and* $G(f; U; M)$ *are the same. In particular, for each* $x_0 \in U$ *the carrier of the representing measure for* $H(\cdot, U)(x_0)$ *is the same as the carrier of the representing measure for* $G(\cdot, U; M)(x_0)$.

Proof. – For any $0 \leq f \in \mathcal{C}(\partial U)$ we know that $(I + t_1 K_1)^{-1}$ applied to $H(f, U)$ decreases with increasing t_1, and thus

$$H(f, U) \geq (I + t_1 K_1)^{-1} H(f, U) \quad \text{for} \quad t_1 \in [0,1 + \varepsilon) .$$

So the zero-set of $G(f, U; M_1) = (I + K_1)^{-1} H(f, U)$ is larger than that of $H(f, U)$. On the other hand, if for $t_1 = 1$ we have

$$[(I + t_1 K_1)^{-1} H(f, U)](x_0) = 0$$

at some point $x_0 \in U$, then since the left-hand side is a decreasing analytic function of t_1 that is always nonnegative, we must have $[(I + t_1K_1)^{-1} H(f, U)](x_0) = 0$ identically in t_1, and so $H(f, U)(x_0) = 0$. Thus $H(f, U)$ and $(I + K_1)^{-1} H(f, U)$ have the same zero-sets. We are now finished if $K_2 = 0$; if not, the fact that

$$(I + K)^{-1} H(f, U) \geqslant (I + K_1)^{-1} H(f, U)$$

shows that $(I + K)^{-1} H(f, U)$ has a smaller zero-set than $H(f, U)$. The same argument applied to the function $(I - t_2K_2)^{-1} H(f, U)$ shows first that $(I - K_2)^{-1} H(f, U)$ has the same zero set as $H(f, U)$ and then that $(I + K)^{-1} H(f, U)$ has a larger zero-set than $H(f, U)$.

Similarly, 3.2.1 gives

COROLLARY 3.2.4. – *For any* $0 \leqslant f \in \mathcal{C}(\partial U)$, $M \geqslant N$ *implies*

$$G(f, U; M) \leqslant G(f, U; N).$$

In particular, $G(f, U; M_1) \leqslant G(f, U; M) \leqslant G(f, U; -M_2)$.

Indeed, $G(f, U; M) = (I + K)^{-1} H(f, U)$

$$\leqslant (I + L)^{-1} H(f, U) = G(f, U; N).$$

THEOREM 3.2.5. – $\mathcal{G}(M)$ *satisfies axioms I through IV of 1.1 above. Moreover, if* $M \geqslant 0$, *the set* "U_x" *of axiom II can be chosen to be a fixed* $\mathcal{H}$*-regular* B*-set, and the basis* "$\mathfrak{U}(x)$" *the family of* $\mathcal{H}$*-regular regions* U *with* $x \in U \subset U_x$, *independently of* M.

Proof. – Axiom I holds by definition of $\mathcal{G}(M)$. To see that axiom II holds, fix $x \in W$, let X be a B-set containing x and contained as a relatively compact subset in an open Z on which the continuous nonnegative superharmonic functions strongly separate points. By 2.1.1 above, there is a unique $S_{(X,Z)}M \in \mathfrak{Q}_Z$ whose canonical image at each point of Z is $\lambda_X M$, and thus whose canonical image at each point of X is M ; clearly $S_{(X,Z)}M \in \mathfrak{P}_Z$ if $M \geqslant 0$, by 1.6.5 above. We can now write $r_{XZ}[S_{(X,Z)}M] = p_1 - p_2$, where p_1 and p_2 belong to $\mathfrak{P}_X$ and $p_2 = 0$ if $M \geqslant 0$, and we are then in the setting established at the beginning of this section 3.2. Now 3.2.2 above supplies us with a basis for the topology of X consisting of sets regular for $\mathcal{G}(M)$, and *a fortiori* with a neighborhood basis for x consisting of

$\mathcal{G}(M)$-regular regions ; all $\mathcal{H}$-regular regions contained in X are admissible if $M \geqslant 0$. Moreover, if we pick U_x as in 3.2.1, an $\mathcal{H}$-regular B-set that is regular for $\mathcal{G}(M)$ (so in particular if $M \geqslant 0$ any such set will do), then there is a positive section of $\mathcal{H}$ defined in a neighborhood of U_x and consequently $H(1, U_x)(y) > 0$ at each $y \in U_x$. By 3.2.3 above, $G(1, U_x; M)(y) > 0$ at each $y \in U_x$ also, and that gives all of axiom II. Axiom III follows immediately from the compactness of the operators $G(\cdot, U; M)$, proved in 3.2.2. To prove that axiom IV holds for $\mathcal{G}(M)$, consider first the case in which $M \geqslant 0$. Then the relation $G(f, U; M) \leqslant H(f, U)$ of 3.2.4 above shows that the nonnegative $\mathcal{H}$-superharmonic functions on U_x, which are known strongly to separate its points, are also $\mathcal{G}(M)$-superharmonic. In the general situation, let M_1 and M_2 be the elements of $\Gamma(X, \mathcal{Q})$ given by the canonical images of p_1 and p_2 respectively, where X, p_1 and p_2 are as they were at the beginning of the proof and $U_x \subseteq X$. We know that the nonnegative $\mathcal{G}(M_1)$-superharmonic functions strongly separate points of U_x and that there is a strictly positive section of $\mathcal{G}(M_1)$ on U_x. By 3.2.4, a nonnegative $\mathcal{G}(M_1)$-harmonic function is $\mathcal{G}(M)$-subharmonic ; therefore a nonnegative $\mathcal{G}(M_1)$-subharmonic function is $\mathcal{G}(M)$-subharmonic, and thus the nonnegative $\mathcal{G}(M)$-subharmonic functions on U_x strongly separate points of U_x. Since there exists a strictly positive section of $\mathcal{G}(M)$ on U_x, the nonnegative $\mathcal{G}(M)$-superharmonic functions strongly separate points of U_x, Q.E.D.

