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ON THE LOG CANONICAL RING OF PROJECTIVE PLT PAIRS WITH THE KODAIRA DIMENSION TWO

by Osamu FUJINO & Haidong LIU (*)

Abstract. — The log canonical ring of a projective plt pair with the Kodaira dimension two is finitely generated.

Résumé. — L'anneau log canonique d'une paire plt projective de dimension de Kodaira deux est finement engendré.

1. Introduction

One of the most important open problems in the theory of minimal models for higher-dimensional algebraic varieties is the finite generation of log canonical rings for lc pairs.

Conjecture 1.1. — *Let (X, Δ) be a projective lc pair defined over $\mathbb{C}$ such that Δ is a $\mathbb{Q}$-divisor on X. Then the log canonical ring*

$$R(X, \Delta) := \bigoplus_{m \geqslant 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

In [12], Yoshinori Gongyo and the first author showed that Conjecture 1.1 is closely related to the abundance conjecture and is essentially equivalent to the existence problem of good minimal models for lower-dimensional varieties. Therefore, Conjecture 1.1 is thought to be a very difficult open problem.

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When (X, Δ) is klt, Shigefumi Mori and the first author showed that it is sufficient to prove Conjecture 1.1 under the extra assumption that $K_X + \Delta$ is big in [13]. Then Birkar–Cascini–Hacon–M$^{\mathrm{c}}$Kernan completely solved Conjecture 1.1 for projective klt pairs in [3]. More generally, in [7], the first author slightly generalized a canonical bundle formula in [13] and showed that Conjecture 1.1 holds true even when X is in Fujiki's class $\mathcal{C}$ and (X, Δ) is klt. We note that Conjecture 1.1 is not necessarily true when X is not in Fujiki's class $\mathcal{C}$ (see [7] for the details). Anyway, we have already established the finite generation of log canonical rings for klt pairs. So we are mainly interested in Conjecture 1.1 for (X, Δ) which is lc but is not klt.

If (X, Δ) is lc, then we have already known that Conjecture 1.1 holds true when $\dim X \leqslant 4$ (see [4]). If (X, Δ) is lc and $\dim X = 5$, then Kenta Hashizume showed that Conjecture 1.1 holds true when $\kappa(X, K_X + \Delta) < 5$ in [14]. We note that

$$\bigoplus_{m \geqslant 0} H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))$$

is always a finitely generated $\mathbb{C}$-algebra when X is a normal projective variety and D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X with $\kappa(X, D) \leqslant 1$. Therefore, the following theorem is the first nontrivial step towards the complete solution of Conjecture 1.1 for higher-dimensional algebraic varieties.

THEOREM 1.2 (Main Theorem). — *Let (X, Δ) be a projective plt pair such that Δ is a $\mathbb{Q}$-divisor. Assume that $\kappa(X, K_X + \Delta) = 2$. Then the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m \geqslant 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

In this paper, we will describe the proof of Theorem 1.2.

We will work over $\mathbb{C}$, the complex number field, throughout this paper. We will freely use the basic notation of the minimal model program as in [5] and [8]. In this paper, we do not use $\mathbb{R}$-divisors. We only use $\mathbb{Q}$-divisors.

2. $\mathbb{Q}$-divisors

Let D be a $\mathbb{Q}$-divisor on a normal variety X, that is, D is a finite formal sum $\sum_i d_i D_i$ where d_i is a rational number and D_i is a prime divisor on X for every i such that $D_i \neq D_j$ for $i \neq j$. We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{\leqslant 1} = \sum_{d_i \leqslant 1} d_i D_i, \quad \text{and} \quad D^{=1} = \sum_{d_i = 1} D_i.$$

We also put

$$\lceil D \rceil = \sum_i \lceil d_i \rceil D_i, \quad \lfloor D \rfloor = -\lceil -D \rceil, \quad \text{and} \quad \{D\} = D - \lfloor D \rfloor,$$

where $\lceil d_i \rceil$ is the integer defined by $d_i \leqslant \lceil d_i \rceil < d_i + 1$. A $\mathbb{Q}$-divisor D on a normal variety X is called a boundary $\mathbb{Q}$-divisor if D is effective and $D = D^{\leqslant 1}$ holds.

Let B_1 and B_2 be two $\mathbb{Q}$-divisors on a normal variety X. Then we write $B_1 \sim_{\mathbb{Q}} B_2$ if there exists a positive integer m such that $mB_1 \sim mB_2$, that is, mB_1 is linearly equivalent to mB_2.

