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ON THE JOINT SPECTRAL RADIUS FOR ISOMETRIES OF NON-POSITIVELY CURVED SPACES AND UNIFORM GROWTH

by Emmanuel BREUILLARD & Koji FUJIWARA (*)

ABSTRACT. — We recast the notion of joint spectral radius in the setting of groups acting by isometries on non-positively curved spaces and give geometric versions of results of Berger–Wang and Bochi valid for δ -hyperbolic spaces and for symmetric spaces of non-compact type. This method produces nice hyperbolic elements in many classical geometric settings. Applications to uniform growth are given, in particular a new proof and a generalization of a theorem of Besson–Courtois–Gallot.

RÉSUMÉ. — Nous généralisons la notion de rayon spectral joint dans le cadre des actions de groupes par isométries sur les espaces à courbure négative ou nulle et nous donnons des versions géométriques des résultats de Berger–Wang et de Bochi valables dans tout espace δ -hyperbolique ainsi que dans les espaces symétriques de type non-compact. Cette méthode permet de produire des éléments hyperboliques dans de nombreuses situations géométriques classiques. Nous donnons par ailleurs des applications à la croissance uniforme ainsi qu'une nouvelle preuve et une généralisation d'un théorème de Besson–Courtois–Gallot.

1. Introduction

This paper is concerned with the following general problem. We are given a group G generated by a finite set S . Suppose that G contains elements with a certain property \mathcal{P} . Can we estimate the shortest length of an element $g \in G$ which has the property \mathcal{P} ? Here the length of an element

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$g \in G$ is the smallest length of a word with letters in S and S^{-1} , which represents g .

For example the property \mathcal{P} could be “being of infinite order”, or “having an eigenvalue of modulus different from 1”, say when G is a matrix group. Or else when G is a subgroup of isometries of a non-positively curved space \mathcal{P} could be : “having positive translation length”. This type of question is ubiquitous in problems dealing with uniform exponential growth. In this paper we give a method to find nice short words in a variety of situations when the group acts on a space with non-positive curvature. As an application we will give several uniform exponential growth results, in particular for groups acting on products of δ -hyperbolic spaces.

This method consists in studying the growth of the *joint minimal displacement* of the generating set S . To fix ideas say (X, d) is a metric space and $S \subset \text{Isom}(X)$ is a finite set of isometries of X . Let $G := \langle S \rangle$ be the group generated by S and $x \in X$ is a point. We define the *joint displacement at x* by

$$L(S, x) := \max_{s \in S} d(x, sx)$$

and the *joint minimal displacement* of S by

$$L(S) := \inf_{x \in X} L(S, x).$$

This quantity appears in many places in geometric group theory, for example in the construction of harmonic maps as in the work of Gromov–Schoen [39] and Korevaar–Schoen [48], or in Kleiner’s proof of Gromov’s polynomial growth theorem [47], and, when X is a Hilbert space, in relation to reduced first cohomology and Hilbert compression as in [28]; it is ubiquitous in the study of free group automorphisms and it is key to constructing limits of representations, e.g. see [9], and in recent work about surface group representations in higher rank Lie groups, e.g. [22].

In fact a more common quantity is the ℓ^2 version of $L(S)$, sometimes called the *energy* of S and defined as the infimum of the averaged squared displacement $\frac{1}{|S|} \sum_{s \in S} d(sx, x)^2$. In this paper however we will exclusively consider $L(S)$, which is the ℓ^∞ version of the energy. One advantage of working with $L(S)$ is that it behaves well under set theoretic powers S^n of S (while the energy is better suited to random walks as expounded for example in Gromov’s paper [38]).

It is natural to also consider the quantities

$$L_k(S) := \min_{1 \leq j \leq k} \frac{1}{j} L(S^j),$$

$$L_\infty(S) := \inf_{k \geq 1} L_k(S) = \lim_{n \rightarrow +\infty} L(S^n)/n.$$

This limit exists by subadditivity (see Section 2 below) and coincides with the limit of $L(S^n, x)/n$ for any point x . We call $L_\infty(S)$ the *asymptotic joint displacement* of S . This quantity is the right geometric analogue of the notion of *joint spectral radius* (see below).

When $S = \{g\}$ is a singleton, $L(\{g\})$ is usually called the *minimal displacement* of g and we call

$$\lambda(g) := L_\infty(\{g\})$$

the *stable length* of the element g . We may also consider the *joint stable length*

$$\lambda(S) := \max_{s \in S} \lambda(s)$$

and the corresponding notion for products

$$\lambda_k(S) := \max_{1 \leq j \leq k} \frac{1}{j} \lambda(S^j),$$

$$\lambda_\infty(S) := \sup_{j \in \mathbb{N}} \frac{1}{j} \lambda(S^j)$$

and compare these quantities to those defined above. The choice of normalization in the definition of $\lambda_k(S)$ is made so as to guarantee the following straightforward inequalities (see Section 2).

LEMMA 1.1 (general nonsense lemma). — *If (X, d) is any metric space and $S \subset \text{Isom}(X)$ a finite set of isometries, then for all $k \in \mathbb{N}$*

$$\lambda(S) \leq \lambda_k(S) \leq \lambda_\infty(S) \leq L_\infty(S) \leq L_k(S) \leq L(S).$$

Moreover $L_\infty(S^k) = kL_\infty(S)$ and $\lambda_\infty(S^k) = k\lambda_\infty(S)$. Finally

$$\lambda_\infty(S) = \limsup_{n \rightarrow +\infty} \frac{1}{n} \lambda(S^n).$$

We will be interested in the following questions: To what extent are these inequalities sharp? What is the growth of $L(S^n)$ as n grows? For which spaces do we always have $\lambda_\infty(S) = L_\infty(S)$?

These quantities are interesting when X is an unbounded metric space, especially in presence of some form of non-positive curvature, for example a CAT(0)-space or a δ -hyperbolic space.

When $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$ and the metric d is given by

$$(1.1) \quad d(g, h) := \log \|g^{-1}h\|,$$

where the norm is the operator norm associated to a Euclidean scalar product on \mathbb{R}^d , then any $S \subset \mathrm{SL}_d(\mathbb{R})$ acts by isometries on X and the quantity $L_\infty(S)$ is the (log of the) well-known *joint spectral radius* of the set of matrices S . This quantity has been first studied by Rota and Strang in 1960 [62] and further by Daubechies–Lagarias [29] in the context of wavelets and iterated function systems and by Berger–Wang [6]. The main result of Berger–Wang [6, Theorem IV] says that

$$(1.2) \quad \lambda_\infty(S) = L_\infty(S).$$

This means that the rate of growth of $\|S^n\|$ is identical to that of the maximal eigenvalue of an element in S^n , where

$$\|S^n\| := \max_{g \in S^n} \|g\|,$$

and a posteriori justifies the terminology “joint spectral radius” for $\exp(L_\infty(S))$ in that it generalizes the classical Gelfand formula expressing the spectral radius (i.e. the modulus of the maximal eigenvalue) as the rate of exponential growth of the norm of powers of a single matrix.

In the literature on the joint spectral radius authors usually consider arbitrary finite sets of matrices $S \subset M_d(\mathbb{R})$ which are not necessarily invertible. In this paper however, because of our geometric point of view, we will focus on the invertible case.

1.1. Geometric Berger–Wang identity and geometric Bochi inequality

One of the leitmotifs of this paper will be to extend the Berger–Wang identity (1.2) and its refinements to geometric actions on non-positively curved spaces X . We will mainly focus on fairly classical spaces such as Euclidean spaces, Hilbert spaces, or non-positively curved spaces such as symmetric spaces of non-compact type. We will also obtain satisfactory results for arbitrary δ -hyperbolic spaces and will mention some properties valid in arbitrary CAT(0)-spaces. We hope that our work may provide motivation to study the joint minimal displacement and Berger–Wang type identities in other geometries of interest, such as the Teichmüller space or the Outer space. We begin with:

THEOREM 1.2 (geometric Berger–Wang identity). — *The Berger–Wang identity (1.2) holds for every finite set S of isometries of X , when X is either a symmetric space of non-compact type, a tree, or an arbitrary δ -hyperbolic space.*

By contrast, we will show in Section 9 that the Berger–Wang identity fails for isometries of a Euclidean space.

In the case of symmetric spaces, this result is closely related to the original Berger–Wang identity for matrices. However in higher rank it does not follow directly (nor does it imply it) as we stress in the following remark. In the case of δ -hyperbolic spaces Theorem 1.2 was recently obtained by Oregon–Reyes [56], whose paper addresses for the first time the question of extending the classical results on the joint spectral radius to other geometric contexts. In this paper δ -hyperbolic spaces are assumed geodesic (see Remark 5.1).

Remark 1.3. — Note that, as usual, symmetric spaces are considered with their defining CAT(0) metric induced by a left-invariant Riemannian metric associated to the Killing form of the group. The quantities studied in this paper such as $L(S)$ and $\lambda(S)$ are sensitive to the choice of metric. For example the classical Berger–Wang identity (1.2) for matrices does not follow from the geometric Berger–Wang identity proved in the above theorem, nor does it imply it, say for $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$. Recall that the distance (1.1) on $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$ is not the usual CAT(0) Riemannian symmetric space metric, but rather a Finsler-type norm-like distance (as in [1] and [59]).

In [10] Bochi gave a different proof of (1.2), which yields a stronger inequality of the form

$$(1.3) \quad \lambda_{k_0}(S) \geq L(S) - C,$$

where k_0 and C are constants depending only on the dimension. The Berger–Wang identity follows immediately by applying the Bochi inequality to S^n and letting n go to infinity. Note that a different proof of Bochi’s inequality was given in [17, Corollary 4.6] and extended in [16] to non-archimedean local fields. In this paper we will prove geometric analogues of the Bochi inequality (1.3) in the geometric settings mentioned in the above theorem. For example:

THEOREM 1.4 (geometric Bochi-type inequality for hyperbolic spaces). *There is an absolute constant $K > 0$ such that the following holds. If (X, d) is a geodesic δ -hyperbolic space, then*

$$L_\infty(S) \geq \lambda_2(S) \geq L(S) - K\delta$$

for every finite set $S \subset \text{Isom}(X)$.

This answers a question raised in [56, Question 6.1].

For isometries of trees (case when $\delta = 0$) we have $\lambda_2(S) = L_\infty(S) = L(S)$. It follows in particular that $\lambda_2(S) = 0$ implies $L(S) = 0$, which is the content of a well-known lemma of Serre about tree isometries ([64, I. Proposition 26]). The proof for δ -hyperbolic spaces is a quasification of the corresponding proof for isometries of trees. It involves in particular a Helly-type theorem for hyperbolic spaces, which we prove in Section 6. See [33] and [23] for a related use of Helly-type theorems in geometric group theory. Finally we note that in [50, Proposition 3.2] Koubi already proved the following closely related consequence of Theorem 1.4 : for $K = 100$ if $L(S) > K\delta$, then S or S^2 contains a hyperbolic isometry of X .

As already mentioned Theorem 1.2 for δ -hyperbolic spaces follows from Theorem 1.4. We now record one more consequence of Theorem 1.4 (see Proposition 13.8 for the proof), which is well-known and due to Gromov [37, 8.1] (see also [56, Theorem 1.10]).

COROLLARY 1.5. — *Let X be a geodesic δ -hyperbolic space and $S \subset \text{Isom}(X)$ a finite set. Assume that $L_\infty(S) = 0$. Then $\langle S \rangle$ either has a bounded orbit on X , or fixes a unique point in ∂X .*

A similar but slightly weaker statement when X is a symmetric space is proven in Section 7, based on the Bochi inequality applied to various linear embeddings of X (see Theorem 7.5 and Proposition 7.7). In particular:

PROPOSITION 1.6. — *Let X be a symmetric space of non-compact type and $S \subset \text{Isom}(X)$ be a finite subset. Then :*

$$\lambda_{k_0}(S) \geq \frac{1}{\sqrt{d}}L(S) - C,$$

where $d \in \mathbb{N}$ is such that X is a convex subspace of $\text{SL}_d(\mathbb{C})/\text{SU}_d(\mathbb{C})$, $k_0 \leq d^2$, and $C > 0$ is a constant depending on d only.

We suspect but were unable to prove that the multiplicative constant $\frac{1}{\sqrt{d}}$ in the above result can be taken to be 1. See Theorem 7.5 and Question 4 in Section 14. On the other hand we will prove in Section 10 that the constant C cannot be taken to be zero. In fact for each n one may find an S with $L_\infty(S) > 0$ but $\lambda_n(S) = 0$ (see Lemma 1.12).

1.2. Growth of joint minimal displacement

A consequence of the geometric Berger–Wang identity is that in order to guarantee the existence of a single element g with positive stable length in the group generated by S , it is enough to show that $L_\infty(S) > 0$. Accordingly this suggests that in order to find a nice hyperbolic element in a power of S , it is important to study the growth rate of $L(S^n)$. In a CAT(0) space a simple geometric argument, similar to one used by V. Lafforgue in [51, Lemma 2] for Hilbert spaces, shows the following:

LEMMA 1.7. — *If X is a CAT(0) metric space and S a finite set of isometries, then*

$$L(S^n) \geq \frac{\sqrt{n}}{2} L(S).$$

The square root n growth behavior is sharp as examples of Cornuier, Tessera and Valette [28] show (see Remark 3.8). Of course if $L_\infty(S) > 0$ then $L(S^n)$ grows linearly, but the above sublinear growth is useful in order for $L(S^n)$ to go above a certain threshold after which the linear growth can start. For example if X is δ -hyperbolic, we will show that $L(S^n) \geq n(L(S) - K\delta)$ for some absolute constant K , so that we get linear growth provided $L(S) > K\delta$. While if X is a symmetric space of non-compact type a consequence of our analysis is:

PROPOSITION 1.8. — *There is $c = c(X) > 0$ such that if S is a finite set of isometries of a symmetric space X of non-compact type*

$$c \cdot \min\{L(S), L(S)^2\} \leq L_\infty(S) \leq L(S).$$

Observe that a consequence of the left hand-side and of Theorem 1.2 is the well-known fact that if a finitely generated group of isometries of X is entirely made of elliptic elements then it fixes a point on X or on its visual boundary (see Corollary 3.13).

When X is a symmetric space of non-compact type, the Margulis lemma asserts the existence of a constant $\epsilon = \epsilon(X) > 0$ depending only on $\dim X$ such that the following holds : every finite set S of isometries of X , which generates a discrete subgroup of $\text{Isom}(X)$, and has $L(S) < \epsilon$ must generate a virtually nilpotent subgroup. Combining this with the previous proposition, one obtains:

COROLLARY 1.9. — *There are constants $c_0, c_1 > 0$ depending only on X , such that if S is a finite set of isometries of a symmetric space X of non-compact type generating a non virtually nilpotent discrete subgroup of $\text{Isom}(X)$, then*

$$c_1 \leq L_\infty(S) \leq L(S) \leq c_0 \cdot L_\infty(S).$$

In [22, Corollary 1.3] Burger and Pozzetti show that in the case of maximal representations of a surface group in the Siegel upper-half space, the above inequalities hold already with $\lambda(S)$ in place of $L_\infty(S)$ and with an explicit value of c_1 .

1.3. Euclidean spaces

When X is a finite dimensional Euclidean space, it is still true that $L(S) = 0$ if and only if $L_\infty(S) = 0$, namely:

PROPOSITION 1.10. — *If S is a finite set of isometries of Euclidean \mathbb{R}^d , then the following are equivalent:*

- (1) S has a common fixed point,
- (2) $L(S) = 0$,
- (3) $L_\infty(S) = 0$.

While it is clear that $L(S) = 0$ implies $L_\infty(S) = 0$, the converse is slightly more subtle, because it can happen that S has no global fixed point, and yet every single element in the group generated by S has a fixed point (see Example 9.1). In particular one can have $L_\infty(S) > 0$ while $\lambda_\infty(S) = 0$. This means that the Berger–Wang identity (1.2) fails for isometries of Euclidean spaces. See Section 3.3.

By contrast, when X is an infinite dimensional Hilbert space one can have $L_\infty(S) = 0$ while $L(S) > 0$. A sublinear cocycle need not be trivial in the reduced first cohomology [28, 3.9], see Section 3.2.

1.4. Large torsion balls and escape from elliptics

Schur proved in 1907 [63] that a subgroup of $\text{GL}_n(\mathbb{C})$ all of whose elements are of finite order must be finite. In [16] the first author proved a uniform version of Schur’s theorem:

THEOREM 1.11 (uniform Schur theorem [16]). — *There is $N = N(d) \in \mathbb{N}$ such that for every finite symmetric subset $S \subset \mathrm{GL}_d(\mathbb{C})$ containing 1 and generating an infinite group, there is an element $g \in S^N$ of infinite order.*

Examples of Grigorchuk and de la Harpe [36] and of Bartholdi and Cornuier [4] show that $N(d)$ must grow to infinity with d . In Section 10 we will show that, by contrast with the case of torsion elements, one cannot escape elliptic elements in general:

LEMMA 1.12 (no uniform escape from elliptics). — *In the real Lie group $\mathrm{SO}(4,1)$, for each $N \in \mathbb{N}$ one can find a pair $S = \{a, b\}$ such that for all words w of length at most N in $a^{\pm 1}, b^{\pm 1}$ the cyclic subgroup $\langle w \rangle$ is bounded, but the subgroup $\langle S \rangle$ is unbounded and even Zariski dense (in particular it does not fix a point on hyperbolic 4-space \mathbb{H}^4 nor on the boundary $\partial\mathbb{H}^4$).*

The proof is based on the Tits alternative and the Borel–Larsen theorem [12] on the dominance of word maps on semisimple algebraic groups.

On the other hand the geometric Bochi-type inequalities proved in this paper give a tool to produce quickly (i.e. in S^n for n small) a non-elliptic element. The time needed to produce this element however depends on $L(S)$. For example Theorem 5.7 in Section 5 shows that, when X is a δ -hyperbolic space, there is $g \in S \cup S^2$ with $L_\infty(g) > 0$ (and in particular of infinite order) provided $L(S) > K\delta$. However $L(S^n)$ may well remain under $K\delta$ for large n , even though $L_\infty(S) > 0$. Examples of δ -hyperbolic groups with large torsion balls in their Cayley graph are easy to construct, see Remark 1.15 below.

Lemma 1.12 also shows that even though the geometric Berger–Wang identity $\lambda_\infty(S) = L_\infty(S)$ holds in symmetric spaces by Theorem 1.2, the additive constant C in the Bochi-type inequality $\lambda_k(S) \geq L_\infty(S)/\sqrt{d} - C$, which is a consequence of Proposition 1.6, cannot be zero.

1.5. Uniform exponential growth (UEG)

If S is a finite generating subset in a group Γ , we define

$$h(S) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log |S^n|,$$

and call $h(S)$ the *entropy* of Γ for S .

The original motivation for the present paper was a theorem of Besson–Courtois–Gallot [7]. They showed that given $a > 0$ and $n \in \mathbb{N}$, there is a constant $c(n, a) > 0$ such that if M is a complete Riemannian manifold of

dimension n with pinched sectional curvature $\kappa_M \in [-a^2, -1]$ and Γ is a discrete group of isometries of M generated by a finite set S , then

$$h(S) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log |S^n| > c(n, a) > 0,$$

provided Γ is not virtually nilpotent.

We will generalize this result and give a new proof of it, seeing it as a fairly direct consequence of the Bochi-type inequality of Theorem 1.4. Our approach provides the additional information that generators of a free semigroup can be found in a bounded ball (a feature the Besson–Courtois–Gallot proof did not yield). The proof will be given in Section 13, but let us briefly explain here how this works. Due to its negative curvature, the manifold M is a δ -hyperbolic metric space for some $\delta > 0$ which is independent of n and a . Once a hyperbolic element in S or a bounded power S^n has been found, a simple ping-pong argument gives generators of a free semigroup (cf. Section 11 below). Thanks to the geometric Bochi inequality for hyperbolic spaces, i.e. Theorem 1.4, in order to find a hyperbolic element, we only need to check that $L(S) > K\delta$ or at least that $L(S^n) > K\delta$ for some controlled n (cf. Theorem 13.1).

The manifold is also CAT(0), so Lemma 1.7 applies, and we see that we only need to rule out the possibility that $L(S)$ is very small. But if we assume that the group Γ is discrete, the Margulis lemma tells us that there is a constant $\epsilon = \epsilon(n, a) > 0$ such that if $L(S) < \epsilon$ and $\langle S \rangle$ is discrete, then $\langle S \rangle$ is virtually nilpotent. This ends the proof.

Using instead the generalized Margulis lemma proved by Green, Tao and the first author in [19] this argument yields the following generalization of the Besson–Courtois–Gallot theorem.

THEOREM 1.13 (UEG for hyperbolic spaces with bounded geometry). *Given $P \in \mathbb{N}$, there is a constant $N(P) \in \mathbb{N}$ such that the following holds. Assume that X is a geodesic δ -hyperbolic space with the property that every ball of radius 2δ can be covered by at most P balls of radius δ . Let S be a finite symmetric set of isometries containing 1. Assume that the subgroup Γ generated by S has discrete orbits. Then either Γ is virtually nilpotent, or S^N contains two elements g, h that are hyperbolic (i.e. $\lambda_\infty(g), \lambda_\infty(h) > 0$) and generate of a free semigroup and in particular:*

$$h(S) > \frac{1}{N} \log 2.$$

The assumption that Γ has discrete orbits means that for every $x, y \in X$ and $r > 0$, the orbit Γx intersects the ball $B(y, r)$ in a finite set. We stress that the constant N depends only on P and not on δ .

1.6. Hyperbolic groups

It is known that a non-elementary hyperbolic group has uniform exponential growth and even uniform exponential growth, that is subgroups have uniform exponential growth. See Koubi [50] as well as [24, 30]. Uniform exponential growth follows immediately from our general result on actions on hyperbolic spaces (see Theorem 13.1). The point is that when X is the Cayley graph of a hyperbolic group, then it is straightforward that $L(S^n) \geq n$ if S generates the group so the Margulis lemma used above in Theorem 1.13 is irrelevant.

THEOREM 1.14 (Growth of hyperbolic groups). — *There is an absolute constant $C_1 > 0$ such that if G is a group with finite generating set S whose Cayley graph $\text{Cay}(G, S)$ is δ -hyperbolic, with $\delta \geq 1$, then either G is finite or virtually cyclic, or S^M contains two hyperbolic elements that are generators of a free semi-group, where M is the least integer larger than $C_1\delta$. In particular there is an absolute constant $C_2 > 0$ such that*

$$h(S) \geq \frac{C_2}{\delta}.$$

We stress that M depends only on δ . This special feature was not explicit in Koubi [50]. While we do not know whether the entropy lower bound is sharp as stated (it is a well-known open problem to determine whether or not there is an absolute constant $c > 0$ (independent of δ) such that $h(S) > c$ for every generating set S of an arbitrary hyperbolic group), the result on the existence of a free semigroup is sharp inasmuch as M must depend on δ (it tends to infinity as δ goes to infinity). This follows from the following well-known observation.

Remark 1.15 (Hyperbolic groups with large torsion balls). — For every $\ell \in \mathbb{N}$, there is $n(\ell) > 0$ such that if $n > n(\ell)$, the two-generated group $\Gamma_{n,\ell}$ with presentation

$$\Gamma_{n,\ell} := \langle a, b \mid r^n = 1, r \in B(\ell) \rangle,$$

where $B(\ell)$ is the set of all reduced words of length at most ℓ in $\{a^{\pm 1}, b^{\pm 1}\}$ satisfies the $C'(1/6)$ small cancellation condition (see [35]) and is therefore a δ -hyperbolic group (with large $\delta = \delta(\ell, n)$). In particular no element of infinite order, let alone generators of a free semigroup, can be found in the ball of radius ℓ in $\Gamma_{\ell,n}$.

