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ON THE NON-VANISHING OF p-ADIC HEIGHTS ON CM ABELIAN VARIETIES, AND THE ARITHMETIC OF KATZ p-ADIC L-FUNCTIONS

by Ashay A. BURUNGALE & Daniel DISEGNI

Abstract. — Let B be a simple CM abelian variety over a CM field E, p a rational prime. Suppose that B has potentially ordinary reduction above p and is self-dual with root number -1. Under some further conditions, we prove the generic non-vanishing of (cyclotomic) p-adic heights on B along anticyclotomic $\mathbf{Z}_p$-extensions of E. This provides evidence towards Schneider's conjecture on the non-vanishing of p-adic heights. For CM elliptic curves over $\mathbf{Q}$, the result was previously known as a consequence of works of Bertrand, Gross–Zagier and Rohrlich in the 1980s. Our proof is based on non-vanishing results for Katz p-adic L-functions and a Gross–Zagier formula relating the latter to families of rational points on B.

Résumé. — Soient B une variété abélienne CM simple sur un corps CM E, p un premier rationnel. On suppose que B a une réduction potentiellement ordinaire au dessus de p et est auto-duale avec signe -1. Sous quelques hypothèses supplementaires, on montre la non-annulation générique des hauteurs p-adiques (cyclotomiques) sur B le long de $\mathbf{Z}_p$-extensions anticyclotomiques de E. Cela confirme partiellement la conjecture de Schneider sur la non-annulation des hauteurs p-adiques. Pour les courbes elliptiques CM sur $\mathbf{Q}$, le résultat était déjà connu comme conséquence de travaux de Bertrand, Gross–Zagier et Rohrlich dans les années 80. Notre preuve est basée sur des résultats de non-annulation pour les fonctions L p-adiques de Katz, et sur une formule de Gross–Zagier qui les relie à des familles de points rationnels sur B.

1. Introduction and statements of the main results

Let B be an abelian variety over a number field E, and let $B^\vee$ be its dual. Let p be a prime and let L be a finite extension of $\mathbf{Q}_p$. The Néron–Tate height pairing on $B(E')_{\mathbf{Q}} \times B^\vee(E')_{\mathbf{Q}}$, where E' is a finite extension of E, admits a p-adic analogue (see e.g. [34])

$$B(E')_{\mathbf{Q}} \times B^\vee(E')_{\mathbf{Q}} \to L \tag{1.1}$$

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depending on the choices of a homomorphism $\ell\colon E^\times\backslash E^\times_{\mathbf{A}^\infty}\to L$ ("p-adic logarithm") and on splittings of the Hodge filtration on $H^1_{\mathrm{dR}}(B/E_v)$ at the primes $v|p$; in the potentially ordinary case under consideration in this paper, there are canonical choices (the "unit root" subspaces) for the Hodge splittings.

While it is a classical result that the Néron–Tate height pairing is non-degenerate, the pairing (1.1) can vanish for some choices of ℓ. Suppose however that $\ell=\ell_{\mathbf{Q}}\circ N_{E/\mathbf{Q}}$ with $\ell_{\mathbf{Q}}$ a p-adic logarithm of $\mathbf{Q}$ such that $\ell_{\mathbf{Q}}|_{1+p\mathbf{Z}_p}$ is nontrivial (we call such ℓ a *cyclotomic* logarithm). Then it is conjectured [39] that (1.1) is non-vanishing. This long-standing conjecture is only known in a few special cases: for CM elliptic curves, thanks to Bertrand [6], and also for elliptic curves over $\mathbf{Q}$ at supersingular primes.(1) The stronger statement that (1.1) is non-degenerate is also conjectured to be true; this is not known in any cases of rank higher than 1. The non-degeneracy conjecture has arithmetic consequences: it allows to formally deduce the p-adic Birch and Swinnerton-Dyer conjecture from the Iwasawa main conjecture for B.(2)

The pairing (1.1) is equivariant for the actions of $\mathrm{Gal}(E'/E)$ and of $K:=\mathrm{End}^0(B)$ on both sides; for our purposes we can then assume that B is simple, that the coefficient field L is sufficiently large and has the structure of a $K\otimes\mathbf{Q}_p$-algebra, and then decompose the pairing into isotypic components

$$B(\chi)\otimes_L B^\vee\left(\chi^{-1}\right)\to L \tag{1.2}$$

for the $\mathrm{Gal}(E'/E)$-action. Here and in the rest of the paper, if R is a $K\otimes\mathbf{Q}_p$-algebra and χ is an $R^\times$-valued character of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$, we define

$$B(\chi):=\left(B(E^{\mathrm{ab}})\otimes_{K\otimes\mathbf{Q}_p}R(\chi)\right)^{\mathrm{Gal}\left(E^{\mathrm{ab}}/E\right)},$$

where $R(\chi)$ is a rank-1 R-module with Galois action by χ.

The most significant result of this paper is the proof that, under some assumptions, the non-vanishing conjecture for (1.2) is true "generically" when B is a p-ordinary CM abelian variety over a CM field E and χ varies among anticyclotomic characters of E unramified outside p.

The method of proof, different from that of previous results on this topic, is automorphic. (In particular, the approach does not involve transcendence arguments.) It combines two ingredients. The first is a pair of nonvanishing

(1) This was observed in [4]; see [31, Section 4.5] for the comparison between the definition of the height pairing used in [4] and the "standard" definition of Zarhin and Nekovář [34].
(2) See [39]. For further applications to the *classical* Birch and Swinnerton–Dyer formula, see [20, 31, 37].

results for Katz p-adic L-functions due to Hida, Hsieh, and the first author (in turn relying on Chai's results on Hecke-stable subvarieties of a mod p Shimura variety [15, 16, 17]). The second ingredient is a Gross–Zagier formula relating derivatives of Katz p-adic L-functions to families of rational points on CM abelian varieties. We deduce this formula from work of the second author, by an argument employed by Bertolini–Darmon–Prasanna [5] in a different context.

In the rest of this section we describe the main results in more detail.

1.1. Non–vanishing of p-adic heights

Let B/E be a simple CM abelian variety over a CM field E, i.e. $K := \mathrm{End}^0(B)$ is a CM field of degree $[K : \mathbf{Q}] = 2\dim B$. Let F be the maximal totally real subfield of E, and let $\eta = \eta_{E/F}$ be the associated character of $F^\times\backslash F^\times_{\mathbf{A}}$.

1.1.1. Assumptions

The abelian variety B is associated with a Hecke character

$$\lambda = (\lambda^\tau)_{(\tau\colon K\hookrightarrow\mathbf{C})}\colon E^\times\backslash E^\times_{\mathbf{A}} \to (K\otimes\mathbf{C})^\times$$

([42, Section 19]). Suppose that λ satisfies the condition

$$\lambda|_{F^\times_{\mathbf{A}}} = \eta|\cdot|^{-1}_{\mathbf{A}}; \tag{1.3}$$

this holds in particular whenever B arises as the base-change of a real-multiplication abelian variety A/F [42, Theorem 20.15]. It implies that the for each τ there is a functional equation with sign $w(\lambda) := \varepsilon(1,\lambda^\tau) \in \{\pm 1\}$ (independent of τ) relating $L(s,\lambda^\tau)$ to $L(2-s,\lambda^\tau)$. We will assume that

$$w(\lambda) = -1. \tag{1.4}$$

Finally, we assume that

(1.5) B has potentially *ordinary* reduction at every prime of E above p.

1.1.2. Anticyclotomic regulators

Let E^-_∞ (respectively E^+_∞) be the anticyclotomic $\mathbf{Z}_p^{[F:\mathbf{Q}]}$-extension (respectively cyclotomic $\mathbf{Z}_p$-extension) of E and, for a prime $\wp$ of F above p, let $E^-_{\wp,\infty} \subset E^-_\infty$ be the $\wp$-anticyclotomic subextension, i.e. the maximal subextension unramified outside the primes above $\wp$ in E; finally, let

$E_\infty := E_\infty^- E_\infty^+$ and $E_{\wp,\infty} = E_{\wp,\infty}^- E_\infty^+$. If $\bullet$ is any combination of subscripts $\emptyset$, $\wp$ and superscripts $\emptyset$, $+$, $-$ (we convene that '$\emptyset$' denotes no symbol), the corresponding infinite Galois group is

$$\Gamma_\bullet := \mathrm{Gal}(E_{\bullet,\infty}/E)\,.$$

Let L be a finite extension of a p-adic completion K_w of K. If $\bullet$ is any set of sub- and superscripts as above, we let

$$\Lambda_\bullet := \mathscr{O}_L[\![\Gamma_\bullet]\!] \otimes L, \quad \mathscr{Y}_\bullet := \mathrm{Spec}\,\Lambda_\bullet.$$

When we want to emphasise the role of a specific choice of L, we will write $\Lambda_{\bullet,\,L}$, $\mathscr{Y}_{\bullet,\,L}$.