Let $f : X \to Y$ be a proper surjective morphism between normal varieties and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then we write $D \sim_{\mathbb{Q}, f} 0$ if there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor B on Y such that $D \sim_{\mathbb{Q}} f^*B$.

Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety X. Let m_0 be a positive integer such that m_0D is a Cartier divisor. Let

$$\Phi_{|mm_0D|} : X \dashrightarrow \mathbb{P}^{\dim |mm_0D|}$$

be the rational map given by the complete linear system $|mm_0D|$ for a positive integer m. We put

$$\kappa(X, D) := \max_m \dim \Phi_{|mm_0D|}(X)$$

if $|mm_0D| \neq \emptyset$ for some m and $\kappa(X, D) = -\infty$ otherwise. We call $\kappa(X, D)$ the Iitaka dimension of D. Note that $\Phi_{|mm_0D|}(X)$ denotes the closure of the image of the rational map $\Phi_{|mm_0D|}$.

Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety X. If $D \cdot C \geqslant 0$ for every curve C on X, then we say that D is nef. If $\kappa(X, D) = \dim X$ holds, then we say that D is big.

In this paper, we will repeatedly use the following well-known easy lemma.

Lemma 2.1. — *Let $\varphi : X \to X'$ be a birational morphism between normal projective surfaces and let M be a nef $\mathbb{Q}$-divisor on X. Assume that $M' := \varphi_* M$ is $\mathbb{Q}$-Cartier. Then M' is nef.*

Proof. — By the negativity lemma, we can write $\varphi^* M' = M + E$ for some effective φ-exceptional $\mathbb{Q}$-divisor E on X. We can easily see that $(M + E) \cdot C \geqslant 0$ for every curve C on X. Therefore, M' is a nef $\mathbb{Q}$-divisor on X'. □

3. Singularities of pairs

Let us quickly recall the notion of singularities of pairs. For the details, we recommend the reader to see [5] and [8].

A pair (X, Δ) consists of a normal variety X and a $\mathbb{Q}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a projective birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E$$

with

$$f_* \left(\sum_E a(E, X, \Delta)E \right) = -\Delta,$$

where E runs over prime divisors on Y. We call $a(E, X, \Delta)$ the discrepancy of E with respect to (X, Δ). Note that we can define the discrepancy $a(E, X, \Delta)$ for any prime divisor E over X by taking a suitable resolution of singularities of X. If $a(E, X, \Delta) \geqslant -1$ (resp. > -1) for every prime divisor E over X, then (X, Δ) is called sub lc (resp. sub klt). If $a(E, X, \Delta) > -1$ holds for every exceptional divisor E over X, then (X, Δ) is called sub plt. It is well known that (X, Δ) is sub lc if it is sub plt.

Let (X, Δ) be a sub lc pair. If there exist a projective birational morphism $f : Y \to X$ from a normal variety Y and a prime divisor E on Y with $a(E, X, \Delta) = -1$, then $f(E)$ is called an lc center of (X, Δ). We say that W is an lc stratum of (X, Δ) when W is an lc center of (X, Δ) or $W = X$.

We assume that Δ is effective. Then (X, Δ) is called lc, plt, and klt if it is sub lc, sub plt, and sub klt, respectively. In this paper, we call $\kappa(X, K_X + \Delta)$ the Kodaira dimension of (X, Δ) when (X, Δ) is a projective lc pair.

4. Preliminary lemmas

In this section, we prepare two useful lemmas. One of them is a kind of connectedness lemma and will play a crucial role in this paper. Another one is a well-known generalization of the Kawamata–Shokurov basepoint-free theorem, which is essentially due to Yujiro Kawamata. We state it explicitly for the reader's convenience.

The following lemma is a key observation. As we mentioned above, it is a kind of connectedness lemma and will play a crucial role in this paper.

Lemma 4.1. — *Let $f : V \to W$ be a surjective morphism from a smooth projective variety V onto a normal projective variety W. Let B_V be a $\mathbb{Q}$-divisor on V such that $K_V + B_V \sim_{\mathbb{Q},f} 0$, (V, B_V) is sub plt, and* $\operatorname{Supp} B_V$ *is a simple normal crossing divisor. Assume that the natural map*

$$\mathcal{O}_W \longrightarrow f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil)$$

is an isomorphism. Let S_i be an irreducible component of $B_V^{=1}$ such that $f(S_i) \subsetneq W$ for $i = 1, 2$. We assume that $f(S_1) \cap f(S_2) \neq \emptyset$. Then $S_1 = S_2$ holds. In particular, we have $f(S_1) = f(S_2)$.