Since $\mathcal{G}(M)$ satisfies the axioms, the minimum principle is valid for $\mathcal{G}(M)$, and so it is meaningful to talk about super- and subharmonic functions using only bases of regular sets. The argument we just gave to show that there was a relation between $\mathcal{G}(M_1)$- and $\mathcal{G}(M)$-super- and subharmonic functions then establishes the following corollary, whose details of proof we omit.

COROLLARY 3.2.6. – *If* $M \geqslant N$, *then every nonnegative* $\mathcal{G}(M)$-*subharmonic function is* $\mathcal{G}(N)$-*subharmonic and every nonnegative* $\mathcal{G}(N)$-*superharmonic function is* $\mathcal{G}(M)$-*superharmonic. The presheaf of local differences of superharmonic functions ("*$\mathcal{R}$*") associated with* $\mathcal{G}(M)$ *is identical with the presheaf* $\mathcal{R}$ *of local differences of* $\mathcal{H}$-*superharmonic functions.*

3.3. Further properties of θ_M ; existence of $\mathcal{G}$(M)-potentials.

PROPOSITION 3.3.1. — *θ_M is an epimorphism of sheaves. A function $g \in \mathcal{R}_X$ (X an open subset of W) is $\mathcal{G}$(M)-superharmonic whenever $\theta_M g$ is a nonnegative element of $\Gamma(X, \mathcal{S})$, and if $M \geqslant 0$ this condition is also necessary.*

Proof. — To prove that θ_M is an epimorphism it will suffice to show that given X, etc., as at the beginning of 3.2 above, and a potential $p \in \mathcal{P}_X$, for any U satisfying the hypotheses of 3.2.1 above we can find $g \in \mathcal{R}_U$ with $g + Kg = p$ on U. But then we need only take

$$g = (I + K)^{-1} p = p - K(I + K)^{-1} p = p - K_1 (I + K)^{-1} p + K_2 (I + K)^{-1},$$

for $(I + K)^{-1}$ sends nonnegative superharmonic functions to nonnegative functions and the K_i send nonnegative functions to nonnegative superharmonic functions.

For the second assertion, suppose $\theta_M g$ is a nonnegative section of $\mathcal{S}$. Then for all sufficiently small U satisfying the hypotheses of 3.2.1 above, the function $g + Kg - H(g | \partial U, U)$ is a potential on U. By 3.2.1,

$$0 \leqslant (I + K)^{-1} [g + Kg - H(g | \partial U, U)] = g - (I + K)^{-1} [H(g | \partial U, U)] = g - G(g | \partial U, U; M),$$

and since the U's satisfying those hypotheses form a basis for the topology of X, g is $\mathcal{G}$(M)-superharmonic. Conversely, suppose $M \geqslant 0$ and g is $\mathcal{G}$(M)-superharmonic, so that we know that

$$0 \leqslant g - G(g | \partial U, U; M) = g - (I + K)^{-1} [H(g | \partial U, U)]$$

for all U satisfying the hypotheses of 3.2.1. Since $M \geqslant 0$, $K = K_1$ is a nonnegative operator, and *a fortiori* so is $I + K$. Hence

$$0 \leqslant (I + K)g - H(g | \partial U, U),$$

and since Kg has boundary values zero,

$$0 \leqslant (g + Kg) - H([g + Kg] | \partial U, U).$$

For any regular $V \subseteq U$ we have $K^V g = K^U g - H(K^U g | \partial V, V)$, and thus we also see that

$$0 \leqslant (g + K^V g) - H([g + K^V g] | \partial V, V) = (g + K^U g) - H([g + K^U g] | \partial V, V),$$

showing that $g + K^U g$ is $\mathcal{H}$-superharmonic in U and thus that $\Delta g + \Lambda_g M \geqslant 0$, Q.E.D.