For $\circ = \wp, \emptyset$ we let

$$\chi_{\mathrm{univ},\,\circ} \colon \Gamma_\circ^- \to \Lambda_\circ^{-,\,\times}$$

be the tautological anticyclotomic character.

We then have a $\Lambda_\circ^-$-module

$$B(\chi_{\mathrm{univ},\circ}) := \left(B(\overline{E}) \otimes \Lambda_\circ^-(\chi_{\mathrm{univ},\circ})\right)^{\mathrm{Gal}\left(\overline{E}/E\right)},$$

whose specialisation at any finite order character $\chi \in \mathscr{Y}^-$ is $B(\chi)$, and height pairings ([36, Section 2.3], see also [35, Section 11])

$$B(\chi_{\mathrm{univ},\circ}) \otimes_{\Lambda_\circ} B^\vee\left(\chi_{\mathrm{univ},\circ}^{-1}\right) \to \Lambda_\circ^- \tag{1.6}$$

associated with choices of a p-adic logarithm ℓ and of Hodge splittings. We suppose that $\ell\colon \mathrm{Gal}(E^{\mathrm{ab}}/E) \to L$ is the cyclotomic logarithm and that the Hodge splittings are given by the unit root subspaces.

Theorem 1.1. — *Let B be a simple CM abelian variety over the CM field E with associated Hecke character λ satisfying* (1.3), (1.4), *and* (1.5)*; suppose that the extension E/F is ramified. Let p be a rational prime, and suppose that $p \nmid 2D_F h_E^-$, where $h_E^- = h_E/h_F$ is the relative class number of E/F and D_F is the absolute discriminant of F. Let $\wp$ be a prime of F above p, and let $L \supset K_w \supset K$ be as above.*

Then for almost all finite-order characters χ of $\Gamma_\wp^-$, the pairing (1.2)

$$B(\chi) \otimes_L B^\vee\left(\chi^{-1}\right) \to L$$

is non-vanishing. Equivalently, the paring (1.6) *for $\circ = \wp$ (hence also for $\circ = \emptyset$) is nonzero.*

Here "almost all" means that the set of finite-order characters $\chi \in \mathscr{Y}_\wp^-$ which fail to satisfy the conclusion of the theorem is not Zariski dense in $\mathscr{Y}_\wp^-$. When $\dim \mathscr{Y}_\wp^- = 1$ (e.g. $F = \mathbf{Q}$), this is equivalent to such set of exceptions being finite.

1.2. Gross–Zagier formula for the Katz p-adic L-function

As recalled in Section 2.4, under our conditions and assuming that the extension $L \supset K_w$ splits E, the character λ (more precisely, its w-adic avatar) has a p-adic CM type $\Sigma_E \subset \mathrm{Hom}(E, L)$ of E. We also identify Σ_E with (i) a choice, for each prime $\wp | p$ of F, of one among the two primes of E above $\wp$, and (ii) an element of $\mathbf{Z}[\mathrm{Hom}(E, L)]$. To the CM type Σ_E is attached the Katz p-adic L-function

$$L_{\Sigma_E} \in \Lambda.$$

It interpolates the values $L(0, \lambda'^{-1})$ for characters λ' whose infinity type lies in a certain region of $\mathbf{Z}[\mathrm{Hom}(E, \mathbf{C}_p)]$; this region is uniquely determined by Σ_E and contains in particular the infinity type Σ_E.

Let $\lambda^*(x) := \lambda(x^c)$, where c denotes the complex conjugation of E/F. The root number assumption (1.4) implies that $L_{\Sigma_E}(\lambda) = L_{\Sigma_E c}(\lambda^*) = L(0, \lambda^{*-1}) = L(0, \lambda^{-1}) = L(1, \lambda) = L(B, 1) = 0$, and more generally that the function $L_{\Sigma_E c,\, \lambda^*} : \chi' \mapsto L_{\Sigma_E c}(\lambda^* \chi')$ vanishes along $\mathscr{Y}^- \subset \mathscr{Y}$. We may then consider the cyclotomic derivative

$$L'_{\Sigma_E c,\, \lambda^*} : \chi \mapsto \frac{\mathrm{d}}{\mathrm{d}s} L_{\Sigma_E c}\left(\lambda^* \chi \cdot \chi_{\mathrm{cyc}}^s\right)\big|_{s=0},$$

for $\chi \in \mathscr{Y}^-$. (Here χ_{cyc} is the p-adic cyclotomic character of $E_{\mathbf{A}}^\times$).

For $\circ = \emptyset, \wp$, let $\mathscr{K}_\circ^-$ be the field of fractions of $\Lambda_\circ^-$ and let $B(\chi_{\mathrm{univ},\circ})_{\mathscr{K}_\circ^-} := B(\chi_{\mathrm{univ},\circ}) \otimes_{\Lambda_\circ^-} \mathscr{K}_\circ^-$; similarly for $B^\vee(\chi_{\mathrm{univ}}^{-1})_{\mathscr{K}_\circ^-}$.

Theorem 1.2. — *Let $\circ = \emptyset$ or $\circ = \wp$. Under the assumptions of Theorem 1.1, there is a 'pair of points'*

$$\mathscr{P} \otimes \mathscr{P}^\vee \in B\left(\chi_{\mathrm{univ},\circ}^{-1}\right)_{\mathscr{K}_\circ^-} \otimes B(\chi_{\mathrm{univ},\circ})_{\mathscr{K}_\circ^-}$$

satisfying

$$\langle \mathscr{P}, \mathscr{P}^\vee \rangle_\circ = L'_{\Sigma_E c,\, \lambda^*}|_{\mathscr{Y}_\circ^-}$$

in $\mathscr{K}_\circ^-$, where $\langle\, , \rangle_\circ$ is the height pairing (1.6), *and we identify $\Gamma^+ \cong \mathbf{Z}_p$ via the cyclotomic logarithm.*

The construction of the points depends on some choices, analogously to how the construction of rational points on an elliptic curve over $\mathbf{Q}$ of analytic rank one depends on the choice of an auxiliary imaginary quadratic field. Like in that situation, it comes from Heegner points and relies on a non-vanishing result for L-functions – in this case, the results of Hida and Hsieh [25, 28] for anticyclotomic Katz p-adic L-functions. Nevertheless the auxiliary setup does not seem to be explored in regard to the cyclotomic derivative.

The following Conjecture 1.3, which can be regarded as analogous to the results of Kolyvagin, would imply that the ambiguity is rather mild.

Conjecture 1.3. — *Let $\circ = \emptyset$ or $\circ = \wp$ for a prime $\mathfrak{p}$ of F. The $\mathscr{K}_\circ^-$-vector spaces $B\left(\chi_{\mathrm{univ},\circ}^{-1}\right)_{\mathscr{K}_\circ^-}$, $B^\vee(\chi_{\mathrm{univ},\circ})_{\mathscr{K}_\circ^-}$ have dimension one.*

When E is an imaginary quadratic field and B is the base-change of an elliptic curve over $\mathbf{Q}$, the conjecture is part of the main result of Agboola and Howard in [2].

Remark 1.4. — It is natural to wonder about the arithmetic significance of the values

$$L_{\Sigma'_E}(\lambda^*) \tag{1.7}$$

for CM types Σ'_E such that

$$\delta(\Sigma'_E) := |\Sigma'_E \cap \Sigma_E| \geqslant 1.$$

We would like to suggest that if $\delta(\Sigma'_E) \leqslant r := \mathrm{ord}_{s=1} L(B,s)$, the cyclotomic order of vanishing of $L_{\Sigma'_E}$ at λ^* (that is, the smallest k such that the cyclotomic derivative $L_{\Sigma'_E}^{(k)}(\lambda^*) \neq 0$) should be

$$\mathrm{ord}_{\mathrm{cyc}} L_{\Sigma'_E,\lambda^*} \stackrel{?}{=} r - \delta(\Sigma_{E'}) \tag{1.8}$$

and that there should be an explicit formula relating

$$L_{\Sigma'_E}^{(r-\delta(\Sigma_{E'}))}(\lambda^*)$$

to a p-adic regulator. When $[E:\mathbf{Q}]$ is a quadratic field and λ comes from an elliptic curve, (1.8) was conjectured by Rubin [38] together with a precise formula, and proved by himself when $r \leqslant 1$.

Theorem 1.2 provides evidence for the general case of (1.8) in one of the cases with $r \leqslant 1$, whereas the other such case is treated, when $[E:\mathbf{Q}] = 2$, by Rubin's formula as generalised by Bertolini–Darmon–Prasanna [5].(3) The particular interest of the case of $[E:\mathbf{Q}] > 2$ lies of course in the possibility of having $\delta(\Sigma'_E) \geqslant 2$: our speculation is also inspired from the recent work of Darmon–Rotger [19] on p-adic L-functions related to certain Mordell–Weil groups of rank 2.