We note that the conclusion given in the previous theorem about $h(S)$ has recently been obtained independently by Besson, Courtois, Gallot and

Sambusetti in their recent preprint [8] without exhibiting a free semi-group, with the following explicit lower bound $h(S) \geq \log 2/(26\delta + 16)$.

Uniform exponential growth also immediately follows from Theorem 13.1. In this case, S may not generate the whole group, and the constant M must depend not only on δ , but also on the size of S . See Corollary 13.4 and Remarks 13.7, 13.6 and 13.5.

1.7. Uniform Tits alternative for groups acting on trees

In [17] Gelander and the first author proved a uniform Tits alternative for linear groups. Namely given a non virtually solvable finitely generated group Γ contained in $\mathrm{GL}_n(k)$ for some field k , there is an integer $N = N(\Gamma)$ such that for every symmetric generating set S of Γ the ball S^N contains a pair of free generators of a free subgroup. The number N was later shown in [15, 16] to be a constant depending only on the dimension n and in particular independent of the group Γ . A natural question arises as to whether or not a similar phenomenon occurs for subgroups of $\mathrm{Isom}(X)$, where X is a tree. We will give the following counter-example:

PROPOSITION 1.16. — *Given $N \in \mathbb{N}$ there are two isometries a, b of a trivalent tree without common fixed point on the tree nor on its boundary, such that for every two words w_1, w_2 of length at most N in a, b and their inverses, the subgroup $\langle w_1, w_2 \rangle$ is not free non-abelian.*

In other words the uniform Tits alternative does not hold for non-elementary subgroups of isometries of a tree. Note that the assumptions imply that $\langle a, b \rangle$ contains a non-abelian free subgroup. This is to be contrasted with the fact (see Proposition 12.1) that under the same assumptions there always is a pair of words of length at most 3, which generates a free semi-group. Our proof is based on an explicit construction of a subgroup of tree automorphisms with prescribed local action. Another example, based on the Baumslag–Solitar groups, was given to us by Jack Button. See Section 12.

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2. The joint minimal displacement

In this section we recall the geometric quantities introduced in the introduction and prove the general inequalities they satisfy, i.e. Lemma 1.1. In this section (X, d) is an arbitrary metric space.

Let S be a finite set of isometries and x a point in X . Recall that we have defined $L(S, x)$, $L(S)$, $L_\infty(S)$ and $\lambda_k(S)$ for $k \in \mathbb{N}_{\geq 1} \cup \infty$ at the beginning of the introduction.

2.1. Proof of Lemma 1.1

First we make the following simple observation:

Claim 1. — If U, V are finite subsets of isometries of X and $x \in X$, then:

$$(2.1) \quad L(UV, x) \leq L(V, x) + L(U, x)$$

Indeed by the triangle inequality:

$$\begin{aligned} L(UV, x) &= \max_{u \in U, v \in V} L(uv, x) \leq \max_{u \in U, v \in V} L(uv, ux) + L(u, x) \\ &\leq \max_{u \in U, v \in V} L(v, x) + L(u, x) \\ &\leq L(V, x) + L(U, x). \end{aligned}$$

From this we get that $n \mapsto L(S^n, x)$ is subadditive, and therefore by the subadditive lemma:

Claim 2. — the following limit exists and is independent of x

$$(2.2) \quad L_\infty(S) := \lim_n \frac{1}{n} L(S^n, x) = \inf_{n \geq 1} \frac{1}{n} L(S^n, x).$$

We take the above as a definition for $L_\infty(S)$. To see that this limit does not depend on the point x simply note that for every finite set S and every pair of points x, y

$$\begin{aligned} L(S, x) &= \max_{s \in S} d(sx, x) \leq \max_{s \in S} \{d(sx, sy) + d(sy, y) + d(y, x)\} \\ &\leq 2d(x, y) + L(S, y) \end{aligned}$$

and hence exchanging the roles of x and y

$$(2.3) \quad |L(S, x) - L(S, y)| \leq 2d(x, y).$$

Claim 3. — For every $n \in \mathbb{N}$ we have

$$(2.4) \quad \frac{1}{n}L(S^n) \leq L(S)$$

Indeed applying (2.1) iteratively we have $L(S^n, x) \leq nL(S, x)$ for all x .

Claim 4. — The sequence $L(S^n)/n$ converges to $L_\infty(S)$ and

$$(2.5) \quad L_\infty(S) = \lim \frac{1}{n}L(S^n) = \inf_{n \geq 1} \frac{1}{n}L(S^n)$$

Indeed, since $L(S^n, x) \geq L(S^n)$ we get immediately from (2.2) that $L_\infty(S) \geq \limsup \frac{1}{n}L(S^n)$. On the other hand $L_\infty(S) \leq \frac{1}{n}L(S^n, x)$ for all x and all n . Minimizing in x we get $L_\infty(S) \leq \frac{1}{n}L(S^n)$ and hence $L_\infty(S) \leq \inf_{n \geq 1} \frac{1}{n}L(S^n) \leq \liminf \frac{1}{n}L(S^n)$.

We conclude immediately that:

Claim 5. — $L_\infty(S^n) = nL_\infty(S)$ for every $n \in \mathbb{N}$.

We now turn to $\lambda(S)$ and $\lambda_k(S)$, which we have defined in the introduction as:

$$\lambda(S) := \max_{s \in S} \lambda(s)$$

and

$$\lambda_k(S) := \max_{1 \leq j \leq k} \frac{1}{j} \lambda(S^j).$$

We can now complete the proof of Lemma 1.1.

Claim 6. — $\lambda(S) \leq L_\infty(S)$.

Indeed (2.2) implies that $L_\infty(s) \leq L_\infty(S)$ for every $s \in S$. It follows that $\lambda(S) \leq L_\infty(S)$ and thus that $\lambda_k(S) \leq L_\infty(S)$ for every $k \in \mathbb{N}$. Finally we have:

Claim 7. — For every $k, n \in \mathbb{N}$ we have

$$(2.6) \quad \lambda_n(S) \leq \frac{1}{k} \lambda_n(S^k) \leq \lambda_{kn}(S).$$

To see this note that $\lambda(s^k) = k\lambda(s)$ for every isometry s and every $k \in \mathbb{N}$. In particular

$$\lambda(S^k) \geq k\lambda(S).$$

From this the left hand side of (2.6) follows easily, while the right hand side is formal.

Now, given $j \in \mathbb{N}$ and applying the left hand side of (2.6) to S^j in place of S , with $n = 1$ and letting k tend to infinity we see that:

$$\frac{1}{j} \lambda(S^j) \leq \limsup_{k \rightarrow +\infty} \frac{1}{k} \lambda(S^k)$$

and hence

$$\lambda_\infty(S) = \sup_{j \geq 1} \frac{1}{j} \lambda(S^j) = \limsup_{k \rightarrow +\infty} \frac{1}{k} \lambda(S^k).$$

To complete the proof of Lemma 1.1 it only remains to verify that $\lambda_\infty(S^k) = k\lambda_\infty(S)$ for every $k \in \mathbb{N}$. This is clear by letting n tend to infinity in Claim 7.

2.2. Joint displacement and circumradius

We define the minimal circumradius $r(S)$ of S to be the lower bound of all positive $r > 0$ such that there exists some $x \in X$ and a ball of radius r which contains $Sx = \{sx; s \in S\}$. This quantity is closely related to $L(S)$ as the following lemma shows:

LEMMA 2.1. — *Suppose $S \subset \text{Isom}(X)$ is a finite set. Then*

$$r(S) \leq L(S) \leq 2r(S).$$

Moreover $r(gSg^{-1}) = r(S)$ for every $g \in \text{Isom}(X)$.

Proof. — If $r > r(S)$, there exists $x \in X$ such that Sx is contained in a ball of radius r . In particular, $d(sx, x) \leq 2r$ for every $s \in S$. This means that $L(S) \leq L(S, x) \leq 2r$. Hence $L(S) \leq 2r(S)$.

Conversely, if x is such that $L(S) \geq L(S, x) - \epsilon$, then $d(sx, x) \leq L(S) + \epsilon$ for all $s \in S$, and in particular Sx belongs to the ball of radius $L(S) + \epsilon$ centered at x . Hence $r(S) \leq L(S) + \epsilon$. Since $\epsilon > 0$ is arbitrary, we get $r(S) \leq L(S)$.

That $r(gSg^{-1}) = r(S)$ for every $g \in \text{Isom}(X)$ follows immediately from the definition of $r(S)$. □

3. General CAT(0) spaces

The goal of this section is to recall some definitions and basic properties of groups acting by isometries on CAT(0) spaces and describe some basic examples, such as Euclidean and Hilbert spaces, where the Berger–Wang (1.2) identity fails. We also relate the vanishing of $L(S)$ to the presence of fixed points.

We first recall the notion of CAT(0) space. A good reference book is [20].

For a geodesic segment σ , we denote by $|\sigma|$ its length.

If $\Delta = (\sigma_1, \sigma_2, \sigma_3)$ is a triangle in a metric space X with σ_i a geodesic segment, a triangle $\bar{\Delta} = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$ in Euclidean \mathbb{R}^2 is called a *comparison*

triangle if $|\sigma_i| = |\bar{\sigma}_i|$ for $i = 1, 2, 3$. A comparison triangle exists if the side-lengths satisfy the triangle inequality.

We say that a triangle Δ is CAT(0) if

$$d(x, y) \leq d(\bar{x}, \bar{y})$$

for all points x, y on the edges of Δ and the corresponding points \bar{x}, \bar{y} on the edges of the comparison triangle $\bar{\Delta}$ in \mathbb{R}^2 . A geodesic space is a CAT(0) space if all triangles are CAT(0). Complete CAT(0) spaces are often called *Hadamard spaces*.

A geodesic metric space is CAT(0) if and only if every geodesic triangle with vertices a, b, c satisfies the following inequality:

$$(3.1) \quad 2d(a, m)^2 \leq d(a, b)^2 + d(a, c)^2 - \frac{1}{2}d(b, c)^2,$$

where m is the midpoint of the side $[b, c]$.

If X is a simply connected, Riemannian manifold whose sectional curvature is non-positive, then it is a CAT(0) space. In particular symmetric spaces of non-compact type are CAT(0) spaces. So are Euclidean and Hilbert spaces.

3.1. Minimal displacement of a single isometry

This section is devoted to the proof of Proposition 3.1 below. This fact is likely to be well-known to experts, but in lack of reference, we decided to include a proof.

Recal that an isometry g is said to be *semisimple* if the infimum is attained in the definition of

$$L(g) := \inf_{x \in X} d(gx, x).$$

When X is a CAT(0) metric space, isometries are classified into three classes (see [20]): g is said to be

- elliptic, if g is semisimple and $L(g) = 0$ (\iff fixes a point in X),
- hyperbolic, if g is semisimple and $L(g) > 0$,
- parabolic otherwise.

It is known that g is elliptic (hyperbolic, parabolic) if and only if g^n is elliptic (hyperbolic, parabolic, resp.) for some $n \neq 0$, [20, II.6.7, II.6.8].

For CAT(0) spaces, it turns out that the minimal displacement coincides with the rate of linear growth of an arbitrary orbit, namely:

PROPOSITION 3.1. — *Let X be a CAT(0) metric space, and g be an isometry. Then we have:*

$$L(g) = \lambda(g).$$

In particular $L(g^n) = nL(g)$ for each $n > 0$ and for any point $x \in X$, we have

$$L(g) = \lim_{n \rightarrow \infty} \frac{1}{n} d(g^n x, x).$$

Proof. — It is enough to prove that

$$L(g^2) = 2L(g),$$

because iterating we will find that $L_\infty(g) = \lim L(g^{2^n})/2^n = L(g)$. Since $L(g^2) \leq 2L(g)$ always (see Lemma 1.1) we only need to verify that $L(g) \leq L(g^2)/2$. For this we have the following simple argument, which we learned from Nicolas Monod. Given $x \in X$ consider the geodesic triangle x, gx, g^2x . The mid-point y between x and gx is mapped under g to the mid-point gy between gx and g^2x . Using a Euclidean comparison triangle, we see from the CAT(0) assumption that

$$d(y, gy) \leq \frac{1}{2} d(x, g^2x).$$

In particular $L(g) \leq \frac{1}{2} L(g^2, x)$ and minimizing in x we obtain what we wanted. \square

3.2. Hilbert spaces and affine isometric actions

Affine isometric actions on Hilbert spaces have been studied by many authors, in particular in connection to Kazhdan's property (T) and the Haagerup property. We refer the reader to the work Cornuier–Tessera–Valette [28] for background; see also the work Korevaar–Schoen [48] and Kleiner [47] for two interesting geometric applications.

When X is a Hilbert space, one can easily relate the vanishing of the quantities $L_\infty(S)$ and $L(S)$ to the cohomological properties of the affine isometric action associated to the finitely generated group $\Gamma := \langle S \rangle$. Let us describe briefly here this connection. The linear part of the Γ action by isometries on X is a unitary representation π of Γ on X . The translation part is a cocycle, i.e. a map $b : \Gamma \rightarrow X$ such that $b(gh) = \pi(g)b(h) + b(g)$ for all $g, h \in \Gamma$; we denote the linear space of cocycles from Γ to X by Z . A cocycle is called a co-boundary if there is $x \in X$ such that $b(g) = \pi(g)x - x$. The closure \bar{B} of the space of B co-boundaries gives rise to the reduced first cohomology group of the action $\bar{H}^1(\pi, X) = Z/\bar{B}$.

It is also natural to study the growth of cocycles. In [28] a cocycle $b : \Gamma \rightarrow X$ is called *sublinear* if $\|b(g)\| = o(|g|_S)$, when the word length $|g|_S$ with respect to the generating set S goes to infinity. It is straightforward to check that b is sublinear when $b \in \overline{B}$ [28, Corollary 3.3].

The relationship with our quantities $L_\infty(S)$ and $L(S)$ is as follows:

- (1) $L(S) = 0$ if and only if b vanishes in \overline{H}^1 , i.e. $b \in \overline{B}$,
- (2) $L_\infty(S) = 0$ if and only if b is sublinear.

We note that there are classes of discrete groups (e.g. polycyclic groups as shown in [28, Theorem 1.1]) for which $L_\infty(S) = 0$ if and only if $L(S) = 0$. While there are others [28, Proposition 3.9], where we may have $L(S) > 0$ and $L_\infty(S) = 0$. We will show below (in Proposition 9.3) that the latter examples can only happen in infinite dimension.

3.3. Euclidean spaces

In this subsection we briefly describe the case when $X = \mathbb{R}^d$ is a Euclidean space. Proofs will be given in Section 9.

PROPOSITION 3.2. — *A subgroup G of $\text{Isom}(\mathbb{R}^d)$ with $d = 2, 3$ all of whose elements have a fixed point must have a global fixed point. In particular $\lambda_\infty(S) = 0$ implies $L(S) = L_\infty(S) = 0$.*

This is no longer true in dimension 4 and higher. Indeed we have:

PROPOSITION 3.3. — *When $d \geq 4$, one can find a finite set S in $\text{Isom}(\mathbb{R}^d)$ such that $L_\infty(S) > 0$ but $\lambda_\infty(S) = 0$.*

The example is given by a subgroup of $\text{Isom}(\mathbb{R}^4)$ generated by two rotations within distinct centers and whose rotation parts generate a free subgroup of $\text{SO}(4, \mathbb{R})$ whose non trivial elements never have 1 as an eigenvalue. See Example 9.1.

Consequently:

COROLLARY 3.4. — *The Berger–Wang identity (1.2) fails on Euclidean spaces of dimension $d \geq 4$.*

In dimension $d = 3$ the Berger–Wang identity holds provided the linear part of the elements of S topologically generate a compact subgroup of $\text{O}(3)$, which is either finite or contains $\text{SO}(3)$. It fails otherwise and already in dimension 2, see Proposition 9.4. However we will show that if a finite set of isometries does not admit a global fixed point, then it always has a positive rate of escape. Namely:

PROPOSITION 3.5. — *If S is a finite set in $\text{Isom}(\mathbb{R}^d)$, then the following are equivalent:*

- (1) $L(S) = 0$,
- (2) $L_\infty(S) = 0$,
- (3) S has a common fixed point.

In other words: if Γ is a group and π a finite dimensional unitary representation, then for any cocycle $b : \Gamma \rightarrow \mathcal{H}_\pi$ the following are equivalent:

- (1) b is in the closure of coboundaries,
- (2) b is sublinear,
- (3) b is a coboundary.

This conveniently complements [28, Corollary 3.7].

3.4. A diffusive lower bound on joint displacement

With just the CAT(0) property one always gets \sqrt{n} growth for the joint displacement $L(S^n)$. This may not seem very surprising as already any random walk is expected to have at least a diffusive behaviour (see Remark 3.10 below). However one interesting feature of the following lower bound is the absence of any additive constant and the linear dependence in terms of $L(SS^{-1})$.

PROPOSITION 3.6. — *Let X be a CAT(0) geodesic metric space and S a finite subset of $\text{Isom}(X)$. For every $n \in \mathbb{N}$ we have:*

$$L(S^n) \geq \frac{\sqrt{n}}{2} L(SS^{-1}).$$

Proof. — Recall that $r(S)$ denotes the infimum over all points x, y of the radius r of the balls centered at y which contain Sx and that (Lemma 2.1) $r(S) \leq L(S)$. By definition of $r(S^n)$ if $r > r(S^n)$, then there exists some $x, y \in X$ such that $S^n x \subset B(y, r)$. This means that $sS^{n-1}x$ lies in $B(y, r)$ for every $s \in S$ and hence $S^{n-1}x$ lies in the intersection of all balls $B(s^{-1}y, r)$ for $s \in S$.

LEMMA 3.7. — *Let X be a CAT(0) geodesic metric space and $B(y, r)$ and $B(z, r)$ two balls of radius $r > 0$. Let m be a mid-point of a geodesic between y and z . Then $B(y, r) \cap B(z, r)$ is contained in the ball centered at m with radius $(r^2 - \frac{1}{4}d(y, z)^2)^{1/2}$.*

Proof. — This is straightforward from the CAT(0) inequality (3.1). \square

Now from this lemma $s_1^{-1}B \cap s_2^{-1}B$ is contained in the ball centered at the mid-point between $s_1^{-1}y$ and $s_2^{-1}y$ and with radius $(r^2 - \frac{1}{4}d(s_1^{-1}y, s_2^{-1}y)^2)^{1/2}$. It follows that $S^{n-1}x$ lies in a ball of that radius.

By definition of $r(S^{n-1})$, it follows that $r(S^{n-1})^2 \leq r^2 - \frac{1}{4}d(s_2s_1^{-1}y, y)^2$. This holds for all $s_1, s_2 \in S$ and all $r > r(S^n)$, thus: $\frac{1}{4}L(SS^{-1}, y)^2 \leq r(S^n)^2 - r(S^{n-1})^2$. We conclude that $L(SS^{-1})^2 \leq 4(r(S^n)^2 - r(S^{n-1})^2)$. Finally, summing over n , we obtain the desired result. \square

The above lower bound is useful to show that even if $L(SS^{-1})$ is very small, there will be some controlled n for which $L(S^n)$ has macroscopic size.

Remark 3.8. — The behaviour in \sqrt{n} of the lower bound is sharp for general CAT(0) spaces. Indeed Cornulier, Tessera and Valette exhibited an affine isometric action of the free group on a Hilbert space with a \sqrt{n} upper bound on cocycle growth, see [28, Proposition 3.9]. When X is a product of symmetric spaces and Euclidean spaces however, then $L(S^n)$ grows linearly provided $L(S) > 0$. This follows from the combination of Propositions 9.3 and 1.8.

Remark 3.9. — In [51] V. Lafforgue gave another proof of Shalom's theorem [65] that a group without property (T) has non-trivial first reduced cohomology. His main lemma is essentially a version of Proposition 3.6 in the case when X is a Hilbert space.

Remark 3.10. — In [38] Gromov investigates the growth of the energy of a random walk on a general CAT(0) space. In particular from his *harmonic growth inequalities* in [38, 3.4] one gets that the average displacement of a random walk grows like \sqrt{n} , which is another way to recover the \sqrt{n} growth in Proposition 3.6.

3.5. Fixed points on the boundary

Let X be a complete CAT(0) geodesic space. Note that the sublevel sets of the function $x \mapsto L(S, x)$ are convex subsets. Note further that any nested sequence of bounded closed convex non-empty sets has non-empty intersection (see e.g. [55, Theorem 14]). In particular we have:

PROPOSITION 3.11. — *If $x \mapsto L(S, x)$ tends to infinity when x leaves every bounded subset of X , then $L(S, x)$ achieves its minimum $L(S)$ at some point $x_0 \in X$.*

If X is locally compact one has:

PROPOSITION 3.12. — *Suppose X is a CAT(0) locally compact geodesic space such that no sublevel sets of $x \mapsto L(S, x)$ is bounded. Then S has a global fixed point on the visual boundary ∂X .*

For this see for example [48, 2.2.1]. Recall that the visual boundary ∂X is the set of equivalence classes of infinite geodesic rays $[0, +\infty) \rightarrow X$, where two rays $(x_t)_t$ and $(y_t)_t$ are equivalent if $d(x_t, y_t)$ is uniformly bounded. We get:

COROLLARY 3.13. — *Suppose X is a CAT(0) locally compact geodesic space. And $S \subset \text{Isom}(X)$ a finite set of isometries such that $L(S) = 0$, then S has a global fixed point in $X \cup \partial X$.*

3.6. CAT(0) and Gromov hyperbolic spaces

If X is only assumed to be CAT(0), then there may be parabolic isometries g with $L(g) > 0$. For example, the product of a parabolic isometry of the hyperbolic plane with a non trivial translation of \mathbb{R} . A less obvious example is given by the warped product $(\exp(-y) + C)dx^2 + dy^2$ on \mathbb{R}^2 , with $C > 0$. This space is CAT(0) and translation $(x, y) \mapsto (x + 1, y)$ is a parabolic isometry with positive displacement C (see [46, Remark 2.4]).

However if we assume additionally that X is Gromov hyperbolic, then parabolic isometries must have zero displacement. More precisely, we have:

PROPOSITION 3.14. — *Let X be a complete CAT(0) space which is Gromov hyperbolic. Let g be an isometry of X with $L(g) > 0$. Then g is hyperbolic in the sense that $L(g)$ is achieved on a geodesic, which is the unique g -invariant geodesic.*

For the definition of Gromov hyperbolic spaces we refer to Section 5 below as well as the books [13, 20, 27]. In this paper Gromov hyperbolic spaces are assumed to be geodesic (i.e. any pair of points can be joined by a geodesic path).

Before we start the proof, we quote some results. Let X be a complete CAT(0) space and ∂X its visual boundary at infinity. Set $\bar{X} = X \cup \partial X$. For a closed convex subset $C \subset X$, let $\bar{C} \subset \bar{X}$ be its closure.

THEOREM 3.15 ([55, Proposition 23]). — *Assume that a complete CAT(0) space X is Gromov hyperbolic. Then for any nested family \mathcal{F} of non-empty closed convex subsets $C \subset X$, the intersection $\bigcap_{C \in \mathcal{F}} \bar{C}$ is non-empty.*

In particular we have the following.

COROLLARY 3.16. — *Let X be a Gromov hyperbolic complete CAT(0) space. Then a parabolic isometry g has a fixed point in $\overline{X} \setminus X$.*

Proof. — The family of sublevel sets of $x \mapsto L(g, x)$ has empty intersection in X , for otherwise g would have a fixed point in X . By the previous theorem, the closures in \overline{X} of these sublevel sets has non-empty intersection in $\overline{X} \setminus X$. Each point in the intersection is fixed by g . \square

We now quote a standard fact on (Gromov) δ -hyperbolic spaces.