Proof. — We note that $B_V^{=1}$ is a disjoint union of smooth prime divisors since (V, B_V) is sub plt and $\operatorname{Supp} B_V$ is a simple normal crossing divisor. We put $C_i = f(S_i)$ for $i = 1, 2$. Then we put $Z = C_1 \cup C_2$ with the reduced scheme structure. By taking some suitable birational modification of V and replacing S_i with its strict transform for $i = 1, 2$, we may further assume that $f^{-1}(Z)$ is a divisor and that $f^{-1}(Z) \cup \operatorname{Supp} B_V$ is contained in a simple normal crossing divisor. Let T be the union of the irreducible components of $B_V^{=1}$ that are mapped into Z by f. Let us consider the following short exact sequence

$$0 \longrightarrow \mathcal{O}_V(A - T) \longrightarrow \mathcal{O}_V(A) \longrightarrow \mathcal{O}_T(A|_T) \longrightarrow 0$$

with $A = \lceil -(B_V^{<1})\rceil$. Then we obtain the long exact sequence

$$\begin{aligned} 0 \longrightarrow f_*\mathcal{O}_V(A - T) \longrightarrow f_*\mathcal{O}_V(A) \longrightarrow f_*\mathcal{O}_T(A|_T)& \\ \xrightarrow{\ \delta\ } R^1 f_*\mathcal{O}_V(A - T) \longrightarrow \cdots .& \end{aligned}$$

Note that

$$A - T - (K_V + \{B_V\} + B_V^{=1} - T) = -(K_V + B_V) \sim_{\mathbb{Q},f} 0.$$

Therefore, by [5, Theorem 6.3 (i)], every nonzero local section of the sheaf $R^1 f_*\mathcal{O}_V(A - T)$ contains in its support the f-image of some lc stratum of $(V, \{B_V\} + B_V^{=1} - T)$. On the other hand, the support of $f_*\mathcal{O}_T(A|_T)$ is contained in $Z = f(T)$. We note that no lc strata of $(V, \{B_V\} + B_V^{=1} - T)$ are mapped into Z by f by construction. Therefore, the connecting homomorphism δ is a zero map. Thus we get a short exact sequence

$$0 \longrightarrow f_*\mathcal{O}_V(A - T) \longrightarrow \mathcal{O}_W \longrightarrow f_*\mathcal{O}_T(A|_T) \longrightarrow 0.$$

Since $f_*\mathcal{O}_V(A - T)$ is contained in $\mathcal{O}_W$ and $f(T) = Z$, we have $f_*\mathcal{O}_V(A - T) = \mathcal{I}_Z$, where $\mathcal{I}_Z$ is the defining ideal sheaf of Z on W. Thus, by the above short exact sequence, we obtain that the natural map $\mathcal{O}_Z \to f_*\mathcal{O}_V(A|_T)$ is an isomorphism. Hence we obtain

$$\mathcal{O}_Z \xrightarrow{\sim} f_*\mathcal{O}_T \xrightarrow{\sim} f_*\mathcal{O}_T(A|_T).$$

In particular, $f : T \to Z$ has connected fibers. Therefore, $f^{-1}(P) \cap T$ is connected for every $P \in C_1 \cap C_2$. Note that T is a disjoint union of smooth prime divisors since $T \leqslant B_V^{=1}$. Thus we get $T = S_1 = S_2$ since $S_1, S_2 \leqslant T$. □

As a corollary of Lemma 4.1, we have:

Corollary 4.2. — *Let $f : V \to W$ be a surjective morphism from a smooth projective variety V onto a normal projective variety W. Let B_V be a $\mathbb{Q}$-divisor on V such that $K_V + B_V \sim_{\mathbb{Q},f} 0$, (V, B_V) is sub plt, and* $\operatorname{Supp} B_V$ *is a simple normal crossing divisor. Assume that the natural map*

$$\mathcal{O}_W \longrightarrow f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil)$$

is an isomorphism. Let S be an irreducible component of $B_V^{=1}$ such that $Z := f(S) \subsetneq W$. We put $K_S + B_S = (K_V + B_V)|_S$ by adjunction. Then (S, B_S) is sub klt and the natural map

$$\mathcal{O}_Z \longrightarrow g_*\mathcal{O}_S(\lceil -B_S\rceil)$$

is an isomorphism, where $g := f|_S$. In particular, Z is normal.