(3) We hope to present a generalisation of this formula in a sequel to the present paper.

1.3. An arithmetic application

For an elliptic curve $E_{/\mathbf{Q}}$ and a prime p, let us refer to the implication

$$\operatorname{corank}_{\mathbf{Z}_p} \operatorname{Sel}_{p^\infty}(E_{/\mathbf{Q}}) = 1 \implies \operatorname{ord}_{s=1} L(s, E_{/\mathbf{Q}}) = 1$$

as the "p-converse theorem" (to the one of Gross–Zagier, Kolyvagin and Rubin). In [13], the authors establish the p-converse theorem in the case of p-ordinary CM elliptic curves (complementing the earlier works [43, 46]). The approach crucially relies on the auxiliary setup introduced in the proof of the main results here, and also on the main results themselves (Theorem 1.1 and Theorem 1.2).

1.4. Context and strategy of proof

When E is an imaginary quadratic field and $B = A_E$ for a CM elliptic curve $A/\mathbf{Q}$, Theorem 1.1 is a consequence (see Rubin [3]) of the aforementioned result of Bertrand together with Mazur's conjecture on the generic non-vanishing of Heegner points along anticyclotomic extensions (which in that case is proved, via the Gross–Zagier formula, by the generic non-vanishing of derivatives of L-functions established by Rohrlich). Our method is rather different (although not distant in spirit): we first deduce the formula of Theorem 1.2 from the p-adic Gross–Zagier formula of the second author [21]. Theorem 1.1, or rather its more precise version Theorem 2.11 below, is then a consequence of Theorem 1.2 and non-vanishing results of the first author [7] (or their refinement in [11]) for the derivatives of Katz p-adic L-functions. As a corollary we recover Mazur's conjecture, which in our case was proved by Aflalo–Nekovář [1] as a generalisation of work of Cornut and Vatsal. Note further that our method would readily adapt to cover the case of (generalised) Heegner cycles upon availability of a suitably general p-adic Gross–Zagier formula for them. The second author expects to present such a formula as part of a forthcoming version of [22].(4)

A parallel approach is followed by the first author in a series of works [9], [10] establishing, without assumptions of complex multiplication, the generic non-vanishing of Heegner points and cycles, or more precisely of (the reduction modulo p of) their images under the Abel–Jacobi map (also see [8]

(4) Note added in proof: this formula has now been proven.

and [12]).(5) The strategy to prove Theorem 1.2 is inspired from the proof of Rubin's formula in [5]. As in [5], we can remark that we have established a result for a motive attached to the group U(1) by making use of p-adic L-functions for $\mathrm{U}(1) \times \mathrm{U}(2)$. Readers with a generous attitude towards mathematical induction might find in this a good omen for future progress.

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2. Proofs

2.1. CM theory

We start by reviewing some basic results in the theory of Complex Multiplication. The classical reference is [42] (see especially Sections 18-20); see also [18, Section 2.5].

Let B be an abelian variety of dimension d over a field E, such that $\mathrm{End}^0(B) = K$ is a CM field of degree $2d$. Denote by $M \subset K$ the maximal totally real subfield. The action of K on $\mathrm{Lie}\, B$ gives, after base-change from E to an extension $\iota \colon E \hookrightarrow C$ which splits K, a CM type (K, Σ) over C, namely $\Sigma = \Sigma(B, \iota)$ is a set of representatives for the action of $\mathrm{Gal}(K/M)$ on $\mathrm{Hom}(K, C)$. Finally, to the CM type (K, Σ) we can associate its reflex CM type (K^*, Σ^*); the reflex field $K^* = K^*(\iota)$ (which depends on ι) comes as a subfield $K^* \subset E$. The set $\Sigma_E := \mathrm{Inf}_{E/K^*} \Sigma^* \subset \mathrm{Hom}(E, C)$, consisting of those embeddings whose restriction to K^* belongs to Σ^*, is a CM type of E. Finally, the CM type $\Sigma(\iota)$ gives rise to a homomorphism $N_{\Sigma(\iota)^*} \colon K^*(\iota)^\times \to K^\times$ called the *reflex norm*. The homomorphism

$$N_{\Sigma_E} := N_{\Sigma(\iota)^*} \circ N_{E/K^*(\iota)} \colon E^\times \to K^\times \tag{2.1}$$

(5) Note that, as p-adic heights factor through the p-adic Abel–Jacobi map, their non-vanishing is a finer statement.

is independent of choices.

A CM type Σ' of K' with values in an extension C of $\mathbf{Q}_p$ splitting E is said to be a *p-adic CM type*[6] if its elements induce pairwise distinct p-adic places of K'. This condition can only be satisfied if all primes $\mathfrak{p}'^+$ of K'^+ (the maximal totally real subfield of K') above p split in K'. We may and will identify a p-adic CM type with a set of primes $\mathfrak{p}'$ of K' containing exactly one prime above each $\mathfrak{p}'^+|p$ of K'^+.

Lemma 2.1. — *Suppose that B has potentially* ordinary *reduction at all primes of E above p. Then:*

1. *for each embedding $\iota_p\colon E \hookrightarrow \overline{\mathbf{Q}}_p$, the set $\Sigma(B, \iota_p)$ is a p-adic CM type of K;*
2. *the prime p is totally split in K^*;*
3. *for each $\iota_p\colon E \hookrightarrow \overline{\mathbf{Q}}_p$, the set $\Sigma^*(\iota_p) := \Sigma(B, \iota_p)^*$ is a p-adic CM type of K^*.*

Proof. — Part (1), which is in fact equivalent to the hypothesis of the lemma, can be checked after base-change from E to a finite extension over which B acquires good reduction. There it becomes a well-known immediate consequence of the Shimura–Taniyama formula [18, (2.1.4.1)]. Part (1) implies part (2) by [29, Proposition 7.1]. Part (3) is implied by part (2). $\square$

The main theorem of Complex Multiplication attaches to B a character

$$\lambda\colon E^\times_{\mathbf{A}^\infty} \to K^\times$$

such that

$$\text{for all } \tau\colon E \hookrightarrow \mathbf{C},\tag{2.2}$$
$$\lambda^\tau := \tau\circ\lambda\cdot\left(N^\tau_{\Sigma_E,\infty}\right)^{-1} : E^\times_{\mathbf{A}} \to \mathbf{C}^\times \quad \text{satisfies } \lambda^\tau_{|E^\times} = 1,$$

$$\lambda(x)\lambda(x)^\rho = |x|^{-1}_{\mathbf{A}^\infty} \quad \text{for all} \quad x \in E^\times_{\mathbf{A}^\infty}.\tag{2.3}$$

Here ρ is the complex conjugation in K and $N^\tau_{\Sigma_E,\infty}\colon E^\times_\infty \to K^\times_\infty \to K^\times_\tau$ is the continuous extension of N_{Σ_E}. We say that λ^τ is an an algebraic Hecke character of infinity type Σ^τ_E, where $\Sigma^\tau_E \subset \mathrm{Hom}(E, \mathbf{C})$ is defined by

$$N^\tau_{\Sigma_E,\infty}(x) = \prod_{\iota\in\Sigma^\tau_E} \iota(x) \quad \text{for all} \quad x \in E^\times_\infty.$$

The L-function of B is

$$L(B, s) = L(s, \lambda) := (L(s, \lambda^\tau))_{\tau\in\mathrm{Hom}(K,\mathbf{C})} \in K\otimes\mathbf{C} \cong \mathbf{C}^{\mathrm{Hom}(K,\mathbf{C})};\tag{2.4}$$

it satisfies a functional equation with centre at $s = 1$.

[6] In some of the literature this is called a *p-ordinary CM type*.

Suppose now that $B = A_E$ for an abelian variety A/F with $\mathrm{End}^0(A) = M$ (the maximal totally real subfield of K). Then $E = K^*F$ [42, Remark 20.5].

Suppose conversely that λ is a Hecke character of E satisfying the conditions (2.2), (2.3) for a CM type (K, Σ). Then by a theorem of Casselman (e.g. [18, Theorem 2.5.2]), there is an abelian variety $B = B_\lambda/E$, unique up to E-isogeny, satisfying (2.4). The abelian variety B is simple if and only if the CM type (K, Σ) is not induced from a CM type of a subfield of K.

2.2. Theta lifts of Hecke characters

Let $\overline{\mathbf{Q}}$ denote an algebraic closure of K, let $\chi_0\colon E^\times\backslash E^\times_{\mathbf{A}} \to \overline{\mathbf{Q}}^\times$ be finite order character, and let ψ be a Hecke character of E with the same CM type as the character λ from the Introduction. We suppose that

$$\chi_0|_{F^\times_{\mathbf{A}}} = \omega := \eta \cdot \psi|_{F^\times_{\mathbf{A}}} \cdot |\ |_{\mathbf{A}_F}. \tag{2.5}$$

Let $K' \subset \overline{\mathbf{Q}}$ be a CM extension of K containing the values of χ_0 and $\psi|_{E^\times_{\mathbf{A}^\infty}}$.

