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EXAMPLES ON LOEWY FILTRATIONS AND K-STABILITY OF FANO VARIETIES WITH NON-REDUCTIVE AUTOMORPHISM GROUPS

by Atsushi ITO (*)

Abstract. — It is known that the automorphism group of a K-polystable Fano manifold is reductive. Codogni and Dervan constructed a canonical filtration of the section ring, called Loewy filtration, and conjectured that the filtration destabilizes any Fano variety with non-reductive automorphism group. In this note, we give a counterexample to their conjecture.

Résumé. — Il est connu que le groupe d'automorphismes d'une variété de Fano K-polystable est réductif. Codogni et Dervan ont construit une filtration canonique de l'anneau des sections, appelée filtration de Loewy, et ont conjecturé que la filtration déstabilise n'importe quelle variété de Fano avec le groupe d'automorphismes non réductif. Dans cette note, nous fournissons un contre-exemple à leur conjecture.

1. Introduction

For a Fano manifold X over $\mathbb{C}$, it is known that X admits Kähler–Einstein metrics if and only if X is K-polystable [2, 3, 4, 5, 6, 14, 18, 19, 20]. The K-polystability of X is defined by using the Donaldson–Futaki invariant $\mathrm{DF}(\mathcal{X},\mathcal{L})$ of a test configuration $(\mathcal{X},\mathcal{L})$ of X. Roughly, X is called *K-polystable* if $\mathrm{DF}(\mathcal{X},\mathcal{L}) \geqslant 0$ for any test configuration of X, and equality holds only for a special type of test configurations, called *of product type*. On the other hand, Matsushima [15] shows that if X admits Kähler–Einstein metrics then the automorphism group $\mathrm{Aut}(X)$ of X is reductive. Hence $\mathrm{Aut}(X)$ is reductive if X is K-polystable. In a recent paper [1],

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this result is generalized for log Fano case, that is, it is shown that a K-polystable log Fano pair over an algebraically closed filed of characteristic 0 have reductive automorphism group.

Let X be a $\mathbb{Q}$-Fano variety over an algebraically closed filed $\mathbb{K}$ of characteristic 0. If $\mathrm{Aut}(X)$ is not reductive, X is not K-polystable by [1]. Then there exists a test configuration $(\mathcal{X}, \mathcal{L})$ of X which *destabilizes* X, i.e. $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$, or $\mathrm{DF}(\mathcal{X}, \mathcal{L}) = 0$ and $(\mathcal{X}, \mathcal{L})$ is not of product type.

By this observation, Codogni and Dervan [8] consider the following question:

Question. — *If* $\mathrm{Aut}(X)$ *is not reductive, can we find a (canonical) destabilizing test configuration* $(\mathcal{X}, \mathcal{L})$ *of* X *related to* $\mathrm{Aut}(X)$*?*

A test configuration $(\mathcal{X}, \mathcal{L})$ can be interpreted as a suitable finitely generated decreasing filtrations $\mathcal{F}_\bullet R = \{\mathcal{F}_i R\}_{i \in \mathbb{Z}}$ of the section ring $R = \bigoplus_{d \geqslant 0} H^0(X, -dK_X)$ by

$$(\mathcal{X}, \mathcal{L}) = \left(\mathrm{Proj}_{\mathbb{A}^1} \bigoplus_i (\mathcal{F}_i R) t^{-i}, \mathcal{O}(1)\right) \to \mathbb{A}^1 = \mathrm{Spec}\, \mathbb{K}[t],$$

where $\mathcal{F}_\bullet R$ is called *finitely generated* if $\bigoplus_i (\mathcal{F}_i R) t^{-i}$ is a finitely generated $\mathbb{K}[t]$-algebra.

Using the action of $\mathrm{Aut}(X)$, Codogni and Dervan construct a canonical filtration $\mathcal{F}^L_\bullet R$ of R, called the *Loewy filtration* of X. In general, we do not know the *multiplicativity* of the Loewy filtration, i.e. $\mathcal{F}_i R \cdot \mathcal{F}_j R \subset \mathcal{F}_{i+j} R$ for any i, j [8, 9]. Hence we do not know whether or not $\bigoplus_i (\mathcal{F}^L_i R) t^{-i}$ is a $\mathbb{K}[t]$-subalgebra of $R[t, t^{-1}]$. However, the Loewy filtration still produces a sequence of test configurations. We note that the multiplicativity holds for toric case [9].

The following is a special case of [8, Conjecture B], i.e. the case when the Loewy filtration is multiplicative and finitely generated:

Conjecture 1.1. — *Let* X *be a* $\mathbb{Q}$*-Fano variety with non-reductive automorphism group. Assume that the Loewy filtration of* X *is multiplicative and finitely generated. Then the induced test configuration* $(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}})$ *destabilizes* X.

We note that they state the conjecture [8, Conjecture B] not only for $\mathbb{Q}$-Fano varieties but also for polarized varieties.

The purpose of this note is to give a counterexample to Conjecture 1.1, and hence to [8, Conjecture B] as follows:

Theorem 1.2. — *There exists a smooth toric Fano 3-fold X with non-reductive automorphism group such that the Loewy filtration is multiplicative, finitely generated and the Donaldson–Futaki invariant* $\mathrm{DF}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}})$ *is positive. In particular, $(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}})$ does not destabilize X.*

In a preliminary version [7] of [8], they also mention *Socle filtrations*, which are "dual" of Loewy filtrations. In Appendix, we also study Socle filtrations and show the following proposition.

Proposition 1.3. — *Let $R = \bigoplus_{d=0}^{\infty} R_d$ be a finitely generated graded integral $\mathbb{K}$-algebra and let U be a unipotent algebraic group which acts on R as a graded $\mathbb{K}$-algebra. Let $\mathcal{G}^S_\bullet R = \{\mathcal{G}^S_i R\}_{i\in\mathbb{Z}}$ be the induced Socle filtration. For $x \in \mathcal{G}^S_i R \setminus \mathcal{G}^S_{i-1} R$ and $y \in \mathcal{G}^S_j R \setminus \mathcal{G}^S_{j-1} R$, it holds that $xy \in \mathcal{G}^S_{i+j} R \setminus \mathcal{G}^S_{i+j-1} R$. In particular, $\mathcal{G}^S_\bullet R$ is multiplicative.*

We note that in this proposition, we do not need to assume that $\operatorname{Proj} R$ is Fano. As we will see in Corollary A.6, the Socle filtration induces a valuation on the function field of $\operatorname{Proj} R$ by Proposition 1.3. Thus we might expect that the Socle filtration is a better candidate for Question in the beginning than the Loewy filtration, and destabilizes any $\mathbb{Q}$-Fano varieties. However, the answer is no, at least for singular X. See Subsection 4.3.

This note is organized as follows. In Section 2, we recall K-stability and Loewy filtrations. In Section 3, we explain some known results about toric varieties. In Section 4, we give a counterexample to Conjecture 1.1. In Appendix, we show a property of Socle filtrations. Throughout this note, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0. We denote by $\mathbb{N}$ the set of all non-negative integers.

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2. K-stability, test configurations, and filtrations

Throughout this section, X is a $\mathbb{Q}$-Fano variety, that is, X is a normal projective variety with at most klt singularities such that the anti-canonical divisor $-K_X$ is $\mathbb{Q}$-Cartier and ample.

2.1. K-stability

Definition 2.1. — *A* test configuration $(\mathcal{X}, \mathcal{L})$ *of X consists of the following data:*

- *a variety $\mathcal{X}$ with a projective morphism $\pi : \mathcal{X} \to \mathbb{A}^1$,*
- *a $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ which is ample over $\mathbb{A}^1$,*
- *a $\mathbb{G}_m$-action on $(\mathcal{X}, \mathcal{L})$ such that π is $\mathbb{G}_m$-equivariant and $(\mathcal{X} \setminus X_0, \mathcal{L}|_{\mathcal{X}\setminus X_0})$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X \times (\mathbb{A}^1 \setminus \{0\}), p_1^*(-K_X))$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ multiplicatively and X_0 is the fiber over $0 \in \mathbb{A}^1$.*

For a test configuration $(\mathcal{X}, \mathcal{L})$ of X, we can define a rational number $\mathrm{DF}(\mathcal{X}, \mathcal{L})$, called the *Donaldson–Futaki invariant* of $(\mathcal{X}, \mathcal{L})$. See [13] for the definition of $\mathrm{DF}(\mathcal{X}, \mathcal{L})$.