Proof. — We can easily check that (S, B_S) is sub klt by adjunction. We consider the following short exact sequence

$$0 \longrightarrow \mathcal{O}_V(\lceil -(B_V^{<1})\rceil - S) \longrightarrow \mathcal{O}_V(\lceil -(B_V^{<1})\rceil) \longrightarrow \mathcal{O}_S(\lceil -B_S\rceil) \longrightarrow 0.$$

Note that $B_V^{<1}|_S = B_S^{<1} = B_S$ holds. By Lemma 4.1, we know that no lc strata of $(V, \{B_V\} + B_V^{=1} - S)$ are mapped into Z by f. By the same argument as in the proof of Lemma 4.1, we obtain that the natural map

$$\mathcal{O}_Z \longrightarrow g_*\mathcal{O}_S(\lceil -B_S\rceil)$$

is an isomorphism. Therefore, the natural map $\mathcal{O}_Z \to g_*\mathcal{O}_S$ is an isomorphism. This implies that Z is normal. □

Lemma 4.3 is well known to the experts. It is a slight refinement of the Kawamata–Shokurov basepoint-free theorem and is essentially due to Yujiro Kawamata (see [15, Lemma 3]).

Lemma 4.3. — *Let (V, B_V) be a projective plt pair and let D be a nef Cartier divisor on V. Assume that $aD - (K_V + B_V)$ is nef and big for some $a > 0$ and that $\mathcal{O}_V(D)|_{\lfloor B_V \rfloor}$ is semi-ample. Then D is semi-ample.*

Proof. — By replacing D with a multiple, we may assume that the linear system $\left|\mathcal{O}_V(mD)|_{\lfloor B_V\rfloor}\right|$ is free for every nonnegative integer m. Since (V, B_V) is plt, the non-klt locus of (V, B_V) is $\lfloor B_V\rfloor$. Therefore, by [8, Corollary 4.5.6], D is semi-ample. □

5. On lc-trivial fibrations

In this section, we recall some results on klt-trivial fibrations in [2] and lc-trivial fibrations in [11] for the reader's convenience. We give only the definition which will be necessary to our purposes.

Let $f : V \to W$ be a surjective morphism from a smooth projective variety V onto a normal projective variety W. Let B_V be a $\mathbb{Q}$-divisor on V such that (V, B_V) is sub lc and $\operatorname{Supp} B_V$ is a simple normal crossing divisor on V. Let P be a prime divisor on W. By shrinking W around the generic point of P, we assume that P is Cartier. We set

$$b_P := \max\left\{t \in \mathbb{Q} \,|\, (V, B_V + tf^*P) \text{ is sub lc over the generic point of } P\right\}.$$

Then we put

$$B_W := \sum_P (1 - b_P)P,$$

where P runs over prime divisors on W. It is easy to see that B_W is a well-defined $\mathbb{Q}$-divisor on W (see the proof of Lemma 5.1 below). We call B_W the discriminant $\mathbb{Q}$-divisor of $f : (V, B_V) \to W$. We assume that the natural map

$$\mathcal{O}_W \longrightarrow f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil)$$

is an isomorphism. In this situation, we have:

Lemma 5.1. — *B_W is a boundary $\mathbb{Q}$-divisor on W.*

We give a detailed proof of Lemma 5.1 for the reader's convenience.

Proof of Lemma 5.1. — We can easily see that there exists a nonempty Zariski open set U of W such that $b_P = 1$ holds for every prime divisor P on W with $P \cap U \neq \emptyset$. Therefore, B_W is a well-defined $\mathbb{Q}$-divisor on W. Since (V, B_V) is sub lc, $b_P \geqslant 0$ holds for every prime divisor P on W. Thus, we have $B_W = B_W^{\leqslant 1}$ by definition. If $b_P > 1$ holds for some prime divisor P on W, then we see that the natural map $\mathcal{O}_W \to f_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil)$ factors through $\mathcal{O}_W(P)$ in a neighborhood of the generic point of P. This is a contradiction. Therefore, $b_P \leqslant 1$ always holds for every prime divisor P on W. This means that B_W is effective. Hence we see that B_W is a boundary $\mathbb{Q}$-divisor on W. □

We further assume that $K_V + B_V \sim_{\mathbb{Q}} f^*D$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on W. We set

$$M_W := D - K_W - B_W,$$

where K_W is the canonical divisor of W. We call M_W the moduli $\mathbb{Q}$-divisor of $K_V + B_V \sim_{\mathbb{Q}} f^*D$. Then we have:

