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ON THE LIPMAN–ZARISKI CONJECTURE FOR LOGARITHMIC VECTOR FIELDS ON LOG CANONICAL PAIRS

by Hannah BERGNER (*)

Abstract. — We consider a version of the Lipman–Zariski conjecture for logarithmic vector fields and logarithmic 1-forms on pairs. Let (X, D) be a pair consisting of a normal complex variety X and an effective Weil divisor D such that the sheaf of logarithmic vector fields (or dually the sheaf of reflexive logarithmic 1-forms) is locally free. We prove that in this case the following holds: if (X, D) is dlt, then X is necessarily smooth and $\lfloor D \rfloor$ is snc. If (X, D) is lc or the logarithmic 1-forms are locally generated by closed forms, then the pair $(X, \lfloor D \rfloor)$ is toroidal.

Résumé. — Nous considérons une version de la conjecture de Lipman–Zariski pour des champs de vecteurs logarithmiques et des 1-formes logarithmiques. Soit (X, D) une paire, où X est une variété complexe normale et D est un diviseur de Weil effectif, tels que le faisceau des champs de vecteurs logarithmiques (ou de façon duale le faisceau des 1-formes logarithmiques réflexives) est localement libre. Nous démontrons le suivant dans ce cas : si (X, D) est dlt, alors X est nécessairement lisse et $\lfloor D \rfloor$ est snc. Si (X, D) est lc ou si les 1-formes logarithmiques sont engendrées localement par des formes fermées, alors la paire $(X, \lfloor D \rfloor)$ est toroïdale.

1. Introduction

The Lipman–Zariski conjecture posed in [19, p. 874] states that every normal complex space with locally free tangent sheaf is smooth. In this paper, we consider a version of this conjecture for the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$, or equivalently the dual sheaf $\Omega_X^{[1]}(\log D)$ of reflexive logarithmic 1-forms, on a pair (X, D), where X is a normal complex quasi-projective variety and D a reduced Weil divisor; for precise definitions see Section 2.

Keywords: Lipman–Zariski conjecture, logarithmic vector fields, logarithmic 1-forms, toroidal varieties.
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Example 1.1 (Snc pair). — Let $X = \mathbb{A}^n$ with coordinates $z_1, \ldots, z_n$ and $D = \{z_1 \cdot \ldots \cdot z_k = 0\}$. Then $z_1 \frac{\partial}{\partial z_1}, \ldots, z_k \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \ldots, \frac{\partial}{\partial z_n}$ form an $\mathcal{O}_X$-basis of the logarithmic tangent sheaf, and the dual sheaf of logarithmic 1-forms is spanned by $\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n$.

More generally, if X is smooth and D is a reduced snc divisor, then the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ and its dual are locally free.

Example 1.2 (Toric variety). — If (X, D) is a pair consisting of a normal toric variety X and a reduced divisor D whose support is the complement of the open torus orbit, then the sheaf $\Omega_X^{[1]}(\log D)$ of reflexive logarithmic 1-forms is free; cf. [21, Section 3.1].

This raises the question in which cases a converse of this is locally true:

Question. — *Let (X, D) be a pair such that the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$, or equivalently $\Omega_X^{[1]}(\log D)$, is locally free. Is (X, D) then necessarily toroidal, i.e. locally of the form as in Example 1.2?*

In general, this is false. Consider for instance the following example:

Example 1.3. — Let $X = \mathbb{A}^2_{\mathbb{C}}$ and $D = \{y^2 = x^3\}$. Then $\Omega_X^{[1]}(\log D)$ is locally free and D is irreducible, but D is not normal.

For smooth varieties X and arbitrary reduced divisors D logarithmic vector fields, logarithmic differential forms and their properties have been studied a lot. Recently, precise conditions under which a reduced divisor D in a smooth variety X such that $\mathcal{T}_X(-\log D)$ is locally free is normal crossing were given in [8] and [3].