Definition 2.2. — *A $\mathbb{Q}$-Fano variety X is called*

1. K-semistable *if for any test configuration $(\mathcal{X}, \mathcal{L})$ of X, we have $DF(\mathcal{X}, \mathcal{L}) \geqslant 0$.*
2. K-polystable *if X is K-semistable and, if $DF(\mathcal{X}, \mathcal{L}) = 0$ for a test configuration $(\mathcal{X}, \mathcal{L})$ of X, then the normalization of $\mathcal{X}$ is isomorphic to a test configuration of product type, i.e. the normalization of $\mathcal{X}$ is isomorphic to $X \times \mathbb{A}^1$ over $\mathbb{A}^1$.*

Let

$$R = \bigoplus_{d\geqslant 0} R_d = \bigoplus_{d\geqslant 0} H^0(X, -dK_X)$$

be the section ring of X. In this note, a *decreasing filtration* $\mathcal{F}_\bullet R$ of R is a sequence of vector subspaces

$$\cdots \supset \mathcal{F}_i R \supset \mathcal{F}_{i+1} R \supset \cdots$$

of R for $i \in \mathbb{Z}$ such that $\mathcal{F}_i R = \bigoplus_{d\geqslant 0} (\mathcal{F}_i R \cap R_d)$ holds for any $i \in \mathbb{Z}$ and $\bigcup_{i\in\mathbb{Z}} \mathcal{F}_i R = R$.

A decreasing filtration $\mathcal{F}_\bullet R$ is called

- *multiplicative* if $\mathcal{F}_i R \cdot \mathcal{F}_j R \subset \mathcal{F}_{i+j} R$ for any i, j. We note that if $\mathcal{F}_\bullet R$ is multiplicative, $\bigoplus_{i\in\mathbb{Z}} (\mathcal{F}_i R) t^{-i}$ has a natural $\mathbb{K}[t]$-algebra structure.
- *finitely generated* if it is multiplicative and the $\mathbb{K}[t]$-algebra $\bigoplus_{i\in\mathbb{Z}} (\mathcal{F}_i R) t^{-i}$ is finitely generated.

An *increasing filtration* $\mathcal{G}_\bullet R$ of R is a sequence of vector subspaces

$$\cdots \subset \mathcal{G}_i R \subset \mathcal{G}_{i+1} R \subset \cdots$$

of R for $i \in \mathbb{Z}$ such that $\mathcal{G}_i R = \bigoplus_{d\geqslant 0} (\mathcal{G}_i R \cap R_d)$ holds for any $i \in \mathbb{Z}$ and $\bigcup_{i\in\mathbb{Z}} \mathcal{G}_i R = R$.

An increasing filtration $\mathcal{G}_\bullet R$ is called

- *multiplicative* if $\mathcal{G}_i R \cdot \mathcal{G}_j R \subset \mathcal{G}_{i+j} R$ for any i, j. In that case, $\bigoplus_{i\in\mathbb{Z}} (\mathcal{G}_i R) t^{i}$ has a natural $\mathbb{K}[t]$-algebra structure.
- *finitely generated* if it is multiplicative and the $\mathbb{K}[t]$-algebra $\bigoplus_{i\in\mathbb{Z}} (\mathcal{G}_i R) t^{i}$ is finitely generated.

If a decreasing filtration $\mathcal{F}_\bullet R$ is finitely generated, we have

$$(\mathcal{X}, \mathcal{L}) := \left(\operatorname{Proj}_{\mathbb{A}^1} \bigoplus_{i\in\mathbb{Z}} (\mathcal{F}_i R) t^{-i}, \mathcal{O}(1)\right) \to \mathbb{A}^1,$$

which is a test configuration of X. We call this $(\mathcal{X}, \mathcal{L})$ the test configuration induced by the finitely generated filtration $\mathcal{F}_\bullet R$. We say that $\mathcal{F}_\bullet R$ *destabilizes* X if so does the induced test configuration $(\mathcal{X}, \mathcal{L})$.

Similarly, if an increasing filtration $\mathcal{G}_\bullet R$ is finitely generated, we have the induced test configuration

$$(\mathcal{X}, \mathcal{L}) := \left(\operatorname{Proj}_{\mathbb{A}^1} \bigoplus_{i\in\mathbb{Z}} (\mathcal{G}_i R) t^{i}, \mathcal{O}(1)\right) \to \mathbb{A}^1.$$

2.2. Loewy and Socle filtrations

Definition 2.3. — *Let U be a unipotent algebraic group, and V be a finite dimensional U-module.*

(1) *The* Loewy filtration $\mathcal{F}^L_\bullet V = \{\mathcal{F}^L_i V\}_{i\in\mathbb{N}}$ *is a decreasing filtration of U-modules defined by*
   (i) $\mathcal{F}^L_0 V = V$,
   (ii) *for $i > 0$, $\mathcal{F}^L_i V$ is the minimal U-submodule of $\mathcal{F}^L_{i-1} V$ such that the quotient $\mathcal{F}^L_{i-1} V / \mathcal{F}^L_i V$ is semisimple, i.e. the action on $\mathcal{F}^L_{i-1} V / \mathcal{F}^L_i V$ is trivial.*

(2) *The* Socle filtration $\mathcal{G}^S_\bullet V = \{\mathcal{G}^S_i V\}_{i\in\mathbb{N}}$ *is an increasing filtration of U-modules defined by*
  (i) $\mathcal{G}^S_0 V = V^U$, *the invariant part of V by the action of U,*
  (ii) *for $i > 0$, $\mathcal{G}^S_i V/\mathcal{G}^S_{i-1} V = (V/\mathcal{G}^S_{i-1} V)^U$.*

Remark 2.4. — Loewy filtrations can be defined for not necessarily unipotent algebraic groups. However, we can reduce the general case to the unipotent case by taking the unipotent radical [8, Lemma 2.3].

Since U is unipotent and V is finite dimensional, $\mathcal{F}^L_i V = \{0\}$ and $\mathcal{G}^S_i V = V$ for $i \gg 0$.

We also note that the indexes of the Socle filtration in Definition 2.3 are shifted by one from those in [7]. More precisely, it is defined as $\mathcal{G}^S_0 V = \{0\}, \mathcal{G}^S_1 V = V^U, \dots$ in [7].

Example 2.5. — Fix $N \in \mathbb{N}$ and set $V_N = \{f \in \mathbb{K}[x] \mid \deg(f) \leqslant N\} \subset \mathbb{K}[x]$. Let U be the additive unipotent algebraic group $\mathbb{G}_a = (\mathbb{K}, +)$, and consider the action of U on V_N by $\alpha \cdot x := x + \alpha$ for $\alpha \in U = \mathbb{K}$. In this case, it holds that

$$\mathcal{F}^L_i V_N = \{f \in V \mid \deg(f) \leqslant N - i\}, \quad \mathcal{G}^S_i V_N = \{f \in V \mid \deg(f) \leqslant i\}.$$

Definition 2.6. — *Let X be a $\mathbb{Q}$-Fano variety and U be the unipotent radical of the automorphism group* $\mathrm{Aut}(X)$ *of X. Then U acts on $R_d = H^0(X, -dK_X)$ for each $d \geqslant 0$.*

(1) *The* Loewy filtration $\mathcal{F}^L_\bullet R$ *of X is a decreasing filtration of R defined by*
  - $\mathcal{F}^L_i R = R$ *for $i < 0$,*
  - $\mathcal{F}^L_i R := \bigoplus_{d\geqslant 0} \mathcal{F}^L_i R_d$ *for $i \geqslant 0$, where $\mathcal{F}^L_\bullet R_d$ is the Loewy filtration of the U-module R_d.*

(2) *The* Socle filtration $\mathcal{G}^S_\bullet R$ *of X is an increasing filtration of R defined by*
  - $\mathcal{G}^S_i R = \{0\}$ *for $i < 0$,*
  - $\mathcal{G}^S_i R := \bigoplus_{d\geqslant 0} \mathcal{G}^S_i R_d$ *for $i \geqslant 0$, where $\mathcal{G}^S_\bullet R_d$ is the Socle filtration of the U-module R_d.*

(3) *If the Loewy filtration $\mathcal{F}^L_\bullet R$ (resp. Socle filtration $\mathcal{G}^S_\bullet R$) is finitely generated, we denote by $(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}})$ (resp. $(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}})$) the induced test configuration of X.*

Remark 2.7. — We note that $\bigcup_{i\in\mathbb{Z}} \mathcal{F}^L_i R = \bigcup_{i\in\mathbb{Z}} \mathcal{G}^S_i R = R$ holds by Remark 2.4. It is not known whether or not the Loewy filtration of a $\mathbb{Q}$-Fano variety is multiplicative in general [9]. At least, if we take a unipotent *subgroup* of $\mathrm{Aut}(X)$ as U in Definition 2.6, the obtained filtration can be non-multiplicative [9].