2.2.1. The abelian variety associated with ψ

By construction, ψ satisfies conditions (2.2), (2.3) for the CM type $(K', \mathrm{Inf}_{K'/K}\,\Sigma)$, and in particular it is associated with an abelian variety $B_{\psi,K'}/E$ of dimension $[K' : \mathbf{Q}]/2$, which is uniquely determined up to E-isogeny, has CM by K', and is K-linearly isogenous to a sum of copies of a simple CM abelian variety. (Note that $B_{\psi,K'}$ depends on the choice of K': if $K' \subset K''$ is a finite extension of CM fields of degree d', then $B_{\psi,K''} \sim B^{\oplus d'}_{\psi,K'}$.) We denote by $B^\natural_\psi$ any one of the simple E-isogeny factors of $B_{\psi,K'}$.

On the other hand, the theta correspondence attaches to ψ a $\mathrm{Gal}(M/\mathbf{Q})$-conjugacy class (for some totally real field $M \subset K'$) of cuspidal automorphic representations $\sigma = (\sigma^{\tau_M})_{(\tau_M \in \mathrm{Hom}(M,\mathbf{C}))}$ of $\mathrm{Res}_{F/\mathbf{Q}}\mathbf{GL}_2$ of parallel weight 2; namely $\sigma^{\tau_M} = \theta(\psi^{\tau'})$ if $\tau'\colon K' \hookrightarrow \mathbf{C}$ satisfies $\tau'|_M = \tau_M$.(7)

Let $A := A_\sigma/F$ be the simple abelian variety associated with $\sigma = \theta(\psi)$, which is determined uniquely up to F-isogeny. Suppose that

$$\varepsilon\left(1/2, \sigma_E \otimes \chi_0^{-1}\right) = -1 \tag{2.6}$$

As discussed in the introductions to [45] or [21], the condition (2.6) guarantees that A can be found as an isogeny factor of the Jacobian of a Shimura

(7) In the terminology of [21, 45], σ is an *M-rational* representation.

curve over F; its endomorphism algebra $\mathrm{End}^0 A$ is a totally real field of dimension $d = \dim A$, which can be identified with M.

For any embedding $\tau_M \colon M \hookrightarrow \mathbf{C}$, we have $L(s, \sigma^{\tau_M}) = L(s, \psi^{\tau'})$ for $\tau' \colon K \hookrightarrow \mathbf{C}$ such that $\tau'|_M = \tau_M$. The following consequences of this identity are proved in [23] (note that the CM type of $\psi^{\tau'}$ is Σ_E):

(1) A acquires complex multiplication by K over some finite extension of F, and in fact, as remarked before, a minimal such extension is $K^* F = E$;
(2) A_E is isogenous to a sum of copies of the abelian variety $B_\psi^\natural$ defined above.

As M is contained in the maximal real subfield of K', the dimension $[M : \mathbf{Q}]$ of A_E divides the dimension $[K' : \mathbf{Q}]/2$ of $B_{\psi, K'}$; hence in fact $B_{\psi, K'}$ is K-linearly isogenous to a sum of copies of A_E, i.e. for some $r \geqslant 1$,

$$B_{\psi, K'} \sim A_E^{\oplus r}. \tag{2.7}$$

2.3. Rankin–Selberg p-adic L-function

Moving to a more general context for this subsection only, let F be a totally real field, E a CM quadratic extension of F. Let M be a number field and let σ be an M-rational automorphic representation of $\mathrm{Res}_{F/\mathbf{Q}} \mathbf{GL}_2$ of parallel weight 2, with central character ω. Let M_v be a p-adic completion of M, let L be a finite extension of M_v, and let

$$\chi_0 \colon E^\times \backslash E^\times_{\mathbf{A}} \to L^\times$$

be a finite-order character satisfying $\chi_0|_{F^\times_{\mathbf{A}}} = \omega$. (We will later specialise to the situation $\sigma = \theta(\psi)$ considered in Section 2.2). We recall the definition of a p-adic Rankin–Selberg L-function on $\mathscr{Y}_L$ attached to the base-change σ_E of σ to E twisted by χ_0^{-1}.

Given a place $\wp | p$ of F, we say that σ is *nearly ordinary at* $\wp$ with unit characters $\alpha_\wp$ if, after possibly enlarging L, there exist characters $\alpha_\wp \colon F_\wp^\times \to \mathscr{O}_L^\times$, such that $\sigma_\wp \otimes L$ is either special $\alpha_\wp \cdot \mathrm{St}$ with character $\alpha_\wp$, or irreducible principal series $\mathrm{Ind}(|\cdot|_\wp \alpha_\wp, \beta_\wp)$ (un-normalised induction) for some other character $\beta_\wp$.

Theorem 2.2. — *Suppose that for all $\wp | p$, σ is nearly ordinary at $\wp$. Then there is a function*

$$L_p\left(\sigma_E \otimes \chi_0^{-1}\right) \in \Lambda$$

characterised by the interpolation property

$$L_p\left(\sigma_E \otimes \chi_0^{-1}\right)(\chi') = e_p\left(\sigma_E^\iota \otimes (\chi_0\chi')^{\iota,-1}\right) \cdot \frac{L^{(p)}\left(1/2, \sigma_E^\iota \otimes (\chi_0\chi')^{\iota,-1}\right)}{\Omega_\sigma^\iota}$$

for all sufficiently p-ramified[8] *finite order characters $\chi' \colon \Gamma \to \overline{\mathbf{Q}}^\times$ and $\iota \colon \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Here*

$$\Omega_\sigma^\iota := L(1, \sigma^\iota, \mathrm{ad}) \ \text{ and } \ e_p\left(\sigma_E^\iota \otimes (\chi_0\chi')^{\iota,-1}\right) = \prod_{\wp | p} e_\wp\left(\sigma_E^\iota \otimes (\chi_0\chi')^{\iota,-1}\right)$$

with

$$e_\wp\left(\sigma_E^\iota \otimes (\chi_0\chi')^{\iota,-1}\right) = \varepsilon\left(0, \iota(\alpha_\wp^{-1}\chi_{0,\mathfrak{p}}\chi'_\mathfrak{p})\right) \cdot \varepsilon\left(0, \iota(\alpha_\wp^{-1}\chi_{0,\mathfrak{p}^c}\chi'_{\mathfrak{p}^c})\right). \tag{2.8}$$

Proof. — This is essentially [21, Theorem A]. Our $L_p(\sigma_E \otimes \chi_0^{-1})$ differs from the one of loc. cit. by the involution $\chi' \mapsto \chi'^{-1}$, the shift χ_0^{-1}, and some algebraic constants. Regarding the interpolation factors, note that if $\chi'_\mathfrak{p}$, $\chi'_{\mathfrak{p}^c}$ are sufficiently ramified then all local L-values in the interpolation formula from [21] are equal to 1. The relation between the Gauß sums used in [21] and our epsilon factors (2.8) follows from e.g. [14, (23.6.2)]. □

2.4. Katz p-adic L-function

From now until the end of this paper, let K, E, Σ_E be as in Section 2.1. Fix a p-adic place w of K, and denote by v be the induced place of the maximal real subfield $M \subset K$. We let $\overline{\mathbf{Q}} \subset \overline{\mathbf{Q}}_p \supset K_w$ be algebraic closures of K and of K_w, and let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$. We also let L be a sufficiently large finite extension of K_w inside $\overline{\mathbf{Q}}_p$ as in the introduction.

2.4.1. p-adic CM type associated with w

Let $N_{\Sigma_E,p}^{(w)} \colon E_p^\times \to K_p^\times \to K_w^\times$ be the continuous extension of N_{Σ_E}. We let

$$\Sigma_E^{(w)} := \left\{\mathfrak{p} \,|p \text{ prime of } E : |\cdot\Big|_w \circ N_{\Sigma_E,p}^{(w)}\Big|_{E_\mathfrak{p}^\times} \text{ is a non-trivial norm on } E_\mathfrak{p}^\times\right\},$$

which, by Lemma 2.1 and the construction of N_{Σ_E}, has the property that $\Sigma_E^{(w)} \sqcup \Sigma_E^{(w)}c$ is the set of all primes $\mathfrak{p}$ of E above p. Hence $\Sigma_E^{(w)}$ is a p-adic CM type of E, identified with a set of embeddings $E \hookrightarrow \overline{\mathbf{Q}}_p$.

[8] See the proof for the precise meaning.