THEOREM 3.17. — *Let X be a δ -hyperbolic space and g an isometry of X . If $L(g)$ is sufficiently large, compared to δ , then g is “hyperbolic” in the sense that there is an infinite quasi-geodesic γ in X which is g -invariant.*

Moreover, let $x \in X$ be any point and m the midpoint of a geodesic segment from x to $g(x)$. Form a g -invariant piecewise geodesic, γ , joining the points $\{g^n(m)\}_{n \in \mathbb{Z}}$ in this order by geodesics. Then, there exists a constant M , which depends only on δ , such that for any points $p, q \in \gamma$, the Hausdorff distance between the part in γ from p to q and a geodesic from p to q is at most M .

Remark 3.18. — The first claim appears in [35, Section 8, Proposition 24] with $L(g) > 26\delta$. The advantage to retake x to m is that we have uniform bounds on the quasi-geodesic constants. In fact, we have $2|m - g(m)| - |m - g^2(m)| \leq 4\delta$, namely, the three points $m, g(m), g^2(m)$ are nearly on a geodesic. It follows that the path γ is a (K, L) -quasi geodesic with constants K, L depending only on ϵ . Once we have that, the existence of M is by the Morse lemma. A detailed argument is, for example, in [34] for $L(g) > 1000\delta$.

For an isometry a of X and a constant C , define

$$\text{Fix}_C(a) = \{x \in X \mid d(x, ax) \leq C\}.$$

This is a closed, possibly empty, set. It is convex if X is CAT(0).

We start the proof of the Proposition 3.14.

Proof. — Since it suffices to show that g^n is hyperbolic for some $n > 0$, by Proposition 3.1, we may assume that $L(g)$ is as large as we want by replacing g by a high power. So, we assume that $L(g)$ is large enough compared to the hyperbolicity constant δ , so that Theorem 3.17 applies to g .

Set $C_0 = L(g)$. We will show that $\text{Fix}_{C_0}(g)$ is not empty. For each $C > C_0$, the set $\text{Fix}_C(g)$ is non-empty convex set which is invariant by g , and is δ -hyperbolic.

By Theorem 3.17, there is a g -invariant quasi-geodesic, but we may assume that this path is contained in $\text{Fix}_C(g)$. Indeed, if we start with a point $x \in \text{Fix}_C(g)$, the point m is also in $\text{Fix}_C(g)$ since it is convex, and so are all points $g^n(m)$. Therefore γ in the theorem is contained in $\text{Fix}_C(g)$. Let us denote this γ by γ_C .

Now, the Hausdorff distance of any two of those quasi-geodesics γ_C , $C > C_0$, is at most $2M + 2\delta = M'$. To see that, observe that the Hausdorff distance of the two quasi-geodesics is finite since they are both g -invariant. Since they are both bi-infinite, in fact the bound is $2M + 2\delta$.

In particular there is a metric ball, $B \subset X$, of radius M' such that for any $C > C_0$, $B \cap \gamma_C \neq \emptyset$, therefore $B \cap \text{Fix}_C(g)$ is not empty.

Set $\mathcal{F} = \{B \cap \text{Fix}_C(g) \mid C > C_0\}$. It is a nested family of bounded convex closed subsets of X . Hence the family has a non-empty intersection, in particular $\bigcap_{C > C_0} \text{Fix}_C(g)$ is not empty, and therefore $\text{Fix}_{C_0}(g)$ is not empty. □

In view of the above propositions, in the case of a complete CAT(0) and δ -hyperbolic space, the classification of isometries becomes:

- elliptic when g fixes a point in X ,
- hyperbolic when $L(g) > 0$ (in this case g fixes a unique geodesic and acts by translation by $L(g)$ on it),
- parabolic when $L(g) = 0$ (in this case g does not fix a point in X but fixes a point in ∂X).

4. Quantitative Serre’s lemma for groups acting on trees

The goal of this section is to prove Proposition 4.4 below, which is an extension of the following well-known lemma due to Serre ([64, I. Proposition 26]). In the next section we will generalize everything to δ -hyperbolic spaces.

LEMMA 4.1 (Serre’s lemma). — *If a, b are isometries of a simplicial tree, such that a, b and ab each have some fixed point, then a and b have a common fixed point.*

We first note the following simple fact:

LEMMA 4.2. — *If a is an isometry of a simplicial tree, then $L(a) = L_\infty(a) = \lambda(a)$.*

Indeed either a fixes a point on the tree, and all quantities vanish, or a is a hyperbolic isometry translating along an axis $\Delta_a = \{x; d(ax, x) = L(a)\}$, so that in particular $L(a^n) = nL(a)$ and thus $\lambda(a) = L(a)$.

PROPOSITION 4.3 (A formula for the joint minimal displacement of a pair). — *Let a, b be isometries of a simplicial tree. Then*

$$L(\{a, b\}) = \max \left\{ L(a), L(b), \frac{L(ab)}{2} \right\} = \lambda_2(\{a, b\}).$$

More generally this formula extends to an arbitrary finite subset of isometries:

PROPOSITION 4.4 (A formula for the joint minimal displacement). — *Let S be a finite set of isometries of a simplicial tree. Then*

$$L(S) = \max_{a, b \in S} \left\{ L(a), \frac{L(ab)}{2} \right\} = \lambda_2(S).$$

Recall that by definition $L(S) = \inf_{x \in X} \max_{s \in S} d(x, sx)$ and $L(g) = L(\{g\})$ for an isometry g . And $L_\infty(S) = \lim L(S^n)/n$.

COROLLARY 4.5 (Growth of joint minimal displacement). — *Let S be a finite set of isometries of a simplicial tree. Then $L(S) = L_\infty(S) = \lambda_2(S)$ and indeed for every $n \in \mathbb{N}$,*

$$L(S^n) = nL(S).$$

We now pass to the proofs of the above statements. The proofs are quite simple and we do not claim much originality here. Even though we were not able to find the above statements in the existing literature, all ingredients in their proofs are well-known and can be found, for example in [26, 3.3].

First we make the following simple observations: Let g is an isometry of a simplicial tree X , let Δ_g be the “axis” of g , $\Delta_g := \{x \in X; d(gx, x) = L(g)\}$ and let x be any point in X .

- (a) we have $d(gx, x) = 2d(x, \Delta_g) + L(g)$,
- (b) $L(g) = d(gm, m)$, where m is the midpoint of the geodesic segment $[x, gx]$.

To see this, consider a point y on Δ_g , which minimizes the distance between x and Δ_g and note that the geodesic between y and gy lies entirely in Δ_g , so that in particular the concatenation of the geodesic segments from x to y and from y to gy remains a geodesic. Furthermore the midpoint between x and gx will coincide with the midpoint between y and gy .

Proof of Proposition 4.3. — Let the point x realize the infimum of $\max\{d(x, ax), d(x, bx)\}$. Assume that $L(\{a, b\}) > \max\{L(a), L(b)\}$. This implies that the axes Δ_a and Δ_b do not intersect. We claim that x must be the midpoint of the geodesic segment $[ax, bx]$ and that $d(x, ax) = d(x, bx)$.

To see this look at the triangle with vertices x, ax and bx . If it is not flat with x the midpoint of $[ax, bx]$, then the geodesics $[x, ax]$ and $[x, bx]$ intersect on some segment near x . But $y \mapsto d(y, ay)$ decreases as the point y moves away from x on this segment. This is a consequence of item (a) above: it decreases strictly unless x is on the axis Δ_a of a , and it decreases until it reaches the midpoint m of the segment $[x, ax]$, where we have $L(a) = d(m, am)$ by item (b) above. The same holds for b . Since $L(\{a, b\}, x) > \max\{L(a), L(b)\}$, we conclude that unless x is on both axes Δ_a and Δ_b , this contradicts the minimality of x .

For the same reason x is also the midpoint of the geodesic segments $[b^{-1}x, ax]$ and $[bx, a^{-1}x]$. By item (b) above applied to x viewed as a midpoint of $[b^{-1}x, ax]$ (noting that $ax = (ab)b^{-1}x$) we get:

$$d(abx, x) = L(ab).$$

But

$$(4.1) \quad d(abx, x) = d(bx, a^{-1}x) = 2d(ax, x) = 2d(bx, x) = 2L(a, b).$$

So this shows that $L(a, b) = L(ab)/2$ and this ends the proof of the proposition. □

We may now extend our formula to an arbitrary finite set of isometries. Recall that for an isometry a of a metric space (X, d) and a number $A \geq 0$ we denote $\text{Fix}_A(a) := \{x \in X; d(ax, x) \leq A\}$.

Proof of Proposition 4.4. — This follows easily from Proposition 4.3. Indeed set $M = \max_{a,b \in S} \{L(a), L(ab)/2\}$. By Proposition 4.3 we have $M \geq L(\{a, b\})$ for every pair $a, b \in S$. Hence $\text{Fix}_M(a)$ and $\text{Fix}_M(b)$ intersect for every pair $a, b \in S$. However recall that each $\text{Fix}_M(a)$ is a convex subset (it is a subtree), and in any tree any collection of subtrees which pairwise intersect non trivially must have a non-empty intersection. Therefore there is a point z such that $d(z, az) \leq M$ for all $a \in S$ and we have established $L(S) \leq M$. The opposite inequality is obvious. □

Proof of Corollary 4.5. — This is obvious combining Lemma 3.6 and Proposition 4.4. □

5. The joint minimal displacement in hyperbolic spaces

In this section we extend the results of Section 4 to isometries of δ -hyperbolic spaces and prove Theorem 1.4 from the introduction, which implies Theorem 1.2 in the case of hyperbolic spaces.

First we recall the definition of δ -hyperbolicity and prove a general lower bound on $L(S^n)$ using the same circumcenter argument as in Proposition 3.6.

Gromov introduced the notion of a δ -hyperbolic metric space in [37]. Recall that a metric space (X, d) is said to be geodesic if any two points can be joined by a geodesic (i.e. length minimizing) continuous path. A geodesic triangle $\Delta = (\sigma_1, \sigma_2, \sigma_3)$ in a geodesic space X is said to be δ -thin if σ_i is contained in the δ -neighborhood of $\sigma_j \cup \sigma_k$ for any permutation of (i, j, k) of $(1, 2, 3)$.

There is another notion that is closely related to δ -thinness. A point c is a δ -center of a geodesic triangle $\Delta = (\sigma_1, \sigma_2, \sigma_3)$ if the distance from c to every σ_i is $\leq \delta$, [13]. It is easy to see that if a (geodesic) triangle is δ -thin, then it has a δ -center. Conversely, if a geodesic triangle has a δ -center, then it is 6δ -thin (see [13, Lemma 6.5]).

The metric space (X, d) is said to be δ -hyperbolic if any geodesic triangle is δ -thin. Good references on the geometry of δ -hyperbolic spaces include [13, 20, 27].

Note that we can also define the δ -hyperbolicity of a geodesic space if every geodesic triangle has a δ -center (for example [13]). As we said, those two definitions are equivalent, but the constant δ may differ, so if we want to stress the difference, we call it δ -center-hyperbolicity.

Remark 5.1. — We note that there is also a notion of δ -hyperbolic space where the space is not required to be geodesic (via the so-called 4-points condition, see [13]). However the geodesic assumption is necessary in our theorems. For example it can be seen easily that results such as Lemma 5.5 or Theorem 5.7 fail when the space is not assumed geodesic. As pointed out in [56, Remark 4.4] one can take for instance the usual hyperbolic disc with a large ball (centered say at the origin) removed. This is still a δ -hyperbolic space in the sense of the 4-points condition with the same δ , but an elliptic isometry centered at the origin will have $\lambda(g) = 0$ while $L(g)$ will be large.

5.1. Using the circumradius to relate joint minimal displacement and asymptotic joint displacement

Here we show:

PROPOSITION 5.2. — *Let X be a δ -hyperbolic geodesic metric space and S a finite subset of $\text{Isom}(X)$. For every $n \in \mathbb{N}$ we have:*

$$\frac{1}{n}L(S^n) \geq \frac{L(SS^{-1})}{2} - 2\delta$$

Proof. — The argument is analogous to that of Proposition 3.6 in the CAT(0) setting. Recall that $r(S)$ denotes the infimum over all points x, y of the radius r of the balls centered at y which contain Sx . Also recall (Lemma 2.1) that

$$r(S) \leq L(S).$$

Let $n \geq 1$ and $r > r(S^n)$. By definition of $r(S^n)$, there exists some $x, y \in X$ such that $S^n x$ is contained in the ball B of radius r centered at y . This means that $sS^{n-1}x$ lies in B for every $s \in S$ and hence $S^{n-1}x$ lies in the intersection of all balls $s^{-1}B$ for $s \in S$. We require the following

LEMMA 5.3. — *Let X be a δ -hyperbolic geodesic metric space and $B(y, r)$ and $B(z, r)$ two balls of radius $r > 0$. Let m be a mid-point of a geodesic between y and z . Then $B(y, r) \cap B(z, r)$ is contained in the ball centered at m with radius $r - \frac{1}{2}d(y, z) + 2\delta$.*

Proof. — This is straightforward from the definition of δ -hyperbolicity. □

Now from this lemma $s_1^{-1}B \cap s_2^{-1}B$ is contained in the ball centered at the mid-point between $s_1^{-1}y$ and $s_2^{-1}y$ and with radius $r - \frac{1}{2}d(s_1^{-1}y, s_2^{-1}y) + 2\delta$. It follows that $S^{n-1}x$ lies in a ball of that radius.

By definition of $r(S^{n-1})$, it follows that $r(S^{n-1}) \leq r - \frac{1}{2}d(s_2s_1^{-1}y, y) + 2\delta$. This holds for all $s_1, s_2 \in S$ and all $r > r(S^n)$, thus: $\frac{1}{2}L(SS^{-1}, y) \leq r(S^n) - r(S^{n-1}) + 2\delta$. We conclude that $\frac{1}{2}L(SS^{-1}) \leq r(S^n) - r(S^{n-1}) + 2\delta$. Finally, summing over n , we obtain the desired result. □

COROLLARY 5.4. — *Let X be a δ -hyperbolic geodesic metric space and S a finite symmetric subset of $\text{Isom}(X)$. We have:*

$$L_\infty(S) \leq \frac{L(S^2)}{2} \leq L_\infty(S) + 2\delta.$$

Proof. — Clear from the combination of Proposition 5.2 and Lemma 1.1. □

This result was obtained quite cheaply using the circumradius. Using a more delicate analysis, based on a refinement of Serre’s lemma, we will prove below in Proposition 5.7 a much stronger result, a Bochi-type inequality, which yields a hyperbolic element of large stable length.

5.2. Powers of a single element

We begin by showing that the stable length of a single element is controlled by its minimal displacement, provided the latter is large enough.

LEMMA 5.5. — *There is a universal constant $C > 0$ such that the following holds. Let g be an isometry in a δ -hyperbolic space. Then for all $m > 0$, $L(g^m) \geq m(L(g) - C\delta)$. In particular:*

$$L(g) - C\delta \leq L_\infty(g) \leq L(g).$$

Proof. — We first prove the lemma for $m = 2$. For the proof we may assume $\delta > 0$. Pick a point y such that $L(g^2, y) \leq L(g^2) + \delta$. Consider the three points y, gy and g^2y . We will show that unless $L(g) = O(\delta)$ this triangle is $O(\delta)$ -flat and gy is $O(\delta)$ away from a mid point of $[y, g^2y]$.

Let q be a point at distance at most δ from all three geodesics joining these points, which is given to us by the δ -hyperbolicity assumption. Since $d(y, gy) = d(gy, g^2y)$ we conclude that $|d(y, q) - d(q, gy)| = O(\delta)$. In a δ -hyperbolic space any two geodesics joining two given points are at distance $O(\delta)$ of each other. Therefore $d(p, q) = O(\delta)$, where p is a mid-point between y and g^2y .

Now if r is a mid-point between y and gy , then r is $O(\delta)$ close to either $[q, gy]$ or to $[y, q]$ according as $d(y, r) \geq d(y, q)$ or not. In the first case $d(r, gr) = O(\delta)$, which implies that $L(g) = O(\delta)$ and there is nothing to prove. In the second case gr will be at most $O(\delta)$ away from $[q, g^2y]$ and we get:

$$|d(y, g^2y) - [d(y, r) + d(r, q) + d(q, gr) + d(gr, g^2y)]| = O(\delta),$$

while $d(r, q) \leq d(y, r) + O(\delta)$ and $d(q, gr) \leq d(gr, g^2y) + O(\delta)$. Combining these two facts we get:

$$2d(r, gr) \leq d(y, g^2y) + O(\delta),$$

from which the inequality $2L(g) \leq L(g^2) + O(\delta)$ follows immediately.

We now pass to the general case, when is an arbitrary integer $m > 2$. First we observe that if $L(g^2) \geq 2(L(g) - C\delta)$ for every isometry g , then a straightforward induction shows that $L(g^{2^n}) \geq 2^n(L(g) - 2C\delta)$ for every g and every integer n . But observe that given any point x the sequence $m \mapsto L(g^m, x)$ is subadditive. In particular, by the subadditive lemma, $\left\{ \frac{L(g^m, x)}{m} \right\}_m$ converges towards

$$\inf_{m \geq 1} \frac{L(g^m, x)}{m} = \lim_{m \rightarrow +\infty} \frac{L(g^m, x)}{m}.$$

Letting m grow along powers of 2 we see that the latter is at least $L(g) - 2C\delta$. This means that $L(g^m, x) \geq m(L(g) - 2C\delta)$ for every integer m and every point x . In particular $L(g^m) \geq m(L(g) - 2C\delta)$ as desired. \square

5.3. Bochi-type formula for hyperbolic spaces

Here we prove Theorem 1.4 from the introduction.

PROPOSITION 5.6 (Joint minimal displacement of a pair). — *There is an absolute constant $K > 0$ such that if $\delta \geq 0$ and (X, d) is a δ -hyperbolic geodesic space, and a, b are two isometries of X , then*

$$L(\{a, b\}) - K\delta \leq \max \left\{ L_\infty(a), L_\infty(b), \frac{L_\infty(ab)}{2} \right\} \leq L(\{a, b\}).$$

Recall our notation used throughout for a finite set S of isometries of a metric space (X, d) we denote by

$$L(S, x) := \max_{s \in S} d(x, sx), \quad L(S) := \inf_{x \in X} L(S, x) \quad \text{and} \quad L_\infty(S) := \lim L(S^n)/n,$$

$$\text{while } \lambda_k(S) := \max_{1 \leq j \leq k} \frac{1}{j} \max_{g \in S^j} L_\infty(g).$$

THEOREM 5.7 (Theorem 1.4 from the introduction). — *There is an absolute constant $K > 0$ such that if $\delta \geq 0$ and (X, d) is a δ -hyperbolic geodesic space, and S is a finite set of isometries of X , then*

$$L(S) - K\delta \leq \lambda_2(S) = \max_{a, b \in S} \left\{ L_\infty(a), \frac{L_\infty(ab)}{2} \right\} \leq L(S).$$

We would like to remark that Koubi [50, Proposition 3.2] proved that if $L(S) > 100\delta$ then S or S^2 contains a hyperbolic isometry of X .

As a consequence of this proposition, we obtain the following strengthening of Proposition 5.2.

COROLLARY 5.8 (Growth of joint minimal displacement). — *There is an absolute constant $K > 0$ such that the following holds. Let S be a finite set of isometries of a δ -hyperbolic space. Let $n \in \mathbb{N}$. Then*

$$n(L(S) - K\delta) \leq \lambda_2(S^n) \leq L(S^n).$$

Moreover $\lambda_\infty(S) = L_\infty(S)$.

We now pass to the proofs. As always in δ -hyperbolic geometry arguments are modeled on the tree case and this is why we decided to include the special case of trees separately even though it is of course implied by the δ -hyperbolic case by setting $\delta = 0$. A key ingredient will be Lemma 5.10 below, which computes the displacement of an isometry in terms of its minimal displacement and the distance to its axis.

Before we embark in the proof we begin by recalling the following basic fact about tree approximation in δ -hyperbolic spaces.

LEMMA 5.9 (see [13, Proposition 6.7]). — Given k points x_1, \dots, x_k in a δ -hyperbolic metric space (X, d) there is a metric tree T embedded in X such that for all i, j

$$(5.1) \quad d(x_i, x_j) \leq d_T(x_i, x_j) \leq d(x_i, x_j) + C_k \delta,$$

where $d_T(x_i, x_j)$ is the length of the geodesic path joining x_i and x_j in T . Here C_k is a constant depending only on k .

Recall that for an isometry a of a metric space (X, d) and a number $A \geq 0$ we denote

$$\text{Fix}_A(a) := \{x \in X; d(ax, x) \leq A\}.$$

LEMMA 5.10. — There are absolute constants $c, K > 0$ with the following property. Let (X, d) be a geodesic δ -hyperbolic metric space. Let $M \geq c\delta$ and $a \in \text{Isom}(X)$. Assume that $\text{Fix}_M(a)$ is non-empty and let $x \notin \text{Fix}_M(a)$. Then

$$(5.2) \quad |2d(x, \text{Fix}_M(a)) + M - d(ax, x)| \leq K\delta,$$

and

$$(5.3) \quad d(q, aq) \leq M + K\delta,$$

for every mid-point q between x and ax .

Proof. — Let $\epsilon > 0$ and pick $y \in \text{Fix}_M(a)$ such that $d(x, y) \leq d(x, \text{Fix}_M(a)) + \epsilon$. We may write:

$$d(ax, x) \leq d(ax, ay) + d(ay, y) + d(y, x) \leq 2d(x, y) + M,$$

so one side of (5.2) follows immediately.

To see the other side consider the four points x, y, ax, ay and apply Lemma 5.9 to these four points with $k = 4$. Let w be the intersection of the three geodesics in T connecting ax, ay and x , and let z be the intersection of the three geodesics in T connecting ax, y and x . First we claim that:

$$(5.4) \quad d(y, z) \leq \epsilon + O(\delta)$$

To see this, first recall that in a δ -hyperbolic space the Gromov product of a pair of points based at a third point is equal up to an error of 4δ to the distance from the third point to the geodesic between the pair (see [13, Lemma 6.2]). Now note that if α is the geodesic joining y and ay , then α is entirely contained in $\text{Fix}_M(a)$. Consequently $d(x, \alpha) \geq d(x, y) - \epsilon$. Hence

$(y, ay)_x \geq d(x, \alpha) - 4\delta \geq d(x, y) - \epsilon - 4\delta$, which unfolding the Gromov product means:

$$d(x, z) + d(z, ay) \geq d(x, ay) \geq d(x, y) + d(y, ay) - 2\epsilon - O(\delta).$$

In view of (5.1), it follows that the same holds with d_T in place of d . However:

$$d_T(x, y) + d_T(y, ay) = d_T(x, z) + d_T(z, ay) + 2d_T(y, z),$$

so we conclude that (5.4) holds.

At this point we note that $d(ay, y) \geq M - 2\epsilon$. Indeed by the intermediate value theorem, we may pick a point u on a geodesic between x and y such that $d(au, u) = M$. Then $d(x, u) \leq d(x, y)$, while $u \in \text{Fix}_M(a)$ so that $d(x, u) \geq d(x, \text{Fix}_M(a)) \geq d(x, y) - \epsilon$. Consequently $d(u, y) \leq \epsilon$ and hence $d(ay, y) \geq d(au, u) - 2\epsilon$ as desired.