On the other hand, we will show that the Socle filtration is multiplicative in Appendix.

Example 2.8. — Let $S \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2 = \operatorname{Proj} \mathbb{K}[X,Y,Z]$ at $[1:0:0]$ and $[0:1:0]$. The Loewy filtration of S is computed in [8, Subsection 3.2] as follows.

The unipotent radical of $\operatorname{Aut}(S)$ consists of matrixes of the form

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for} \quad \alpha, \beta \in \mathbb{K},$$

which acts on $\mathbb{K}[X,Y,Z]$ by

$$X \mapsto X + \alpha Z, \quad Y \mapsto Y + \beta Z, \quad Z \mapsto Z. \tag{2.1}$$

Since $-K_S = 3H - E_1 - E_2$, where H is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ and E_1, E_2 are the exceptional divisors, we have

$$R_d = H^0(S, -dK_S) = \langle X^a Y^b Z^{3d-a-b} \,|\, 0 \leqslant a, b \leqslant 2d, a+b \leqslant 3d \rangle.$$

In [8], it is shown that $\mathcal{F}_i^L R_d = \langle X^a Y^b Z^{3d-a-b} \,|\, 0 \leqslant a, b \leqslant 2d, a+b \leqslant 3d - i \rangle$ for $i \geqslant 0$, and hence $\mathcal{F}_\bullet^L R$ is finitely generated. In this example, $\mathrm{DF}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}}) < 0$ holds as computed in [8].

For the Socle filtration $\mathcal{G}_\bullet^S R$, we need to compute the invariant part of the action of U. By (2.1), an element in $R_d = \langle X^a Y^b Z^{3d-a-b} \,|\, 0 \leqslant a, b \leqslant 2d, a+b \leqslant 3d \rangle$ is invariant if and only if it is a polynomial of Z. Hence we have

$$\mathcal{G}_0^S R_d = R_d^U = \langle Z^{3d} \rangle.$$

For $\mathcal{G}_1^S R_d$, we need to consider the action on

$$R_d / \mathcal{G}_0^S R_d = \langle X^a Y^b Z^{3d-a-b} \,|\, 0 \leqslant a, b \leqslant 2d, a+b \leqslant 3d \rangle / \langle Z^{3d} \rangle.$$

Since $(R_d / \mathcal{G}_0^S R_d)^U = \langle XZ^{3d-1}, YZ^{3d-1}, Z^{3d} \rangle / \langle Z^{3d} \rangle$, we have $\mathcal{G}_1^S R_d = \langle XZ^{3d-1}, YZ^{3d-1}, Z^{3d} \rangle$. Inductively, it holds that

$$\mathcal{G}_i^S R_d = \langle X^a Y^b Z^{3d-a-b} \,|\, 0 \leqslant a, b \leqslant 2d, a+b \leqslant \min\{i, 3d\} \rangle$$

In this example, the Socle filtration is essentially the same as the Loewy filtration. More precisely, $\mathcal{G}_i^S R_d = \mathcal{F}_{3d-i}^L R_d$ holds for any i, d and hence $(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}})$ coincides with $(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}})$. In Section 4, we will give examples where the Loewy and Socle filtrations are different.

3. Toric varieties

Let $M \simeq \mathbb{Z}^n$ be a lattice of rank n, and N be the dual lattice of M. An n-dimensional lattice polytope $P \subset M_{\mathbb{R}} := M \otimes \mathbb{R}$ is called *reflexive* if P contains $0 \in M$ in its interior and the dual polytope

$$P^* := \{v \in N_{\mathbb{R}} := N \otimes \mathbb{R} \,|\, \langle u, v\rangle \geqslant -1 \text{ for any } u \in P\}$$

is a lattice polytope as well. A reflexive polytope $P \subset M_{\mathbb{R}}$ defines an n-dimensional Gorenstein toric Fano variety X by

$$(X, -K_X) = (\operatorname{Proj} \mathbb{K}[\Gamma_P], \mathcal{O}(1)),$$

where $\Gamma_P = \{(d, u) \in \mathbb{N} \times M \,|\, u \in dP\}$ and $\mathbb{K}[\Gamma_P] = \bigoplus_{(d,u)\in\Gamma_P} \mathbb{K}\chi^{(d,u)}$ is the semigroup ring graded by $\mathbb{N}$. In particular, it holds that

$$H^0(X, -dK_X) = \bigoplus_{u\in dP\cap M} \mathbb{K}\chi^{(d,u)}.$$

In the rest of this section, $P \subset M_{\mathbb{R}}$ is a reflexive polytope and X is the corresponding Gorenstein toric Fano variety.

3.1. Toric test configurations

Let $f : P \to \mathbb{R}$ be a piecewise linear concave function with rational coefficients. As is well known, f induces a test configuration of X as follows.

Consider a decreasing filtration $\mathcal{F}^f_\bullet R$ of the section ring $R = \mathbb{K}[\Gamma_P]$ by

$$\mathcal{F}^f_i R = \langle \chi^{(d,u)} \,|\, (d, u) \in \Gamma_P, f(u/d) \geqslant i/d\rangle.$$

This filtration $\mathcal{F}^f_\bullet R$ is multiplicative by the concavity of f, and finitely generated since f is piecewise linear with rational coefficients. Hence $\mathcal{F}^f_\bullet R$ induces a test configuration $(\mathcal{X}_f, \mathcal{L}_f)$ of X.

Similarly, a piecewise linear convex function $g : P \to \mathbb{R}$ with rational coefficients induces a finitely generated increasing filtration $\mathcal{G}^g_\bullet R$ by

$$\mathcal{G}^g_i R = \langle \chi^{(d,u)} \,|\, (d, u) \in \Gamma_P, g(u/d) \leqslant i/d\rangle.$$

In particular, $\mathcal{G}^g_\bullet R$ induces a test configuration $(\mathcal{X}_g, \mathcal{L}_g)$ of X.

We note that $(\mathcal{X}_f, \mathcal{L}_f) = (\mathcal{X}_g, \mathcal{L}_g)$ holds if $g = C - f$ for some rational number C.

Remark 3.1. — If f is an integral affine function, we can define an isomorphism

$$\bigoplus_{i\in\mathbb{Z}} (\mathcal{F}^f_i R)t^{-i} \to R[t] \quad : \quad \chi^{(d,u)}t^{-i} \mapsto \chi^{(d,u)}t^{-i+d\cdot f(u/d)}$$

and hence $\mathcal{X}_f \simeq X \times \mathbb{A}^1$ over $\mathbb{A}^1$, i.e. the induced test configuration $(\mathcal{X}_f, \mathcal{L}_f)$ is of product type.

Similarly, the test configuration $(\mathcal{X}_g, \mathcal{L}_g)$ is of product type if g is an integral affine function.

Other than the Donaldson–Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{L})$, there exists another invariant $\mathrm{Ding}(\mathcal{X}, \mathcal{L})$ introduced in [2], called the *Ding invariant* of $(\mathcal{X}, \mathcal{L})$, which also can be used to define K-stability.