Lemma 2.3. — *Every prime of F above p splits in E, and the CM type $\Sigma_E^{(w)}$ is a p-adic CM type of E.*

Proof. — The first assertion is Lemma 2.1 part (2). The second assertion follows from Lemma 2.1 part (3), after noting that $\Sigma_E^{(w)} = \mathrm{Inf}_{E/K^*} \Sigma^*(\iota_p)$ for any $\iota_p \colon E \hookrightarrow \overline{\mathbf{Q}}_p$ inducing a prime in Σ_E. $\square$

2.4.2. p-adic Hecke characters

Assume that L splits E, and let

$$\chi\colon E^\times \backslash E^\times_{\mathbf{A}^\infty} \to L^\times$$

be a locally algebraic p-adic character. We say that χ is a Hecke character of p-adic infinity type $k \in \mathbf{Z}[\mathrm{Hom}(E, L)]$ if there exists an open subgroup $U \subset E^\times_{\mathbf{A}^\infty}$ such that

$$\chi(t) = \prod_{\sigma \in \mathrm{Hom}(E,L)} t_\sigma^{-k_\sigma}$$

for $t \in U$. Note that this definition differs from the one in some of the literature (e.g. [27] and [25]) by a sign in the exponents; it agrees with [5].

If χ is a locally algebraic character of p-adic infinity type k and $\iota\colon L \to \mathbf{C}$ is an embedding, we can define the ι-avatar $\chi^\iota\colon E^\times_{\mathbf{A}} \to \mathbf{C}^\times$ similarly to [32, Definition 1.5]. The embedding ι induces an isomorphism $\mathrm{Hom}(E, L) \to \mathrm{Hom}(E, \mathbf{C})$ by $\sigma \mapsto \iota \circ \sigma$; the infinity type $k^\iota \in \mathbf{Z}[\mathrm{Hom}(E, \mathbf{C})]$ of χ^ι corresponds to k under this bijection. The association $\chi \to \chi^\iota$ defines a bijection between locally algebraic Hecke characters over E with values in L and arithmetic Hecke characters over E in the usual sense.

For example, if λ is as in Section 2.1, the character

$$\lambda^{(w)}(x) := \lambda(x) N^{(w)}_{\Sigma_E,\, p}(x_p)^{-1} \colon E^\times \backslash E^\times_{\mathbf{A}^\infty} \to K_w^\times$$

is a Hecke character of p-adic infinity type $\Sigma_E^{(w)}$.[(9)] The place w being fixed, in the rest of this paper we will often abuse notation by simply writing λ in place of $\lambda^{(w)}$.

2.4.3. p-adic L-function

Let now L be an extension of K_w splitting E, and let Σ_E be a p-adic CM type of E over some L. Let $E_\infty^\sharp$ be a finite extension of E_∞ contained in E^{ab}, and let $\Gamma^\sharp := \mathrm{Gal}(E^\sharp_\infty/E)$. Let $\Lambda^\sharp := \mathbf{Z}_p[\![\Gamma^\sharp]\!]_L$, $\mathscr{Y}^\sharp := \mathrm{Spec}\,\Lambda^\sharp$.

(9) Strictly speaking this assertion holds after considering $\lambda^{(w)}$ as valued in some extension $L \supset K_w$ splitting E.

By [27, 30] and the first assertion of Lemma 2.3, there is an element

$$L_{\Sigma_E} \in \Lambda^\sharp$$

uniquely characterised by the interpolation property that we now describe. The domain of interpolation consists of locally algebraic p-adic Hecke characters $\lambda'\colon \Gamma^\sharp \to \overline{\mathbf{Q}}_p^\times$ with infinity type

$$k\Sigma_E^{(w)} + \kappa(1-c)$$

for $k \in \mathbf{Z}$, $\kappa \in \mathbf{Z}[\Sigma_E^{(w)}]$ such that

(i) $k \geqslant 1$, or

(ii) $k \leqslant 1$ and $k\Sigma_E^{(w)} + \kappa \in \mathbf{Z}_{>0}[\Sigma_E^{(w)}]$.

The interpolation property is then the following. There exist explicit p-adic periods

$$\Omega_{\Sigma_E} = (\Omega_{\Sigma_E,\,\mathfrak{p}})_{\mathfrak{p}\,\in\,\Sigma_E} \in \left(\bar{\mathbf{Z}}_p^\times\right)^{\Sigma_E}$$

and, for each complex CM type $\Sigma_{E,\infty}$ of E, complex periods

$$\Omega_{\Sigma_{E,\,\infty}} = (\Omega_{\Sigma_{E,\,\infty},\tau})_{\tau\,\in\,\Sigma_{E,\,\infty}} \in \left(\mathbf{C}^\times\right)^{\Sigma_{E,\infty}}$$

(both defined in [27, (4.4)]) such that for any character $\lambda'\colon \Gamma \to \overline{\mathbf{Q}}_p^\times$ in the domain of interpolation, and any $\iota\colon L(\lambda') \hookrightarrow \mathbf{C}$, we have[10]

$$\iota\left(\frac{L_{\Sigma_E}(\lambda')}{\Omega_{\Sigma_E}^{k+2\kappa}}\right) = e_p\left((\lambda')^{-1}\right)^\iota\Big) \cdot \frac{L^{(p)}\left(0,((\lambda')^{-1})^\iota\right)}{\Omega_{\Sigma_{E,\,\iota},}^{k+2\kappa}} \cdot \frac{\pi^\kappa\Gamma_\Sigma(k\Sigma_{E,\,\iota}+\kappa)}{(\Im\vartheta)^\kappa} \cdot \frac{[\mathscr{O}_E^\times : \mathscr{O}_F^\times]}{\sqrt{|D_F|}}$$

In the interpolation formula, $\Sigma_{E,\iota}$ is the complex CM type induced from $\Sigma_E^{(w)}$ via ι, and we then identify κ with $\kappa^\iota \in \mathbf{Z}_{\geqslant 0}[\Sigma_{E,\iota}]$; when $k \in \mathbf{Z}$ appears in the exponent of one of the periods Ω_Σ it is considered as $k \cdot \sum_{\tau\,\in\,\Sigma_E^\iota} \tau$. If $\chi = (\lambda')^{-1}$ is ramified at all $\mathfrak{p}|p$, the p-Euler factor is given by

$$e_p(\chi^\iota) = \prod_{\mathfrak{p}\,\in\,\Sigma_E^{(w)}} e(\chi_\mathfrak{p}^\iota)$$

for (dropping all superscripts ι)

$$e(\chi_\mathfrak{p}) = \frac{L(0,\chi_\mathfrak{p})}{\varepsilon(0,\,\chi_\mathfrak{p})L\left(1,\chi_\mathfrak{p}^{-1}\right)} \tag{2.9}$$

[10] Note that we are ignoring interpolation factors at places away from p appearing elsewhere in the literature, since those, while non-integral, can be interpolated by polynomial functions on $\mathscr{Y}^\sharp$.

Finally, $\Gamma_\Sigma(k\Sigma_{E,\iota}+\kappa)=\prod_{\tau\in\Sigma_{E,\iota}}\Gamma(k+\kappa_\tau)$ for the usual Γ-function, and $\vartheta\in E$ as in [28, Section 3.1]. All local epsilon factors in this paper are understood with respect to some uniform choice of additive characters of $E_\mathfrak{p}$ of level one for all $\mathfrak{p}|p$.

For a later consideration, we fix a sufficiently large extension $E^\sharp$ as above and consider the restriction of L_{Σ_E} to certain open subsets of $\mathscr{Y}^\sharp$: if λ_0 is a p-adic Hecke character factoring through $\Gamma^\sharp$ with values in L, and Σ_E is a p-adic CM type, we define

$$L_{\Sigma_E,\,\lambda_0}\in\Lambda$$

by

$$L_{\Sigma_E,\,\lambda_0}(\chi'):=L_{\Sigma_E}(\lambda_0\chi').$$

We will henceforth drop the superscript w from the notation for the character $\lambda^{(w)}$.

2.5. Factorisation of the Rankin–Selberg p-adic L-function

Let $\sigma:=\theta(\psi)$ as in Section 2.2. There is a factorisation

$$L\left(s-1/2,\sigma_E\otimes(\chi_0\chi')^{-1}\right)=L\left(s,\psi(\chi_0\chi')^{-1}\right)L\left(s,\psi^*(\chi_0\chi')^{-1}\right)\qquad(2.10)$$

of complex (more precisely $K\otimes\mathbf{C}$-valued) L-functions, valid for algebraic Hecke characters χ' over E. It implies the following factorisation of p-adic L-functions.

Lemma 2.4. — *Let $L_p(\sigma_E\otimes\chi_0^{-1})$ be the p-adic L-function associated with $\sigma=\theta(\psi)$ and the embedding $M\subset M_v\subset K_w$, where w is the place of K fixed above and v its restriction to M. Let L be a finite extension of K_w splitting E. We have*

$$L_p\left(\sigma_E\otimes\chi_0^{-1}\right)\doteq\frac{L_{\Sigma_E,\,\psi\chi_0^{*-1}}}{\Omega_{p,\,\Sigma_E}}\cdot\frac{L_{\Sigma_E c,\,\psi^*\chi_0^{*-1}}}{\Omega_{p,\,\Sigma_E c}}\qquad(2.11)$$

in $\Lambda=\Lambda_L$, where we use the symbol $\doteq$ to signify an equality which holds up to multiplication by a constant in $\overline{\mathbf{Q}}^\times$.