Now two cases occur, according as w belongs to the tree geodesic $[ay, z]_T$ or to $[z, x]_T$. In the first case, by symmetry the previous argument also shows that

$$(5.5) \quad d(ay, w) \leq \epsilon + O(\delta)$$

Combining (5.4) and (5.5) (and using (5.1) then yields

$$\begin{aligned} d(ax, x) &\geq d(ax, w) + d(w, z) + d(z, x) - O(\delta) \\ &\geq d(ax, ay) + d(ay, y) + d(y, x) - 2\epsilon - O(\delta) \\ &\geq 2d(x, \text{Fix}_M(a)) + M - 6\epsilon - O(\delta), \end{aligned}$$

from which the remaining side of (5.2) is immediate since ϵ is arbitrary.

We can also observe now from the second inequality above that any mid-point p between y and ay satisfies $|d(x, p) - d(ax, p)| = O(\delta)$. The δ -hyperbolicity of the space then implies that $d(p, q) = O(\delta)$ for every mid-point q between x and ax . On the other hand $d(p, ap) \leq M$ (because $y \in \text{Fix}_M(a)$). Therefore $d(q, aq) \leq M + O(\delta)$. And (5.3) holds.

In the second case, z is also the intersection of the tree geodesics between the three points y, ay and ax and the previous argument once again shows that

$$(5.6) \quad d(ay, z) \leq \epsilon + O(\delta)$$

This means (using (5.4)) that $d(ay, y) \leq 2\epsilon + O(\delta)$ and hence that $M \leq 4\epsilon + O(\delta)$. So this case happens only when $M = O(\delta)$, a case we discard by choosing c large enough. □

Proof of Proposition 5.6. — Let $\epsilon > 0$ be arbitrary and let x be a point in X such that $L(\{a, b\}, x) \leq L(\{a, b\}) + \epsilon$. At the end of the argument we will let ϵ tend to 0. Let K be a large absolute constant, whose value we will specify later. Without loss of generality, we may assume that $L(\{a, b\}) > \max\{L(a), L(b)\} + K\delta + 3\epsilon$, for otherwise there is nothing to prove.

We can apply Lemma 5.10 to a with $M_a = \max\{L(a) + \epsilon, K\delta\}$. Let \bar{x} be a point in $\text{Fix}_{M_a}(a)$ such that $d(x, \bar{x}) \leq d(x, \text{Fix}_{M_a}(a)) + \epsilon$. Then (5.2) of Lemma 5.10 shows that the piecewise geodesic path between $x, \bar{x}, a\bar{x}$ and ax is geodesic up to an error $O(\delta) + \epsilon$. More precisely if y is any point on a geodesic between x and \bar{x} , then (5.2) gives

$$\begin{aligned} d(ay, y) &\leq d(ay, a\bar{x}) + d(a\bar{x}, \bar{x}) + d(\bar{x}, y) \\ &\leq 2d(\bar{x}, y) + L(a) + \epsilon + O(\delta) \\ (5.7) \qquad &\leq d(ax, x) - 2d(x, y) + \epsilon + O(\delta) \end{aligned}$$

Now we may conclude that $|d(ax, x) - d(bx, x)| \leq K\delta + 3\epsilon$. To see this assume by contradiction that $d(x, ax) > d(x, bx) + 2t + \epsilon$ with $2t > 2\epsilon + K\delta$ and $t < d(x, \text{Fix}_{M_a}(a))$. We may choose y at distance t from x on the geodesic $[x, \bar{x}]$. Then $L(\{a, b\}, x) = d(ax, x)$, while $d(by, y) \leq d(bx, x) + 2t < d(ax, x) - \epsilon$ and $d(ay, y) \leq d(ax, x) - 2d(x, y) + \epsilon + K\delta$ by the above inequality, so $d(ay, y) < d(ax, x) - \epsilon$ so

$$L(\{a, b\}, y) < L(\{a, b\}, x) - \epsilon,$$

which contradicts the choice of x . This argument is valid if $2d(x, \text{Fix}_{M_a}(a)) > 2\epsilon + K\delta$. But we are always in this case, because otherwise $L(\{a, b\}, x) \leq d(x, ax) \leq M_a + 2\epsilon + K\delta$, which is in contradiction with our assumption that $L(\{a, b\}) > \max\{L(a), L(b)\} + K\delta + 3\epsilon$.

Now we claim that x is at most $O(\delta) + \epsilon$ away from the midpoint between ax and bx . By the above discussion it is enough to show that it is that close to the geodesic between ax and bx . Let us verify this claim. Without loss of generality we may assume that $d(x, \text{Fix}_{M_a}(a)) \geq d(x, \text{Fix}_{M_b}(b))$. Now consider the triangle with vertices x, ax and bx . It is δ -thin by hyperbolicity. Thus there is a point z at distance at most δ from all three geodesics. In particular the geodesic between z and x is in the $O(\delta)$ neighborhood of both the geodesic between x and ax and the geodesic between x and bx . If y is any point on $[x, z]$ such that $d(x, y) \leq d(x, \text{Fix}_{M_b}(b)) + \epsilon$. Then we also have $d(x, y) \leq d(x, \text{Fix}_{M_a}(a)) + \epsilon$ and the above inequality (5.7) for a and its analogue for b applied at y show that

$$L(\{a, b\}, y) \leq L(\{a, b\}, x) - 2d(x, y) + \epsilon + O(\delta),$$

and thus, given our choice of x ,

$$d(x, y) \leq \epsilon + O(\delta).$$

Choosing y as far from x as possible, we see that this means either that $d(x, z) \leq \epsilon + O(\delta)$ and we have proved our claim, or that $d(x, \text{Fix}_{M_b}(b)) \leq 2\epsilon + O(\delta)$. Let us rule out the second case.

If $d(x, \text{Fix}_{M_b}(b)) \leq 2\epsilon + O(\delta)$ the entire segment $[x, bx]$ lies in $\text{Fix}_{M'_b}(b)$, where $M'_b = M_b + 2\epsilon + O(\delta) \leq L(b) + 3\epsilon + O(\delta)$. So again if y is any point on $[x, z]$ such that $d(x, y) \leq d(x, \text{Fix}_{M_a}(a)) + \epsilon$, then not only $d(ay, y) \leq d(ax, x) - 2d(x, y) + \epsilon + O(\delta)$ but also $d(by, y) \leq M'_b \leq L(b) + 3\epsilon + O(\delta)$. Now we may choose our numerical constant $K > 0$ so large that the implied constant in the big O in the two last inequalities is at most $K/2$ say. Then $d(by, y) \leq L(\{a, b\}) - K\delta/2$ and $d(ay, y) \leq L(\{a, b\}) - 2d(x, y) + 2\epsilon + K\delta/2$ so

$$L(\{a, b\}, y) \leq L(\{a, b\}) + \max\{-K\delta/2, 2\epsilon + K\delta/2 - 2d(x, y)\},$$

and we conclude that

$$2d(x, y) \leq 2\epsilon + K\delta/2.$$

Again choosing y as far from x as possible, we see that this means that $d(x, \text{Fix}_{M_a}(a)) \leq 2\epsilon + K\delta/2$. In particular $d(ax, x)$ and $d(bx, x)$ are both at most $\max\{M_a, M_b\} + 4\epsilon + K\delta \leq \max\{L(a), L(b)\} + 5\epsilon + 2K\delta$. The last quantity is $< L(\{a, b\})$ by assumption and this is impossible. This ends the proof of the claim.

We have thus shown that x is at most $O(\delta) + \epsilon$ away from the midpoint between ax and bx . The same reasoning applies to the segments between $b^{-1}x$ and ax , as well as between bx and $a^{-1}x$. If m is the midpoint of $[b^{-1}x, ax]$ then (5.3) of Lemma 5.10 shows that

$$L(ab) \leq d(abm, m) \leq L(ab) + \epsilon + O(\delta),$$

so $d(abx, x) \leq L(ab) + O(\delta) + 2\epsilon$. On the other hand

$$\begin{aligned} 2d(x, bx) &\leq d(bx, a^{-1}x) + O(\delta) + 2\epsilon \\ &\leq d(abx, x) + O(\delta) + 2\epsilon \\ &\leq L(ab) + O(\delta) + 4\epsilon. \end{aligned}$$

And similarly $2d(x, bx) \leq L(ab) + O(\delta) + 4\epsilon$, so that

$$L(\{a, b\}) \leq L(\{a, b\}, x) \leq \frac{1}{2}L(ab) + O(\delta) + 2\epsilon.$$

Letting ϵ tend to zero, the proof is complete. □

Proof of Theorem 5.7. — The right hand side inequality is obvious. The left hand side follows easily from Proposition 5.6 combined with Helly's theorem for hyperbolic spaces, namely Theorem 6.1. Indeed pick $\epsilon > 0$ and set

$$M = \max_{a,b \in S} \{L(a), L(ab)/2\} + K\delta + \epsilon,$$

where K is as in Proposition 5.6. This proposition tells us that $\text{Fix}_M(a)$ and $\text{Fix}_M(b)$ intersect non trivially for every pair of isometries $a, b \in S$. From our Helly-type theorem for hyperbolic spaces Theorem 6.1 we conclude that the 28δ -neighborhoods of all $\text{Fix}_M(a)$, $a \in S$, intersect non-trivially. Each such neighborhood is clearly contained in $\text{Fix}_{M+56\delta}(a)$. We have established that $L(S) \leq M + 56\delta$. Letting ϵ tend to 0, this ends the proof. \square

Proof of Corollary 5.8. — Recall from Claim 7 equation (2.6) in the proof of Lemma 1.1 in Section 2 that

$$n\lambda_2(S) \leq \lambda_2(S^n).$$

Now by Theorem 5.7 we get the desired inequalities:

$$n(L(S) - K\delta) \leq n\lambda_2(S) \leq \lambda_2(S^n) \leq L(S^n) \leq nL(S).$$

If we now apply Theorem 5.7 to S^n we get

$$L(S^n) - K\delta \leq \lambda_2(S^n)$$

Letting n tend to infinity we also get $\lambda_\infty(S) = L_\infty(S)$. \square

We give a consequence of Proposition 5.8 and Lemma 1.7.

COROLLARY 5.11. — *Let X be CAT(0) and δ -hyperbolic. Then for any $\epsilon > 0$, there exists an integer $N = N(\delta, \epsilon) > 0$ with the following property. For any finite set $S \subset \text{Isom}(X)$ with $1 \in S$, one of the following holds:*

- (1) $L(S) < \epsilon$,
- (2) there is $g \in S^N$ such that $L_\infty(g) \geq L(S)$.

Proof. — By Theorem 5.7, $L(S) \leq \lambda_2(S) + K\delta$. Also, for any $n > 0$, $L(S^n) \leq \lambda_2(S^n) + K\delta$. But since X is CAT(0), we also have $\frac{\sqrt{n}}{2}L(S) \leq L(S^n)$ by Lemma 1.7. Combining those two inequalities, and taking n large enough, we obtain the desired conclusion. \square

6. Helly type theorem for hyperbolic spaces

It is well-known and easy to prove that any family of convex subsets of a tree with non-empty pairwise intersection must have a non-empty

intersection. In this section we prove the following extension of this fact to hyperbolic spaces:

THEOREM 6.1 (Helly for hyperbolic spaces). — *Let (X, d) be a δ -hyperbolic geodesic metric space. Let $(C_i)_i$ be a family of convex subsets of X such that $C_i \cap C_j \neq \emptyset$ for all i, j . Then the intersection of all $(C_i)_{28\delta}$ is non-empty.*

Recall that for a subset $E \subset X$ we denote by $(E)_t$ the t -neighborhood of E , i.e.

$$(E)_t := \{x \in X; d(x, E) \leq t\}.$$

The proof is inspired by [25]. We begin with a lemma:

LEMMA 6.2. — *Let C_1 and C_2 be convex subsets of X with non empty intersection. Let z be a point in X . Let x_i be a point in C_i such that $d(z, x_i) \leq d(z, C_i) + \delta$. Assume that $d(z, x_1) \geq d(z, x_2) - \delta$. Then $d(x_1, C_2) \leq 28\delta$.*

Proof. — Pick $u \in C_1 \cap C_2$. It is a standard fact about δ -hyperbolic spaces that the distance to a geodesic and the Gromov product of the end points are equal within an error 4δ (see [13, Lemma 6.2]). This means that $(u, x_1)_z \geq d(z, ux_1) - 4\delta \geq d(z, x_1) - 5\delta$. Unfolding the Gromov product, we get:

$$d(z, u) + 10\delta \geq d(u, x_1) + d(x_1, z)$$

The same holds for x_2 in place of x_1 . This means that the paths u, x_1, z and u, x_2, z are almost geodesic (i.e. 10δ -taut in the terminology of [13, Chapter 6]). Hence they must be very close to each other. Applying [13, Lemma 6.4] we see that both paths are within 27δ of each other. In particular $d(x_1, C_2) \leq d(x_1, ux_2) \leq 28\delta$, which is what we wished for. \square

Proof of Theorem 6.1. — Pick any point $z \in X$. The point of the proof is that the previous lemma identifies one specific point that must be in the intersection. Let $x_i \in C_i$ as in the previous lemma. Without loss of generality, we may assume that $d(x_1, z) \geq d(x_i, z) - \delta$ for all indices i . Then we may apply the lemma to all pairs C_1 and C_i . And conclude that x_1 belongs to the desired intersection. This ends the proof. \square

7. Symmetric spaces of non-compact type

In this section we prove the Berger–Wang identity for symmetric spaces of non-compact type, as well as the Bochi-type inequality stated in the

introduction. In particular we establish Proposition 1.8 and Theorem 1.2 for these spaces.

So X is assumed to be a symmetric space of non-compact type associated to a real semisimple algebraic group G . The space (X, d) is then CAT(0) and the distance is G -invariant. We refer to [20] for background on these spaces and to [44, 60] for finer properties. We only recall here the following important example:

Example 7.1. — Let P_d be the symmetric space associated to $G = \mathrm{SL}_d(\mathbb{C})$ and $K = \mathrm{SU}(d, \mathbb{C})$, that is $X = P_d = G/K$. Recall the Cartan decomposition $G = KAK$, where A is the Lie subgroup of diagonal matrices with positive real entries. The distance on P_d is left G -invariant and is given by the following simple formula:

$$(7.1) \quad d(gx_0, x_0) = d(ax_0, x_0) = \sqrt{(\log a_1)^2 + \cdots + (\log a_d)^2},$$

where $g = kak \in KAK$, x_0 is the point in P_d fixed by the maximal compact subgroup K and $a = \mathrm{diag}(a_1, \dots, a_d)$.

This example is important also because every symmetric space of non-compact type arises as a convex (totally geodesic) subspace of some P_d . In fact if M is an arbitrary symmetric space of non-compact type, then $\mathrm{Isom}(M)$ is a linear semisimple Lie group with finitely many connected components and thus embeds as a closed subgroup of some $\mathrm{SL}_d(\mathbb{C})$ for some d . By the Karpelevich–Mostow theorem the connected component of the identity $\mathrm{Isom}(M)^0$, which is a semisimple Lie group, admits a convex totally geodesic orbit within the symmetric space P_d , which is isometric to M . In particular every isometry of M extends to an isometry of P_d .

We also recall here that, since (X, d) is a locally compact CAT(0) space we know that $g \in \mathrm{Isom}(X)$ satisfies $L(g) = 0$ if and only if $L_\infty(g) = 0$ and if and only if g fixes a point in $\bar{X} = X \cup \partial X$ (Corollary 3.13). This is also equivalent to saying that the eigenvalues of g under some (or any) faithful linear representation of $\mathrm{Isom}(X)$ have modulus 1.

To begin with, we recall the following fact:

PROPOSITION 7.2 (subgroups of elliptics). — *Suppose X is a symmetric space of non-compact type and $S \subset \mathrm{Isom}(X)$ a finite set of isometries. Then the following are equivalent:*

- (1) $\lambda_\infty(S) = 0$,
- (2) $L(S) = 0$,

Moreover in this case S fixes a point in \bar{X} .

Note the contrast with Euclidean spaces, where the analogous statement fails (see Section 9). This proposition, or at least some variant, is likely well-known, but lacking a reference we will include a proof for the reader's convenience.

Remark 7.3. — Observe that $L(S) = L(S \cup S^{-1})$. So the conditions of the previous proposition are also equivalent to $\lambda_\infty(S \cup S^{-1}) = 0$, which amounts to say that $L(g) = 0$ for every g in the subgroup generated by S .

Remark 7.4. — In Section 10 we will show that we cannot replace $\lambda_\infty(S)$ by $\lambda_k(S)$ for some finite k independent of S in the above proposition.

We now pass to the Berger–Wang identity.

THEOREM 7.5 (geometric Berger–Wang). — *Let X be a symmetric space of non-compact type. For every $\epsilon > 0$ there is $k = k(X, \epsilon) \in \mathbb{N}$ such that for every finite set $S \subset \text{Isom}(X)$ one has:*

$$\lambda_k(S) \geq (1 - \epsilon)L_\infty(S) - \epsilon.$$

In particular the Berger–Wang identity holds, i.e. $\lambda_\infty(S) = L_\infty(S)$.

We also relate the joint minimal displacement to the asymptotic joint displacement and find the following general inequality:

PROPOSITION 7.6. — *Let X be a symmetric space of non-compact type viewed as a convex subspace of $P_d := \text{SL}_d(\mathbb{C})/\text{SU}_d(\mathbb{C})$, and let $S \subset \text{Isom}(X)$ be a finite subset, then :*

$$(7.2) \quad \frac{1}{\sqrt{d}}L(S) - \log \sqrt{d} \leq L_\infty(S) \leq L(S).$$

Similarly using the Bochi inequality (see Proposition 7.11 below) we will show:

PROPOSITION 7.7. — *Let X be a symmetric space of non-compact type viewed as a convex subspace of $P_d := \text{SL}_d(\mathbb{C})/\text{SU}_d(\mathbb{C})$, and let $S \subset \text{Isom}(X)$ be a finite subset, then :*

$$\lambda_{k_0}(S) \geq \frac{1}{\sqrt{d}}L(S) - C,$$

where $C > 0$ is a constant depending on d only and $k_0 \leq d^2$.

Remark 7.8. — Bochi's original proof of his inequality (Proposition 7.11 below) gave a worse estimate on k_0 (exponential in d). A different proof of Bochi's inequality with the above d^2 bound was given in [17, Corollary 4.6]. Furthermore the dependence of C on d is not effective in Bochi's proof

from [10]. In [14] we make effective the argument from [17, Corollary 4.6] and give an explicit estimate on the constant.

Remark 7.9. — Since rank one symmetric spaces are Gromov hyperbolic the stronger inequality:

$$\lambda_{k_0}(S) \geq L(S) - C$$

holds in these cases even with $k_0 = 2$ according to Theorem 1.4. It is likely that a similar linear lower bound (with no multiplicative constant in front of $L(S)$) holds for general symmetric spaces as well (although not necessarily with $k_0 = 2$). The best we could do in this direction however is the lower bound from Theorem 7.5.

Before passing to the proofs of the above results, we would like to explain the connection with the well-studied notion of *joint spectral radius* of a finite set of matrices.

Remark 7.10 (Connection with the joint spectral radius). — Recall that if $S \subset M_d(\mathbb{C})$ is a finite set of matrices, then the joint spectral radius $R(S)$ of S (in the sense of Rota and Strang [62]) is defined as

$$R(S) = \lim_{n \rightarrow +\infty} \|S^n\|_2^{1/n},$$

where $\|Q\|_2 := \max_{g \in Q} \|g\|_2$ for $Q \subset M_d(\mathbb{C})$ and $\|g\|_2$ is the operator norm of g acting on Hermitian \mathbb{C}^d . As it turns out, when $S \subset \mathrm{SL}_d(\mathbb{C})$, we can interpret $R(S)$ as $\exp L_\infty(S)$ for a suitably defined left invariant distance on the homogeneous space $P_d^\infty := \mathrm{SL}_d(\mathbb{C})/\mathrm{SU}_d(\mathbb{C})$. This distance is defined by:

$$(7.3) \quad d(gx_0, x_0) = \max_i \{\log a_i\}$$

where $g = kak \in KAK$, x_0 is the point in P_d^∞ fixed by the maximal compact subgroup $K = \mathrm{SU}_d(\mathbb{C})$ and $a = \mathrm{diag}(a_1, \dots, a_d) \in A$ the subgroup of diagonal matrices with positive real entries. Note that even though P_d^∞ and the symmetric space P_d discussed in Example 7.1 have the same underlying space $\mathrm{SL}_d(\mathbb{C})/\mathrm{SU}_d(\mathbb{C})$, the distance is not the same.

Observe that as we have defined it:

$$d(gx_0, x_0) = \log \|g\|_2,$$

and hence we have:

$$L_\infty^{P_d^\infty}(S) = \log R(S),$$

where $L_\infty^{P_d^\infty}(S)$ is the $L_\infty(S)$ defined in the introduction for the metric space $X = P_d^\infty$.

Similarly the largest modulus $\Lambda(g)$ of an eigenvalue of $g \in \text{SL}_d(\mathbb{C})$ is the limit $\lim \|g^n\|^{1/n}$, that is $\exp L_\infty^{P_d^\infty}(g)$. Therefore we see that for a finite subset S in $\text{SL}_d(\mathbb{C})$ the Berger–Wang identity

$$R(S) = \limsup_{n \rightarrow +\infty} \max\{\Lambda(g), g \in S^n\}^{1/n}$$

proved in [6] is simply the statement:

$$L_\infty^{P_d^\infty}(S) = \lambda_\infty^{P_d^\infty}(S).$$

Comparing (7.1) and (7.3) we see that all quantities pertaining to P_d are comparable to the corresponding quantities pertaining to P_d^∞ . In particular

$$L_\infty^{P_d^\infty}(S) \leq L_\infty^{P_d}(S) \leq \sqrt{d} \cdot L_\infty^{P_d^\infty}(S).$$

In [10] Bochi gave a different proof of the Berger–Wang identity, which gave much more, namely an eigenvalue lower bound in terms of $R(S)$. He proved:

PROPOSITION 7.11 (Bochi inequality [10, Theorem B]). — *There are constants $c = c(d) > 0$ and $k_0 = k_0(d) \in \mathbb{N}$ such that if $S \subset M_d(\mathbb{C})$ is a finite set of matrices, then*

$$c \cdot R(S) \leq \max_{j \leq k_0} \max_{g \in S^j} \Lambda(g)^{1/j} \leq R(S).$$

It is also known [10] that

$$R(S) = \inf_{\|\cdot\|} \|S\|$$

where the infimum is taken over the operator norm associated to an arbitrary choice of (not necessarily hermitian) norm on the real vector space \mathbb{C}^d . Since by John’s ellipsoid theorem every two operator norms are equivalent up to a factor $\sqrt{2d}$, we have that

$$R(S) \leq \inf_{x \in \text{GL}_d(\mathbb{C})} \|xSx^{-1}\|_2 \leq \sqrt{2d} \cdot R(S).$$

Combining this with (7.1) and (7.3) we obtain in particular the following inequalities between the joint minimal displacement, the asymptotic joint displacement and the joint spectral radius of a finite subset S of $\text{SL}_d(\mathbb{C})$

$$(7.4) \quad \log R(S) \leq L^{P_d}(S) \leq \sqrt{d} \log(\sqrt{2d}R(S))$$

and, replacing S by S^n and passing to the limit:

$$(7.5) \quad \log R(S) \leq L_\infty^{P_d}(S) \leq \sqrt{d} \log(R(S))$$

We now turn to the proofs of the statements above.