For toric test configurations, the following formulas are known:

Theorem 3.2 ([13], [21, Theorem 5, Proposition 7]). — *Under the above notation, it holds that*

$$\begin{aligned}
\mathrm{DF}(\mathcal{X}_f, \mathcal{L}_f) &= n\left(\frac{1}{\mathrm{vol}(P)}\int_P f(u)\,\mathrm{d}u - \frac{1}{\mathrm{vol}(\partial P)}\int_{\partial P} f(u)\,\mathrm{d}\sigma\right),\\
\mathrm{Ding}(\mathcal{X}_f, \mathcal{L}_f) &= f(0) - \frac{1}{\mathrm{vol}(P)}\int_P f(u)\,\mathrm{d}u,\\
\mathrm{DF}(\mathcal{X}_g, \mathcal{L}_g) &= n\left(-\frac{1}{\mathrm{vol}(P)}\int_P g(u)\,\mathrm{d}u + \frac{1}{\mathrm{vol}(\partial P)}\int_{\partial P} g(u)\,\mathrm{d}\sigma\right),\\
\mathrm{Ding}(\mathcal{X}_g, \mathcal{L}_g) &= -g(0) + \frac{1}{\mathrm{vol}(P)}\int_P g(u)\,\mathrm{d}u,
\end{aligned}$$

where du is the Euclidean measure on $M_{\mathbb{R}}$ and $d\sigma$ is the boundary measure on ∂P induced by the lattice M. The volumes $\mathrm{vol}(P), \mathrm{vol}(\partial P)$ are with respect to $du, d\sigma$ respectively.

Furthermore, it holds that

$$\mathrm{DF}(\mathcal{X}_f, \mathcal{L}_f) \geqslant \mathrm{Ding}(\mathcal{X}_f, \mathcal{L}_f) \quad (\textit{resp. } \mathrm{DF}(\mathcal{X}_g, \mathcal{L}_g) \geqslant \mathrm{Ding}(\mathcal{X}_g, \mathcal{L}_g)),$$

and the equality holds if and only if f (resp. g) is radically affine, where we say that a function $\varphi : P \to \mathbb{R}$ is radically affine if $\varphi(tu) - \varphi(0) = t(\varphi(u) - \varphi(0))$ for any $t \in [0, 1]$ and $u \in \partial P$.

3.2. Automorphism groups

The automorphism group of toric varieties are studied by [10, 11, 12, 17], etc. For simplicity, we only consider the Gorenstein Fano case here.

Let $v_1, \ldots, v_N \in N_{\mathbb{R}}$ be all the vertices of P^*. Then we have

$$P = \{u \in M_{\mathbb{R}} \mid \langle u, v_i\rangle \geqslant -1 \text{ for all } i\}.$$

We denote by D_i the torus invariant prime divisor on X corresponding to v_i.

Let $S = \mathbb{K}[x_1, \ldots, x_N]$, which is called the *Cox ring* of X, be the polynomial ring whose variables correspond to the prime divisors $D_1, \ldots, D_N$ on X. Hence a torus invariant effective Weil divisor $D = \sum a_i D_i$ corresponds to the monomial $x_1^{a_i} x_2^{a_2} \ldots x_N^{a_N} \in S$, which is denoted by x^D.

Under this notation, S is the direct sum of $\mathbb{K}x^D$ for all torus invariant effective Weil divisors D. Hence the Cox ring is graded by the Chow group $A^1(X)$ of X by

$$S = \bigoplus_{\alpha \in A^1(X)} S_\alpha = \bigoplus_{\alpha \in A^1(X)} \left(\bigoplus_{[D]=\alpha} \mathbb{K}x^D \right),$$

where $[D]$ is the class of D in $A^1(X)$.

We note that the monomial $\chi^{(d,u)} \in H^0(X, -dK_X)$ defines an effective torus invariant divisor $\sum_i (\langle u, v_i \rangle + d) D_i \in |-dK_X|$. Thus we can naturally identify $H^0(X, -dK_X)$ with $S_{[-dK_X]}$. Hence the section ring $R = \mathbb{K}[\Gamma_P]$ can be identified with the subring of S

$$\bigoplus_{d \geqslant 0} S_{[-dK_X]} \subset S.$$

Definition 3.3. — *An element $m \in M$ is called a* root *of P if there exists some i such that $\langle m, v_i \rangle = -1$ and $\langle m, v_j \rangle \geqslant 0$ for any $j \neq i$. In other words, $m \in M$ is a root if and only if m is contained in the relative interior of a facet F of P.*

A root m is called semisimple *if $-m \in M$ is a root as well. Otherwise, m is called* unipotent.

We note that $-m$ is called a root in [10, 12] for a root m in Definition 3.3. We follow the notation in [16].

Example 3.4. — The reflexive polytope in Figure 3.1 has two semisimple roots ▲ and two unipotent roots ★.

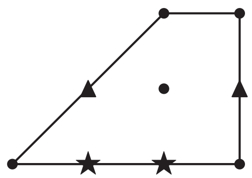

Figure 3.1.

For each root $m \in M$, we have a corresponding group homomorphism $y_m : \mathbb{G}_a \to \mathrm{Aut}(X)$, and the unipotent radical U of $\mathrm{Aut}(X)$ is generated by $\bigcup_m y_m(\mathbb{G}_a)$, where we take the union over all the unipotent roots of P.

Recall the definition of y_m. Let i be the unique index with $\langle m, v_i \rangle = -1$ as in Definition 3.3. Then D_i is linearly equivalent to the effective Weil divisor $D = \sum_{j \neq i} \langle m, v_j \rangle D_j$. For each $\alpha \in \mathbb{K}$, we have an automorphism of S defined by

$$x_i \mapsto x_i + \alpha x^D, \quad x_j \mapsto x_j \ \text{ for } \ j \neq i,$$

which preserves the $A^1(X)$-grading. This induces an automorphism $y_m(\alpha) \in \mathrm{Aut}(X)$.

4. Examples

Let $P \subset M_{\mathbb{R}}$ be a reflexive polytope and X be the corresponding Gorenstein toric Fano variety. In this section, we only consider examples with the simplest non-trivial unipotent radical, that is, we assume that there exists a unique unipotent root m of P throughout this section. Hence the unipotent radical U of $\mathrm{Aut}(X)$ is isomorphic to $\mathbb{G}_a$ via the group homomorphism y_m. In this case, the Loewy and Socle filtrations and the Donaldson–Futaki invariants of them are described as follows.

Let F be the unique facet of P containing m. Without loss of generality, we may assume $M = M' \times \mathbb{Z}$ for $M' \simeq \mathbb{Z}^{n-1}$, $m = (0, -1) \in M' \times \mathbb{Z}$, and $F = F' \times \{-1\}$ for a lattice polytope $F' \subset M'_{\mathbb{R}}$. By [16, Lemma 5.9], there exists a piecewise linear concave function $h : F' \to \mathbb{R}$ such that

$$P = \{(u', t) \in F' \times \mathbb{R} \,|\, -1 \leqslant t \leqslant h(u')\}. \tag{4.1}$$

Example 4.1. — For the reflexive polytope $P \subset \mathbb{R}^2$ in Figure 4.1, $F' = [-1, 1] \subset \mathbb{R}$ and $h(u') = 1 - |u'|$.

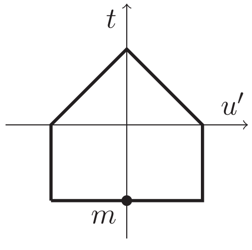

Figure 4.1.

For $u' \in dF' \cap M'$, set

$$\begin{aligned} R_d^{u'} &= \langle \chi^{(d,u)} \,|\, u = (u', l) \in dP \cap M \rangle \\ &= \langle \chi^{(d,u)} \,|\, u = (u', l) \text{ with } l \in \mathbb{Z}, -d \leqslant l \leqslant \lfloor dh(u'/d) \rfloor \rangle \subset R_d. \end{aligned}$$

By (4.1), we have a decomposition

$$R_d = \bigoplus_{u' \in dF' \cap M'} R_d^{u'}$$

as vector spaces. In fact, this is a decomposition as U-modules by the following lemma:

Lemma 4.2. — *For any $u' \in dF' \cap M'$, $R_d^{u'}$ is a U-submodule of R_d. Furthermore $R_d^{u'}$ is isomorphic to $V_{\lfloor dh(u'/d) \rfloor + d}$ in Example 2.5 as U-modules.*

Proof. — As in Subsection 3.2, let $D_1, \ldots, D_N$ be all the torus invariant prime divisors on X. We may assume that the facet $F \ni m = (0, -1)$ corresponds to D_1. Recall that $\alpha \in \mathbb{K} = U \subset \mathrm{Aut}(X)$ acts on the Cox ring S by

$$x_1 \mapsto x_1 + \alpha x^D, \quad x_i \mapsto x_i \ \text{ for } \ i \geqslant 2,$$

where $D = \sum_{i \geqslant 2} \langle m, v_i \rangle D_i$.