In this article, we study the pairs (X, D) with locally free sheaf $\Omega_X^{[1]}(\log D)$, where X is allowed to be singular. If $D = \sum_i a_i D_i$, $a_i \in \mathbb{Q}$, is an effective Weil divisor, let $\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i$ denote its rounddown. We completely answer the above question for pairs $(X, \lfloor D \rfloor)$ such that there is a pair (X, D) that is dlt or lc, or such that the sheaf $\Omega_X^{[1]}(\log \lfloor D \rfloor)$ of reflexive logarithmic 1-forms is locally generated by closed forms:

Theorem 1.4 (cf. Theorems 4.1, 5.14, 6.8). — *Let (X, D) be a pair consisting of a normal quasi-projective variety X and a divisor $D = \sum_i a_i D_i$, where D_i are distinct prime divisors, $a_i \in \mathbb{Q}$ and $0 \leqslant a_i \leqslant 1$. Assume that $\Omega_X^{[1]}(\log \lfloor D \rfloor)$ is locally free. Then the following holds:*

(a) *If the sheaf $\Omega_X^{[1]}(\log \lfloor D \rfloor)$ of reflexive logarithmic 1-forms is locally generated by closed forms, then $(X, \lfloor D \rfloor)$ is toroidal.*

(b) *If the pair (X, D) is dlt, then X is smooth and $\lfloor D \rfloor$ is an snc divisor. If (X, D) is lc, then $(X, \lfloor D \rfloor)$ is toroidal.*

Recall that we call a pair (X, D) toroidal if X is locally (in the analytic topology) isomorphic to a toric variety Y and D is a reduced divisor corresponding to the complement of the open torus orbit in Y. A consequence of Theorem 1.4 and [7, Theorem 1.4.2] is the same result for Du Bois pairs, which is stated in Corollary 6.9.

In the special case of a projective lc pair (X, D) with globally free sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$, the main result of [24] on compact Kähler manifolds with trivial logarithmic tangent bundle has direct implications for the geometry of (X, D):

Corollary 1.5 (cf. Corollary 6.3). — *Let (X, D) be an lc pair such that X is projective. Then the sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is free if and only of there is a semi-abelian variety T which acts on X with $X \setminus \lfloor D\rfloor$ as an open orbit.*

Outline of the article

First some definitions and notation in the context of pairs and logarithmic vector fields and 1-forms are recalled in Section 2. In Section 3, some facts about extension of logarithmic differential forms, flows of vector fields on varieties, and residues of logarithmic 1-forms are collected.

The case of dlt pairs is considered in Section 4. In Section 5, we study pairs whose sheaf of reflexive logarithmic 1-forms is locally generated by closed forms, and use globalisation techniques in order to obtain local embeddings into toric varieties. Finally, the case of lc pairs is considered in Section 6. The statement for lc pairs in Theorem 1.4 (b) is proven by reducing to part (a) of the theorem. If the singular locus of an lc pair (X, D) consists of isolated points, we prove that locally there exist closed reflexive logarithmic 1-forms spanning the sheaf of logarithmic 1-forms; see Proposition 6.5. Then an argument using hyperplane sections is used to reduce to this case and thus the setting as in Theorem 1.4 (a).

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2. Definitions and Notation

Convention. — *Throughout, we work over the complex numbers and all varieties are complex algebraic varieties. We will also work with the induced structure of a complex space on an algebraic variety, and open neighbourhoods are allowed to be open neighbourhoods in the analytic* $\mathbb{C}$*-topology.*

2.1. Pairs

In the following Definition 2.1, we fix the notation for a few important definitions in the context of pairs. Definitions and more details may be found in [18, Chapter 2].

Definition 2.1 (Pair). — *A pair (X, D) is a pair consisting of a normal quasi-projective complex variety X and a divisor $D = \sum_i a_i D_i$, where $D_1, \dots, D_k$ are distinct prime divisors, $a_i \in \mathbb{Q}$, and $0 \leqslant a_i \leqslant 1$.*

The rounddown $\lfloor D \rfloor$ of the divisor D is defined as $\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i$. The pair (X, D) is called snc if X is smooth and D is snc, that is, all intersections $D_{i_1} \cap \ldots \cap D_{i_k}$ are smooth.

The singular locus $Z = (X, D)_{\mathrm{sing}}$ of a pair (X, D) is the smallest closed subset $Z \subset X$ such $(X \setminus Z, D|_{X \setminus Z})$ is snc.

Note that Definition 2.1 is slightly less general than [18, Definition 2.25] since we put the additional assumption that $0 \leqslant a_i \leqslant 1$ on the coefficients a_i of the divisor D.