Proof of Proposition 7.2. — That (2) implies (1) is clear by Lemma 3.6. For the implication (1) implies (2), we first reduce to the case when $X = P_d$ is the symmetric space of Example 7.1. As described in this example, X embeds as totally geodesic subspace of P_d for some d and $\text{Isom}(X)$ embeds in $\text{Isom}(P_d)$. Since P_d is $\text{CAT}(0)$ and X is a closed convex subset, the nearest point projection π_X from P_d to X is well-defined and it is a distance non-increasing map [20, Proposition II.2.4]. Consequently for $x \in P_d$ and $S \subset \text{Isom}(X)$ we have $L(S, \pi_X(x)) = L(S, x)$. This means that, when $S \subset \text{Isom}(X)$, all quantities $L(S)$, $L_\infty(S)$, $\lambda_k(S)$ coincide when defined in X or in P_d . So now we assume as we may that $X = P_d$.

Recall Burnside’s theorem (as in [5, 1.2] for example) : if $G \leq GL_d(\mathbb{C})$ is an irreducible subgroup, then there are finitely many $g_1, \dots, g_{d^2} \in G$ and $t_1, \dots, t_{d^2} \in M_d(\mathbb{C})$ such that $x = \sum \text{tr}(g_i x) t_i$ for every $x \in M_d(\mathbb{C})$. So if g has eigenvalues of modulus 1 for each $g \in G$, then G is contained in a bounded part of $M_d(\mathbb{C})$. Note that Burnside’s theorem holds just as well if G is a semi-group containing 1 in place of a subgroup: indeed the linear span in $M_d(\mathbb{C})$ of a semigroup in $GL_d(\mathbb{C})$ is invariant under the subgroup it generates so it must be all of $M_d(\mathbb{C})$ provided the subgroup acts irreducibly.

We now show (1) implies (2) for P_d . For $g \in \text{Isom}(P_d)$ the condition $L(g) = 0$ is equivalent to the requirement that all eigenvalues of g have modulus 1. Taking a composition series for $\langle S \rangle$, Burnside’s theorem implies that $\langle S \rangle$ is conjugate to a bloc upper-triangular subgroup of $GL_d(\mathbb{C})$, and bounded (i.e. relatively compact) in each bloc. Conjugating by an appropriate diagonal element, we can conjugate S into any neighborhood of the (compact) bloc diagonal part. This means that choosing $y \in P_d$ we can make $L(S, y)$ arbitrarily small. Therefore $L(S) = 0$ as desired. \square

Proof of Theorem 7.5. — As in the proof of Proposition 7.2 we may assume that X is the symmetric space P_d of Example 7.1. For $g \in \text{SL}_d(\mathbb{C})$ with eigenvalues a_1, \dots, a_d ordered in such a way that $|a_1| \geq \dots \geq |a_d|$, we set

$$j(g) = (\log |a_1|, \dots, \log |a_d|)$$

and we observe that

$$(7.6) \quad L_\infty(g) = \sqrt{(\log |a_1|)^2 + \dots + (\log |a_d|)^2} = \|j(g)\|_2.$$

If $g = k_1 a k_2$, with $k_1, k_2 \in K = \text{SU}_d(\mathbb{C})$ and a diagonal, then we set $\kappa(g) = j(a)$ and note that $\|g\| = \|\kappa(g)\|_2$.

We will use Bochi’s inequality in various irreducible representations of $\text{SL}_d(\mathbb{C})$. in the spirit of Kostant’s paper [49].

Recall that irreducible linear representations of $SL_d(\mathbb{C})$ are parametrized by a highest weight $\mathbf{n} := (n_1, \dots, n_d)$, where the n_i 's are integers satisfying $n_1 \geq \dots \geq n_d \geq 0$. We denote the associated representation by $(\pi_{\mathbf{n}}, V_{\mathbf{n}})$. We may find a hermitian scalar product and an orthonormal basis of the representation space $V_{\mathbf{n}}$ of $\pi_{\mathbf{n}}$ in which $\pi_{\mathbf{n}}(K)$ is unitary and $\pi_{\mathbf{n}}(a)$ diagonal for every diagonal $a \in SL_d(\mathbb{C})$. Then $\pi_{\mathbf{n}}(a)$ has maximal eigenvalue equal to its operator norm and equal to

$$\langle \mathbf{n}, j(a) \rangle := n_1 \log |a_1| + \dots + n_d \log |a_d|.$$

Furthermore:

$$(7.7) \quad \log \|\pi_{\mathbf{n}}(g)\| = \langle \mathbf{n}, \kappa(g) \rangle,$$

$$(7.8) \quad \log \Lambda(\pi_{\mathbf{n}}(g)) = \langle \mathbf{n}, j(g) \rangle.$$

Fix $\epsilon > 0$. Note that there are finitely many integer vectors $\mathbf{n}_1, \dots, \mathbf{n}_m$ in the quadrant $\mathcal{Q} = \{x \in \mathbb{R}^d; x_1 \geq \dots \geq x_d \geq 0\}$ such that for all $x \in \mathcal{Q}$ we have:

$$(7.9) \quad \sup_{1 \leq i \leq m} \left\langle \frac{\mathbf{n}_i}{\|\mathbf{n}_i\|}, x \right\rangle \geq \|x\|(1 - \epsilon).$$

Indeed just pick rational points forming an ϵ -net near the unit sphere in \mathcal{Q} .

Let $L_{\infty}^{\mathbf{n}}(S)$ be the asymptotic joint displacement of $\pi_{\mathbf{n}}(S)$ with respect to uniform norm on $V_{\mathbf{n}}$. Namely

$$L_{\infty}^{\mathbf{n}}(S) := \lim_k \frac{1}{k} \max_{g \in S^k} \log \|\pi_{\mathbf{n}}(g)\|$$

and similarly

$$\lambda_k^{\mathbf{n}}(S) := \max_{j \leq k} \frac{1}{j} \max_{g \in S^k} \log \Lambda(\pi_{\mathbf{n}}(g)).$$

Note that

$$(7.10) \quad \lambda_k^{\mathbf{n}}(S) \leq \lambda_k(S) \|\mathbf{n}\|$$

because

$$\log \Lambda(\pi_{\mathbf{n}}(g)) = \langle \mathbf{n}, j(g) \rangle \leq \|j(g)\|_2 \|\mathbf{n}\| = L_{\infty}(g) \|\mathbf{n}\|.$$

Now using (7.9) we may write for each $k \geq 1$

$$(7.11) \quad L_{\infty}(S) \leq \frac{1}{k} \max_{g \in S^k} \|\kappa(g)\| \leq \frac{1}{k} \frac{1}{(1 - \epsilon)} \max_{g \in S^k} \sup_{1 \leq i \leq m} \left\langle \kappa(g), \frac{\mathbf{n}_i}{\|\mathbf{n}_i\|} \right\rangle$$

$$(7.12) \quad \leq \frac{1}{(1 - \epsilon)} \sup_{1 \leq i \leq m} \frac{1}{\|\mathbf{n}_i\|} \frac{1}{k} \max_{g \in S^k} \log \|\pi_{\mathbf{n}_i}(g)\|.$$

Passing to the limit as k tends to infinity we get:

$$(7.13) \quad L_\infty(S) \leq \frac{1}{(1-\epsilon)} \sup_{1 \leq i \leq m} \frac{L_\infty^{\mathbf{n}_i}(S)}{\|\mathbf{n}_i\|}.$$

Then the Bochi inequality for the joint spectral radius (Proposition 7.11) implies that for each \mathbf{n} there is an integer $k_{\mathbf{n}}$ and a positive constant $C_{\mathbf{n}}$ such that for all S ,

$$L_\infty^{\mathbf{n}}(S) \leq \lambda_{k_{\mathbf{n}}}^{\mathbf{n}}(S) + C_{\mathbf{n}}.$$

Setting $K = \max\{k_{\mathbf{n}_i}, i = 1, \dots, m\}$ and $C = \max\{C_{\mathbf{n}_i}/\|\mathbf{n}_i\|; i = 1, \dots, m\}$ we get from (7.13) and (7.10)

$$(1-\epsilon)L_\infty(S) \leq \sup_{1 \leq i \leq m} \frac{\lambda_K^{\mathbf{n}_i}(S)}{\|\mathbf{n}_i\|} + C \leq \lambda_K(S) + C.$$

Finally we may choose the smallest integer n such that $C/n < \epsilon$ and, changing S into S^n , from Claim 7 equation (2.6) in the proof of Lemma 1.1 we obtain:

$$(1-\epsilon)L_\infty(S) \leq \lambda_{Kn}(S) + \epsilon. \quad \square$$

Proof of Proposition 7.6. — This is just the combination of (7.4) and (7.5) given that, as argued in the proof of Proposition 7.2 $L(S)$ and $L_\infty(S)$ defined in X coincide with their counterpart in P_d . \square

Proof of Proposition 7.7. — This follows from Proposition 7.11 and (7.5) after we note that $\log \Lambda(g) \leq L_\infty(g)$ for $g \in \text{SL}(\mathbb{C})$. The bound $k_0 \leq d^2$ follows from the different proof of Bochi’s inequality given in [17, Corollary 4.6]. \square

We end this section by recording two consequences of the above analysis (compare with Corollary 5.11).

COROLLARY 7.12 (Escaping elliptic elements in symmetric spaces). — *Let X be a symmetric space of non-compact type and $\epsilon > 0$, then there is $N(\dim X, \epsilon) > 0$ such that for every finite subset $S \subset \text{Isom}(X)$ with $1 \in S$ one of the following holds:*

- (1) $L(S) < \epsilon$,
- (2) there is $g \in S^N$ such that $L(g) \geq L(S)$.

Proof. — Since X is CAT(0), Theorem 3.6 shows that $L(S^n) \geq \frac{\sqrt{n}}{2}L(S)$ for every integer n . In particular by Proposition 7.7, if $L(S) \geq \epsilon$ and $\sqrt{N} \gg 2\sqrt{d}(C/\epsilon + 1)$, then there is $g \in S^N$ such that $L(g) \geq L(S)$ as desired. \square

Recall that for every symmetric space X , there exists a positive constant $\epsilon > 0$, the Margulis constant of X , such that if S is a finite set of isometries of X generating a discrete subgroup of $\text{Isom}(X)$ and if $L(S) < \epsilon$, then the

subgroup generated by S is virtually nilpotent (see [21, 61]). Hence we also get:

COROLLARY 7.13. — *Let X be a symmetric space of non-compact type, then there is $N > 0$ such that for every finite subset $S \subset \text{Isom}(X)$ containing 1 and generating a discrete subgroup, which is not virtually nilpotent, there is $g \in S^N$ such that $L(g) \geq L(S)$.*

Proof. — Apply Corollary 7.12 with ϵ being the Margulis constant of X . □

Remark 7.14. — The best $N = N(\epsilon, d)$ for which Corollary 7.12 holds tends to infinity as ϵ goes to 0. For the same reason the constant C in Proposition 7.7 cannot be taken to be 0 even at the cost of decreasing the multiplicative constant in front of $L(S)$. An example showing this in $SL(2, \mathbb{R})$ is given below in Section 8.

We can now give the

Proof of Proposition 1.8. — This follows from the combination of Proposition 7.6 and Lemma 1.7. We embed X as a closed convex subspace of the symmetric space P_d from Example 7.1 for some d . Let n be the smallest positive integer such that $L(S)\sqrt{n}/2 \geq 2\sqrt{d} \log d$. If $n = 1$, then $L(S) \geq 4\sqrt{d} \log \sqrt{d}$ and thus the inequality in Proposition 7.6 yields $L_\infty(S) \geq \frac{3}{4}L(S)/\sqrt{d}$. When $n \geq 2$, Lemma 1.7 and Proposition 7.6 imply

$$nL_\infty(S) \geq \frac{1}{\sqrt{d}}L(S^n) - \log \sqrt{d} \geq \frac{\sqrt{n}}{2\sqrt{d}}L(S) - \log \sqrt{d} \geq \frac{\sqrt{n}}{4\sqrt{d}}L(S)$$

On the other hand if $n \geq 2$, then the minimality of n implies that $L(S)\sqrt{n} \leq 16\sqrt{d} \log \sqrt{d}$ and the result follows. □

As shown in [17, Corollary 4.6] Proposition 7.11 also holds for Bruhat–Tits buildings. In fact it was observed and shown later in [16, Lemma 2.1] that in this case the constant c can be taken to be 1, that is the joint spectral radius is equal to the renormalized maximal eigenvalue, namely:

$$R(S) = \max_{j \leq k_0} \max_{g \in S^j} \Lambda(g)^{1/j},$$

if $S \subset M_d(k)$ is a finite set of matrices and k a non-archimedean local field. The same argument as in the proof of Proposition 7.7 readily implies:

LEMMA 7.15 (Bochi inequality for Bruhat–Tits buildings). — *Let X be a Bruhat–Tits building associated to a non-archimedean local field k and a semisimple algebraic group \mathbf{G} of dimension d . Then for every finite subset $S \subset \mathbf{G}(k)$ containing 1 there is $g \in S^{O(d^2)}$ such that*

$$L(g) \geq L(S)/\sqrt{d}.$$

Similarl the argument in the proof of Theorem 7.5 shows that $L_\infty(S) = \lambda_\infty(S)$ if $S \subset \mathbf{G}(k)$. An immediate consequence is also the following:

THEOREM 7.16 (Escaping elliptic elements in Bruhat–Tits buildings). *Let X as in Theorem 7.15. There is $N = N(d) \in \mathbb{N}$, such that for every finite subset $S \subset \mathbf{G}(k)$ containing 1 we have:*

- (1) *either $L(S) = 0$ and S fixes a point in X ,*
- (2) *or $L(S) > 0$ and there is $g \in S^N$ such that $L(g) \geq L(S)$.*

Compare with Corollary 7.12 and note the absence of ϵ . This result can be seen as a quantitative version of the fact, proved in [60], that if a subgroup of $G(k)$ is made entirely of elliptic elements, then it must fix a point in X . In particular we can always escape from elliptic elements in bounded time. By contrast, we will show in Section 10 that this property fails in certain symmetric spaces of non-compact type.

Even though results such as Theorems 7.12 and 7.16 fail for general CAT(0) spaces (as they fail already for Euclidean spaces $X = \mathbb{R}^d$) it is worth investigating for which classes of CAT(0) spaces they hold. For example:

QUESTION. — *Does Theorem 7.16 hold for the isometry group of an arbitrary affine building? does it hold for isometries of a finite dimensional CAT(0) cube complex?*

We note that [45] answers this question positively for CAT(0) square complexes.

8. Almost elliptics in $\mathrm{PSL}_2(\mathbb{R})$ with large displacement

The purpose of this section is to prove Lemma 1.12 from the introduction, which gives a simple example showing that Theorem 1.4 is best possible. More generally we will show:

PROPOSITION 8.1. — *Let X be a symmetric space of non-compact type. Then there is no $N = N(X) > 0$ such that for every finite set $S \subset \mathrm{Isom}(X)$ with $1 \in S = S^{-1}$, there is $g \in S^N$ such that $L(g) \geq L(S)$.*

This shows that the additive term $-K\delta$ is necessary (i.e. that we cannot take $K = 0$) in Theorem 1.4, and the $N(d, \epsilon)$ must tend to infinity as $\epsilon \rightarrow 0$ in Corollary 7.12. Note that this is in stark contrast to the non-archimedean case (see Theorem 7.16).

The proof is based on the fact that zooming in near a point in the hyperbolic plane $X = \mathbb{H}^2$, the metric becomes almost Euclidean, while on the Euclidean plane we can explicitly construct a set S with the desired properties. We begin by giving this construction:

Example 8.2. — Let \mathbb{R}^2 be the Euclidean plane. Subgroups of $\text{Isom}(\mathbb{R}^2)$ which consist only of elliptic elements must have a global fixed point in \mathbb{R}^2 . Indeed the commutator of two non trivial rotations with different fixed points is a non-trivial translation.

However the following is an example showing that we may have $L(S) = 1$, while no translation with significant minimal displacement can be found in S^N for any fixed N .

Take small numbers $x_1, x_2 > 0$ and set $\theta_i \in (0, \pi)$ so that $2 \sin(\theta_i/2) = x_i$. Assume that θ_1 and θ_2 are independent (i.e. that $1, \frac{\theta_1}{2\pi}$ and $\frac{\theta_2}{2\pi}$ are \mathbb{Q} -linearly independent). Let $S := \{1, R_1^{\pm 1}, R_2^{\pm 1}\}$, where R_1 is the rotation around the point $p_1 := (-x_1^{-1}, 0)$ and angle θ_1 , and R_2 the rotation around the point $p_2 := (x_2^{-1}, 0)$ and angle θ_2 .

Note that due to the independence assumption on θ_1 and θ_2 the linear parts of R_1 and R_2 generate a subgroup of $\text{SO}(2, \mathbb{R})$ which is free abelian of rank 2. Consequently any word in R_1 and R_2 which gives a translation in $\text{Isom}(\mathbb{R}^2)$ must belong to the commutator subgroup of the free group.

Moreover $L(S) = 1$ because $\text{Fix}_1(R_1) = \{p; |R_1 p - p| \leq 1\}$ and $\text{Fix}_2(R_2) = \{p; |R_2 p - p| \leq 1\}$ are the discs with radius x_1^{-1} and x_2^{-1} respectively centered around p_1 and p_2 respectively. They intersect at $(0, 0)$.

However, any word $w = w(R_1, R_2)$ in R_1 and R_2 which is a translation in $\text{Isom}(\mathbb{R}^2)$ belongs to the commutator subgroup of the free group, and hence the powers of R_1 in this word sum up to 0 and so do the powers of R_2 . As a result, for every $\epsilon > 0$ and every given $N \in \mathbb{N}$, one can choose small but positive x_1 and x_2 so that any such w of length at most N maps the origin $(0, 0)$ at a distance at most ϵ from itself. Hence $L(w) \leq \epsilon$ while $L(S) = 1$.

Proof of Proposition 8.1. — It is enough to prove this in the case when $X = \mathbb{H}^2$ is the hyperbolic plane, because \mathbb{H}^2 always embeds as a closed convex subspace of X stabilized by a copy of $\text{PSL}(2, \mathbb{R})$ in $\text{Isom}(X)$.

Suppose, by way of contradiction, that there is $N \in \mathbb{N}$ such that $\max_{S^N} L(g) \geq L(S)$ for every finite subset $S \subset \text{Isom}(\mathbb{H}^2)$ containing 1.

Let x_1, x_2 be two small positive numbers to be determined later. On the hyperbolic plane $X = \mathbb{H}^2$ consider a base point x_0 and a geodesic through x_0 . For every $\epsilon > 0$ let $p_1(\epsilon)$ and $p_2(\epsilon)$ be points on this geodesic on opposite sides of the base point x_0 such that $d(x_0, p_i(\epsilon)) = \epsilon x_i^{-1}$. Let $R_i(\epsilon)$ be the

hyperbolic rotation fixing $p_i(\epsilon)$ and of angle $\theta_i(\epsilon) \in (0, \pi)$ defined in such a way that $d(R_i(\epsilon)x_0, x_0) = \epsilon$.

Also note that the sets $\{x \in \mathbb{H}^2; d(R_i(\epsilon)x, x) \leq \epsilon\}$ are two hyperbolic disc centered at $p_i(\epsilon)$ that intersect only at the base point x_0 . It follows that

$$(8.1) \quad L(S_\epsilon) = L(S_\epsilon, x_0) = \epsilon.$$

Consider the renormalized pointed metric space $(X_\epsilon, d_\epsilon, x_0) := (X, d/\epsilon, x_0)$. In the Gromov–Hausdorff topology for pointed metric spaces, this family of metric spaces converges, as ϵ tends to 0, to the Euclidean plane $(\mathbb{R}^2, 0)$ with its standard Euclidean metric. The points $p_1(\epsilon)$ and $p_2(\epsilon)$ converge to two points p_1 and p_2 , which, after choosing coordinates, can be assumed to be $(-x_1^{-1}, 0)$ and $(x_2^{-1}, 0)$. Moreover the hyperbolic rotations $R_i(\epsilon)$ converge to their Euclidean counterpart R_i based at p_i with angle θ_i defined by $\|R_i(0, 0)\| = 1$, i.e. $x_i^{-1} = 2 \sin(\theta_i/2)$ as in Example 8.2. Consequently any word of length at most N in $S_\epsilon := \{1, R_1(\epsilon)^{\pm 1}, R_2(\epsilon)^{\pm 1}\}$ converges to the isometry of \mathbb{R}^2 given by the same word with letters R_1 and R_2 . In particular:

$$\lim_{\epsilon \rightarrow 0} w(R_1(\epsilon), R_2(\epsilon))x_0 = w(R_1(\epsilon), R_2(\epsilon))(0, 0),$$

and

$$(8.2) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} L(w(R_1(\epsilon), R_2(\epsilon)), x_0) = L(w(R_1, R_2), (0, 0)),$$

where $L(g, x)$ is the displacement of g at x in \mathbb{H}^2 .

Now by our assumption for each ϵ there is a word $w = w_\epsilon$ of length at most N such that

$$(8.3) \quad L(w(R_1(\epsilon), R_2(\epsilon))) \geq L(S_\epsilon) = \epsilon.$$

There are boundedly many words of length at most N , so letting ϵ tend to 0 along a certain sequence only we may assume that w is independent of ϵ . Then there are two options according as $w(R_1, R_2)$ is a translation or a rotation. Suppose first that w is a translation. We have:

$$\frac{1}{\epsilon} L(w(R_1(\epsilon), R_2(\epsilon)), x_0) \geq \frac{1}{\epsilon} L(w(R_1(\epsilon), R_2(\epsilon))) \geq 1,$$

while by (8.2) the left handside converges (as ϵ goes to zero along the subsequence) to $L(w(R_1, R_2), (0, 0))$. However, $w(R_1, R_2)$ being a translation means that the word w belongs to the commutator of the free group and the powers of R_1 sum to 0 as do the powers of R_2 (due to the independence of θ_1 and θ_2 as in Example 8.2). This implies that $L(w(R_1, R_2), (0, 0)) < 1/100$

provided x_1 and x_2 are chosen larger than some absolute constant. This is a contradiction.

We are left with the case when $w(R_1, R_2)$ is a rotation, with center say $c \in \mathbb{R}^2$. Since $(X, d/\epsilon, x_0)$ Gromov–Hausdorff converges to Euclidean \mathbb{R}^2 , there must be some point $c(\epsilon) \in \mathbb{H}^2$ which converges to c . This also means that

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} d(w(R_1(\epsilon), R_2(\epsilon))c(\epsilon), c(\epsilon)) = d(w(R_1, R_2)c, c) = 0.$$

In particular:

$$L(w(R_1(\epsilon), R_2(\epsilon))) = o(\epsilon),$$

which is in contradiction with (8.3). This ends the proof. □

9. Euclidean spaces and linear escape of cocycles

In this section we assume that the metric space X is the Euclidean space \mathbb{R}^d . We will prove the results stated in Section 3 regarding isometric actions on \mathbb{R}^d . In particular we will show an example of a finitely generated group of $\text{Isom}(\mathbb{R}^d)$ without global fixed point all of whose elements are elliptic. And we will show that every affine isometric action on \mathbb{R}^d without global fixed point has positive drift, i.e. that cocycles that are not co-boundaries have positive linear rate of escape.

We begin with the former.