Fix $u' \in dF' \cap M'$. For simplicity, set $\chi_l = \chi^{(d,u)} \in R_d^{u'}$ for $u = (u', l)$ with $-d \leqslant l \leqslant \lfloor dh(u'/d) \rfloor$. By the identification of $R = \mathbb{K}[\Gamma_P]$ with $\bigoplus_{d \geqslant 0} S_{[-dK_X]}$ in Subsection 3.2, $\chi_l \in R_d^{u'} \subset R$ is identified with

$$X_l := \prod_{i=1}^{N} x_i^{\langle u, v_i \rangle + d} \in S.$$

Since $v_1 = (0, 1) \in N' \times \mathbb{Z}$, where N' is the dual lattice of M', we have $\langle u, v_1 \rangle + d = l + d$. Hence

$$X_l = x_1^{l+d} \prod_{i=2}^{N} x_i^{\langle u, v_i \rangle + d} \in S,$$

which is mapped to

$$(x_1 + \alpha x^D)^{l+d} \prod_{i=2}^{N} x_i^{\langle u, v_i \rangle + d}$$

by the action of $\alpha \in \mathbb{K}$. Since $x^D = \prod_{i=2}^N x_i^{\langle m, v_i\rangle}$,

$$\begin{aligned}(x_1 + \alpha x^D)^{l+d} &= \sum_{j=0}^{l+d} \binom{l+d}{j} \alpha^j x_1^{l+d-j} x^{jD} \\ &= \sum_{j=0}^{l+d} \binom{l+d}{j} \alpha^j x_1^{l+d-j} \prod_{i=2}^N x_i^{j\langle m, v_i\rangle}.\end{aligned}$$

Thus by the action of $\alpha \in \mathbb{K}$, X_l is mapped to

$$\begin{aligned}&(x_1 + \alpha x^D)^{l+d} \prod_{i=2}^N x_i^{\langle u, v_i\rangle + d} \\ &\qquad = \left(\sum_{j=0}^{l+d} \binom{l+d}{j} \alpha^j x_1^{l+d-j} \prod_{i=2}^N x_i^{j\langle m, v_i\rangle}\right) \prod_{i=2}^N x_i^{\langle u, v_i\rangle + d} \\ &\qquad = \sum_{j=0}^{l+d} \binom{l+d}{j} \alpha^j x_1^{l+d-j} \prod_{i=2}^N x_i^{\langle u+jm, v_i\rangle + d} \\ &\qquad = \sum_{j=0}^{l+d} \binom{l+d}{j} \alpha^j X_{l-j},\end{aligned}$$

where the last equality follows from $u + jm = (u', l) + j(0, -1) = (u', l-j)$. In particular, $\langle X_l \mid -d \leqslant l \leqslant \lfloor dh(u'/d)\rfloor\rangle \subset S$ is closed under the action of U. Hence so is $R_d^{u'} = \langle \chi_l \mid -d \leqslant l \leqslant \lfloor dh(u'/d)\rfloor\rangle \subset R_d$, i.e. $R_d^{u'}$ is a U-submodule.

By the above argument,

$$R_d^{u'} \to V_{\lfloor dh(u'/d)\rfloor + d} \quad : \quad \chi_l \mapsto x^{l+d} \tag{4.2}$$

is an isomorphism as U-modules. □

Lemma 4.3. — *Under the above setting, for $u = (u', l) \in dP \cap M$, $\chi^{(d,u)} \in R_d$ is contained in $\mathcal{F}_i^L R_d$ if and only if $l \leqslant dh(u'/d) - i$.*

On the other hand, $\chi^{(d,u)} \in R_d$ is contained in $\mathcal{G}_i^S R_d$ if and only if $l \leqslant i - d$.

Proof. — By (4.1), this lemma holds for $i < 0$. Hence we may assume $i \geqslant 0$.

We use the notation in the proof of Lemma 4.2. By Lemma 4.2, we have a decomposition $R_d = \bigoplus_{u' \in dF' \cap M'} R_d^{u'}$ as U-modules. Hence $\mathcal{F}_i^L R_d = \bigoplus_{u' \in dF' \cap M'} \mathcal{F}_i^L R_d^{u'}$ holds.

Since $\mathcal{F}_i^L V_{\lfloor dh(u'/d)\rfloor+d} = \langle x^j \mid 0 \leqslant j \leqslant \lfloor dh(u'/d)\rfloor + d - i\rangle$ by Example 2.5, we have

$$\mathcal{F}_i^L R_d^{u'} = \langle \chi_l \mid -d \leqslant l \leqslant \lfloor dh(u'/d)\rfloor - i\rangle$$

by (4.2). Thus $\chi_l = \chi^{(d,u)}$ for $u = (u', l)$ is contained in $\mathcal{F}_i^L R_d$ if and only if $l \leqslant \lfloor dh(u'/d)\rfloor - i$, which is equivalent to $l \leqslant dh(u'/d) - i$ since l and i are integers.

Similarly, we have $\mathcal{G}_i^S R_d = \bigoplus_{u' \in dF' \cap M'} \mathcal{G}_i^S R_d^{u'}$ and

$$\mathcal{G}_i^S R_d^{u'} = \langle \chi_l \mid -d \leqslant l \leqslant -d + i\rangle.$$

Hence $\chi_l = \chi^{(d,u)}$ is contained in $\mathcal{G}_i^S R_d$ if and only if $l \leqslant -d + i$. □

Proposition 4.4. — *Under the above setting, the Loewy filtration $\mathcal{F}_\bullet^L R$ of X coincides with the decreasing toric filtration $\mathcal{F}_\bullet^f R$ induced by the concave function f defined as*

$$f : P \to \mathbb{R}, \quad (u', t) \mapsto h(u') - t.$$

On the other hand, the Socle filtration $\mathcal{G}_\bullet^S R$ of X coincides with the increasing toric filtration $\mathcal{G}_\bullet^g R$ induced by the affine (hence convex) function g defined as

$$g : P \to \mathbb{R}, \quad (u', t) \mapsto t + 1.$$

In particular, both $\mathcal{F}_\bullet^L R$ and $\mathcal{G}_\bullet^S R$ are finitely generated and induce test configurations $(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}})$ and $(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}})$, respectively. Furthermore, $(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}})$ is of product type.

Proof. — By the definition of toric filtrations, for $u = (u', l) \in dP \cap M$, $\chi^{(d,u)} \in R_d$ is contained in $\mathcal{F}_i^f R_d$ if and only if

$$i/d \leqslant f(u/d) = f(u'/d, l/d) = h(u'/d) - l/d,$$

which is equivalent to $l \leqslant dh(u'/d) - i$. Hence $\mathcal{F}_\bullet^f R$ coincides with the Loewy filtration $\mathcal{F}_\bullet^L R$ by Lemma 4.3.

Similarly, $\chi^{(d,u)} \in R_d$ is contained in $\mathcal{G}_i^g R_d$ if and only if

$$i/d \geqslant g(u/d) = g(u'/d, l/d) = l/d + 1,$$

which is equivalent to $l \leqslant i - d$. Hence $\mathcal{G}_\bullet^g R$ coincides with the Socle filtration $\mathcal{G}_\bullet^S R$ by Lemma 4.3.