COROLLARY 3.3.2.

$$0 \longrightarrow \mathcal{G}(M) \longrightarrow \mathcal{R} \xrightarrow{\theta_M} \mathcal{Q} \longrightarrow 0$$

is a fine resolution of $\mathcal{G}(M)$. *One thus has the cohomology exact sequence*

$$0 \longrightarrow \Gamma(W, \mathcal{G}(M)) \longrightarrow \Gamma(W, \mathcal{R}) \xrightarrow{\theta_M} \Gamma(W, \mathcal{Q}) \xrightarrow{\delta} H^1(W, \mathcal{G}(M)) \longrightarrow 0 .$$

Proof. – That $\mathcal{R} \longrightarrow \mathcal{Q} \longrightarrow 0$ is exact is precisely the fact that θ_M is an epimorphism of sheaves, and the cohomology exact sequence follows from a universal property of fine resolutions [6, 2. Lemma, p. 177].

COROLLARY 3.3.3. – *For any* $0 < g \in \Gamma(W, \mathcal{R})$ *there exists an* $M_0 \in \Gamma(W, \mathcal{Q})$ *for which* g *is* $\mathcal{G}(M_0)$*-superharmonic, and* M_0 *may be taken greater than any preassigned section of* $\mathcal{Q}$. *In particular,* M_0 *can be found for which* g *is* $\mathcal{G}(M_0)$*-superharmonic but not* $\mathcal{G}(M_0)$*-harmonic on any open set.*

Proof. – By the proposition it will suffice to find $M_0 \geqslant 0$ for which $\Delta g + \Lambda_g M_0 > 0$, and that is equivalent to $\Lambda_g M_0 > -\Delta g$, or to $M_0 > -\Lambda_{1/g}(\Delta g)$. Since we can write $\Delta g = M_1 - M_2$ with the $M_i \geqslant 0$ $(i = 1,2)$ and find M_3 that is strictly positive, it will suffice to take $M_0 \geqslant M_3 + \Lambda_{1/g}(M_1 + M_2)$.

THEOREM 3.3.4. – *If* W *is compact, then for all sufficiently large* $M \geqslant 0$ *there is a positive continuous* $\mathcal{G}(M)$*-potential on* W.

Proof. – For each $M \geqslant 0$, let A_M denote the intersection of the zero-sets of the continuous $\mathcal{G}(\mathrm{M})$-potentials. By the construction used in 1.3.4 above (take a sequence $\{p_n\}_{n=1}^{\infty}$ of continuous $\mathcal{G}(\mathrm{M})$-potentials that is uniformly dense in the cone of all such potentials, and choose multipliers $\alpha_n > 0$ for which $\sum_{n=1}^{\infty} \alpha_n p_n$ converges uniformly on W) it is easy to see that $\mathrm{A_M}$ is itself a zero-set. If $\mathrm{M} \geqslant \mathrm{N}$, then every nonnegative $\mathcal{G}(\mathrm{N})$-superharmonic function is $\mathcal{G}(\mathrm{M})$-superharmonic, and every nonnegative $\mathcal{G}(\mathrm{M})$-harmonic function $\mathcal{G}(\mathrm{N})$-subharmonic ; thus every $\mathcal{G}(\mathrm{N})$-potential is a $\mathcal{G}(\mathrm{M})$-potential, and so $\mathrm{A_M} \subseteq \mathrm{A_N}$. Thus the correspondence $\mathrm{M} \longrightarrow \mathrm{A_M}$ from the cone of nonnegative global sections of $\mathcal{Z}$ to closed subsets of W is decreasing, and to prove the assertion of the theorem it will thus suffice to show that $\cap \{\mathrm{A_M} : 0 \leqslant \mathrm{M} \in \Gamma(\mathrm{W}, \mathcal{Z})\} = \emptyset$, by the finite intersection property.

Suppose otherwise, and call the intersection A. If x_1 and x_2 are distinct points of A, an easy application of 1.5.3 above shows that we can find positive functions g_1 and g_2 in $\Gamma(\mathrm{W}, \mathcal{R})$ that strongly separate them ; for suitably chosen $\mathrm{M_0}$ we can be sure that $\mathcal{G}(\mathrm{M_0})$ admits a positive superharmonic g_0, and for suitably chosen $\mathrm{M_1} \geqslant \mathrm{M_0}$, g_1 and g_2 will also be $\mathcal{G}(\mathrm{M_1})$-super armonic. Let $g_1 = h_1 + p_1$ and $g_2 = h_2 + p_2$ be the decompositions of g_1 and g_2 into their $\mathcal{G}(\mathrm{M_1})$-harmonic and $\mathcal{G}(\mathrm{M_1})$-potential parts, respectively. Since both p_1 and p_2 vanish on A, it must be the case that h_1 and h_2 strongly separate x_1 and x_2. Both h_1 and h_2 are $\mathcal{G}(\mathrm{M_0})$-subharmonic by 3.2.6, and since $\mathcal{G}(\mathrm{M_0})$ possesses a positive superharmonic function, they have $\mathcal{G}(\mathrm{M_0})$-superharmonic majorants. Hence they have least $\mathcal{G}(\mathrm{M_0})$-harmonic majorants f_1 and f_2 respectively. The functions $f_1 - h_1$ and $f_2 - h_2$ are $\mathcal{G}(\mathrm{M_0})$-potentials on W ; hence they vanish on A, and since $h_i = f_i - (f_i - h_i)$ $(i = 1,2)$, the functions f_1, f_2 strongly separate x_1 and x_2. This shows that the $\mathcal{G}(\mathrm{M_0})$-harmonic functions on W strongly separate points of A, since the matter is trivial if A has only one point.