Example 9.1 (A subgroup of rotations without global fixed point). — This is an example of a subgroup of $\text{Isom}(\mathbb{R}^{2n})$ for each $n \geq 2$ without global fixed point in \mathbb{R}^{2n} all of whose elements are elliptic. A similar example for $n = 2$ for isometries of $\mathbb{R}^4 \simeq \mathbb{C}^2$ with linear part in $\text{SU}_2(\mathbb{C})$ is due to Bass, answering a question of Kaplansky ([5, Counterexample 1.10]) and was later generalized by Kabenyuk in [43], who gave a complete characterization of those connected Lie groups admitting a dense subgroup entirely made of elliptic elements. We thank Misha Kapovich for directing us to this paper. The example below will be used in Section 10 in order to construct a Zariski dense subgroup of $\text{Isom}(\mathbb{H}^{2n})$ whose Cayley graph has balls of large radius made exclusively of elliptic elements.

Consider $S = \{1, A^{\pm 1}, B^{\pm 1}\}$, where A and B are two rotations fixing different fixed points $p_A \neq p_B$. We may choose the linear parts R_A and R_B in $O(2n)$ of A and B in such a way that the group they generate is a free group whose non-trivial elements never have 1 as an eigenvalue. This fact follows easily from Borel’s theorem on the dominance of words maps [12, 52] in simple algebraic groups (here $\text{SO}(2n)$) and from the fact

that in even dimension a generic rotation does not have 1 as an eigenvalue. To see this consider that, due to the dominance of word maps, the preimage in $\mathrm{SO}(2n) \times \mathrm{SO}(2n)$ of the elements having 1 as an eigenvalue via any word map is an algebraic subvariety of positive co-dimension, hence has empty interior. By Baire's theorem a Baire generic pair will lie outside the (countable) union of these subvarieties when the word ranges over all reduced words on two letters in the free group.

A consequence of this property is that every element in the subgroup Γ of $\mathrm{Isom}(\mathbb{R}^{2n})$ generated by A and B is elliptic (i.e. fixes a point in \mathbb{R}^{2n}). Indeed, since R_A and R_B generate a free subgroup, every non trivial element of Γ has a non trivial linear part and this linear part does not have 1 as an eigenvalue. But such isometries must fix a point. So $L(g) = 0$ for every $g \in \Gamma$, while clearly $L(S) \geq \frac{1}{2}|p_A - p_B| \min|\theta|$, where the minimum is taken over the angles θ of R_A and R_B (i.e. the numbers in $[0, \pi]$ such that $\exp(\pm i\theta)$ are the eigenvalues of R_A and R_B).

This example is to be contrasted with Proposition 3.2, which we restate here:

PROPOSITION 9.2. — *A subgroup G of $\mathrm{Isom}(\mathbb{R}^d)$ with $d = 2, 3$ all of whose elements have a fixed point must have a global fixed point.*

Proof. — Let G_0 be the index two subgroup of orientation preserving isometries. When $d = 2$, the commutator subgroup $[G_0, G_0]$ is made of translations. Hence G_0 is abelian. Two commuting isometries must preserve the fixed point set of each other. It follows that G_0 is either trivial or has a unique global fixed point, which must then be fixed by G (since G_0 is normal in G).

When $d = 3$ generic orientation preserving isometries are skew translations and do not have fixed points, in fact isometries with a fixed point form a closed subset of empty interior. In particular the closure H of G_0 is a proper closed subgroup of $\mathrm{Isom}(\mathbb{R}^3)^0$. Say $\mathrm{SO}(3, \mathbb{R})$ denotes the stabilizer of the origin in \mathbb{R}^3 . If the linear part of H is not all of $\mathrm{SO}(3, \mathbb{R})$ it must either be finite or conjugate to $\mathrm{SO}(2, \mathbb{R})$. In the latter case all elements in H have parallel axes and H must preserve an affine plane (the orthogonal to the axes) and be either trivial or have a unique global fixed point by the $d = 2$ case; in this case G fixes this point. If the linear part of H is finite, then every element of H has bounded order forcing G_0 itself (and hence G) to be finite (Burnside) and hence to have a global fixed point. Finally if the linear part of H is all of $\mathrm{SO}(3, \mathbb{R})$, then H must be conjugate to $\mathrm{SO}(3, \mathbb{R})$ (and hence have the origin as its unique global fixed point, which

must then be fixed by all of G). To see this note first the following simple geometric fact: if g, h are two rotations of angle π with disjoint axis, then gh is a skew rotation with axis the common perpendicular to the axes of g and h and translation length twice the distance between the axes. Now pick $h \in H$ a rotation of angle π and axis Δ . If $g\Delta$ intersects Δ for all $g \in H$, then any two lines of the form $g\Delta, g \in H$, intersect. This means either that the H -orbit of Δ is made of all lines through a single point (and then this point is the unique global fixed point), or that it lies in an affine plane (the one spanned by any two of the lines) and that H preserves that plane, a fact not compatible with the assumption that its linear part is all of $SO(3, \mathbb{R})$. □

Now we show that non trivial cocycles have linear rate of escape.

PROPOSITION 9.3. — *For any $d \geq 2$ and $S \subset \text{Isom}(\mathbb{R}^d)$ a finite set, then $L_\infty(S) = 0$ if and only if S has a global fixed point in \mathbb{R}^d .*

Proof. — First we give a proof of the easier fact that $L(S) = 0$ implies the existence of a global fixed point. Indeed note that the fixed point set $\text{Fix}(s)$ of an isometry s of \mathbb{R}^d is an affine subspace and, when $\text{Fix}(s)$ is non-empty, the sublevel sets $\text{Fix}_t(s) = \{x \in \mathbb{R}^d; d(x, sx) \leq t\}$ are convex subsets of \mathbb{R}^d that are products of $\text{Fix}(s)$ by the unit ball for a positive definite quadratic form on the orthogonal of $\text{Fix}(s)$. In particular there is $c_s > 0$ such that $\text{Fix}_t(s) \subset (\text{Fix}(s))_{c_s t}$ for all $t > 0$. Similarly the intersection of all $(\text{Fix}(s))_t$ for s varying in S and all $t > 0$ coincides with the intersection of all $\text{Fix}(s), s \in S$ (note that for any two affine subspaces there is $c > 0$ such that the intersection of their t -neighborhoods is contained in the ct -neighborhood of their intersection). So we obtain that S has a global fixed point.

Now we will show the slightly more delicate fact that the absence of global fixed point implies $L_\infty(S) > 0$. To do this we will first assume that Γ is dense in a Lie subgroup of the form $R \ltimes V$, where R is a closed Lie subgroup of $O(V)$ and $V \leq \mathbb{R}^d$ is a non-zero vector subspace. Afterwards we will reduce to this case.

For $\epsilon > 0$ consider a finite covering of the unit sphere in V by Euclidean balls of radius ϵ . In each ball pick an element $\gamma \in \Gamma$ whose translation part t_γ belongs to this ball and whose rotation part r_γ belongs to the ball of radius ϵ around the identity in R , i.e. $\|r_\gamma - 1\| \leq \epsilon$ for the operator norm on endomorphisms of V . This gives us a finite list $\gamma_1, \dots, \gamma_N$ of elements of Γ . They all belong to S^k for some integer k .

Now we make the following observation. If $x \in V \setminus \{0\}$ and $g = (r_g, t_g) \in \text{Isom}(V)$, then

$$\|gx\|^2 = \|r_g x + t_g\|^2 = \|x\|^2 + \|t_g\|^2 + 2\langle r_g x, t_g \rangle.$$

In particular:

$$\|gx\|^2 \geq \|x\|^2 + \|t_g\|^2 + 2\langle r_g x, t_g \rangle \geq \left(\|x\| + \frac{1}{2} \right)^2$$

provided $\frac{1}{2} \leq \|t_g\|$ and $\langle \frac{r_g x}{\|r_g x\|}, t_g \rangle \geq \frac{1}{2}$. Now if t_g lies in the same ϵ -ball as $x/\|x\|$ and $\|r_g - 1\| \leq \epsilon$, then these conditions are satisfied, provided ϵ is small enough ($\epsilon = 1/10$ does it). So we have shown that given any $x \in V \setminus \{0\}$, there is one $\gamma_i \in S^k$ such that

$$\|\gamma_i x\| \geq \|x\| + \frac{1}{2}.$$

Starting with any point $x_0 \in V \setminus \{0\}$ this immediately implies that for all $n \geq 1$,

$$L(S^{nk}, x_0) \geq \frac{n}{2},$$

and thus $L_\infty(S) \geq 1/2k > 0$ as desired.

We now explain how to reduce to the above special case. Let H be the closure of Γ in $\text{Isom}(\mathbb{R}^d)$. The Lie subgroup H has the form $R \times (\Delta \oplus V)$, there $V \leq \mathbb{R}^d$ is a vector subspace and Δ is a discrete subgroup of \mathbb{R}^d contained in the orthogonal of V . The isometric H -action on $\Delta \oplus V$ factorizes modulo Δ to an isometric H -action on V . It is enough to show that $L_\infty(S) > 0$ in the quotient action. The quotient action contains all translations from V . So by the analysis above it is enough to know that V is non-zero. Note that, since Δ is discrete and invariant under conjugation by R , there is a finite index subgroup of R commuting with Δ . So if V were trivial, there is a finite index subgroup Γ_0 of Γ such that any element g in this finite index subgroup will satisfy $t_{g^n} = nt_g$. It follows that $L_\infty(S) > 0$ unless Γ_0 lies in R . But then Γ_0 (and hence Γ taking the barycenter of a finite orbit) has a global fixed point. \square

Finally we mention the following result, which completes the picture regarding the failure of the Berger–Wang identity for isometries of \mathbb{R}^d for $d \leq 3$. Its proof is very close to that of the two previous propositions, so we leave it to the reader.

PROPOSITION 9.4. — *The Berger–Wang identity $\lambda_\infty(S) = L_\infty(S)$ holds for all finite subsets $S \subset \text{Isom}(\mathbb{R}^d)$ if and only if $d = 1$. When $d = 2$ it holds for all symmetric finite subsets but fails in general, and when $d = 3$ it holds*

for subsets S generating a subgroup whose rotation part is either finite or contains a dense subgroup of $\mathrm{SO}(3, \mathbb{R})$, but fails in general.

10. Escaping elliptic isometries of symmetric spaces

The goal of this section is mainly to provide certain counter-examples showing that in general one cannot escape elliptic elements in bounded time when taking products of isometries of symmetric spaces.

In this section we say that an element $g \in \mathrm{Isom}(X)$ is a *generalized elliptic element* if $L_\infty(g) = 0$. Since here X will be $\mathrm{CAT}(0)$, this is equivalent to the condition $L(g) = 0$ by Proposition 3.1. For symmetric spaces of non-compact type, this condition is equivalent to requiring that all eigenvalues of g (under some or any faithful linear representation of $\mathrm{Isom}(X)$) have modulus one.

We recalled in Proposition 7.2 the well-known fact that any subgroup of isometries of a symmetric space X of non-compact type, which is made exclusively of generalized elliptic elements, must fix a point in X or its boundary. So if G is a subgroup of $\mathrm{Isom}(X)$ generated by a finite set S we have $L(S) = 0$ if and only if $L(g) = 0$ for all $g \in G$. And if this happens then G fixes a point either in X or on the visual boundary ∂X (see Proposition 3.13).

The following natural question then arises:

QUESTION. — *Do we always escape from elliptics (or generalized elliptics) in bounded time? namely does there exist $N = N(X) \in \mathbb{N}$ such that for every finite symmetric set $S \subset \mathrm{Isom}(X)$ containing 1, either $L(S) = 0$ or there is $g \in S^N$ such that $L(g) > 0$?*

This section is devoted to answering this question. The answer, for hyperbolic spaces, is a little surprising:

PROPOSITION 10.1. — *The above question has a positive answer if X is an n -dimensional hyperbolic \mathbb{H}^n , when $n = 2, 3$, but a negative answer when $n \geq 4$.*

So if X is hyperbolic n -space with $n \geq 4$, then there are subgroups of isometries that pretend to be elliptic on a ball of arbitrarily large radius, even though they have no global fixed point in X nor on its boundary (and we will even build Zariski-dense examples when n is even). Further below we will also answer the question completely for arbitrary symmetric spaces of non-compact type.

Similarly a subgroup of isometries of the Euclidean plane or 3-space, all of whose elements are rotations, must have a global fixed point (Proposition 3.2 above). But this is no longer the case in \mathbb{R}^n for $n \geq 4$ by Example 9.1. So the dimension threshold is the same as for hyperbolic spaces.

Note that Example 9.1 can of course be embedded in a symmetric space X (e.g. by viewing $\text{Isom}(\mathbb{R}^{2n})$ as a bloc upper triangular subgroup of $SL_{2n+1}(\mathbb{R})$). This group will be made of elliptic elements only. Even though it will not fix a point in the symmetric space X , it will fix a point on the boundary (see Corollary 3.13). In particular $L(S) = 0$ in this example (even though $L(S)$ and $L_\infty(S)$ are strictly positive, when the group is viewed as a subgroup of isometries of Euclidean space).

In the case of $SL_2(\mathbb{C})$ a simple matrix computation yields the following avatar of Serre's lemma (Lemma 4.1):

PROPOSITION 10.2. — *Let $a, b \in SL_2(\mathbb{C})$. Assume that a, b and ab have all their eigenvalues of modulus 1. Then one of the following holds:*

- (1) *a and b can be simultaneously conjugated into $SU_2(\mathbb{C})$,*
- (2) *a and b can be simultaneously conjugated into the subgroup of upper triangular matrices with eigenvalues of modulus 1,*
- (3) *$[a, b] := aba^{-1}b^{-1}$ is loxodromic (i.e. its eigenvalues have modulus $\neq 1$).*

Since $PSL_2(\mathbb{R}) = \text{Isom}(\mathbb{H}^2)^0$ and $PSL_2(\mathbb{C}) = \text{Isom}(\mathbb{H}^3)^0$ the first part of Proposition 10.1 follows easily.

Proof. — We first recall the following well-known fact:

Claim. — For any $a, b \in SL_2(\mathbb{C})$ write $x = \frac{1}{2} \text{tr}(a)$, $y = \frac{1}{2} \text{tr}(b)$, $z = \frac{1}{2} \text{tr}(ab)$. Then

$$\frac{1}{2} \text{tr}([a, b]) = 2(x^2 + y^2 + z^2) - 4xyz - 1.$$

The proof is omitted: it is a simple computation using Cayley–Hamilton. One writes $a^2 - 2xa + 1 = 0$ and similarly for b^2 and $(ab)^2$ in order to expand any word in a and b as a linear combination of a , b , ab , aba and bab , then one takes the trace.

Note that a matrix $u \in SL_2(\mathbb{C})$ has eigenvalues of modulus 1 if and only if $\text{tr}(u)$ is real and belongs to $[-2, 2]$.

Suppose first that neither a nor b is diagonalizable. Then after changing a and b into their opposite if necessary, we may assume that both a and b are unipotent, hence have trace 2, that is $x = y = 1$. Then $\frac{1}{2} \text{tr}[a, b] = 2z^2 - 4z + 3$. This achieves its minimum at $z = 1$ only and the minimum

is 1. This means that $[a, b]$ is loxodromic unless $z = 1$. In some basis the matrices of a and b read:

$$a = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

We may assume that $t \neq 0$. We compute easily $\text{tr}(ab) = \text{tr}(b) + t\gamma$. Hence if $[a, b]$ is not loxodromic, we conclude that $t\gamma = 0$. It follows that $\gamma = 0$, which is case (2) in the proposition.

We may thus assume that a is diagonalizable, and $a \notin \{\pm 1\}$. In some basis the matrices of a and b now read:

$$a = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ and } b = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\cos(2\theta) \neq 1$. Then we compute:

$$\frac{1}{2} \text{tr}[a, b] = \cos(2\theta) + \alpha\delta(1 - \cos(2\theta)).$$

If $[a, b]$ is not loxodromic, then $\text{tr}[a, b]$ is real in $[-2, 2]$. Consequently $\alpha\delta \in \mathbb{R}$. Since $\alpha + \delta = \text{tr}(b)$ is real, we conclude that α and δ are complex conjugates and $\alpha\delta = |\alpha|^2$. Since $\text{tr}[a, b] \leq 2$, the above formula forces $|\alpha| \leq 1$. If $|\alpha| = 1$, then $\alpha\delta = 1$ and thus $\beta\delta = 0$, which means that we are in case (2) of the proposition.

We may thus assume that $|\alpha| < 1$ and also that $\beta\delta \neq 0$. Up to conjugating simultaneously a and b by a diagonal matrix $\text{diag}(t, t^{-1})$, for a suitable real $t > 0$, we may assume that $\gamma = -\bar{\beta}$. Indeed:

$$t^{-2}\gamma = -\overline{t^2\beta} \iff t^4 = -\gamma/\bar{\beta} = (1 - |\alpha|^2)/|\beta|^2.$$

But now $\delta = \bar{\alpha}$ and $\gamma = -\bar{\beta}$. This means that $b \in \text{SU}_2(\mathbb{C})$. So we are in case (1) of the proposition. This ends the proof. □

To prove the second part of Proposition 10.1, we need to construct a counter-example in $\text{Isom}(\mathbb{H}^4)$.

Example 10.3 (No escape in hyperbolic 4-space). — Let A, B be the two rotations in $\text{SO}(4)$ constructed in Example 9.1. They generate a free subgroup of $\text{SO}(4)$ whose non trivial elements do not have 1 as an eigenvalue. Now observe that the union of all conjugates of $\text{SO}(4)$ inside $\text{SO}(4, 1)$ has non-empty interior. One way to see this is to argue that this set is definable in real algebraic geometry and has the same dimension as $\text{SO}(4, 1)$ itself, because the absolute (complex) ranks of $\text{SO}(4)$ and of $\text{SO}(4, 1)$ coincide (they are equal to 2). Moreover an element of $\text{SO}(4)$ lies in the interior if and only if 1 is not one of its eigenvalues. This implies that for every integer $N \geq 1$ there are neighborhoods $\mathcal{U}_N(A)$ and $\mathcal{U}_N(B)$ in $\text{SO}(4, 1)$ of A

and B respectively such that $w(a, b)$ lies in the interior of elliptic elements for every non trivial reduced word w of length at most N in the free group on two letters, and every choice of a in $\mathcal{U}_N(A)$ and b in $\mathcal{U}_N(B)$. But we may choose such a pair a, b so that it generates a Zariski-dense subgroup of $\mathrm{SO}(4, 1)$. Indeed the set of pairs generating a non-Zariski dense subgroup of a semisimple algebraic group is a proper closed subvariety of the product ([40, Theorem 3.3] or [18, Theorem 4.1]). A Zariski-dense subgroup cannot be bounded, for otherwise it would be contained in a conjugate of the maximal compact subgroup $\mathrm{SO}(4)$. So by Proposition 7.2 it must contain a non-elliptic element.

Remark 10.4 (real closed fields). — This counter-example shows that Proposition 7.2 is very special to the field of real numbers. It does not hold for a general real closed field, or say for an ultrapower K of the reals. Over such fields K the example yields a Zariski-dense subgroup of $\mathrm{SO}(4, 1)(K)$ all of whose elements are contained in some conjugate of $\mathrm{SO}(4)(K)$.

The above counter-example can be made to work (with the exact same proof) in any symmetric space X for which the elliptic elements of $\mathrm{Isom}(X)^0$ have non-empty interior. Namely for each $N \in \mathbb{N}$ one can find pairs generating a Zariski-dense subgroup with the entire N -ball of the Cayley graph contained in the set of elliptic elements. Elliptic elements have non-empty interior if and only if the fundamental rank $\mathrm{rk}_{\mathbb{C}} G - \mathrm{rk}_{\mathbb{C}} K$ vanishes, where $G = \mathrm{Isom}(X)^0$ and K is a maximal compact subgroup of G (by [58, Example 3] this condition is equivalent to the vanishing of the Euler characteristic of the compact dual of X). An equivalent condition is to ask that G contains a Cartan subgroup (i.e. maximal abelian made of semisimple elements), which is compact (see e.g. [42]). For example if $X = \mathbb{H}^n$ this happens if and only if n is even. On the other hand, if X is such that elliptic elements have empty interior in $\mathrm{Isom}(X)^0$, then it is always possible to escape them, and in fact escape from the set of generalized elliptic elements, in bounded time, because they are contained in a proper algebraic subvariety of $\mathrm{Isom}(X)^0$.

PROPOSITION 10.5 (general symmetric spaces). — *Let $X = G/K$ be a symmetric space of non-compact type and $G = \mathrm{Isom}(X)^0$. If $\mathrm{rk}_{\mathbb{C}} G > \mathrm{rk}_{\mathbb{C}} K$, then there is $N = N(X) \in \mathbb{N}$ such that for every finite set $S \subset G$ generating a Zariski-dense subgroup of G , S^N contains an element g with an eigenvalue of modulus different from 1.*

Proof. — An element $g \in G$ is generalized elliptic (i.e. $L_\infty(g) = 0$) if and only if all its eigenvalues under some or any faithful linear representation of G have modulus 1. This can be read off the characteristic polynomial of g by asking that its real irreducible factors are either $X \pm 1$ or $X^2 + 2bX + 1$, where $b \in [-1, 1]$. Therefore generalized elliptic elements form a definable set in real algebraic geometry [11] whose Zariski-closure is a subvariety of positive co-dimension in G . Then the lemma follows by “escape from subvarieties” (see [32] and [19, Lemma 3.11]). \square

11. Producing free semi-groups by Ping-Pong

We prove a proposition concerning free semi-groups in δ -hyperbolic geometry. A hyperbolic isometry g of a δ -hyperbolic space X has two fixed points, which are in the ideal boundary of X . We denote the fixed point set by $\text{Fix}(g)$.

PROPOSITION 11.1. — *For $\delta \geq 0$, there is an absolute (numerical) constant Δ with the following property. Let X be a δ -hyperbolic space, and g, h isometries of X . Suppose $L(g), L(h) > \Delta\delta$. (Then g, h are hyperbolic isometries.) Assume $\text{Fix}(g) \neq \text{Fix}(h)$.*

Then the pair $\{g, h\}$, maybe after taking inverses of one or both of those, generates a free semi-group.

This proposition appears as Proposition 7.1 in [7] when X is a Hadamard manifold with sectional curvature $K \leq -1$. The case when X is a simplicial tree was treated by Bucher and de la Harpe in [41, Lemma]. In that case, $\delta = 0$. We give a proof of Proposition 11.1 since we do not know a reference in this context. It is not optimal, but $\Delta \leq 10000$. We follow the strategy in [41] and use the following well-known Ping-Pong lemma. Finding suitable ping-pong sets A, B in the δ -hyperbolic setting is more complicated than in the tree case.

LEMMA 11.2. — *Let X be a set and g, h injective maps from X to X . Suppose there are non-empty subsets A, B in X such that*

$$A \cap B = \emptyset, g(A \cup B) \subset A, h(A \cup B) \subset B.$$

Then g, h generate a free semi-group.

We will use axes of g, h to define A, B . We believe the following result is well known to specialists, but do not know a reference exactly for this statement, so we will give an argument at the end of this section.

LEMMA 11.3. — *If g is an isometry of a δ -hyperbolic space X with $L(g) > 1000\delta$, then there is a g -invariant, piecewise-geodesic, γ , parametrized by the arc-length, such that*

- (1) γ is a $(\frac{9}{10}, 24\delta)$ -quasi-geodesic.
- (2) For any points $x, y \in \gamma$, the Hausdorff distance between a geodesic $[x, y]$ and the sub-path of γ from x to y is at most 12δ .

We call this quasi-geodesic γ an axis of g in this section. The fixed point set $\text{Fix}(g)$ consists of the two end-points of γ , which are $\gamma(\pm\infty) = \lim_{t \rightarrow \pm\infty} \gamma(t)$, respectively for $+$ and $-$.