Since toric filtrations $\mathcal{F}_\bullet^f R$ and $\mathcal{G}_\bullet^g R$ are finitely generated, we obtain test configurations $(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}}) = (\mathcal{X}_f, \mathcal{L}_f)$ and $(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}}) = (\mathcal{X}_g, \mathcal{L}_g)$. The last statement follows from Remark 3.1 since $g = t + 1$ is integral affine. □

Since $P = \{(u', t) \in F' \times \mathbb{R} \mid -1 \leqslant t \leqslant h(u')\}$, roughly Proposition 4.4 states that the Loewy (resp. Socle) filtration is determined by the distance from the top facets of P defined by h (resp. the distance from the bottom facet $F = F' \times \{-1\}$).

By Theorem 3.2, we can compute the Donaldson–Futaki invariant and the Ding invariant of these test configurations as follows:

Corollary 4.5. — *It holds that*

$$\mathrm{Ding}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}}) = \frac{1}{\mathrm{vol}(P)} \int_P (h(0) - h(u') + t) \mathrm{d}u' \mathrm{d}t,$$

$$\mathrm{DF}(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}}) = \mathrm{Ding}(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}}) = \frac{1}{\mathrm{vol}(P)} \int_P t \, \mathrm{d}u' \mathrm{d}t.$$

If $h : F' \to \mathbb{R}$ is radically affine, $\mathrm{DF}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}}) = \mathrm{Ding}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}})$ *holds.*

Proof. — This follows from Theorem 3.2 and Proposition 4.4. We note that $g(u', t) = t + 1$ is affine, in particular, radically affine. On the other hand, $f(u', t) = h(u') - t$ is radically affine if and only if so is h. □

In all the following examples, h is radically affine and hence $\mathrm{DF}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}}) = \mathrm{Ding}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}})$ holds.

4.1. A singular toric del Pezzo surface

In this subsection, we give a counterexample to Conjecture 1.1 with singular X.

Let $P \subset \mathbb{R}^2$ be the reflexive polytope in Figure 4.1. We note that the corresponding X is a singular del Pezzo surface of degree 6 with one ordinary double point. In this case, $F' = [-1, 1]$ and $h : F' \to \mathbb{R}$ is defined by $h(x) = 1 - |x|$ as stated in Example 4.1. Since h is radically affine, we have

$$\mathrm{DF}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}}) = \frac{1}{\mathrm{vol}(P)} \int_P (|x| + t) \mathrm{d}x \mathrm{d}t = \frac{2}{9} > 0,$$

$$\mathrm{DF}(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}}) = \frac{1}{\mathrm{vol}(P)} \int_P t \, \mathrm{d}x \mathrm{d}t = -\frac{2}{9} < 0.$$

by Corollary 4.5. Hence the Loewy filtration does not destabilize X, but the Socle filtration does.

4.2. A smooth toric Fano 3-fold

In this subsection, we show Theorem 1.2, i.e. we give a counterexample to Conjecture 1.1 with smooth X.

The reflexive polytope

$$F' = \mathrm{Conv}((1,1),(0,1),(-2,-1),(1,-1)) \subset \mathbb{R}^2$$

in Figure 4.2 corresponds to the Hirzebruch surface $\Sigma_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1))$. Let X be the smooth toric Fano 3-fold obtained as the blow-up of $\Sigma_1 \times \mathbb{P}^1$ along $C \times \{p\}$, where $C \subset \Sigma_1$ is the torus invariant section with $(C^2) = 1$ and $p \in \mathbb{P}^1$ is a torus invariant point. Since $\Sigma_1 \times \mathbb{P}^1$ corresponds to $F' \times [-1,1]$, the polytope P corresponding to X is written as

$$P = \left\{(x,y,t) \in F' \times \mathbb{R} \subset \mathbb{R}^3 \mid -1 \leqslant t \leqslant h(x,y) := \min\{1, 1+y\}\right\}.$$

We note that P has two semisimple roots and one unipotent root $m = (0,0,-1)$.

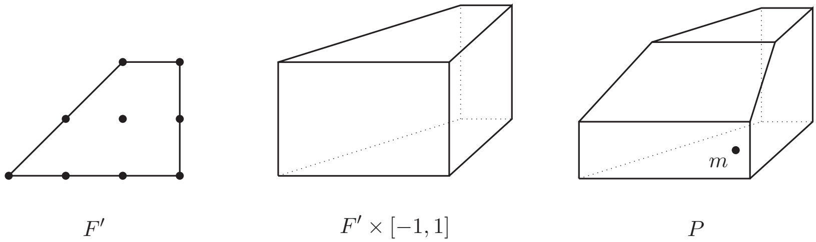

Figure 4.2.

Since h is radically affine, we have

$$\mathrm{DF}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}}) = \frac{1}{\mathrm{vol}(P)} \int_P (\max\{0,-y\} + t)\mathrm{d}x\mathrm{d}y\mathrm{d}t = \left(\frac{20}{3}\right)^{-1} \cdot \frac{7}{8} = \frac{21}{160},$$

$$\mathrm{DF}(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}}) = \frac{1}{\mathrm{vol}(P)} \int_P t\,\mathrm{d}x\mathrm{d}y\mathrm{d}t = \left(\frac{20}{3}\right)^{-1} \cdot \left(-\frac{7}{8}\right) = -\frac{21}{160}.$$

Proof of Theorem 1.2. — The above X satisfies the conditions in the theorem. □

4.3. A singular toric Fano 3-fold

For examples in Subsections 4.1 and 4.2, the invariant $\mathrm{DF}(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}})$ is negative, and hence the Socle filtration destabilizes X.

As we will see in Appendix, the Socle filtration is the filtration induced from a valuation on the function field of X, and hence multiplicative in general. Thus we might expect that the Socle filtration destabilizes any $\mathbb{Q}$-Fano varieties.

However, the answer is no, at least for singular X. The following is an example of a Gorenstein toric Fano 3-fold with non-reductive automorphism group such that $\mathrm{DF}(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}}) = \mathrm{Ding}(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}}) > 0$.

Let $F' \subset \mathbb{R}^2$ be the hexagon with vertexes $(1,0)$, $(0,1)$, $(-1,1)$, $(-1,0)$, $(0,-1)$, $(1,-1)$. We define a function $h : F' \to \mathbb{R}$ by

$$h(x,y) = \begin{cases} 1-2x & (x \geqslant 0) \\ 1-x & (x \leqslant 0) \end{cases}$$

for $(x,y) \in F'$. The polytope $P \subset \mathbb{R}^3$ in Figure 4.3 is defined by h and (4.1). We can check that P is reflexive, and $m = (0,0,-1) \in P$ is the unique unipotent root. By Corollary 4.5, we can compute

$$\mathrm{DF}(\mathcal{X}_{\mathrm{Loe}}, \mathcal{L}_{\mathrm{Loe}}) = \left(\frac{16}{3}\right)^{-1} \cdot \left(-\frac{3}{8}\right) = -\frac{9}{128} < 0,$$
$$\mathrm{DF}(\mathcal{X}_{\mathrm{Soc}}, \mathcal{L}_{\mathrm{Soc}}) = \left(\frac{16}{3}\right)^{-1} \cdot \frac{3}{8} = \frac{9}{128} > 0.$$

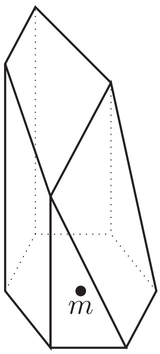

Figure 4.3.

Appendix A. On Socle filtrations

Let $R = \bigoplus_{d=0}^{\infty} R_d$ be a finitely generated graded integral $\mathbb{K}$-algebra and set $X = \operatorname{Proj} R$. We do not assume that X is Fano. Let U be a unipotent algebraic group which acts on R as a graded $\mathbb{K}$-algebra. By exactly the same definition as Definition 2.3, we can define the Socle filtration $\mathcal{G}^S_\bullet R$ of R.

Recall that an increasing filtration $\mathcal{G}_\bullet R$ is multiplicative if and only if $\mathcal{G}_i R \cdot \mathcal{G}_j R \subset \mathcal{G}_{i+j} R$ holds for any i, j.