We claim that the restriction of $\Gamma(\mathrm{W}, \mathcal{G}(\mathrm{M_0}))$ to A is a sublattice of $\mathcal{C}(\mathrm{A})$. To see this, observe that since $\mathcal{G}(\mathrm{M_0})$ possesses a positive superharmonic function g_0 whose potential part must vanish on A, there exists an $0 \leqslant h_0 \in \Gamma(\mathrm{W}, \mathcal{G}(\mathrm{M_0}))$ that does not vanish on A. Thus to prove our claim it suffices to show that if $f, g \in \Gamma(\mathrm{W}, \mathcal{G}(\mathrm{M_0}))$ are nonnegative, then $f \wedge g$ is the restriction to A of an element of

$\Gamma(W, \mathcal{G}(M_0))$. But $f \wedge g = h + p$ for uniquely determined

$$h \in \Gamma(W, \mathcal{G}(M_0))$$

and $\mathcal{G}(M_0)$-potential p ; since p vanishes on A, $h|A = (f \wedge g)|A$.

Since $\Gamma(W, \mathcal{G}(M_0))|A$ is a strongly separating sublattice of $\mathcal{C}(A)$ and $\Gamma(W, \mathcal{G}(M_0))$ is finite-dimensional (an easy consequence of axiom III), the Stone-Weierstrass theorem shows that $\mathcal{C}(A)$ is finite-dimensional ; A must thus be finite. We can thus take a point $x_0 \in A$ and find an $\mathcal{H}$-regular B-set U that contains x_0 but has no other points of A in a neighborhood of its closure. Since

$$A \cap \overline{U} = \cap \{A_M \cap \overline{U} : M_0 \leqslant M \in \Gamma(W, \mathcal{Q})\} = \{x_0\},$$

for sufficiently large $M \geqslant M_0$ one must have $A_M \cap \overline{U} \subseteq U$. Take such an M, and let p be a $\mathcal{G}(M)$-potential on W whose zero-set is A_M ; then $p|\overline{U}$ is zero only at interior points of U. However, U is a regular set for $\mathcal{G}(M)$ on which the nonnegative $\mathcal{G}(M)$-superharmonic functions separate points and a strictly positive $\mathcal{G}(M)$-harmonic function exists, by 3.2.5 above ; so we have a contradiction to the minimum principle for $\mathcal{G}(M)$, Q.E.D.

Remark. – The situation is somewhat simpler in the subelliptic case. Indeed, if $\mathcal{G}(M_0)$ admits a positive superharmonic nonharmonic function g_0, then its potential part p_0 is a nonzero superharmonic function on W and therefore is strictly positive on W, by 1.8.2. In particular, the compactness of W, which is used so strongly in the proof above, is not needed.

4. The index-zero theorem and related results.

The assumption that the base space W is compact is hereby made, once and for all, for this entire section.

4.1. The index-zero theorem.

LEMMA 4.1.1. – *The space $\Gamma(W, \mathcal{R})$ has a unique Fréchet-space topology finer than the topology of pointwise convergence (and* *thus finer than the uniform norm topology). If* M *is a nonnegative section of* $\Gamma(W, \mathfrak{L})$ *so large that* $\mathfrak{G}(M)$ *admits strictly positive potentials, and if* $\mathfrak{D}_W(M)$ *denotes the space of differences of* $\mathfrak{G}(M)$*-potentials on* W, *topologized as in 2.2.2 above, then the Fréchet-space topology of* $\Gamma(W, \mathcal{R})$ *is that it receives under identification with* $\mathfrak{D}_W(M)$, *so* $\Gamma(W, \mathcal{R})$ *is a Banach space.*

Proof. – The assertions of the first sentence above are immediate consequences of the closed-graph theorem : the identity mapping between any two topologies satisfying the description of the lemma is necessarily closed and thus bicontinuous, and the mapping from such a topology to the uniform-norm topology is necessarily closed. It remains to prove the existence of such a topology. By 2.1.2 above (applied to $\mathfrak{G}(M)$), $\mathfrak{D}_W(M) = \Gamma(W, \mathcal{R})$, and the $|||\cdot|||$-topology on $\mathfrak{D}_W(M)$ is a Banach-space topology which is finer than the uniform-norm topology by 2.2.2 above, Q.E.D.