Proof. — We evaluate both sides of the proposed equality at finite order characters χ' of Γ which are sufficiently ramified in the sense that, for all primes $\mathfrak{p}|p$ of E, $\chi_\mathfrak{p}$ has conductor larger than the conductors of $\psi_\mathfrak{p}$ and $\chi_{0,\mathfrak{p}}$. This is sufficient as such characters are dense in $\mathscr{Y}^-$ (cf. [21, Lemma 10.2.1]).

Note first the self-duality relation

$$\lambda'^{*} = \lambda'^{-1} |\cdot|_{\mathbf{A}_E}^{-1} \tag{2.12}$$

(where $|\cdot| = |\cdot|_{\mathbf{A}_E}$), valid for both $\lambda' = \psi\chi_0^{-1}$ and $\lambda' = \psi\chi_0^{*-1}$ (this follows from (2.5)). Evaluating (2.10) at $s = 1$, we find

$$\begin{aligned} L\left(1/2, \sigma_E \otimes (\chi_0\chi')^{-1}\right) &= L\left(0, \psi|\cdot|\,(\chi_0\chi')^{-1}\right) L\left(0, \psi^*|\cdot|\,(\chi_0\chi')^{-1}\right) \\ &= L\left(0, (\psi^*\chi_0^{*-1}\chi')^{-1}\right) L(0, \left(\psi\chi_0^{*-1}\,\chi')^{-1}\right). \end{aligned}$$

Therefore the L-values agree with the ones being interpolated by the p-adic L-functions in (2.11). We now compare the local Euler-like interpolation factors and the complex periods.

Recall from e.g. [26, p. 119] that, for $\wp\mathscr{O}_E = \mathfrak{p}\mathfrak{p}^c$, we have

$$\sigma_\wp \simeq \pi(\psi_\mathfrak{p}, \psi_{\mathfrak{p}^c})$$

By the construction of $\Sigma_E^{(w)}$ and the Shimura–Taniyama formula for Hecke characters (see [18, Proposition A.4.7.4 (ii)], cf. also the paragraph after Example A 4.8.3 ibid.), the character $\psi_\mathfrak{p}$ has values in w-adic units if and only if $\mathfrak{p} \notin \Sigma_E^{(w)}$, equivalently $\mathfrak{p} \in \Sigma_E^{(w)}c$.

Denoting by $\wp$ a fixed prime of F above p and by $\mathfrak{p}$ the unique prime in $\Sigma_E^{(w)}$ above $\wp$, it follows that $\alpha_\wp = \psi_{\mathfrak{p}^c}$ under the identification $E_{\mathfrak{p}^c} = F_\wp$. Then, under our assumption on the ramification of χ', we have

$$e_\wp\left(\sigma_E \otimes \chi_0^{-1}, \chi'\right) = \varepsilon\left(0, \psi_{\mathfrak{p}^c}^{-1}\chi_{0,\,\mathfrak{p}}\chi'_\mathfrak{p}\right) \cdot \varepsilon\left(0, \psi_{\mathfrak{p}^c}^{-1}\chi_{0,\,\mathfrak{p}^c}\chi'_{\mathfrak{p}^c}\right) \tag{2.13}$$

whereas the Katz interpolation factors above $\wp$ are

$$\varepsilon\left(0, \psi_\mathfrak{p}^{-1}\chi_{0,\,\mathfrak{p}^c}\chi_\mathfrak{p}'^{-1}\right)^{-1} \cdot \varepsilon\left(0, \psi_\mathfrak{p}^{-1}\chi_{0,\,\mathfrak{p}}\chi_{\mathfrak{p}^c}'^{-1}\right)^{-1}.$$

By the functional equation

$$\varepsilon(s,\chi)^{-1} = \varepsilon(1-s,\chi^{-1})\chi(-1)$$

valid for any character χ of a local field (e.g. [14, (23.4.2)]) and the self-dualities (2.12), these equal

$$\begin{aligned} &\varepsilon\left(0, |\cdot|\psi_\mathfrak{p}\chi_{0,\mathfrak{p}^c}^{-1}\chi'_\mathfrak{p}\right) \cdot \varepsilon\left(0, |\cdot|\psi_\mathfrak{p}\chi_{0,\mathfrak{p}}^{-1}\chi'_{\mathfrak{p}^c}\right) \\ = {}&\varepsilon\left(0, \psi_{\mathfrak{p}^c}^{-1}\chi_{0,\mathfrak{p}}\chi'_\mathfrak{p}\right) \cdot \varepsilon\left(0, \psi_{\mathfrak{p}^c}^{-1}\chi_{0,\mathfrak{p}^c}\chi'_{\mathfrak{p}^c}\right) \end{aligned}$$

matching (2.13).

In regard to periods, we first note that the periods in the asserted equality are independent of χ'. As

$$\mathrm{Ad}(\sigma) \simeq \eta \oplus \mathrm{Ind}_E^F\left(\psi(\psi^*)^{-1}\right),$$

we have

$$L(1,\sigma^\iota,\mathrm{ad}) = L(1,\eta)L\left(1,\psi(\psi^*)^{-1}\right) = L(1,\eta)\,L\left(0,(\psi(\psi^*)^{-1})^D\right)$$

The infinity type of $\iota(\psi(\psi^*)^{-1})^D$ is $2\Sigma_E^\iota$. From the algebraicity of Hecke L-values due to Shimura, we have

$$L\left(0,(\psi(\psi^*)^{-1})^D\right) \doteq \Omega^2_{\Sigma_E^\iota}$$

(see [40] and [27, Section 4], especially [27, pp. 215–16]). Moreover, we have a period relation

$$\Omega^1_{\Sigma_E^\iota} \doteq \Omega^1_{\Sigma_E^\iota\circ c}$$

(see [41] and [42, Theorem. 32.5]).

It follows that

$$\Omega^\iota_\sigma \doteq \Omega^2_{\Sigma_E^\iota} \doteq \Omega_{\Sigma_E^\iota}\Omega_{\Sigma_E^\iota\circ c}.$$

□

2.6. Construction of an auxiliary character

We will now look for a character χ_0 suitable for our purposes. If a Hecke character λ' of $E^\times_{\mathbf{A}}$ satisfies (2.12), then its functional equation relates $L(s,\lambda')$ with $L(1-s,\lambda'^{-1}) = L(2-s,\lambda'^*) = L(2-s,\lambda')$; the sign of this functional equation is the root number

$$w(\lambda') := \varepsilon(1,\lambda') \in \{\pm 1\}.$$

We say that finite-order character $\chi\colon E^\times\backslash E^\times_{\mathbf{A}^\infty} \to \overline{\mathbf{Q}}^\times$ is *anticyclotomic* if it satisfies the following two conditions, which are equivalent by [24, Lemma 5.31]:

(1) $\chi^* = \chi^{-1}$;

(2) there exists a finite-order character $\chi_0\colon E^\times\backslash E^\times_{\mathbf{A}^\infty} \to \overline{\mathbf{Q}}^\times$ such that

$$\chi = \chi_0/\chi_0^*. \tag{2.14}$$

Lemma 2.5. — *Let λ be a Hecke character satisfying* (2.12). *Suppose that the extension E/F is ramified. Then there exist an anticyclotomic finite-order character $\chi = \chi_0/\chi_0^*\colon E^\times\backslash E^\times_{\mathbf{A}} \to \overline{\mathbf{Q}}^\times$ such that the root number*

$$w(\lambda\chi) = +1.$$

Proof. — Suppose first that λ satisfies the following condition. (We will later reduce to this case.)

$(*)$ there is a prime $\wp$ of $E,$ ramified over $F,$ such that $\mathrm{ord}_{\wp}(\mathfrak{C})$ is odd,

where $\mathfrak{C}$ is the conductor of λ. In particular, the norm ideal $N_{E/F}(\mathfrak{C})$ is not a square. For a quadratic character χ' over F, let $\chi'_E = \chi' \circ N_{E/F}$ be the corresponding Hecke character over E. By definition, χ'_E is a quadratic Hecke character over E and also anticyclotomic. For the latter, note that

$$\chi'_E(a)\chi'^{*}_E(a) = \chi'\big(N_{E/F}(a)\big)\chi'\big(N_{E/F}(c(a))\big) = \chi'\big(N_{E/F}(a)\big)^2 = 1.$$

We consider twists of λ by characters of the form χ'_E with the conductor of χ'_E prime to $\mathfrak{C}$. Recall that the twist of a self-dual character by an anticyclotomic character is again self-dual. To prove existence of twist with change in the root number, it thus suffices to show that χ' can be chosen so that $\chi'(N_{E/F}(\mathfrak{C}))$ takes value 1 or -1. The sufficiency follows from the explicit root number formula for the twist $\lambda\chi'_E$ in [44, (3.4.6)]. As the norm ideal $N_{E/F}(\mathfrak{C})$ is not a square, the existence of desired χ' follows readily.