Lemma A.1. — *Under the above setting, the Socle filtration $\mathcal{G}^S_\bullet R$ is multiplicative.*

To show this lemma, we use the Lie algebra $\mathfrak{u}$ of U. Since U acts on R as a $\mathbb{K}$-algebra, any $D \in \mathfrak{u}$ acts on R as a $\mathbb{K}$-derivation, i.e. $Dc = 0$ for any $c \in \mathbb{K}$ and

$$D(xy) = (Dx)y + x(Dy)$$

holds for any $x, y \in R$. By induction, for any $D_1, \dots, D_N \in \mathfrak{u}$ it holds that

$$D_N \cdots D_1(xy) = \sum_{(\varepsilon_1,\dots,\varepsilon_N)\in\{0,1\}^N} (D_N^{\varepsilon_N} \cdots D_1^{\varepsilon_1} x)(D_N^{1-\varepsilon_N} \cdots D_1^{1-\varepsilon_1} y), \tag{A.1}$$

where $D^0 x = x$ by convention.

Lemma A.2. — *For any $i, d \geqslant 0$, it holds that*

$$\mathcal{G}^S_i R_d = \{x \in R_d \mid D_{i+1} \dots D_1 x = 0 \text{ for any } D_1, \dots, D_{i+1} \in \mathfrak{u}\}.$$

Proof. — For $i = 0$, $x \in R_d$ is contained in the invariant part $\mathcal{G}^S_0 R_d = (R_d)^U$ if and only if $Dx = 0$ for any $D \in \mathfrak{u}$. Hence the statement holds for $i = 0$. By the induction on i, this lemma follows. □

Proof of Lemma A.1. — Take $x \in \mathcal{G}^S_i R$ and $y \in \mathcal{G}^S_i R$ for $i, j \in \mathbb{Z}$. We need to show $xy \in \mathcal{G}^S_{i+j} R$. Since $\mathcal{G}^S_k R = \{0\}$ for $k < 0$ by definition, $xy = 0 \in \mathcal{G}^S_{i+j} R$ holds if i or j is negative. Hence we may assume $i, j \geqslant 0$.

Set $N = i + j + 1$ and take any $D_1, \dots, D_N \in \mathfrak{u}$. It suffices to show $D_N \dots D_1(xy) = 0$ by Lemma A.2. For each $(\varepsilon_1, \dots, \varepsilon_N) \in \{0,1\}^N$, one of $\sum \varepsilon_k \geqslant i + 1$ or $\sum(1 - \varepsilon_k) \geqslant j + 1$ must hold. Hence $D_N^{\varepsilon_N} \cdots D_1^{\varepsilon_1} x = 0$ or $D_N^{1-\varepsilon_N} \cdots D_1^{1-\varepsilon_1} y = 0$ holds by Lemma A.2. By (A.1), we have $D_N \dots D_1(xy) = 0$. □

In fact, we can show the following proposition, which refines Lemma A.1.

Proposition A.3 (=Proposition 1.3). — *Let $x \in \mathcal{G}^S_i R \setminus \mathcal{G}^S_{i-1} R$ and $y \in \mathcal{G}^S_j R \setminus \mathcal{G}^S_{j-1} R$ for $i, j \geqslant 0$. Then $xy \in \mathcal{G}^S_{i+j} R \setminus \mathcal{G}^S_{i+j-1} R$ holds.*

Proof. — Since $xy \in \mathcal{G}^S_{i+j} R$ by Lemma A.1, what we need to show is $xy \notin \mathcal{G}^S_{i+j-1} R$. By Lemma A.2, it is enough to find $D_1, \dots, D_{i+j} \in \mathfrak{u}$ such that $D_{i+j} \cdots D_1(xy) \neq 0$.

Consider the set Φ which consists of sequences of non-negative integers $(a_k)_{k=1}^\infty$ satisfying

- $\sum_{k=1}^\infty a_k = i$. In particular, there exists m such that $a_k = 0$ for any $k \geqslant m + 1$.
- For the above m, there exist $D_1, \dots, D_m \in \mathfrak{u}$ such that $D_m^{a_m} \cdots D_1^{a_1} x \neq 0$.

We note that a_k could be zero even if $k \leqslant m$. For simplicity, we denote $(a_k)_{k=1}^{\infty}$ by $(a_1, \dots, a_m)$ if $a_k = 0$ for any $k \geqslant m+1$.

Since $x \notin \mathcal{G}^S_{i-1}R$, $D_i \cdots D_1(x) \neq 0$ for some $D_1, \dots, D_i \in \mathfrak{u}$. Hence $(\underbrace{1, 1, \dots, 1}_{i})$ is contained in Φ. In particular, $\Phi \neq \emptyset$.

Let $(a_1, \dots, a_m) = (a_1, \dots, a_m, 0, 0, \dots) \in \Phi$ be the maximum element with respect to the lexicographical order. Take and fix $D_1, \dots, D_m \in \mathfrak{u}$ with $D_m^{a_m} \cdots D_1^{a_1} x \neq 0$.

Consider another set $\Phi' \subset \mathbb{N}^m$ defined as follows: $(a'_1, \dots, a'_m) \in \mathbb{N}^m$ is contained in Φ' if and only if

- $n := j - (a'_1 + \dots + a'_m) \geqslant 0$ and $D'_n \dots D'_1 D_m^{a'_m} \cdots D_1^{a'_1} y \neq 0$ for some $D'_1, \dots, D'_n \in \mathfrak{u}$.

Since $y \notin \mathcal{G}^S_{j-1}R$, $D'_j \cdots D'_1 y \neq 0$ for some $D'_1, \dots, D'_j$. Hence $(\underbrace{0, 0, \dots, 0}_{m})$ is contained in Φ'. In particular, $\Phi' \neq \emptyset$.

Let $(a'_1, \dots, a'_m) \in \Phi'$ be the maximum element with respect to the lexicographical order. Take and fix $D'_1, \dots, D'_n \in \mathfrak{u}$ with $D'_n \dots D'_1 D_m^{a'_m} \cdots D_1^{a'_1} y \neq 0$ for $n = j - (a'_1 + \dots + a'_m)$.

To prove $xy \notin \mathcal{G}^S_{i+j-1}R$, it suffices to show

$$D'_n \cdots D'_1 D_m^{a_m + a'_m} \cdots D_1^{a_1 + a'_1}(xy) \neq 0 \tag{A.2}$$

since $\sum_{k=1}^m (a_k + a'_k) + n = \sum_{k=1}^m a_k + (n + \sum_{k=1}^m a'_k) = i + j$.

By (A.1), $D'_n \cdots D'_1 D_m^{a_m + a'_m} \cdots D_1^{a_1 + a'_1}(xy)$ is equal to

$$\begin{aligned}\sum_{\boldsymbol{\alpha}, \boldsymbol{\varepsilon}} c_{\boldsymbol{\alpha}, \boldsymbol{\varepsilon}} (D'^{\,\varepsilon_n}_n \cdots D'^{\,\varepsilon_1}_1 D_m^{\alpha_m} \cdots D_1^{\alpha_1} x) \\ \times (D'^{\,1-\varepsilon_n}_n \cdots D'^{\,1-\varepsilon_1}_1 D_m^{a_m + a'_m - \alpha_m} \cdots D_1^{a_1 + a'_1 - \alpha_1} y),\end{aligned} \tag{A.3}$$

where the sum is taken over all $(\boldsymbol{\alpha}, \boldsymbol{\varepsilon}) = (\alpha_1, \dots, \alpha_m, \varepsilon_1, \dots, \varepsilon_n)$ with

$$\alpha_k \in \{0, 1, \dots, a_k + a'_k\}, \quad \boldsymbol{\varepsilon} \in \{0, 1\}^n$$

and the coefficient $c_{\boldsymbol{\alpha}, \boldsymbol{\varepsilon}} \in \mathbb{N}$ is

$$c_{\boldsymbol{\alpha}, \boldsymbol{\varepsilon}} = \prod_{k=1}^m \binom{a_k + a'_k}{\alpha_k}.$$