Theorem 5.2. — *There exist a birational morphism $p : W' \to W$ from a smooth projective variety W' and a nef $\mathbb{Q}$-divisor $M_{W'}$ on W' such that $p_*M_{W'} = M_W$.*

Theorem 5.2 is a special case of [11, Theorem 3.6], which is a generalization of [2, Theorem 2.7]. When W is a curve, we have:

Theorem 5.3 ([2, Theorem 0.1]). — *If $\dim W = 1$ and (V, B_V) is sub klt, then M_W is semi-ample.*

As an easy consequence of Theorem 5.3, we have:

Corollary 5.4. — *If $\dim W = 1$, (V, B_V) is sub klt, and D is nef, then D is semi-ample.*

Proof. — If $\deg D > 0$, then D is ample. In particular, D is semi-ample. From now on, we assume that D is numerically trivial. By definition, $D = K_W + B_W + M_W$. Since B_W is effective by Lemma 5.1 and M_W is nef by Theorem 5.2, W is $\mathbb{P}^1$ or an elliptic curve. If $W = \mathbb{P}^1$, then $D \sim_{\mathbb{Q}} 0$. Of course, D is semi-ample. If W is an elliptic curve, then $D \sim M_W$, that is, D is linearly equivalent to M_W. In this case, D is semi-ample by Theorem 5.3. Anyway, D is always semi-ample. □

Corollary 5.5 is a key ingredient of the proof of the main theorem: Theorem 1.2.

Corollary 5.5. — *If $\dim W = 2$, (V, B_V) is sub plt, (W, B_W) is plt, and D is nef and big, then D is semi-ample.*

Proof. — Let Z be an irreducible component of $\lfloor B_W \rfloor$. Then, by the definition of B_W and Lemma 4.1, there exists an irreducible component S of $B_V^{=1}$ such that $f(S) = Z$. Therefore, by Corollary 4.2, the natural map $\mathcal{O}_Z \to g_*\mathcal{O}_S(\lceil -B_S \rceil)$ is an isomorphism, where $K_S + B_S = (K_V + B_V)|_S$ and $g = f|_S$. Note that (S, B_S) is sub klt and that $K_S + B_S \sim_{\mathbb{Q}} g^*(D|_Z)$. Thus, $D|_Z$ is semi-ample by Corollary 5.4. On the other hand, by Theorem 5.2, M_W is nef since $M_W = D - (K_W + B_W)$ is $\mathbb{Q}$-Cartier and W is a normal projective surface (see Lemma 2.1). Therefore, $2D - (K_W + B_W) = D + M_W$ is nef and big. Thus we obtain that D is semi-ample by Lemma 4.3. □

We close this section with a short remark on recent preprints [9] and [10].

Remark 5.6. — In [9], the first author introduced the notion of basic slc-trivial fibrations, which is a generalization of that of lc-trivial fibrations, and got a much more general result than Theorem 5.2 (see [9, Theorem 1.2]). In [10], we established the semi-ampleness of M_W for basic slc-trivial fibrations when the base space W is a curve (see [10, Corollary 1.4]). We strongly recommend the interested reader to see [9] and [10].

6. Minimal model program for surfaces

In this section, we quickly see a special case of the minimal model program for projective plt surfaces.

We can easily check the following lemma by the usual minimal model program for surfaces. We recommend the interested reader to see [6] for the general theory of log surfaces.

Lemma 6.1. — *Let (X, B) be a projective plt surface such that B is a $\mathbb{Q}$-divisor and let M be a nef $\mathbb{Q}$-divisor on X. Assume that K_X+B+M is big. Then we can run the minimal model program with respect to K_X+B+M and get a sequence of extremal contraction morphisms*

$$(X, B+M) =: (X_0, B_0+M_0) \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{k-1}} (X_k, B_k+M_k) =: (X^*, B^*+M^*)$$

with the following properties:

(i) *each φ_i is a $(K_{X_i}+B_i+M_i)$-negative extremal birational contraction morphism,*
(ii) *$K_{X_{i+1}} = \varphi_{i*}K_{X_i}$, $B_{i+1} = \varphi_{i*}B_i$, and $M_{i+1} = \varphi_{i*}M_i$ for every i,*
(iii) *M_i is nef for every i, and*
(iv) *$K_{X^*}+B^*+M^*$ is nef and big.*