Notation 2.2. — We use the abbreviations klt, plt, dlt, and lc for Kawamata log terminal, purely log terminal, divisorially log terminal and log terminal. For definitions of these notions see [18, Definition 2].

Definition 2.3 (Log resolution). — *Let (X, D) be a pair. A log resolution of (X, D) is a proper surjective birational morphism $\pi : \widetilde{X} \to X$ defined on a smooth variety $\widetilde{X}$ such that its exceptional divisor $E = \mathrm{Exc}(\pi)$ is of pure codimension 1 and the divisor $\mathrm{Exc}(\pi) + \overline{D}$ is snc, where $\overline{D}$ is the strict transform of D, and $E = \mathrm{Exc}(\pi)$ is endowed with the induced reduced structure.*

We will furthermore only consider log resolutions which are strong in the sense that π induces an isomorphism $\widetilde{X} \setminus (\pi^{-1}(Z)) \to X \setminus Z$ outside the singular set $Z = (X, D)_{\mathrm{sing}}$ of (X, D).

2.2. Logarithmic 1-forms

The notion of a logarithmic 1-form is essential for this article. For the theory of logarithmic differential forms see [22], and for more specific aspects about reflexive forms compare also [10, Section 2.E].

Notation 2.4 (Sheaves of 1*-forms).* — Let (X, D) be a pair. We denote the sheaf of Kähler differential 1-forms on X by Ω^1_X and the sheaf of reflexive differential 1-forms by $\Omega^{[1]}_X$.

The sheaf of Kähler logarithmic 1-forms is denoted by $\Omega^1_X(\log\lfloor D\rfloor)$ and the sheaf of reflexive logarithmic 1-forms by $\Omega^{[1]}_X(\log\lfloor D\rfloor)$.

Remark 2.5. — We have $\Omega^{[1]}_X = (\Omega^1_X)^{**} = \iota_*(\Omega^1_{X_{\mathrm{reg}}})$, where $\iota : X_{\mathrm{reg}} \hookrightarrow X$ denotes the inclusion of the smooth locus X_{reg} of X. A reflexive 1-form on an open subset $U \subseteq X$ is thus simply given by a 1-form on the smooth part $U \cap X_{\mathrm{reg}}$.

Similarly, a reflexive logarithmic 1-form on U is given by logarithmic 1-form on $U' = U \setminus (U, \lfloor D\rfloor|_U)_{\mathrm{sing}}$. Recall that a rational 1-form σ on U' is logarithmic if σ is regular on $U' \setminus \lfloor D\rfloor$ and σ and $d\sigma$ have at most first order poles along each irreducible component of $\lfloor D\rfloor$.

On $X \setminus \lfloor D\rfloor$ the notion of reflexive 1-forms and of reflexive logarithmic 1-forms coincide and we have $\Omega^{[1]}_X \cong \Omega^{[1]}_X(\log\lfloor D\rfloor)$.

Definition 2.6. — *We say that a reflexive logarithmic* 1*-form on a pair* (X, D) *is closed if its restriction to the smooth locus of* $X \setminus D$*, where it is a regualar* 1*-form, is closed in the usual sense.*

Convention. — *Throughout the article, we always consider reflexive (logarithmic)* 1*-forms and thus a (logarithmic)* 1*-form shall always mean a reflexive (logarithmic)*1*-form.*

2.3. Vector fields

Dual to the notion of 1-forms, there is the notion of vector fields on a variety (cf. [22, Definition 1.4]):

Notation 2.7 (Tangent sheaf). — Recall that a vector field on a normal variety X is a $\mathcal{O}_X$-linear derivation $\mathcal{O}_X \to \mathcal{O}_X$ of sheaves. A logarithmic vector field on a pair (X, D) is a vector field ξ on X which is tangent to $\lfloor D\rfloor$ in the sense that $\xi(I_{\lfloor D\rfloor}) \subseteq I_{\lfloor D\rfloor}$, where $I_{\lfloor D\rfloor}$ denotes the ideal sheaf of $\lfloor D\rfloor$. We denote the sheaf of vector fields on X (or tangent sheaf of X) by $\mathcal{T}_X$, and the sheaf of logarithmic vector fields (or logarithmic tangent sheaf) on a pair (X, D) by $\mathcal{T}_X(-\log\lfloor D\rfloor)$.