If $\sum_{k=1}^m \alpha_k + \sum_{l=1}^n \varepsilon_l > i$, it holds that $D'^{\,\varepsilon_n}_n \cdots D'^{\,\varepsilon_1}_1 D_m^{\alpha_m} \cdots D_1^{\alpha_1} x = 0$ by $x \in \mathcal{G}^S_i R$. If $\sum_{k=1}^m \alpha_k + \sum_{l=1}^n \varepsilon_l < i$,

$$D'^{\,1-\varepsilon_n}_n \cdots D'^{\,1-\varepsilon_1}_1 D_m^{a_m + a'_m - \alpha_m} \cdots D_1^{a_1 + a'_1 - \alpha_1} y = 0$$

by $y \in \mathcal{G}^S_j R$ and $\sum_{k=1}^m (a_k + a'_k - \alpha_k) + \sum_{l=1}^n (1-\varepsilon_l) > j$. Hence it suffices to take the sum in (A.3) over $(\boldsymbol{\alpha}, \boldsymbol{\varepsilon})$ with

$$\sum_{k=1}^m \alpha_k + \sum_{l=1}^n \varepsilon_l = i. \tag{A.4}$$

Assume that $(\boldsymbol{\alpha}, \boldsymbol{\varepsilon})$ satisfies (A.4). By the definition of Φ, ${D'_n}^{\varepsilon_n} \cdots {D'_1}^{\varepsilon_1} D_m^{\alpha_m} \cdots D_1^{\alpha_1} x = 0$ if $(\alpha_1, \ldots, \alpha_m, \varepsilon_1, \ldots, \varepsilon_n) \notin \Phi$. Since $(a_1, \ldots, a_m) \in \Phi$ is the maximum element, it suffices to take the sum in (A.3) over $(\boldsymbol{\alpha}, \boldsymbol{\varepsilon})$ with (A.4) and

$$(\alpha_1, \ldots, \alpha_m, \varepsilon_1, \ldots, \varepsilon_n) \leqslant (a_1, \ldots, a_m). \tag{A.5}$$

By the definition of Φ', ${D'_n}^{1-\varepsilon_n} \cdots {D'_1}^{1-\varepsilon_1} D_m^{a_m + a'_m - \alpha_m} \cdots D_1^{a_1 + a'_1 - \alpha_1} y = 0$ if $(a_1 + a'_1 - \alpha_1, \ldots, a_m + a'_m - \alpha_m) \notin \Phi'$. Since $(a'_1, \ldots, a'_m) \in \Phi'$ is the maximum element, it suffices to take the sum in (A.3) over $(\boldsymbol{\alpha}, \boldsymbol{\varepsilon})$ with (A.4), (A.5) and

$$(a_1 + a'_1 - \alpha_1, \ldots, a_m + a'_m - \alpha_m) \leqslant (a'_1, \ldots, a'_m). \tag{A.6}$$

Assume that the index $(\boldsymbol{\alpha}, \boldsymbol{\varepsilon})$ satisfies (A.4), (A.5), and (A.6). Then $\alpha_1 \leqslant a_1$ and $a_1 + a'_1 - \alpha_1 \leqslant a'_1$ hold. Hence α_1 must be a_1.

Since $\alpha_1 = a_1$, we have $\alpha_2 \leqslant a_2$ and $a_2 + a'_2 - \alpha_2 \leqslant a'_2$, which imply $\alpha_2 = a_2$. Repeating this, $(\alpha_1, \ldots, \alpha_m)$ must coincide with $(a_1, \ldots, a_m)$. By (A.4) and $\sum_{k=1}^m a_k = i$, we have $\boldsymbol{\varepsilon} = (0, \ldots, 0)$.

After all, the index $(\boldsymbol{\alpha}, \boldsymbol{\varepsilon})$ which we need to take is only $((a_1, \ldots, a_m), (0, \ldots, 0))$. Hence $D'_n \cdots D'_1 D_m^{a_m + a'_m} \cdots D_1^{a_1 + a'_1}(xy)$ is equal to

$$c_{(a_1, \ldots, a_m), (0, \ldots, 0)} (D_m^{a_m} \cdots D_1^{a_1} x)(D'_n \ldots D'_1 D_m^{a'_m} \cdots D_1^{a'_1} y),$$

which is nonzero since both $D_m^{a_m} \cdots D_1^{a_1} x$ and $D'_n \ldots D'_1 D_m^{a'_m} \cdots D_1^{a'_1} y$ are non-zero elements in the integral domain R, and $c_{(a_1, \ldots, a_m), (0, \ldots, 0)} \neq 0$. Thus $xy \notin \mathcal{G}^S_{i+j-1} R$ follows. $\square$

Proposition A.3 implies that the Socle filtration induces a valuation on the function field of X as follows.

Definition A.4. — *For $x \in R$, we set*

$$\iota(x) = \inf\{i \in \mathbb{Z} \mid x \in \mathcal{G}^S_i R\} \in \{-\infty\} \cup \mathbb{N}.$$

We note that $\{i \in \mathbb{Z} \mid x \in \mathcal{G}^S_i R\} \neq \emptyset$ for any $x \in R$ since $\bigcup_i \mathcal{G}^S_i R = R$, and $\iota(x) = -\infty$ if and only if $x = 0$, and $\iota(c) = 0$ for $c \neq 0 \in \mathbb{K} \subset R_0$. For $x, y \in R$, $\iota(xy) = \iota(x) + \iota(y)$ holds by Proposition A.3.

Definition A.5. — *Let $K(X)$ be the function field of X. We define a function $v : K(X) \to \mathbb{Z} \cup \{\infty\}$ by*

$$v\left(\frac{x}{y}\right) = -\iota(x) + \iota(y)$$

for $d \geqslant 0, x, y \in R_d, y \neq 0$.

Corollary A.6. — *The above function v is well-defined. Furthermore, v is a valuation which is trivial on $\mathbb{K}$.*

Proof. — For the well-definedness, we need to check

(1) for $x, y \in R_d, y \neq 0$, $-\iota(x) + \iota(y) \in \mathbb{Z} \cup \{\infty\}$.
(2) if $x/y = x'/y' \in K(X)$, it holds that $-\iota(x) + \iota(y) = -\iota(x') + \iota(y')$.

As in Definition A.4, $\iota(y) \in \mathbb{N}$ if $y \neq 0$. Since $-\iota(x)$ is in $\mathbb{Z} \cup \{\infty\}$, we have $-\iota(x) + \iota(y) \in \mathbb{Z} \cup \{\infty\}$. Thus (1) follows.

For (2), if $x/y = x'/y' \in K(X)$, we have $xy' = x'y \in R$. Then

$$\iota(x) + \iota(y') = \iota(xy') = \iota(x'y) = \iota(x') + \iota(y)$$

by Proposition A.3. Hence $-\iota(x) + \iota(y) = -\iota(x') + \iota(y')$ holds. Thus v is well-defined.

By Lemmas A.1, A.2 and Proposition A.3, we can check that v satisfies the definition of valuation, i.e.

- $v(0) = \infty$ and $v(x) \neq \infty$ for $x \in K(X) \setminus 0$.
- $v(x+y) \geqslant \min\{v(x), v(y)\}$ for $x, y \in K(X)$, with equality if $v(x) \neq v(y)$.
- $v(xy) = v(x) + v(y)$ for $x, y \in K(X)$.
- $v(a) = 0$ for $a \in \mathbb{K} \setminus 0$.

□

Example A.7. — Let P be a reflexive polytope with a unique unipotent root $m = (0, -1) \in M' \times \mathbb{Z}$ such that

$$P = \{(u', t) \in F' \times \mathbb{R} \,|\, -1 \leqslant t \leqslant h(u')\}$$

for some F', h as in Section 4. In this case, the valuation induced by the Socle filtration $\mathcal{G}^S_\bullet R$ is the toric valuation corresponding to $(0, -1) \in N' \times \mathbb{Z}$. We note that this is *not* the divisorial valuation ord_D, which corresponds to $(0, 1) \in N' \times \mathbb{Z}$, for the prime divisor $D \subset X$ corresponding to the facet $F = F' \times \{-1\}$ of P.

For example, for the singular del Pezzo surface in Subsection 4.1, the valuation v is nothing but the divisorial valuation ord_E, where E is the exceptional divisor of the blow-up of the ordinary double point in X.

Recall that the function g in Section 4 corresponding to the Socle filtration is not only concave but also affine, contrary to the convex function f by Proposition 4.4. The affineness is due to Corollary A.6.

The author does not know whether or not the valuation v is the divisorial valuation for some prime divisor over X in general.

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