Proof. — If $K_{X_i}+B_i+M_i$ is not nef, then we can take an ample $\mathbb{Q}$-divisor H_i and an effective $\mathbb{Q}$-divisor Δ_i on X_i such that $K_{X_i}+B_i+M_i+H_i \sim_{\mathbb{Q}} K_{X_i}+\Delta_i$, (X_i, Δ_i) is plt, and $K_{X_i}+\Delta_i$ is not nef. By the cone and contraction theorem, we can construct a $(K_{X_i}+\Delta_i)$-negative extremal birational contraction morphism $\varphi_i : X_i \to X_{i+1}$. Since H_i is ample, φ_i is a $(K_{X_i}+B_i+M_i)$-negative extremal contraction morphism. Moreover, since M_i is nef, φ_i is $(K_{X_i}+B_i)$-negative. Therefore, (X_{i+1}, B_{i+1}) is plt by the negativity lemma. In particular, X_{i+1} is $\mathbb{Q}$-factorial. By Lemma 2.1, we obtain that $M_{i+1} = \varphi_{i*}M_i$ is nef. Since K_X+B+M is big by assumption, this minimal model program terminates. Then we finally get a model (X^*, B^*+M^*) such that $K_{X^*}+B^*+M^*$ is nef and big. □

If we put $\varphi := \varphi_{k-1} \circ \cdots \circ \varphi_0 : X \to X^*$, then we have

$$K_X + B + M = \varphi^*(K_{X^*} + B^* + M^*) + E$$

for some effective φ-exceptional $\mathbb{Q}$-divisor E on X by the negativity lemma.

We will use Lemma 6.1 in the proof of the main theorem: Theorem 1.2.

7. Proof of the main theorem: Theorem 1.2

In this section, let us prove the main theorem: Theorem 1.2.

Let (X, Δ) be a projective plt (resp. lc) pair such that Δ is a $\mathbb{Q}$-divisor. Assume that $0 < \kappa(X, K_X + \Delta) < \dim X$. Then we consider the Iitaka fibration

$$f := \Phi_{|m_0(K_X+\Delta)|} : X \dashrightarrow Y$$

where m_0 is a sufficiently large and divisible positive integer. By taking a suitable birational modification of $f : X \dashrightarrow Y$, we get a commutative diagram:

$$\begin{array}{ccc} X & \xleftarrow{\ g\ } & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[\ h\ ]{} & Y' \end{array}$$

which satisfies the following properties:

(i) X' and Y' are smooth projective varieties,
(ii) g and h are birational morphisms, and
(iii) $K_{X'} + \Delta' = g^*(K_X + \Delta)$ such that $\operatorname{Supp} \Delta'$ is a simple normal crossing divisor on X'.

We note that $(X', (\Delta')^{>0})$ is plt (resp. lc) and that

$$H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)) \simeq H^0(X', \mathcal{O}_{X'}(\lfloor m(K_{X'} + (\Delta')^{>0}) \rfloor))$$

holds for every nonnegative integer m. Therefore, for the proof of the finite generation of the log canonical ring $R(X, \Delta)$, we may replace (X, Δ) with $(X', (\Delta')^{>0})$ and assume that the Iitaka fibration $f : X \dashrightarrow Y$ with respect to $K_X + \Delta$ is a morphism between smooth projective varieties. By construction, $\dim Y = \kappa(X, K_X + \Delta)$ and $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta|_{X_{\overline{\eta}}}) = 0$, where $X_{\overline{\eta}}$ is the geometric generic fiber of $f : X \to Y$.

By [1, Theorem 2.1, Proposition 4.4, and Remark 4.5], we can construct a commutative diagram

$$\begin{array}{ccccc} U_{X'} & \hookrightarrow & X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U_{Y'} & \hookrightarrow & Y' & \xrightarrow[h]{} & Y \end{array}$$

such that g and h are projective birational morphisms, X' and Y' are normal projective varieties, the inclusions $U_{X'} \subset X'$ and $U_{Y'} \subset Y'$ are toroidal embeddings without self-intersection satisfying the following conditions:

(a) f' is toroidal with respect to $(U_{X'} \subset X')$ and $(U_{Y'} \subset Y')$,
(b) f' is equidimensional,
(c) Y' is smooth,
(d) X' has only quotient singularities, and
(e) there exists a dense Zariski open set U of X such that g is an isomorphism over U, $U_{X'} = g^{-1}(U)$, and $U \cap \Delta = \emptyset$.