Remark 2.8 (Compare also [11, Section 3.1] *and references therein).* The tangent sheaf $\mathcal{T}_X$ and the logarithmic tangent sheaf $\mathcal{T}_X(-\log\lfloor D\rfloor)$ are reflexive sheaves. Therefore, a vector field on a normal variety X could also be defined to be a vector field on the smooth locus X_{reg} of X, and $\mathcal{T}_X$ as $\mathcal{T}_X = \iota_*(T_{X_{\mathrm{reg}}})$ if $\iota : X_{\mathrm{reg}} \hookrightarrow X$ denotes again the inclusion of the smooth locus.

The sections of the logarithmic tangent sheaf $\mathcal{T}_X(-\log\lfloor D\rfloor)$ of a pair (X, D) are precisely those vector fields on X whose flows (in the sense of Section 3.2) stabilise $\lfloor D\rfloor$ as a set.

Vector fields and 1-forms are dual, and we have $\mathcal{T}_X = (\Omega^1_X)^* = (\Omega^{[1]}_X)^*$ and $\mathcal{T}_X(-\log\lfloor D\rfloor) = (\Omega^{[1]}_X(\log\lfloor D\rfloor))^*$. In particular, the logarithmic tangent sheaf $\mathcal{T}_X(-\log\lfloor D\rfloor)$ is (locally) free if and only if $\Omega^{[1]}_X(\log\lfloor D\rfloor))$ is (locally) free.

3. Methods

3.1. Extension of differential forms

The extension of logarithmic forms on pairs to log resolutions is an important tool. The following result was proven in [10]:

Theorem 3.1 (Extension Theorem, [10, Theorem 1.5]). — *Let the pair (X, D) be an lc pair, $\pi : \widetilde{X} \to X$ a log resolution, and let $\widetilde{D}$ be the largest reduced divisor contained in the support of $\pi^{-1}(W)$, where W is the smallest closed subset such that $(X \setminus W, D|_{X\setminus D})$ is klt. Then the sheaf $\pi_*\Omega^p_{\widetilde{X}}(\log \widetilde{D})$ is reflexive for any $p \leqslant \dim X$.* □

This means that logarithmic forms defined on the regular part of the pair (X, D), i.e. the largest open subset $Y \subseteq X$ such that Y is smooth and $D|_Y$ is an snc divisor, extend to any log resolution.

3.2. Flows of vector fields on varieties

A useful result when studying vector fields on complex varieties is the following theorem by Kaup on the existence of local flows of holomorphic vector fields on complex spaces:

Theorem 3.2 (Existence of flows, [14, Satz 3]). — *Let X be a normal complex space and ξ a holomorphic vector field on X. Then the local flow of ξ exists, in other words, there is an open subset $\Omega \subseteq \mathbb{C} \times X$ such that*

(1) *the set* Ω *contains* $\{0\}\times X$ *and for each* $x \in X$, $\Omega_x = \{t \in \mathbb{C} \mid (t,x) \in \Omega\} \subseteq \mathbb{C}$ *is connected, and*
(2) *there exists a holomorphic map* $\varphi : \Omega \to X$ *with* $\varphi(0,x) = x$ *for all* $x \in X$ *and* $\frac{d}{dt} f(\varphi(t,-)) = \xi(f)(\varphi(t,-))$ *for any holomorphic function* f *defined on an open subset of* X. □

Even though vector fields on a variety X can in general not be pulled back by a morphism $f : Y \to X$ to vector fields on Y, the existence of local flows allows us to lift vector fields on a variety to the functorial resolution of singularities as in [16, Theorems 3.35, 3.36]. A detailed description of this procedure can be found in [9, Section 4.2].

Proposition 3.3. — *Let* (X,D) *be a pair and* $\pi : \widetilde{X} \to X$ *the functorial log resolution of the pair. Let* $E = \operatorname{Exc}(\pi)$ *denote the exceptional divisor of* π *and set* $\widetilde{D} = E + \overline{D}$, *where* $\overline{D}$ *is the strict transform of* D. *Then we have*

$$\mathcal{T}_X \cong \pi_*(\mathcal{T}_{\tilde{X}}) \cong \pi_*(\mathcal{T}_{\tilde{X}}(-\log E))$$

and also

$$\mathcal{T}_X(-\log\lfloor D\rfloor) \cong \pi_*(\mathcal{T}_{\tilde{X}}(-\log\lfloor \widetilde{D}\rfloor)).$$