LEMMA 4.1.2. – *Let* M *be a nonnegative section of* $\mathfrak{L}$ *on* W, *so large that* $\mathfrak{G}(M)$ *admits strictly positive potentials. If* $\Gamma(W, \mathcal{R})$ *is given its Banach-space topology and* $\Gamma(W, \mathfrak{L})$ *given its usual topology, then the mapping* $\theta_M : \Gamma(W, \mathcal{R}) \longrightarrow \Gamma(W, \mathfrak{L})$ *is a topological isomorphism.*

Proof. – By 2.1.2 above, again applied to $\mathfrak{G}(M)$, $\Gamma(W, \mathfrak{G}(M)) = 0$ and $H^1(W, \mathfrak{G}(M)) = 0$. By the cohomology exact sequence of 3.3.2 above, $\theta_M : \Gamma(W, \mathcal{R}) \longrightarrow \Gamma(W, \mathfrak{L})$ is 1-1 and onto. It is a consequence of the closed graph theorem that the following condition is sufficient for the bicontinuity asserted by the lemma : if $\{r_n\}_{n=1}^{\infty}$ is a sequence in $\Gamma(W, \mathcal{R})$ converging to $r \in \Gamma(W, \mathcal{R})$ in the Banach-space topology of $\Gamma(W, \mathcal{R})$, $\{N_n\}_{n=1}^{\infty}$ is the sequence in $\Gamma(W, \mathfrak{L})$ defined by $N_n = \theta_M r_n$, and $N_n \longrightarrow 0$ in the topology of $\Gamma(W, \mathfrak{L})$, then $r = 0$. To see that this condition holds, fix $x \in W$, let U be a regular B-region containing x, let V be an open neighborhood of x with compact closure contained in U, and for each regular X with $x \in X$ and $\overline{X} \subseteq V$, let K^X be the kernel on X defined by $r_{XV}[S_{(V,U)}M]$. It is easy to check that the Laplacian of

$$r_n - H(r_n | \partial X, X) + K^X r_n - r_{XV}[S_{(V,U)} N_n]$$

is zero throughout X, and that it continuously takes the boundary values zero ; thus it is identically zero in X, i.e.

$$r_n - \mathrm{H}(r_n \mid \partial \mathrm{X}, \mathrm{X}) + \mathrm{K}^{\mathrm{X}} r_n = r_{\mathrm{XV}}[\mathrm{S}_{(\mathrm{V},\mathrm{U})} \mathrm{N}_n]$$

throughout X. The left side depends continuously on r_n in the uniform norm on $\mathcal{C}_0(\mathrm{X})$, and by 2.2.7 the right side converges to zero in a topology stronger than uniform convergence on X, as $n \longrightarrow \infty$ and $\mathrm{N}_n \longrightarrow 0$. Thus in the limit we have

$$r - \mathrm{H}(r \mid \partial \mathrm{X}, \mathrm{X}) + \mathrm{K}^{\mathrm{X}} r = 0$$

and since X was an arbitrary regular neighborhood of x, the value of $\theta_{\mathrm{M}} r$ at x is zero. Since x was arbitrary, r is a $\mathfrak{G}(\mathrm{M})$-harmonic function on W ; but $\Gamma(\mathrm{W}, \mathfrak{G}(\mathrm{M})) = 0$, so $r = 0$, Q.E.D.

LEMMA 4.1.3. – *Let* E *be a Banach space, and let* $\mathrm{S} \in \mathcal{L}(\mathrm{E})$ *be factorable through a space* $\mathcal{C}(\mathrm{X})$, *where* X *is a compact Hausdorff space. If* $\mathrm{S} = \mathrm{TU}$, *where* $\mathrm{U} \in \mathcal{L}(\mathrm{E}, \mathcal{C}(\mathrm{X}))$ *is weakly compact and* $\mathrm{T} \in \mathcal{L}(\mathcal{C}(\mathrm{X}), \mathrm{E})$ *is continuous, then* S^2 *is compact, and* $\mathrm{I} - \mathrm{S}$ *is a Fredholm operator of index zero (i.e., the dimension of its null space and the codimension of its range are finite and equal).*

Proof. – Write $\mathrm{S}^2 = \mathrm{T}(\mathrm{UT})\mathrm{U}$. Since U sends the unit ball of E to a relatively weakly compact set in $\mathcal{C}(\mathrm{X})$ and UT sends weakly compact sets in $\mathcal{C}(\mathrm{X})$ to strongly compact sets [5, Thm. 4, p. 494], S^2 is compact. The assertion that $\mathrm{I} - \mathrm{S}$ is Fredholm of index zero is a paraphrase of [5, Thm. 6, p. 579] ; in the context of [5] it follows from that theorem and the Fredholm alternative [5, p. 609]. Another way of looking at this assertion is the following : since

$$(\mathrm{I} - t\mathrm{S})(\mathrm{I} + t\mathrm{S}) = \mathrm{I} - t^2 \mathrm{S}^2 ,$$

$\mathrm{I} - t\mathrm{S}$ is invertible modulo compact operators on E for every t. Thus $\mathrm{I} - t\mathrm{S}$ is Fredholm for every t, and moreover belongs to the same component of the Fredholm operators as I. Since I has index zero, $\mathrm{I} - \mathrm{S}$ has index zero, i.e. the null space and the cokernel of $\mathrm{I} - \mathrm{S}$ have the same dimension.