Although it is not treated explicitly in [1], we can make $g : X' \to X$ satisfy condition (e) (see Remark 7.1).

Remark 7.1. — In this remark, we will freely use the notation in [1]. For condition (e), it is sufficient to prove that there exists a Zariski open set U of X such that $U_{X'} = m_X^{-1}(U)$ and that m_X is an isomorphism over $U_{X'}$ in [1, Theorem 2.1]. Precisely speaking, we enlarge Z and may assume that $X \setminus Z$ is a Zariski open set of the original X in [1, 2.3], and enlarge Δ suitably in [1, 2.5]. Then we can construct $m_X : X' \to X$ such that U is a Zariski open set of X, $U_{X'} = m_X^{-1}(U)$, and $m_X : U_{X'} \to U$ is an isomorphism.

By condition (e), we have $\operatorname{Supp} \Delta' \subset X' \setminus U_{X'}$, where Δ' is a $\mathbb{Q}$-divisor defined by $K_{X'} + \Delta' = g^*(K_X + \Delta)$. We note that $(X', (\Delta')^{>0})$ is plt (resp. lc) and that

$$H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)) \simeq H^0(X', \mathcal{O}_{X'}(\lfloor m(K_{X'} + (\Delta')^{>0}) \rfloor))$$

holds for every nonnegative integer m. Therefore, by replacing $f : X \to Y$ and (X, Δ) with $f' : X' \to Y'$ and $(X', (\Delta')^{>0})$, respectively, we may assume that $f : X \to Y$ satisfies conditions (a), (b), (c), (d), and $\operatorname{Supp} \Delta \subset X \setminus U_X$, where $(U_X \subset X)$ is the toroidal structure in (a).

Since $\kappa(X, K_X + \Delta) > 0$, we can take a divisible positive integer a such that

$$H^0(X, \mathcal{O}_X(a(K_X + \Delta))) \neq 0.$$

Therefore, there exists an effective Cartier divisor L on X such that

$$a(K_X+\Delta)\sim L.$$

We put

$$G := \max\{N \mid N \text{ is an effective } \mathbb{Q}\text{-divisor on } Y \text{ such that } L \geqslant f^*N\}.$$

Then we set

$$D := \frac{1}{a}G \quad \text{and} \quad F := \frac{1}{a}(L - f^*G).$$

By definition, we have

$$K_X+\Delta \sim_{\mathbb{Q}} f^*D + F.$$

Lemma 7.2. — *For every nonnegative integer i, the natural map*

$$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X(\lfloor iF \rfloor)$$

is an isomorphism.

Proof. — By definition, F is effective. Therefore, we have a natural nontrivial map

$$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X(\lfloor iF \rfloor)$$

for every nonnegative integer i. By $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta|_{X_{\overline{\eta}}}) = 0$, we have $\kappa(X_{\overline{\eta}}, F|_{X_{\overline{\eta}}}) = 0$. Thus, we see that $f_*\mathcal{O}_X(\lfloor iF \rfloor)$ is a reflexive sheaf of rank one since f is equidimensional. Moreover, since Y is smooth, $f_*\mathcal{O}_X(\lfloor iF \rfloor)$ is invertible. Let P be any prime divisor on Y. By construction, $\operatorname{Supp} F$ does not contain the whole fiber of f over the generic point of P. Therefore, we obtain that $\mathcal{O}_Y \to f_*\mathcal{O}_X(\lfloor iF \rfloor)$ is an isomorphism in codimension one. This implies that the natural map

$$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X(\lfloor iF \rfloor)$$

is an isomorphism for every nonnegative integer i. □

By construction and Lemma 7.2, there exists a divisible positive integer r such that $r(K_X+\Delta)$ and rD are Cartier and that

$$H^0(X, \mathcal{O}_X(mr(K_X+\Delta))) \simeq H^0(Y, \mathcal{O}_Y(mrD))$$

holds for every nonnegative integer m. In particular, D is a big $\mathbb{Q}$-divisor on Y. We put $B := \Delta - F$ and consider $K_X + B \sim_{\mathbb{Q}} f^*D$. Let $p : V \to X$ be a birational morphism from a smooth projective variety V such that

$K_V + B_V = p^*(K_X + B)$ and that $\operatorname{Supp} B_V$ is a simple normal crossing divisor.

$$\begin{array}{ccc} V & \xrightarrow{p} & X \\ & \searrow^{\pi} & \downarrow{\scriptstyle f} \\ & & Y \end{array}$$