□

The idea for the proof of this proposition (as in [9, Section 4.2]) is to consider the local flow of the given vector field on X, which exists by Theorem 3.2. Since the functorial resolution commutes with smooth morphisms and as the flow map $\varphi : \Omega \to X$ is a smooth morphism, it can be lifted to a local action $\widetilde{\varphi}$ on $\widetilde{X}$, which then induces a vector field $\widetilde{\xi}$ on $\widetilde{X}$. The flow map of a vector field on an algebraic variety is not necessarily algebraic, but in general a holomorphic map of complex spaces, one also needs to consider resolution of complex spaces at this point, see e.g. [16, Theorem 3.45], and do the procedure for these.

Remark 3.4. — Proposition 3.3 means that every vector field ξ on X can be lifted to $\widetilde{X}$ in the sense that there is a vector field $\widetilde{\xi}$ on $\widetilde{X}$ whose restriction to $\widetilde{X} \setminus (\pi^{-1}((X,D)_{\mathrm{sing}})) \cong X \setminus (X,D)_{\mathrm{sing}}$ coincides with the restriction of ξ to $X \setminus (X,D)_{\mathrm{sing}}$. The flow of $\widetilde{\xi}$ stabilises the exceptional divisor E and is thus logarithmic with respect to E.

If a vector field ξ on X is logarithmic with respect to D, then $\widetilde{\xi}$ is logarithmic with respect to $\widetilde{D} = E + \overline{D}$.

Moreover, if the flow φ of a vector field on some variety Y stabilises a divisor E, then the flow actually has to stabilise every irreducible component E_j of E, i.e. $\varphi_t(E_j) = E_j$ for any $t \in \mathbb{C}$ such that φ_t is defined on a neighbourhood of E, since we have $\varphi_0(E_j) = \mathrm{id}(E_j) = E_j$ for each irreducible component E_j and by continuity this implies $\varphi_t(E_j) = E_j$ for all suitable t.

Conversely, logarithmic vector fields on a log resolution of singularities of a pair always induce logarithmic vector fields on the pair itself:

Remark 3.5. — If (X, D) is a pair and $\pi : \widetilde{X} \to X$ is any log resolution (as in Definition 2.3) with exceptional divisor E and $\overline{D}$ the strict transform of D, $\widetilde{D} = \overline{D} + E$, then any logarithmic vector field $\widetilde{\xi} \in \mathcal{T}_{\tilde{X}}(-\log\lfloor\widetilde{D}\rfloor)$ on $\widetilde{X}$ induces a logarithmic vector field $\xi \in \mathcal{T}_X(-\log\lfloor D\rfloor)$ on X with $\pi^* \circ \xi = \widetilde{\xi} \circ \pi^*$. This can be seen as follows:

Since π induces an isomorphism $\widetilde{X} \setminus \pi^{-1}(Z) \to X \setminus Z$ outside the singular set $Z = (X, D)_{\mathrm{sing}}$ of (X, D), the vector field $\widetilde{\xi}$ naturally induces a vector field ξ on $X \setminus Z$ and ξ is logarithmic with respect to the divisor $\lfloor D\rfloor \cap (X \setminus Z) = \pi(\lfloor\widetilde{D}\rfloor \cap (\widetilde{X} \setminus \pi^{-1}(Z)))$. This vector field ξ then extends to a vector field ξ on all of X which is logarithmic with respect to the divisor $\lfloor D\rfloor$ because the singular set $Z = (X, D)_{\mathrm{sing}}$ of (X, D) has at least codimension 2 in X and the logarithmic tangent sheaf is reflexive (cf. Remark 2.8). By construction we have $\pi^* \circ \xi = \widetilde{\xi} \circ \pi^*$ on $X \setminus Z$ and by the identity principle this holds on all of X.