THEOREM 4.1.4 *(index-zero theorem). – For any* $\mathrm{M} \in \Gamma(\mathrm{W}, \mathfrak{D})$, *the dimensions of the vector spaces* $\Gamma(\mathrm{W}, \mathfrak{G}(\mathrm{M})) = \mathrm{H}^0(\mathrm{W}, \mathfrak{G}(\mathrm{M}))$ *and* $\mathrm{H}^1(\mathrm{W}, \mathfrak{G}(\mathrm{M}))$ *are finite and equal.*

Proof. – Take $\mathrm{M}_0 \in \Gamma(\mathrm{W}, \mathfrak{D})$ with $\mathrm{M}_0 \geqslant \mathrm{M}$ and so large that

$\mathcal{G}(M_0)$ admits strictly positive potentials. Define the linear transformation $V : \Gamma(W, \mathcal{R}) \longrightarrow \Gamma(W, \mathcal{Q})$ by setting $Vr = \Lambda_r[M_0 - M]$. Since V can be factored into the composition of the "identity" mapping of $\Gamma(W, \mathcal{R})$ into $\mathcal{C}(W)$ and the mapping $r \longrightarrow \Lambda_r[M_0 - M]$ of $\mathcal{C}(W)$ into $\Gamma(W, \mathcal{Q})$, and the latter mapping is weakly compact by 2.2.8 above, V can be factored through a weakly compact linear transformation defined on the space $\mathcal{C}(W)$, a space of continuous functions on a compact Hausdorff space. If we use θ_{M_0} (a topological isomorphism by 4.1.2 above) to identify $\Gamma(W, \mathcal{R})$ and $\Gamma(W, \mathcal{Q})$, then θ_M is identified with $\theta_{M_0}^{-1}[\theta_{M_0} - V] = I - \theta_{M_0}^{-1}V$. The mapping $\theta_{M_0}^{-1}V$ satisfies the hypotheses for "S" of 4.1.3 above, so θ_M is identified with a Fredholm operator of index zero : its null space is precisely $\Gamma(W, \mathcal{G}(M))$, and its range is a subspace of $\Gamma(W, \mathcal{R})$ of finite codimension, equal to the dimension of $\Gamma(W, \mathcal{G}(M))$. Applying θ_{M_0} again maps $\Gamma(W, \mathcal{R})$ onto $\Gamma(W, \mathcal{Q})$ and the range of $\theta_{M_0}^{-1}\theta_M$ onto $\theta_M[\Gamma(W, \mathcal{R})]$, so

$$\dim H^1(W, \mathcal{G}(M)) = \dim [\Gamma(W, \mathcal{Q})/\theta_M[\Gamma(W, \mathcal{R})]] =$$
$$= \dim H^0(W, \mathcal{G}(M)) = \dim \Gamma(W, \mathcal{G}(M)),$$

as desired (the first equality follows from the cohomology exact sequence of 3.3.2 above), Q.E.D.

In this essentially perturbation-theoretic setting, it may be appropriate to state the following corollary.

COROLLARY 4.1.5. – *In the setting of 4.1.4 above, for any* $N \in \Gamma(W, \mathcal{Q})$ *the dimensions of the spaces* $\Gamma(W, \mathcal{G}(M + tN))$ *and/or* $H^1(W, \mathcal{G}(M + tN))$ *are constant on* R *(as functions of t) except at a discrete set of points.*

Proof. – Observe that the section $M \in \Gamma(W, \mathcal{Q})$ of the theorem is arbitrary, so that for all real t, the operator :

$$\theta_{M+tN} : \Gamma(W, \mathcal{R}) \longrightarrow \Gamma(W, \mathcal{Q})$$

is a Fredholm operator of index zero. One can admit complex t (at least for small imaginary parts, which is all we need) simply by complexifying all the Banach spaces involved. By [8, Thm. 5.31,

p. 241] the dimensions of the null space $\Gamma(W, \mathfrak{G}(M + tN))$ and the cokernel $\Gamma(W, \mathfrak{D})/\theta_{M+tN}[\Gamma(W, \mathcal{R})] \cong H^1(W, \mathfrak{G}(M + tN))$ of θ_{M+tN} are constant in a punctured neighborhood in the complex plane of each t, and therefore are constant on R except at a discrete set of points, Q.E.D.

COROLLARY 4.1.6. – *Let* $N \geqslant 0$ *be such that* $\mathfrak{G}(M + tN)$ *admits a strictly positive potential for some t. Then* $\Gamma(W, \mathfrak{G}(M + tN)) = 0$ *except for a discrete set of values of t, and for all sufficiently large t.*

Proof. – If $\mathfrak{G}(M + t_1N)$ admits a strictly positive potential, then that potential is also a potential for $\mathfrak{G}(M + tN)$ with $t > t_1$, and so $\Gamma(W, \mathfrak{G}(M + tN)) = 0$ for $t > t_1$, by the minimum principle. The assertion then follows from 4.1.5 above.