It is obvious that $K_V + B_V \sim_{\mathbb{Q}} \pi^*D$ holds. Since $p_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil) \subset \mathcal{O}_X(kF)$ for some divisible positive integer k, the natural map $\mathcal{O}_Y \to \pi_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil)$ is an isomorphism. For any prime divisor P on Y, we put

$$b_P := \max\{t \in \mathbb{Q} \,|\, (X, B + tf^*P) \text{ is sub lc over the generic point of } P\}.$$

Then we set

$$B_Y := \sum_P (1 - b_P)P$$

as in Section 5. Since $K_V + B_V = p^*(K_X + B)$ and the natural map $\mathcal{O}_Y \to \pi_*\mathcal{O}_V(\lceil -(B_V^{<1})\rceil)$ is an isomorphism, B_Y is the discriminant $\mathbb{Q}$-divisor of $\pi : (V, B_V) \to Y$ and is a boundary $\mathbb{Q}$-divisor on Y (see Lemma 5.1). By construction, we have $b_P = 1$ if $P \cap U_Y \neq \emptyset$, where $(U_Y \subset Y)$ is the toroidal structure in (a). Therefore, $\operatorname{Supp} B_Y \subset Y \setminus U_Y$.

From now on, we assume that (X, Δ) is plt and $\kappa(X, K_X + \Delta) = 2$. Then (V, B_V) is sub plt and Y is a smooth projective surface. As in Section 5, we write

$$D = K_Y + B_Y + M_Y,$$

where M_Y is the moduli $\mathbb{Q}$-divisor of $K_V + B_V \sim_{\mathbb{Q}} \pi^*D$. As we saw above, $\operatorname{Supp} B_Y \subset Y \setminus U_Y$, where $(U_Y \subset Y)$ is the toroidal structure in (a). In particular, this means that $\operatorname{Supp} B_Y$ is a simple normal crossing divisor on Y because Y is smooth. By Lemma 4.1, $\lfloor B_Y \rfloor$ is a disjoint union of some smooth prime divisors. Therefore, (Y, B_Y) is plt. By Lemma 6.1, there exists a projective birational contraction morphism $\varphi : Y \to Y'$ onto a normal projective surface Y' such that $D' = K_{Y'} + B_{Y'} + M_{Y'}$ is nef and big and that $D = \varphi^*D' + E$ for some effective φ-exceptional $\mathbb{Q}$-divisor E on Y. Of course, D', $K_{Y'}$, $B_{Y'}$, and $M_{Y'}$ are the pushforwards of D, K_Y, B_Y, and M_Y by φ, respectively.

$$\begin{array}{ccc} V & & \\ {\scriptstyle \pi}\downarrow & \searrow^{\pi'} & \\ Y & \xrightarrow[\varphi]{} & Y' \end{array}$$

By replacing V birationally, we may further assume that the union of $\operatorname{Supp} B_V$ and $\operatorname{Supp} \pi^*E$ is a simple normal crossing divisor on V. We consider

$$K_V + B_V - \pi^*E \sim_{\mathbb{Q}} \pi'^*D'.$$

We note that the natural map

$$\mathcal{O}_{Y'} \longrightarrow \pi'_*\mathcal{O}_V(\lceil -(B_V - \pi^*E)^{<1}\rceil)$$

is an isomorphism since $\pi_*\mathcal{O}_V(\lceil -(B_V - \pi^*E)^{<1}\rceil) \subset \mathcal{O}_Y(kE)$ for some divisible positive integer k and $\mathcal{O}_{Y'} \xrightarrow{\sim} \varphi_*\mathcal{O}_Y(kE)$. By construction, $(Y', B_{Y'})$ is plt (see Lemma 6.1) and $B_{Y'}$ is the discriminant $\mathbb{Q}$-divisor of $\pi' : (V, B_V - \pi^*E) \to Y'$. Therefore, by Corollary 5.5, D' is semi-ample. Thus, we obtain that

$$\bigoplus_{m\geqslant 0} H^0(Y, \mathcal{O}_Y(\lfloor mD\rfloor)) \simeq \bigoplus_{m\geqslant 0} H^0(Y', \mathcal{O}_{Y'}(\lfloor mD'\rfloor))$$

is a finitely generated $\mathbb{C}$-algebra. This implies that the log canonical ring $R(X, \Delta)$ of (X, Δ) is also a finitely generated $\mathbb{C}$-algebra.

Hence we have finished the proof of Theorem 1.2.

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