Let $\pi_\gamma : X \rightarrow \gamma$ be the nearest points projection. This is a coarse map in the sense that the image of each point is not a point but a set in X , but it is uniformly bounded:

Claim. — For every $x \in X$,

$$\text{diam } \pi_\gamma(x) \leq 30\delta.$$

Proof. — If $\delta = 0$ then X is a tree and the conclusion holds, so that we assume $\delta > 0$. Let $M(\delta) = 12\delta$ be the constant in Lemma 11.3. Let $x \in X$ and $y, z \in \pi_\gamma(x)$. We show $|y - z| \leq 2M + 6\delta = 30\delta$. Assume not. Let w be the midpoint of $[y, z]$, and take v on $[x, y]$ (or on $[x, z]$) with $|w - v| \leq \delta$. Also take $w' \in \gamma$ with $|w - w'| \leq M$. Then $|w' - v| \leq M + \delta$. On the other hand, $|y - v| \geq M + 3\delta - \delta$, so that $|x - v| \leq |x - y| - M - 2\delta$. We then have $|x - w'| \leq |x - v| + |v - w'| \leq |x - y| - \delta < |x - y|$, so that $y \in \gamma$ is not a nearest point from x ($w' \in \gamma$ is closer), a contradiction. \square

Since γ is g -invariant, the map π_γ is g -equivariant: $g\pi_\gamma(x) = \pi_\gamma(gx)$. For convenience, we choose a point from the set $\pi_\gamma(x)$ and denote this point by $\pi_\gamma(x)$. We arrange π_γ in this new definition to be g -equivariant.

Proof of Proposition 11.1. — We set $\Delta = 10000$. We also assume $\delta > 0$ since if $\delta = 0$ then X is a tree and this case is essentially treated in [41]. We will define subsets A, B in X that satisfies the assumption of Lemma 11.2 for g, h , or maybe one or both of their inverses. Then we are done by the lemma.

Let γ, σ be axes of g, h from Lemma 11.3. We arrange the direction (to which the parameter increases) of γ, σ as follows. If $\text{Fix}(g) \cap \text{Fix}(h)$ is non-empty (then it consists of one point by our assumption), then we arrange $\gamma(-\infty) = \sigma(-\infty)$. Notice that this case is equivalent to the projection $\pi_\gamma(\sigma)$ being unbounded. Now, if $\pi_\gamma(\sigma)$ is bounded but its diameter is longer than 500δ , then it means that near $\pi_\gamma(\sigma)$, σ runs parallel to γ in a bounded ($\leq 30\delta$) neighborhood of σ (use Lemma 11.3(2)), so that

we arrange the directions of γ, σ coincide in this part. Otherwise, we put directions randomly. In any case, $\gamma(\infty) \neq \sigma(\infty)$.

Since g, h are hyperbolic, there are $L \neq 0$ with $g(\gamma(t)) = \gamma(t + L)$ for all t , and $K \neq 0$ with $h(\sigma(t)) = \sigma(t + K)$ for all t . Changing g or h or both to their inverses if necessary, we may assume that $L > 0, K > 0$. We will prove the proposition for the pair g, h .

We choose two base points $P \in \gamma, Q \in \sigma$ in the following way.

Case (a). — If the distance between γ, σ is more than 100δ , then choose a shortest geodesic τ between them, and set $P = \tau \cap \gamma, Q = \tau \cap \sigma$.

Case (b). — If the distance between γ, σ is at most 100δ , then let P be the last point (in terms of the parameter t) in γ that is contained in $(\sigma)_{100\delta}$, which denotes the 100δ -neighborhood of σ . Likewise, let Q be the last point in σ that is contained in $(\gamma)_{100\delta}$. Note that $|P - \pi_\sigma(P)| = |Q - \pi_\gamma(Q)| = 100\delta$.

In either case, by adding constants to parameters, we may assume $P = \gamma(0), Q = \sigma(0)$.

Now we define two subsets in X . Set

$$A = \left\{ x \in X \mid \pi_\gamma(x) \in \gamma \left(\left[\frac{1}{2} \min(L(g), L(h)), \infty \right) \right) \right\}$$

$$B = \left\{ x \in X \mid \pi_\sigma(x) \in \sigma \left(\left[\frac{1}{2} \min(L(g), L(h)), \infty \right) \right) \right\}$$

We want to show the following. Then we are done by Lemma 11.2.

$$(11.1) \quad A \cap B = \emptyset, \quad g(A \cup B) \subset A, \quad h(A \cup B) \subset B.$$

For two points $x, y \in X$, we write $x \sim y$ if $|x - y| \leq 1000\delta$. We first show:

- (I) $\pi_\sigma(P) \sim Q, \pi_\gamma(Q) \sim P$.
- (II) If $x \in B$ then $\pi_\gamma(x) \sim P$. If $x \in A$ then $\pi_\sigma(x) \sim Q$.

We use the following lemma on the nearest points projection in a δ -hyperbolic space to prove (I) and (II). If α is a geodesic, then the lemma is well-known. We do not know a reference for our setting, so we give an argument at the end of the section. We denote the Hausdorff distance between two sets Y, Z in X by $d_H(Y, Z)$. A geodesic between two points $x, y \in X$ is denoted by $[x, y]$.

LEMMA 11.4.

- (1) Let α be an axis from Lemma 11.3, $x \in X$, and $w \in \alpha$. Let $z = \pi_\alpha(x)$. Then $d_H([x, z] \cup [z, w], [x, w]) \leq 16\delta$. Moreover, suppose $y \in X$ with $w = \pi_\alpha(y)$. If $|z - w| \geq 40\delta$, then $d_H([x, z] \cup [z, w] \cup [w, y], [x, y]) \leq 18\delta$.

(2) If $|x - y| \leq D$, then $|\pi_\alpha(x) - \pi_\alpha(y)| \leq D + 36\delta$.

We start the proof of (I) and (II). Note that if $x \in A$, then $|\pi_\gamma(x) - P| \geq \frac{9}{10}5000\delta - 24\delta = 4476\delta$ since $L(g), L(h) \geq 10000\delta$ and γ is an axis of g . Similarly, if $x \in B$, then $|\pi_\sigma(x) - Q| \geq 4476\delta$.

First, we discuss the case (a) in terms of P, Q , ie, $|P - Q| > 100\delta$. Then (I) is trivial since (we can arrange that) $\pi_\sigma(P) = Q$ and $\pi_\gamma(Q) = P$.

For (II), we show that if $x \in B$ then $\pi_\gamma(x) \sim P$. Set $y = \pi_\sigma(x)$ and $z = \pi_\gamma(x)$. We want to show $|z - P| \leq 1000\delta$. Since $Q = \pi_\sigma(P), y = \pi_\sigma(x)$ and $|Q - y| \geq 4476\delta$, we have $d_H([x, y] \cup [y, Q] \cup [Q, P], [x, P]) \leq 18\delta$ by Lemma 11.4. Also $d_H([x, z] \cup [z, P], [x, P]) \leq 16\delta$ since $z = \pi_\gamma(x)$. So, $d_H([x, y] \cup [y, Q] \cup [Q, P], [x, z] \cup [z, P]) \leq 34\delta$. To argue by contradiction, assume $|P - z| > 1000\delta$. Let $R \in [P, Q]$ be the point with $|P - R| = 100\delta$. Then there is $R' \in [P, z]$ with $|R - R'| \leq 34\delta$. Then there is $R'' \in \gamma$ with $|R' - R''| \leq 12\delta$, so that $|R - R''| \leq 46\delta < |R - P|$. But then $|R'' - Q| \leq |Q - R| + |R - R''| < |Q - R| + |R - P| = |Q - P|$. This is a contradiction since P must be the nearest point on γ from Q (but we found R'' is closer). We are done for the first claim of (II). The second one is similar (just switch the roles), and we do not repeat. Case (a) is finished.

Suppose we are in case (b). For two points $x, y \in \gamma$, we write $x \leq y$ if $x = y$, or y appears after x in terms of the parameter t . We write $x < y$ if $x \leq y$ and $x \neq y$. We use the similar notation for points on σ .

We argue (I). By the way we chose P, Q , we have $\pi_\sigma(P) \leq Q$ and $\pi_\gamma(Q) \leq P$. By Lemma 11.4, $d_H([Q, \pi_\sigma(P)] \cup [\pi_\sigma(P), P], [Q, P]) \leq 16\delta$ and $d_H([Q, \pi_\gamma(Q)] \cup [\pi_\gamma(Q), P], [Q, P]) \leq 16\delta$. So, $d_H([Q, \pi_\sigma(P)] \cup [\pi_\sigma(P), P], [Q, \pi_\gamma(Q)] \cup [\pi_\gamma(Q), P]) \leq 32\delta$.

To argue by contradiction, suppose $|\pi_\sigma(P) - Q| > 1000\delta$. Then since $|Q - \pi_\gamma(Q)| = 100\delta$ and $|P - \pi_\sigma(P)| = 100\delta$, we have $|\pi_\gamma(Q) - P| > 800\delta$. But this forces that the orientations of γ and σ are opposite along the geodesic $[\pi_\gamma(Q), P]$ (draw a thin rectangle for $\pi_\gamma(Q), Q, \pi_\sigma(P), P$), which is a contradiction. We showed $|\pi_\sigma(P) - Q| \leq 1000\delta$. The other inequality is proved similarly. (I) is shown.

We argue (II). As before we only discuss the first one. Set $y = \pi_\sigma(x)$ and $z = \pi_\gamma(x)$. We want to show $|P - z| \leq 1000\delta$. Since $\pi_\sigma(P) \leq Q$, we have $|y - \pi_\sigma(P)| \geq 4476\delta$. So, $d_H([P, \pi_\sigma(P)] \cup [\pi_\sigma(P), y] \cup [y, x], [P, x]) \leq 18\delta$. Also, $d_H([P, z] \cup [z, x], [P, x]) \leq 16\delta$. Hence, $d_H([P, z] \cup [z, x], [P, \pi_\sigma(P)] \cup [\pi_\sigma(P), y] \cup [y, x]) \leq 34\delta$.

Now, to argue by contradiction, assume $|P - z| > 1000\delta$. Let $R \in [P, z]$ be the point with $|P - R| = 500\delta$. Pick a point $R' \in \gamma$ with $P < R' < z$ and $|R - R'| \leq 12\delta$ by Lemma 11.3. Also, pick a point $R'' \in [P, \pi_\sigma(P)] \cup$

$[\pi_\sigma(P), y] \cup [y, x]$ with $|R - R''| \leq 34\delta$. Notice that in fact $R'' \in [\pi_\sigma(P), y]$ since $|P - \pi_\sigma(P)| = 100\delta$. Then pick a point $R''' \in \sigma$ with $|R'' - R'''| \leq 12\delta$. But this implies $|R''' - R'| \leq (12 + 34 + 12)\delta = 58\delta$, so that $R' \in (\sigma)_{100\delta}$. This is a contradiction since $P < R'$ on γ (P must be the last such point). (II) is shown. We finished the proof of (I) and (II).

We now show (11.1) using (I) and (II).

Clearly both A and B are not empty. To see $A \cap B$ is empty, take any point $x \in B$. Then by (II), $\pi_\gamma(x) \sim P$, ie, $|\pi_\gamma(x) - P| \leq 1000\delta$. But if $x \in A$ then as we said $|\pi_\gamma(x) - P| \geq 4476\delta$, so $x \notin A$, hence we are done. $g(A) \subset A$ is immediate from the g -equivariance of π_γ . To see $g(B) \subset A$, let $x \in B$. Since $\pi_\gamma(x) \sim P$ we have $g(\pi_\gamma(x)) \sim g(P)$. Also $g(P) \in \gamma \cap A$ implies $|P - g(P)| \geq \frac{9}{10}10000\delta - 24\delta = 8976\delta$, hence $|P - g(\pi_\gamma(x))| \geq 7976\delta$. This implies $g(\pi_\gamma(x)) \in \gamma \cap A$. But since $g(\pi_\gamma(x)) = \pi_\gamma(g(x))$, it follows $g(x) \in A$ from the definition of A . We showed $g(B) \subset A$, hence $g(A \cup B) \subset A$. $h(A \cup B) \subset B$ is similar (switch the roles) and we do not repeat. (11.1) is proved. □

We give a proof of Lemma 11.3. A similar statement appears as Lemma 1 in [34], in which we only assume g is hyperbolic, with no large lower bound of $L(g)$, and find a g -invariant piecewise geodesic that satisfies (2) with a constant bigger than 12δ . One can not expect (1) holds in that case.

Proof of Lemma 11.3. — If $\delta = 0$, then X is a tree and there is a g -invariant geodesic, γ . This is a desired path, so we assume $\delta > 0$ in the rest.

If we take any point $x \in X$ and join its g -orbit in the obvious order by geodesics, we obtain a g -invariant, piecewise geodesic. This path is always a quasi-geodesic but we do not have uniform bounds on the quasi-geodesic constants. But we choose m to be the midpoint of $[x, gx]$, then define a g -invariant, piecewise geodesic:

$$\gamma = \bigcup_{n \in \mathbb{Z}} g^n([m, gm]).$$

The merit of m is that $[m, gm]$ is very close to $L(g)$, and the “bumps” at points $g^n(m)$ on γ are small. We first show γ satisfies (2) using $L(g) \geq 1000\delta$. Take any two points $x, y \in \gamma$. Using the g -action if necessary, we may assume $x \in [m, g(m)], y \in [g^{n-1}(m), g^n(m)]$ for some $n > 1$. (If $n = 1$, then (2) is trivial.) The first observation is that $g^i(m) \in N_{2\delta}([m, g^n(m)])$ for all $0 \leq i \leq n$. This is an easy exercise by induction on n , and we leave it to readers (see Remark). From this it follows $x, y \in N_{3\delta}([m, g^n(m)])$.

Next, we show:

Claim. — $g(m), \dots, g^{n-1}(m)$ are all in $N_{10\delta}([x, y])$.

To see that, take points on $[m, g^n(m)]$ that are close to those points:

$$\bar{x}, \overline{g(m)}, \overline{g^2(m)}, \dots, \overline{g^{n-1}(m)}, \bar{y} \in [m, g^n(m)]$$

with $|x - \bar{x}| \leq 3\delta$, $|g(m) - \overline{g(m)}| \leq 2\delta, \dots, |g^{n-1}(m) - \overline{g^{n-1}(m)}| \leq 2\delta$, $|y - \bar{y}| \leq 3\delta$. Take $[\bar{x}, \bar{y}]$ to be the subpath of $[m, g^n(m)]$. Notice that the points $\overline{g^2(m)}, \dots, \overline{g^{n-2}(m)}$ are contained in $[\bar{x}, \bar{y}]$ since $L(g) \geq 1000\delta$, but possibly $\overline{g(m)}$ or $\overline{g^{n-1}(m)}$ is not in $[\bar{x}, \bar{y}]$, ie, $\overline{g(m)} \in [m, \bar{x}]$ or $\overline{g^{n-1}(m)} \in [\bar{y}, g^n(m)]$. But this exceptional case happens only when $g(m)$ is close to x , or $g^{n-1}(m)$ is close to y . We finish this case first. So suppose $\overline{g(m)} \in [m, \bar{x}]$. Then pick a point $z \in [m, x]$ with $|\overline{g(m)} - z| \leq 4\delta$ using $|x - \bar{x}| \leq 3\delta$. It implies that $|g(m) - z| \leq 2\delta + 4\delta = 6\delta$, hence $|x - g(m)| \leq 6\delta$. This is a desired bound and the claim is shown for $g(m)$. Similarly, we have $|y - g^{n-1}(m)| \leq 6\delta$.

Now we go back to the general case, ie, $\overline{g^i(m)} \in [\bar{x}, \bar{y}]$, so that $g^i(m) \in N_{2\delta}([\bar{x}, \bar{y}])$. But, $d_H([x, y], [\bar{x}, \bar{y}]) \leq 8\delta$ since $|x - \bar{x}|, |y - \bar{y}| \leq 3\delta$. It then implies that $g^i(m)$ is in $N_{10\delta}([x, y])$. The claim is shown.

Finally, let $\gamma(x, y)$ denote the subpath of γ between x, y then the above claim implies that $d_H([x, y], \gamma(x, y)) \leq 12\delta$ since $\gamma(x, y)$ is a piecewise geodesic joining the points in the claim. (2) is shown.

We prove (1). Remember that $g^i(m), 1 \leq i \leq n-1$, are the points that are contained in $\gamma(x, y)$, the subpath of γ between x, y . Then by (2), pick points $x_i \in [x, y], 1 \leq i \leq n-1$, with $|x_i - g^i(m)| \leq 12\delta$. Let $x = \gamma(t), y = \gamma(s)$. Then,

$$\begin{aligned} &|x - y| \\ &= |x - x_1| + |x_1 - x_2| + \dots + |x_{n-1} - y| \\ &\geq (|x - g(m)| - 12\delta) + (|g(m) - g^2(m)| - 24\delta) + \dots \\ &\quad + (|g^{n-2}(m) - g^{n-1}(m)| - 24\delta) + (|g^{n-1}(m) - y| - 12\delta) \\ &\geq \frac{1000\delta - 24\delta}{1000\delta} (|x - g(m)| + |g(m) - g^2(m)| + \dots + |g^{n-1}(m) - y|) - 24\delta \\ &\geq \frac{1000 - 24}{1000} |t - s| - 24\delta \geq \frac{9}{10} |t - s| - 24\delta. \end{aligned}$$

Note that we need -24δ since we do not have $|x - g(m)|, |g^{n-1}(m) - y| \geq 1000\delta$. □

Remark 11.5. — This claim appears in the beginning of the proof of Lemma 1 in [34], where we do not assume $L(g) \geq 1000\delta$, so that we take a power of g, g^N , with $L(g^N) \geq 1000\delta$, then prove the same claim for g^N .

We now give a proof of Lemma 11.4.

Proof of Lemma 11.4. — If $\delta = 0$, then X is a tree and the lemma is trivial, so we assume $\delta > 0$.

We prove the first claim. If $|x - z| \leq 14\delta$, then the conclusion is trivial, so assume $|x - z| > 14\delta$. Let $z' \in [x, z]$ with $|z - z'| = 14\delta$. $z' \in N_\delta([x, w] \cup [z, w])$, but in fact $z' \in N_\delta([x, w])$. This is because if $z' \in N_\delta([z, w])$, then $z' \in N_{13\delta}(\alpha)$ since α is an axis, which is impossible since $d(z', \alpha) = d(z', z) = 14\delta$. So, pick $z'' \in [x, w]$ such that $|z'' - z'| \leq \delta$, then $|z - z''| \leq 15\delta$. It implies that $d_H([z, w], [x, z] \cup [z, w]) \leq 16\delta$, and we are done. An argument for the second claim is similar. Drawing a geodesic quadrilateral for x, z, w, y and using $|z - w|$ is long, we find $z'', w'' \in [x, y]$ with $|z - z''|, |w - w''| \leq 16\delta$. Then the conclusion easily follows. We omit details. (1) is finished.

(2) is a consequence of (1). Indeed, suppose $|\pi_\alpha(x) - \pi_\alpha(y)| > D + 36\delta$. By (1), there must be points $Z, W \in [x, y]$ with $|Z - \pi_\alpha(x)|, |W - \pi_\alpha(y)| \leq 18\delta$. That implies that $|Z - W| > D$, so that $|x - y| > D$, a contradiction. \square

12. Uniform Tits alternative for groups acting on trees

There is no uniform Tits alternative for groups acting on trees. This section is devoted to an example showing this. To begin with, we recall the following:

PROPOSITION 12.1. — *If S is a finite set of isometries of a tree with no global fixed point on the tree nor on its boundary, then one may find a pair a, b in $(S \cup S^{-1})^3$ generating a free sub-semigroup.*

Proof. — We give a quick proof for the readers convenience (e.g. see also [54, Theorem 1]). This assumption implies that $L(S) > 0$. Proposition 4.4 implies that there is $g \in S \cup S^2$ with $L(g) > 0$. Since g is a hyperbolic isometry it fixes exactly two points on the boundary of the tree. Say x is the forward fixed point. Our assumption implies that there is $s \in S$ with $sx \neq x$. Now we may apply Proposition 11.1 to the pair $\{g, sgs^{-1}\}$ (take inverses if necessary) and get the desired conclusion. \square

So we can find a free sub-semigroup quickly. We now show that by contrast we may not be able to quickly find a pair generating a free subgroup. This is Proposition 1.16 from the Introduction, which we restate here.

PROPOSITION 12.2. — *For every $N \in \mathbb{N}$ one can find a pair $S = \{a, b\}$ of automorphisms of the 3-valent homogeneous tree T such that the subgroup generated by S contains a free subgroup and has no global fixed point on T nor on the boundary ∂T , but no pair of elements in the N -ball $B(N) := (\{1\} \cup S \cup S^{-1})^N$ generates a free subgroup.*

Proof. — Let a and b be two hyperbolic elements with minimal displacement 1 and whose axes intersect on a finite segment $[p, q]$ of length L on which they translate in the same direction. It is clear that a^{L+1} and b^{L+1} generate a free subgroup. Indeed they play ping-pong on the tree : the attracting and repelling neighborhoods being the four connected components of $T \setminus \{p, q\}$ different from that containing the open segment (p, q) . In particular the subgroup $\Gamma := \langle a, b \rangle$ generated by a and b does not fix a point in T nor on the boundary ∂T .

Fix a labelling of the edges of T using the alphabet $\{1, 2, 3\}$. Now choose a and b as above with the further property that a and b preserve the cyclic ordering of the edges at every vertex, except possibly at the end-points p and q of the intersection of the axes and a and b . To see that it is possible to do this note first that if we start with one edge of T there is a unique bi-infinite geodesic through this edge such that any two consecutive edges on the geodesic are labelled by consecutive labels and thus there is a unique tree automorphism a that preserves the cyclic order and acts by unit translation on this bi-infinite geodesic Δ_a . This gives a . To find b do the same on a bi-infinite geodesic Δ_b such that $\Delta_a \cap \Delta_b$ is a segment $[p, q]$ of length L . It is possible to obtain the isometry b (in a unique way) so that b preserves the cyclic order at every vertex, except that we need to reverse that order at exactly two points : the end-points p and q .

Now let c and d be any two words of length at most $L/2$ in a, b and their inverses. The commutator $[c, d]$ must fix the point p . We claim that $[c, d]$ has finite order (and thus c, d cannot generate a free subgroup). Indeed $[c, d]$ preserves the cyclic ordering at any vertex which is at distance at least R from either p or q , where R is the maximum distance from either p or q or the image of either p or q under any word of length at most $2L$ in a, b and their inverses. Since $[c, d]$ fixes p some power $[c, d]^n$ of $[c, d]$ will fix pointwise the entire ball of radius R around p . But since the edge ordering outside is also preserved, $[c, d]^n$ must be trivial. This ends the proof. \square

Remark 12.3. — In [2, 12.1] it is shown that if S is a finite set of tree automorphisms, generators of a free subgroup can be found in $(\{1\} \cup S \cup S^{-1})^N$ with controlled N , provided $\langle S \rangle$ has not fixed global point on the tree nor on its boundary and satisfies a certain acylindricity condition.

Remark 12.4. — As was pointed out to us by Yves de Cornulier, the above isometries a, b belong to the group of tree automorphisms with prescribed local action and finitely many singularities. These groups, denoted by $G(F, F')$, in [53] have attracted a lot of attention lately. In our example $F' = \text{Sym}(3)$ and $F = \mathbb{Z}/3\mathbb{Z}$.