4.2. Flux and duality. – All of the results of [11, § 4] and [12, § 3, Prop. 3.8 ff. and § 4] concerning the determination of $H^1(W, \mathcal{H})$ can be deduced from the results above. Let us recall the definition of the inductive topology of $H^1(W, \mathcal{H})$ [12, Def. 2.4] :

DEFINITION 4.2.1. – *For each Cousin pair* (A, U) *in* W, *let* $j_{(A,U)} : \mathcal{H}_{U\setminus A} \longrightarrow H^1(W, \mathcal{H})$ *be the natural linear mapping of* $\mathcal{H}_{U\setminus A}$, *identified with* $Z^1(N(\{W\setminus A, U\}), \mathcal{H})$, *into* $H^1(W, \mathcal{H})$. *The inductive topology on* $H^1(W, \mathcal{H})$ *is the inductive (locally convex) topology [10, p. 54] generated by the Fréchet spaces* $\mathcal{H}_{U\setminus A}$ *and the linear mappings* $j_{(A,U)}$ *as* (A, U) *ranges over all Cousin pairs in* W *for which* U *is a* B-*set.*

The proof of [12, Prop. 3.8] is valid in the present setting ; it gives us a proof of the following proposition :

PROPOSITION 4.2.2 : The conditions

a) $H^1(W, \mathcal{H})$ *is a Hausdorff* LTS *in the inductive topology* ;

b) $H^1(W, \mathcal{H})$ *is finite-dimensional* ;

c) *There is a cover* $\mathfrak{U}$ *of* W *for which the natural map*

$$H^1(N(\mathfrak{U}), \mathcal{H}) \longrightarrow H^1(W, \mathcal{H})$$

is surjective

hold in the present setting

Indeed, the proof that b) $\Longrightarrow$ c) $\Longrightarrow$ a) given in [12, Prop. 3.8] is just as valid with the present assumptions as it was there, and we know that b) holds by 4.1.4 above. Moreover, since the Hausdorff topology on a finite-dimensional vector space is unique, we have the following corollary.

COROLLARY 4.2.3. – *The inductive topology on* $H^1(W, \mathcal{H})$ *is identical with the quotient topology of* $\Gamma(W, \mathcal{Q})/\Delta[\Gamma(W, \mathcal{R})]$. *In particular, if* F *is an element of the dual space of* $H^1(W, \mathcal{H})$, *the linear functional* $j_{(A,U)} \circ F$ *that it induces on* $\mathcal{H}_{U\setminus A}$, *where* (A, U) *is a Cousin pair in* W *for which* U *is a* B-*set, is continuous in the topology of uniform convergence on compacta.*

The "dimensionality" results of [11, § 4] can be derived immediately. The special rôle of the function 1 disappears.

PROPOSITION 4.2.4. – *Suppose* $(W, \mathcal{H})$ *is subelliptic and that there exists a nonzero nonnegative* $\mathcal{H}$-*superharmonic continuous function defined on all of* W. *If this function is not harmonic, then* $\Gamma(W, \mathcal{H}) = 0$ *and* $H^1(W, \mathcal{H}) = 0$; *if it is harmonic,*

$$\dim \Gamma(W, \mathcal{H}) = \dim H^1(W, \mathcal{H}) = 1 .$$

Proof. – If h and p are the harmonic and potential parts of the nonzero nonnegative continuous superharmonic function given in the hypotheses, then exactly one of h and p is nonzero : if $p \neq 0$ then it is strictly positive on W, by 1.8.2 above, so a suitable multiple of p majorizes h and $h = 0$. If $p > 0$, the fact that $\Gamma(W, \mathcal{H}) = 0$ and $H^1(W, \mathcal{H}) = 0$ is Theorem 2.1.2 above. If $h \neq 0$, then h is strictly positive on W by 1.8.2 above, and an easy application of 1.8.2 also shows that h generates $\Gamma(W, \mathcal{H})$; thus $\dim \Gamma(W, \mathcal{H}) = 1$ and so $\dim H^1(W, \mathcal{H}) = 1$ by 4.1.4 above, Q.E.D.

The flux functionals of [11, § 3] are probably of less interest in this context than they are in the context of normal structures, so we will only give a brief description of their construction and the verification of their properties in the present setting. First we observe that whenever $\mathcal{H}$ is subelliptic and $\mathcal{H}_W$ has a positive generator, the dual space of $H^1(W, \mathcal{H})$ also has a positive generator in a suitable sense. Indeed, let F be a generator of the dual space of $H^1(W, \mathcal{H})$; since $H^1(W, \mathcal{H}) \cong \Gamma(W, \mathcal{Q})/\Delta[\Gamma(W, \mathcal{R})]$ and the codimension of