To reduce to condition $(*)$, it suffices, given λ, to find $\lambda' = \lambda\chi_1/\chi_1^*$ satisfying $(*)$. Let $\wp$ be a prime of E ramified in E/F and let $\delta := \mathrm{ord}_{\wp}(2)$. Let r be an odd integer greater than $\delta + 2$ and the exponent of $\wp$ in the conductor of λ. Let χ_1 be a finite order character of $E_{\mathbf{A}}^{\times}/E^{\times}$ such that the exponent of $\wp$ in the conductor of χ_1 is exactly $r - \delta$. As $(1 + \wp^{t-\delta}\mathscr{O}_E)^2 = 1 + \wp^t\mathscr{O}_E$ for all $t \geqslant \delta + 1$, the conductor of $\chi_{1,\wp}^2$ is exactly $\wp^r$. Letting $\chi = \chi_1/\chi_1^*$, and denoting by ϖ a uniformiser at $\wp$, for any $t \geqslant \delta + 1$ we have for any $t \geqslant (r - \delta)/2$:

$$\chi_{\wp}\left(1 + \varpi^t a\right) = \chi_{1,\wp}\left(1 + \varpi^t a\right)\chi_{1,\wp}\left(1 - \varpi^t a\right)^{-1} = \chi_{1,\wp}\left(1 + \varpi^t a\right)^2.$$

It follows that the conductor of $\chi_{\wp}$ is the same as the conductor of χ_1^2, that is, $\wp^r$; by our choice of r the same is true of $\lambda\chi$, and in particular λ' satisfies $(*)$. $\square$

Remark 2.6. — Let $\chi = \chi_0\chi_0^{*-1}$ be as in the Lemma 2.5 and let $\psi := \lambda\chi_0$, where λ is as fixed in the Introduction. Then ψ satisfies (2.5); by the analogous formula to (2.10), $\sigma := \theta(\psi)$ has root number -1 (i.e., it satisfies (2.6)).

2.7. p-adic Gross–Zagier formula

We recall a formula relating Rankin–Selberg p-adic L-functions to points on abelian varieties; we will later deduce from it an analogous formula for

Katz p-adic L-functions. Let σ, $A = A_\sigma$, and χ_0 be as in Theorem 2.2, and assume that E/F splits at each prime above p. and that

$$\varepsilon\left(1/2, \sigma_E \otimes \chi_0^{-1}\right) = -1. \tag{2.15}$$

Theorem 2.7. — *Let $\circ = \emptyset$ or $\circ = \wp$ for a prime $\wp|p$ of F. There is a 'pair of points'*

$$\mathscr{P}_{\chi_0} \otimes \mathscr{P}^\vee_{\chi_0} \in A_E\left(\chi_0^{-1}\chi_{\mathrm{univ},\circ}^{-1}\right) \otimes_{\Lambda_\circ^-} A_E^\vee(\chi_0\chi_{\mathrm{univ},\circ}) \otimes_{\Lambda_\circ^-} \mathscr{K}_\circ^-,$$

such that

$$\left\langle \mathscr{P}_{\chi_0} \otimes \mathscr{P}^\vee_{\chi_0} \right\rangle = L_p'\left(\sigma_E \otimes \chi_0^{-1}\right)$$

in $\mathscr{K}_\circ^-$. Here $\langle x \otimes y\rangle := \langle x, y\rangle$ is the big height pairing relative to the cyclotomic logarithm as in (1.6).

Proof. — This follows from [21, Theorem C.4]. Consider the scheme $\mathscr{Y}_{V^p}/L$ corresponding to the rigid space with that name in loc. cit. (in the sense that our $\mathscr{Y}_{V^p}$ is the spectrum of the ring of bounded functions on the space $\mathscr{Y}_{V^p}$ of [21]), for a choice of level $V^p \subset E^\times_{\mathbf{A}^{p\infty}}$; it parametrises continuous p-adic characters $\widetilde{\chi}$ of $E^\times_{\mathbf{A}^\infty}/E^\times V^p$ satisfying $\omega\widetilde{\chi}|_{F^\times_{\mathbf{A}^\infty}} = \mathbf{1}$. We denote by

$$\widetilde{\chi}_{\mathrm{univ}} \colon E^\times_{\mathbf{A}^\infty}/E^\times V^p \to \mathscr{O}(\mathscr{Y}_{\mathscr{V}})^\times$$

the universal character. Assume that $\omega|_{V^p} = \mathbf{1}$. Then we may identify $\mathscr{Y}^-$ with the connected component $\mathscr{Y}^\circ_{\chi_0} \subset \mathscr{Y}_{V^p}$ containing the character χ_0^{-1} via

$$\chi \mapsto \widetilde{\chi} = \chi_0^{-1}\chi^{-1}. \tag{2.16}$$

Using the notation of loc. cit. with the addition of a tilde, the p-adic Gross–Zagier formula proved there has the form

$$\left\langle \widetilde{\mathscr{P}}^+(f^{+,p}), \widetilde{\mathscr{P}}^-(f^{-,p})^\iota \right\rangle = \widetilde{L}_p'(\sigma_E) \cdot \widetilde{\mathscr{Q}}\left(f^{+,p}, f^{-,p}\right)\big|_{\mathscr{Y}_{\chi_0}} \quad \text{in } \Lambda^-$$

up to an explicit and nonzero rational constant. Here ι is the involution $\widetilde{\chi} \mapsto \widetilde{\chi}^{-1}$ on $\mathscr{Y}_{V^p}$, and the

$$\widetilde{\mathscr{P}}^\pm\left(f^{\pm,p}\right) \in A^\pm\left((\widetilde{\chi}_{\mathrm{univ}})^{\pm 1}\right)$$

are families of Heegner points associated with E/F and (the limits of certain sequences of) parametrisations $f^{\pm,p}$ of A and $A^\vee$ by a (tower of) Shimura curves. The term $\widetilde{\mathscr{Q}}(\cdot,\cdot) \in \Lambda^-$ is a product of local terms at primes not dividing p.

By results of Tunnell and Saito explained in [21, Introduction], under the assumption (2.15) for each $\chi \in \mathscr{Y}_{\chi_0}$ we may find families of Shimura curve parametrisations $f^{\pm,p}$ such that $\widetilde{\mathscr{Q}}(f^{+,p}, f^{-,p})(\chi) \neq 0$. Applying this result to a character χ in the image $\mathscr{Y}_{\chi_0,\circ}$ of $\mathscr{Y}_\circ^-$ under the isomorphism

$\mathscr{Y}^- \to \mathscr{Y}_{\chi_0}$, we find $f^{\pm,p}$ such that $\widetilde{\mathscr{Q}}(f^{+,p}, f^{-,p})|_{\mathscr{Y}_{\chi_0,\circ}} \neq 0$. Up to constants in $L^\times$, we have

$$L'_p\left(\sigma_E \otimes \chi_0^{-1}\right)(\chi) = -\widetilde{L}'_p(\sigma_E)\left(\chi_0^{-1}\chi^{-1}\right).$$

Then we may choose, using the identification (2.16)

$$\mathscr{P}_{\chi_0} \otimes \mathscr{P}^\vee_{\chi_0} := -\widetilde{\mathscr{Q}}\left(f^{+,p}, f^{-,p}\right)|^{-1}_{\mathscr{Y}_{\chi_0,\circ}} \cdot \widetilde{\mathscr{P}}\left(f^+\right)|_{\mathscr{Y}_{\chi_0,\circ}} \otimes \widetilde{\mathscr{P}}\left(f^-\right)^\iota|_{\mathscr{Y}_{\chi_0,\circ}}.$$

There are four conditions to be verified in order to be able to invoke the result of [21].(11) The first one is (weaker than) the potential ordinariness of A, which can be verified after base-change to E where it becomes the converse to Lemma 2.1 part (1). The second one is that all primes of F above p split in E: this is satisfied by Lemma 2.3. Finally, the conditions on the central character and root number are satisfied by Remark 2.6. □

2.8. Non-vanishing of p-adic L-functions

Let λ be the character fixed in the Introduction.

Theorem 2.8. — *Let $\chi = \chi_0/\chi_0^*$ be as in Lemma 2.5. For every $\wp|p$, the restriction of the anticyclotomic Katz p-adic L-function*

$$L_{\Sigma_E,\,\lambda\chi_0^*\chi_0^{-1}}|_{\mathscr{Y}_\wp^-}$$

does not vanish.