Remark 12.5. — We learned from Jack Button that the Baumslag–Solitar group $\Gamma = BS(2, 3) := \langle a, b \mid ba^3b^{-1} = a^2 \rangle$ also yields a similar example. Indeed this group is a non-trivial HNN extension and as such acts on its Bass–Serre tree, which is a 5-valent homogenous tree. This action has no global fixed point on the tree nor on its boundary. Vertex stabilizers are conjugates of $\langle a \rangle$, hence abelian. However it is easy to check that given $N \in \mathbb{N}$ for every large enough n no pair of words of length at most N in the elements b and a^{2^n} can generate a free subgroup: indeed there is an arbitrarily long portion of the axis of b which is fixed by a^{2^n} if n is arbitrarily large. In particular commutators of short words in b and a^{2^n} commute. An interesting additional feature of this construction is that b and a^{2^n} still generate Γ . And this group is not uniformly non-amenable: see [2].

13. Application to uniform exponential growth

In this section we discuss applications to the exponential growth of groups. We recall some definitions. Let Γ be a group and S a finite set in Γ . Assume that $1 \in S$ and $S = S^{-1}$. Set

$$h(S) := \lim_{n \rightarrow \infty} \frac{1}{n} \log |S^n|.$$

write $h(S, \Gamma)$ to indicate that the subset S generates Γ . We denote by $\langle S \rangle$ the subgroup generated by S . We recall that the quantities $L_\infty(S)$ and $L(S)$ were defined in the Introduction. Let Γ be a finitely generated group. Set

$$h(\Gamma) = \inf_S \{h(S); \langle S \rangle = \Gamma\},$$

where S runs over finite generating subsets. If $h(\Gamma) > 0$ we say Γ has *uniform exponential growth*, of *growth rate* $h(\Gamma)$.

13.1. Trichotomy for actions on hyperbolic spaces

The following result is a consequence of the Bochi-type inequality for hyperbolic spaces (Theorem 1.4) and of the construction of ping-pong pairs from Section 11.

THEOREM 13.1 (Trichotomy for actions on hyperbolic spaces). — *There is an absolute constant $C > 1$ such that the following holds. Let X be a δ -hyperbolic geodesic space and $S \subset \text{Isom}(X)$ a finite set with $1 \in S$. Assume $S = S^{-1}$. Then one of the following holds:*

- (1) $L(S) \leq C\delta$ on X .
- (2) $\langle S \rangle$ leaves a set of two points in the boundary of X invariant. Moreover, $\langle S \rangle$ contains a hyperbolic isometry g , and $\text{Fix}(g) \subset \partial X$ is the invariant set.
- (3) If $N > C$ is an integer, then S^N contains two elements, which are hyperbolic isometries and are generators of a free semi-group. In particular, $h(S) \geq (\log 2)/N > 0$.

Proof. — Assume $C > K + 1$, where K is the numerical constant from Theorem 5.7 (and Corollary 5.8). If we are not in case (1), then $L(S) > (K + 1)\delta$. Since by Corollary 5.8, for all $n > 0$, $n(L(S) - K\delta) \leq \lambda_2(S^n)$, so that $n\delta < \lambda_2(S^n)$. It implies that for each $n > 0$ there exists $g \in S^{2n}$ with $2n\delta < L_\infty(g)$.

Set N_0 be the smallest integer with $\Delta \leq N_0$, where Δ is the constant from Proposition 11.1. Then there is $g \in S^{2N_0}$ with $\Delta\delta \leq L_\infty(g)$. Since $L_\infty(g) \leq L(g)$, we have $\Delta\delta \leq L(g)$ and g is hyperbolic on X .

Now, if S preserves $\text{Fix}(g)$, then we are in (2), otherwise, there must be $s \in S$ such that the proposition applies to $g, sgs^{-1} \in S^{2N_0+2}$ (take inverses if necessary), and we are in (3) provided $C > 2N_0 + 2$. \square

13.2. Uniform exponential growth of hyperbolic groups

We now prove uniform, and uniform exponential growth of hyperbolic groups.

First we show Theorem 1.14 from the introduction, that δ -hyperbolic groups have uniform exponential growth depending only on δ , in the form of a corollary of our trichotomy Theorem 13.1 applied to hyperbolic groups. This slightly improves on Koubi's result [50] and Champetier–Guirardel [24], where a further dependence on the size of the generating set was required. We then discuss uniform exponential growth.

COROLLARY 13.2 (Growth of hyperbolic groups). — *There is an absolute constant $C_1 > 0$ with the following property. If (G, S) is a non-elementary, δ -hyperbolic group, then S^M contains two hyperbolic elements that are generators of a free semi-group, where M is the least integer larger than $C_1\delta$. In particular, $h(S) \geq (\log 2)/M$.*

We start with a simple lemma.

LEMMA 13.3. — *Let G be an infinite group generated by a finite symmetric set S with Cayley graph $\Gamma(G, S)$. Then for the (left) action of G on $\Gamma(G, S)$, we have for all $n > 0$, $L(S^n) \geq n$.*

Proof. — Fix $n > 0$. For any $g \in G$, we need to show that there is $h \in S^n g$ such that $d(g, h) \geq n$. Equivalently this amounts to ask that $h \in S^n g$ but $h \notin gS^{n-1}$. Suppose for contradiction that $S^n g$ is contained in gS^{n-1} , then $|S^n| \leq |S^{n-1}|$. But since $S^{n-1} \subset S^n$, this means that $S^n = S^{n-1}$. This implies that $\langle S \rangle$ is finite. \square

Proof of Corollary 13.2. — Let $C > 0$ be the absolute constant from Theorem 13.1. (We may assume both C and δ are integers.) For each $n > 0$, $L(S^n) \geq n$ by Lemma 13.3. Let n be the least integer larger than $C\delta$. Theorem 13.1 applied to S^n implies that either S^{Nn} contains generators of a free semigroup, or S^n and hence $\langle S \rangle$ leaves invariant a pair of distinct points on the boundary ∂X of the Cayley graph $X = \Gamma(G, S)$. But G acts with dense orbits on its boundary, so the second case cannot occur. Now set $C_1 = NC$. \square

We now give a uniform exponential growth result for subgroups of hyperbolic groups, which is a direct consequence of Theorem 13.1.

COROLLARY 13.4 (Uniform growth of subgroups in a hyperbolic group). *There exists an absolute constant (integer) $C_1 > 0$ with the following property. Let G be a group generated by a finite symmetric set T so that the Cayley graph $\Gamma(G, T)$ is δ -hyperbolic for some integer $\delta > 0$. Then for every finite subset $S \subset G$ with $1 \in S$ and $S = S^{-1}$, either $\langle S \rangle$ is finite or virtually cyclic, or there is $g \in G$ such that $gSg^{-1} \subset T^{C_1\delta}$, or $S^{\lceil C_1\delta \rceil}$ contains two generators of free semi-group.*

Proof. — This is immediate from Theorem 13.1 as the condition $L(S) \leq C\delta$ translates by saying that there is $g \in G$ such that $gSg^{-1} \subset T^{C\delta}$. \square

We note that although it does not appear explicitly in the literature as far as we know the last corollary is hardly new as explained in the remarks below.

Remark 13.5. — Without a bound on $|T|$ it can happen that a large power of S stays inside $T^{C\delta}$ and indeed for every $\ell \geq 1$ one can easily build a 2-hyperbolic group containing pairs of elements generating a non virtually cyclic subgroup so that every word of length at most ℓ in these two elements are torsion. For example consider the example of Remark 1.15. The group $\Gamma_{n,\ell}$ is δ -hyperbolic with generators $\{a, b\}$ for some $\delta = \delta(n, \ell) > 0$.

Consider the same group with generating set $T := \{1, a^{\pm 1}, b^{\pm 1}\}^{\lceil \delta \rceil}$. Then $(\Gamma_{n,\ell}, T)$ is 2-hyperbolic, but every word of length at most ℓ in $S := \{a, b\}$ is of order dividing n .

Remark 13.6. — Note that if $\langle S \rangle$ is not virtually cyclic we can find an upper bound on the least n such that $L(S^n) > C_1\delta$, and hence on the least n for which S^n contains generators of a free semigroup. Indeed we can get $n < |T^{\lceil C_1\delta \rceil}|$, because if $L(S^n) \leq C_1\delta$, then $|S^n| \leq |T^{\lceil C_1\delta \rceil}|$. But $|S^n| \geq n$ unless $\langle S \rangle$ is finite, hence the upper bound on n . Note that this bound depends not only on δ , but also on $|T|$. A closely related result appears in Arzhantseva–Lysienok [3] where it is shown that S^n contains generators of a free subgroup with $n \leq C(G)$, for some (effective but not explicit) constant $C(G)$ depending only on G . As the referee pointed out to us, there are only finitely many hyperbolic groups with given number of generators and given δ , therefore this shows that we have recovered the cited result from [3] with the explicit bound $C(G) = |T^{\lceil C_1\delta \rceil}|$.

Remark 13.7. — In [50] Koubi obtains the same result in the case when $S = T$ is a generating subset of G . In [24] Champetier and Guirardel show a related result: there is an explicit n depending only on δ and $|T|$ such that given any $f, g \in G$ either at least one of the pairs $\{f^n, g^n\}$ or $\{f^n, g^{-n}\}$ generates a free semigroup, or f^n and g^n commute.

In a very recent work Delzant and Steenbock [31] improve on Corollary 13.4 and on the above mentioned work of Arzhantseva–Lysienok and give an explicit entropy lower which improves as $|S|$ gets larger.

13.3. Actions by virtually nilpotent groups

We give a proof of the following that is stated as Corollary 1.5.

PROPOSITION 13.8. — *Let X be a geodesic δ -hyperbolic space and $S \subset \text{Isom}(X)$ a finite set. Assume that $L_\infty(S) = 0$. Then $\langle S \rangle$ either has a bounded orbit on X , or fixes a unique point in ∂X .*

Proof. — For every $n > 0$ we have $L_\infty(S^n) = 0$ since $nL_\infty(S) = L_\infty(S^n)$. So, $\lambda_2(S^n) = 0$ by Lemma 1.1. Apply Theorem 1.4 to S^n , we obtain $K\delta \geq L(S^n)$.

So, for each $n > 0$, there is a point $x_n \in X$ such that $L(S^n, x_n) \leq K\delta$. We will show that $\Gamma = \langle S \rangle$ has a bounded orbit, or fixes a point in ∂X .

We recall an elementary fact. Let y_n be an infinite sequence of points in the δ -hyperbolic space X . Then there is a subsequence y_{m_n} such that either

y_{m_n} converges to a point in ∂X , or it (coarsely) “rotates” about some point $x \in X$ in the sense that any geodesic $[y_{m_n}, y_{m_k}]$, $m_n \neq m_k$, intersects the ball $B(x, 20\delta)$.

By this fact, there are two cases for the sequence x_n : there is an infinite subsequence x_{m_n} such that either

- (i) it converges to a point $x \in \partial X$, or
- (ii) it rotates, ie, there exist a point $x \in X$ such that for any $m_n < m_k$, the distance between x and the geodesic $[x_{m_n}, x_{m_k}]$ is at most 20δ .

Then, (i) implies that the point x is fixed by Γ . We discuss the uniqueness later. In the case (ii), since both x_{m_n}, x_{m_k} , $m_n < m_k$ are moved by S^{m_n} at most by $K\delta$, each point on $[x_{m_n}, x_{m_k}]$ is moved by S^{m_n} at most by, say, $K\delta + 10\delta$. It implies that S^{m_n} moves x by at most $40\delta + (K + 10)\delta$. This is for any $n > 0$, so the Γ -orbit of x is bounded.

Now we argue that if there are at least two fixed points in ∂X , then Γ has a bounded orbit in X . Indeed, if there are three fixed points, then a Γ -orbit must be bounded since X is hyperbolic. If there are exactly two fixed points in ∂X , then join those two points by a quasi-geodesic γ in X . Then γ is coarsely invariant by Γ , ie, there is a constant C such that for any $g \in \Gamma$, the Hausdorff-distance between γ and $g(\gamma)$ is at most C . Notice that Γ does not contain any hyperbolic isometry since $L_\infty(S) = 0$. It implies that there is a constant D , which depends on C and δ , such that every point on γ is moved by at most D by any element in Γ , ie, Γ -orbit is bounded. □

COROLLARY 13.9 (Virtually nilpotent groups). — *Let X be a δ -hyperbolic space, $S \subset \text{Isom}(X)$ a finite set with $1 \in S$ and $S = S^{-1}$. Assume $\Gamma = \langle S \rangle$ is virtually nilpotent. Then one of the following holds:*

- (i) Γ has a bounded orbit in X ,
- (ii) Γ fixes a unique point $x \in \partial X$,
- (iii) Γ leaves invariant a set of two points in ∂X , which is the fixed point set of some hyperbolic isometry $g \in \Gamma$.

Proof. — This is again well-known. We deduce it from Theorem 13.1 applied to S as follows. The case (3) does not happen since $h(S) = 0$ since Γ has polynomial growth. The case (2) implies the conclusion (iii). So assume S does not satisfy (2). Then S satisfies (1), ie, $L(S) < (K + 1)\delta$.

We also apply Theorem 13.1 to S^n for each $n > 1$. Again, (3) does not happen, and (2) is desirable, so we assume $L(S^n) < (K + 1)\delta$. But this implies $L_\infty(S) = 0$. Apply Proposition 13.8, and we are done. □

13.4. Spaces with bounded packing

In this paragraph, we prove Theorem 1.13 from the Introduction. We will establish in fact the slightly more precise statement of Theorem 13.11 below.

For a metric space X , a subgroup Γ in $\text{Isom}(X)$ is said to be *discrete* if it has discrete orbits, namely if for any point $x \in X$ and a bounded subset $Y \subset X$, $\Gamma x \cap Y$ is finite.

To state our main result, we recall one definition from [19]. A metric space X has *bounded packing* with packing constant $P > 0$ if every ball of radius 2 in X can be covered by at most P balls of radius 1.

Here is an elementary lemma we use later.

LEMMA 13.10. — *If a geodesic space has bounded packing for P , then any ball of radius $n > 0$, which is an integer, is covered by at most P^{n-1} balls of radius 1.*

Proof. — We argue by induction on n . For $n = 1$, the claim is trivial. Suppose the claim holds for $n > 0$. Take a ball B of radius $n + 1$, and let B' be the ball of radius n with the same center. By assumption, cover B' by at most P^n balls of radius 1. Now for each of those balls, take the ball of radius 2 with the same center. Those balls of radius 2 cover B (here we are using that the space is geodesic). Also, each ball of radius 2 is covered by at most P balls of radius 1. So, by collecting all of those balls of radius 1, B is covered by at most $P \cdot P^n$ balls of radius 1. \square

In [19] balls of radius 4 instead of 2 were used to define the bounded packing property. This is a minor change, which only affects the constant P according to this lemma.

In the following theorem we state the bounded packing property in terms of balls of radius δ and 2δ , which is more natural for a δ -hyperbolic space.

THEOREM 13.11. — *Given P , there is $N(P)$ with the following property. Let X be a geodesic δ -hyperbolic space, with $\delta > 0$, such that every ball of radius 2δ is covered by at most P balls of radius δ .*

Let S be a finite subset in $\text{Isom}(X)$ with $S = S^{-1}$ and assume that $\Gamma = \langle S \rangle$ is a discrete subgroup of $\text{Isom}(X)$. Then either Γ is virtually nilpotent, or S^N contains two generators of a free semi-group, and in particular:

$$h(S) \geq \frac{1}{N} \log 2.$$

Moreover, if Γ is virtually nilpotent, then either (i) Γ is finite, (ii) fixes a unique point in ∂X , or (iii) Γ is virtually cyclic and contains a hyperbolic isometry g such that $\text{Fix}(g)$ in ∂X is invariant by Γ .

The following theorem by Breuillard–Green–Tao, which improved Gromov’s theorem on groups with polynomial growth, is a key ingredient of the argument.

THEOREM 13.12 ([19, Corollary 11.2]). — *For $Q \geq 1$ there is a constant $C(Q)$ with the following property. Let S be a finite generating set of a group G with $1 \in S$. Suppose there exists a finite subset A in G such that $|A^2| \leq Q|A|$ and $S^{C(Q)} \subset A$. Then G is virtually nilpotent.*

Here is a useful consequence.

COROLLARY 13.13. — *For integers $P, J > 0$, set $k = C(P^{2J})$, where C is from Theorem 13.12. Suppose X is a geodesic space and has bounded packing P . Let S be a finite set in $\text{Isom}(X)$ with $S = S^{-1}$ such that $\Gamma = \langle S \rangle$ is discrete.*

If $L(S^k) < J$, then $\langle S \rangle$ is virtually nilpotent.

Proof. — For a point $x \in X$ define

$$S_J(x) = \{\gamma \in \Gamma \mid d(x, \gamma x) \leq J\}.$$

By assumption, there is $x \in X$ with $L(S^k, x) < J$, so $S^k \subset S_J(x)$. Set $A = S_J(x)$. A is finite since Γ is discrete. We have $A^2 \subset S_{2J}(x)$ by triangle inequality.

Since X has bounded packing for P , $|A^2| \leq P^{2J}|A|$. Indeed, let B be the ball of radius $2J$ centered at x in X . Then $A^2(x) \subset B$. By Lemma 13.10, B is covered by balls of radius 1: B_1, \dots, B_k with $k \leq P^{2J}$. Now choose $a_i \in A^2$, if it exists, with $a_i(x) \in B_i$ for each i . Now for any $a \in A^2$, since $a(x) \in B$, there is B_i with $a(x) \in B_i$, so that $d(a_i(x), a(x)) \leq 2$. This means $a_i^{-1}a \in A$, so that $a \in a_i A$. Since $a \in A^2$ was arbitrary, and $|a_i A| = |A|$, we find $|A^2| \leq k|A| \leq P^{2J}|A|$.

On the other hand, by definition, $S^{C(P^{2J})} = S^k \subset S_J(x) = A$. Now by Theorem 13.12 with $Q = P^{2J}$, $\langle S \rangle$ is virtually nilpotent. □

Proof of Theorem 13.11. — By scaling the metric of X by a constant, we assume that X is 1-hyperbolic. By our assumption, X has bounded packing property for P with respect to the new metric. Set $k = C(P^{2(K+1)})$ as in Corollary 13.13 for $J = K + 1$.

There are two cases.

Case 1: $L(S^k) < K + 1$. — In this case $\langle S \rangle$ is virtually nilpotent by Corollary 13.13 applied to S^k with $J = K + 1$.

Case 2: $L(S^k) \geq K + 1$. — Apply Theorem 13.1 to S^k with $\delta = 1$. Set $N_0 = N(1)$, where $N(1)$ is the constant from Theorem 13.1. Since we are

in Case 2, (1) does not happen. If (2) happens then $\langle S \rangle$ is virtually cyclic since $\langle S \rangle$ is discrete. If (3) happens then S^{kN_0} contains two elements that generate a free semi-group. The constant k depends only on P . (K does not depend on anything.) Set $N = kN_0$ and we are done.

To show the moreover part, we apply Corollary 13.9. If Γ has a bounded orbit, then Γ must be finite since the action is discrete. If there is $g \in \Gamma$ that is hyperbolic such that $\text{Fix}(g)$ is invariant by Γ , then $\langle g \rangle$ has finite index in Γ since Γ is discrete, so that Γ is virtually cyclic. Otherwise, Γ fixes a unique point $x \in \partial X$. \square

Using Theorem 13.11, we give a quick proof of the following theorem by Besson–Courtois–Gallot.

THEOREM 13.14 ([7, Theorem 1.1]). — *Let X be a d -dimensional, simply connected Riemannian manifold with curvature $-a^2 \leq K \leq -1$. Let $\Gamma = \langle S \rangle$ be a finitely generated discrete subgroup of $\text{Isom}(X)$ with $S = S^{-1}$. Then either Γ is virtually nilpotent, or S^N contains two generators of free semigroup, in particular, $h(S) \geq \frac{1}{N} \log 2$, where the constant N depends only on d and a .*

In other words, unless Γ is virtually nilpotent, Γ has uniform exponential growth, and the growth rate depends only on d, a .

Before we start the proof, we quote a well-known fact (see the paragraph in [19] before Corollary 11.19).

LEMMA 13.15. — *Let $d \geq 1$ be an integer, and $a \geq 0$. Then there exists $K(d, a) \geq 1$ with the following property. Suppose M is a d -dimensional complete Riemannian manifold with a Ricci curvature lower bound $\text{Ric} \geq -(d-1)a^2$. Then M has bounded packing $K(d, a)$.*

For readers' convenience we give an outline of an argument.

Outline of proof of Lemma 13.15. — Let $B(x, R)$ denote the ball of radius R in M centered at x . By the Bishop–Gromov inequality, $\sup_{x \in M} \frac{\text{vol}(B(x, 4))}{\text{vol}(B(x, 1/2))}$ is bounded from above by a number K that depends only on d, a .

We will show $B(x, 2)$ is covered by at most K balls of radius 1. Let L be the maximal number of disjoint balls of radius $1/2$ in $B(x, 2)$. Let $B(y, 1/2)$ be the one of maximal volume, v , among them. Since $B(x, 2)$ is contained in $B(y, 4)$, we have $Lv \leq \text{vol}(B(x, 2)) \leq Kv$. In particular, $L \leq K$. But the balls of radius 1 centered at the same points cover $B(x, 2)$. We are done. \square

Proof of Theorem 13.14. — Since X is simply connected and $K \leq -1$, X is CAT(0) and δ -hyperbolic for, say, $\delta = 2$. Since $-a^2 \leq K$ and the dimension of X is d , we have $\text{Ric} \geq -(d-1)a^2$. By Lemma 13.15, X has bounded packing for the constant $K(d, a)$. Namely, any ball of radius 2 is covered by at most K balls of radius 1. So, any ball of radius 4 is covered by at most K^3 balls of radius 1 by Lemma 13.10.

Set $N = N(K(d, a)^3)$, where N on the right hand side is the function from Theorem 13.11. Since the assumption of Theorem 13.11 is satisfied by X for $\delta = 2$ and $P = K(d, a)^3$, either Γ is virtually nilpotent, or S^N contains two generators of free semigroup, in particular, $h(S) \geq \frac{1}{N} \log 2$. \square

14. Questions

1. — Let X be the metric completion of the Teichmüller space of a surface Σ with the Weil–Petersson metric. X is a complete CAT(0) space. Do we have a Bochi-type inequality (see Theorem 1.4)? To be concrete, let a, b be the Dehn twists along curves α, β . Then they are elliptic isometries. Assume that a, b do not commute (ie, the geometric intersection number of α, β is not 0), then $\text{Fix}(a), \text{Fix}(b)$ are disjoint. Do we have a Bochi-type inequality for the set $\{a, b\}$? We remark that there is a uniform positive lower bound, which depends on Σ , on the distance between $\text{Fix}(a), \text{Fix}(b)$.

2. — Does there exist an absolute constant $c > 0$ such that $h(S) > c$ for every generating set S of an arbitrary non elementary word hyperbolic group (independently of δ , see [57, Question 2.1]).

3. — Can we remove the multiplicative constant in Proposition 7.7? that is given a symmetric space X of non-compact type, does there exist $k, C > 0$ such that $\lambda_k(S) \geq L(S) - C$ for every finite set S of isometries of X . Does this hold also for isometries of Bruhat–Tits buildings or more generally of any Euclidean building?

4. — The proof of the geometric Berger–Wang identity (Theorem 7.5) and the Bochi-type inequality (Proposition 7.7) for symmetric spaces of non-compact type relies on the Bochi inequality for matrices, thus eventually on linear algebra (see the other proof given in [17]). It would be very interesting to find a geometric proof, akin to our proof of Theorem 1.4 for hyperbolic spaces. Perhaps this could shed light on Question 4.

5. — Is there a geometric Bochi inequality for isometries of a CAT(0) cube complex? what about isometries of median spaces?

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