$\Delta[\Gamma(W, \mathcal{R})]$ is 1, F can be thought of as a linear functional on $\Gamma(W, \mathcal{Q})$ whose null space is precisely $\Delta[\Gamma(W, \mathcal{R})]$. Without loss of generality one can take F in such a way that $F(M) > 0$ for some $0 \leqslant M \in \Gamma(W, \mathcal{Q})$; but we claim that then F is a strictly positive functional on $\Gamma(W, \mathcal{Q})$. Indeed, if $0 \leqslant N \in \Gamma(W, \mathcal{Q})$ but $F(N) \leqslant 0$, then for suitable $\beta \geqslant 0$ one will have $F(\beta M + N) = 0$. Since the null space of F is $\Delta[\Gamma(W, \mathcal{R})]$, one can find $g \in \Gamma(W, \mathcal{R})$ for which $\Delta g = \beta M + N$. But $\mathcal{H}_W$ has a strictly positive generator, and by adding a suitable multiple of it to g (which does not disturb Δg) we could cause g to be nonnegative and possess a zero. But g is superharmonic, because $\Delta g = \beta M + N \geqslant 0$; thus $g = 0$ by 1.8.2, and so $\beta M + N = 0$. This says that $N = -\beta M \leqslant 0$, so $N = 0$, and that proves that F is strictly positive.

If we now define a flux functional by setting $\Psi_{(A,U)}[s] = F[j_{(A,U)}s]$ for each Cousin pair (A , U) and each $s \in \mathcal{H}_{U \setminus A}$, one can verify the properties of [11, Thm. (3.8)]. Indeed, (1) and (2) of that theorem hold automatically, so it suffices to check (3) (reference to a normal structure in the theorem is vacuous in the present context). To do this it suffices to show that if (A , U) is a Cousin pair and s a superharmonic function on U with support contained in A which is continuous on U, then $\Psi_{(A,U)}[s \mid U \setminus A] \geqslant 0$, with equality if and only if s is harmonic in U. By the definition of $j_{(A,U)}$ it is easy to see that $j_{(A,U)}[s \mid U \setminus A]$ is the coset of $\Delta[\Gamma(W, \mathcal{R})]$ in $\Gamma(W, \mathcal{Q})$ to which the section that is Δs in U and zero outside A belongs ; therefore, $\Psi_{(A,U)}[s \mid U \setminus A] = F[j_{(A,U)}[s \mid U \setminus A]] = F[\Delta s] \geqslant 0$, with equality iff $\Delta s = 0$, by the strict positivity of F --and that is what one wants. Similarly one can check [11, Cor. 3.10] : (1) is the continuity of $j_{(A,U)} \circ F$ proved in 4.2.3 above (reference to $\Psi^c_{(A,U)}$ is vacuous in the present context), (2) is irrelevant in the absence of a normal structure, and (3) is trivial since F is already defined on $H^1(W, \mathcal{H})$ and therefore induces a well-defined element of the dual of $H^1(N(\mathfrak{U}), \mathcal{H})$ for every cover $\mathfrak{U}$ of W. To conclude the treatment of flux functionals, we observe that if the present flux functional is used to make the definition [11, Def. (4.2)] of the total charge distribution of an element of $\Gamma(W, \mathcal{Q})$, then it is easy to see that for any $M \in \Gamma(W, \mathcal{Q})$ and Borel set $E \subseteq W$, $(\tau M)(E) = F[\lambda_E M]$. The countable additivity of τM then follows directly from the countable additivity of

$$E \longrightarrow \lambda_E M \in \Gamma(W, \mathcal{Q})$$

proved in 2.2.7 above. Since $\int 1 \, d(\tau M) = F[\lambda_W M] = F[M]$ and the null space of F is $\Delta[\Gamma(W, \mathcal{R})]$, a necessary and sufficient condition for M to belong to $\Delta[\Gamma(W, \mathcal{R})]$ is that $\int 1 \, d(\tau M) = 0$, just as in [11, Thm. (4.3)].

Finally, we consider the relation between the results of the present paper and the theory of adjoint sheaves of [7]. Suppose, therefore, that $(W, \mathcal{H})$ satisfies the Brelot axioms [3] and the hypotheses of the adjoint-sheaf theory [7] locally ; then a global adjoint sheaf $\mathcal{H}^*$ for $\mathcal{H}$ is available, as in [12], and the results of the last-named paper can be employed. As 4.2.2 above observes, the equivalent conditions of [12, Prop. 3.8] hold as a consequence of 4.1.4 above. By [12, Prop. 2.5], since $H^1(W, \mathcal{H})$ is Hausdorff in the inductive topology, it and $\Gamma(W, \mathcal{H}^*)$ are identifiable with each other's dual spaces. In particular, these spaces have the same dimension, and so we have the correct axiomatic version of the classical index-zero theorem :

THEOREM 4.2.5. – *If* $(W, \mathcal{H})$ *satisfies locally the assumptions of the Hervé adjoint-sheaf theory and* $\mathcal{H}^*$ *is the global adjoint sheaf of* $\mathcal{H}$ *(as in* [12, § 1]*), then* $\dim \Gamma(W, \mathcal{H}) = \dim \Gamma(W, \mathcal{H}^*)$.

Indeed, $\Gamma(W, \mathcal{H}^*)$ and $H^1(W, \mathcal{H})$ are each other's duals, and $\dim H^1(W, \mathcal{H}) = \dim \Gamma(W, \mathcal{H})$ by 4.1.4 above.

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