Proof. — By construction, $w(\lambda\chi_0^*\chi_0^{-1}) = +1$. The Theorem 2.8 thus follows from [11, Theorem B] combined with the main results of [25] and [28]. Here we only use the hypothesis $p \nmid 2D_F$. □

Theorem 2.9. — *For every $\wp|p$, the restriction of the cyclotomic derivative*

$$L'_{\Sigma_E c,\,\lambda^*}|_{\mathscr{Y}_\wp^-}$$

does not vanish.

Proof. — Recall from the Introduction that λ^* is self-dual with infinity type $\Sigma_E c$ and root number -1. The Theorem 2.9 thus follows from [11, Thmeorem C] combined with the main result of [7]. Here we use the hypothesis $p \nmid 2D_F h_E^-$. □

(11) It is crucial here that in [21] the sets of ramified primes of E/F and of bad-reduction primes for A are not required to be disjoint.

2.9. Proofs of main theorems

We introduce the useful category

$$\mathcal{CM}_{E,\,(K,\,\Sigma)},$$

described as follows. The objects are abelian varieties B over E of dimension equal to $\frac{d}{2}[K:\mathbf{Q}]$ for some $d \geqslant 1$, together with an inclusion $i\colon R \hookrightarrow \mathrm{End}^0(B)$ of a K-algebra R of dimension d such that the type of i is $(K, d\Sigma)$. For two objects $B = (B, R, i)$, $B' = (B, R', i')$ of $\mathcal{CM}_{E,\,(K,\,\Sigma)}$, let $R^\circ \subset R$ and $R^{\circ'} \subset R'$ be finite-index subrings in the integral closure of $\mathscr{O}_K$ in R, R' whose image by i, i' is contained in $\mathrm{End}(B)$, $\mathrm{End}(B')$ respectively. Let

$$\mathrm{preHom}_{\mathcal{CM}_{E,\,(K,\,\Sigma)}}(B, B')$$

be the set of pairs of morphisms (f, γ) with $f\colon B \to B'$, $\gamma\colon R \to R'$ such that $i'(\gamma(r)) \circ f = f \circ i(r)$ for any $r \in R$. This is a module over a sufficiently small order $\mathscr{O}$ in K, and we let

$$\mathrm{Hom}_{\mathcal{CM}_{E,\,(K,\,\Sigma)}}(B, B') := \mathrm{preHom}_{\mathcal{CM}_{E,\,(K,\,\Sigma)}}(B, B') \otimes_{\mathscr{O}} K.$$

If (B, i, R) is an object of $\mathcal{CM}_{E,\,(K,\,\Sigma)}$ and T is a finite-dimensional K-algebra, then Serre's construction provides a well-defined isomorphism class $(B \otimes_K T, i \otimes \mathrm{id}_T, R \otimes_K T)$ of objects in $\mathcal{CM}_{E,\,(K,\,\Sigma)}$, with action by the K-algebra $R \otimes_K T$.

Note that in $\mathcal{CM}_{E,\,(K,\,\Sigma)}$ any object (B, i, R) is isomorphic to one such that $\mathrm{End}(B)$ contains the integral closure R^{int} of $\mathscr{O}_K$ in R: namely, if $R^\circ \subset R^{\mathrm{int}}$ is an order conatined in $\mathrm{End}(B)$ we may take $B' := \mathrm{Hom}_{R^\circ}(R^{\mathrm{int}}, B)$. Given an object (B, i, R) of $\mathcal{CM}_{E,\,(K,\,\Sigma)}$ and a finite-order character $\chi\colon \mathrm{Gal}(H_\chi/E) \to K^\times$ we define the twist

$$B \otimes_K K(\chi)$$

(an object of $\mathcal{CM}_{E,\,(K,\,\Sigma)}$ with the R-action of induced from the one on B) as follows. Assume that $R \supset \mathscr{O}_K$, which as noted above is not restrictive. Then we may regard χ as an element in $H^1(\mathrm{Gal}(H_\chi/E), \mathscr{O}_K^\times) \subset H^1(\mathrm{Gal}(H/E), \mathrm{Aut}_K(B))$; the abelian variety $B \otimes_K K(\chi)$ is the corresponding inner twist (denoted by B^χ in [5]), so that for any finite character $\chi'\colon \mathrm{Gal}(\overline{E}/E) \to K^\times$ we have $(B \otimes K(\chi)(\chi') = B(\chi\chi')$.

Let us now return to our usual setting, so that $B = B_\lambda$. It is an object of $\mathcal{CM}_{E,(K,\Sigma)}$ with $R = K$. (Note that, as the validity of the statements we are interested in is invariant under K-linear isogenies, it is appropriate to work in this category.) Let $\chi_0\colon \mathrm{Gal}(\overline{E}/E) \to K'^\times$ be a finite-order character, where K' is the CM extension of K fixed above. Let $\psi := \lambda\chi_0^{-1}$ and let

$A = A_\sigma$ be the abelian variety associated with $\sigma = \theta(\psi)$ as in 2.2. The abelian varieties A_σ and $B_{\psi, K'}$ have CM by K'.

Lemma 2.10. — *There is an isomorphism in $\mathcal{CM}_{E,(K',\Sigma)}$*

$$f\colon B_\lambda \otimes_K K' \to B_{\psi,K'} \otimes_{K'} K'\left(\chi_0^{-1}\right) \cong A_E^{\oplus r} \otimes_K K'\left(\chi_0^{-1}\right).$$

Proof. — The second isomorphism is (2.7). The proof of the first one, based on Casselman's theorem, is entirely analogous to the proof of [5, Lemma 2.9]. □

Proof of Theorem 1.2. — We prove the p-adic Gross–Zagier formula of Theorem 1.2. Let χ_0 be as in Lemma 2.5 for $\lambda' := \lambda^*$, let $A = A_{\theta\left(\lambda\chi_0^{-1}\right)}$ and let

$$\mathscr{P}_{\chi_0} \otimes \mathscr{P}^\vee_{\chi_0} \in A_E\left(\chi_0^{-1}\chi_{\mathrm{univ},\circ}^{-1}\right) \otimes_{\Lambda_\circ^-} A_E^\vee(\chi_0\chi_{\mathrm{univ},\circ}) \otimes_{\Lambda_\circ^-} \mathscr{K}_\circ^-$$

be as in Theorem 2.7. Let f be as in Lemma 2.10 and let

$$\mathscr{P} \otimes \mathscr{P}^\vee := L_{\Sigma_E c,\, \lambda^*\chi_0\chi_0^{*-1}}\Big|_{\mathscr{Y}_\circ^-}^{-1} \cdot \left(f^{-1} \otimes f^\vee\right)\left(i_1\left(\mathscr{P}^\vee_{\chi_0} \otimes \mathscr{P}^\vee_{\chi_0}\right)\right)\Big|_{\mathscr{Y}_\circ^-},$$

which is an element of $B(\chi_{\mathrm{univ},\circ}^{-1})_{\mathscr{K}_\circ^-} \otimes B(\chi_{\mathrm{univ},\circ})_{\mathscr{K}_\circ^-}$. by Theorem 2.8.

By the projection formula for heights [33], for any $P_1 \in B(\overline{E})$, $P_2 \in B^\vee(\overline{E})$ we have

$$\left\langle f^{-1}(P_1), f^\vee(P_2)\right\rangle_B = \langle P_1, P_2\rangle_{B_{\psi,\, K'}} = \langle P_1, P_2\rangle_{A_E^{\oplus r}}.$$

As maps of finitely generated $\Lambda_\circ^-$-modules are determined by their specialisations at finite order characters, this implies the analogous result for big height pairings. Then Theorem 1.2 follows from Theorem 2.7 and the factorisation (2.10). □

Proof of Theorem 1.1. — We state and prove the following slightly more precise version of Theorem 1.1. Recall from the Introduction that we say that a property P holds *for almost all* finite-order characters in $\mathscr{Y}_\circ^-$ if the set of those χ not satisfying P is not Zariski dense in $\mathscr{Y}_\circ^-$.

We keep the assumption of Theorem 1.1. □

Theorem 2.11. — *For almost all finite-order $\chi \in \mathscr{Y}_\circ^-$, we have*

$$L'_{\Sigma_E}(\lambda\chi) \neq 0,$$

the specialisation $\mathscr{P} \otimes \mathscr{P}^\vee(\chi)$ is a well-defined and non-zero element of $B(\chi^{-1}) \otimes B^\vee(\chi)$, and

$$\langle \mathscr{P} \otimes \mathscr{P}^\vee(\chi)\rangle \neq 0.$$

Proof. — The first assertion is equivalent to Theorem 2.9. That the points are generically well-defined at χ amounts to the assertion that

$$L_{\Sigma_E, \lambda\chi_0^* \chi_0^{-1}}|_{\mathscr{Y}_\wp^-} \neq 0,$$

which is Theorem 2.8. Finally the non-vanishing of p-adic heights follows from the other assertions and the Gross–Zagier formula of Theorem 1.2. □

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