As a consequence of Proposition 3.3 and the extension result for logarithmic 1-forms, we get the following:

Corollary 3.6. — *Let (X, D) be an lc pair and $\pi : \widetilde{X} \to X$ the functorial log resolution, denote its exceptional divisor by E and set $\widetilde{D} = E + \overline{D}$, where $\overline{D}$ is the strict transform of D. Let $U \subseteq X$ be an open subset such that the restriction of the sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ to U is free (or equivalently, the restriction of $\mathcal{T}_X(-\log\lfloor D\rfloor)$ to U is free). Then the sheaves $\Omega_{\tilde{X}}^{[1]}(\log\lfloor\widetilde{D}\rfloor)$ and $\mathcal{T}_{\tilde{X}}(-\log\lfloor\widetilde{D}\rfloor)$ are free when restricted to $\pi^{-1}(U)$.*

Proof. — Since $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ and $\mathcal{T}_X(-\log\lfloor D\rfloor)$ are dual to each other, one of them is (locally) free if and only if the other one is. Since the question is local, we may assume that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is free.

Let $\sigma_1, \ldots, \sigma_n$ be logarithmic 1-forms on X spanning $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ and let $\xi_1, \ldots, \xi_n$ be logarithmic vector fields which span $\mathcal{T}_X(-\log\lfloor D\rfloor)$ and are dual to $\sigma_1, \ldots, \sigma_n$, i.e. $\sigma_i(\xi_j) = \delta_{ij}$. By Theorem 3.1, the logarithmic 1-forms $\sigma_1, \ldots, \sigma_n$ can be extended to logarithmic 1-forms $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ on $\widetilde{X}$. Let $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ denote the lifts of $\xi_1, \ldots, \xi_n$ to $\widetilde{X}$. Since π is an isomorphism onto its image when restricted to $\pi^{-1}(X \setminus (X, D)_{\mathrm{sing}})$, we have $\widetilde{\sigma}_i(\widetilde{\xi}_j) = \sigma_i(\xi_j) = \delta_{ij}$ on the open dense subset $\pi^{-1}(X \setminus (X, D)_{\mathrm{sing}}) \cong (X \setminus$

$(X, D)_{\mathrm{sing}}$) of $\widetilde{X}$ and thus on all of $\widetilde{X}$. Consequently, the logarithmic 1-forms $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ span $\Omega^{[1]}_{\tilde{X}}(\log\lfloor\widetilde{D}\rfloor)$, and $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ span $\mathcal{T}_{\tilde{X}}(-\log\lfloor\widetilde{D}\rfloor)$. ☐

3.3. Residues of logarithmic 1-forms

If D is a smooth hypersurface in a complex manifold X, then the residue map for logarithmic 1-forms with respect to D gives an exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \mathcal{O}_D \to 0.$$

This can directly be generalised to the case of an snc divisor D in a complex manifold or smooth variety (and also logarithmic p-forms). In general however, a residue sequence like this for logarithmic differential forms on an arbitrary pair does not exist.

If (X, D) is a pair such that X is smooth and D is the sum of irreducible prime divisors, the residue of a logarithmic p-form can be defined as described in [22, Section 2], but it is in general not holomorphic but meromorphic.

Proposition 3.7 ([22, Section 1.1 and Lemma 2.2]). — *Let X be a complex manifold, D a hypersurface in X locally defined by the reduced equation $h(z) = 0$ for a holomorphic function h. If σ is a logarithmic 1-form on X, then locally there are holomorphic functions g_1, g_2, and a holomorphic 1-form η such that*

$$g_1\sigma = g_2\frac{dh}{h} + \eta.$$

The functions g_1 and g_2 are in general not unique, but the restriction to D of their ratio $\frac{g_2}{g_1}$ gives rise to a well-defined meromorphic function $\mathrm{res}(\sigma)$ on the normalisation $\widetilde{D}$ of D. ☐

This allows us to define a residue map as follows:

Definition 3.8. — *Let X be a complex manifold, D a reduced hypersurface in X, and let $\rho : \widetilde{D} \to D$ denote the normalisation of D. We define the residue map as*

$$\Omega^1_X(\log D) \to \rho_*(\mathcal{M}_{\tilde{D}}), \quad \sigma \mapsto \mathrm{res}(\sigma),$$

where $\mathcal{M}_{\tilde{D}}$ denotes the sheaf of meromorphic functions on $\widetilde{D}$.

Remark 3.9. — Similarly to the residue of a logarithmic 1-form, the residue of logarithmic p-forms can be defined, and for any logarithmic form σ we have

$$\mathrm{res}(d\sigma) = d(\mathrm{res}(\sigma)).$$

Remark 3.10. — Recently, precise characterisations under which a reduced divisor D in a complex manifold X is normal crossing under the assumption that $\mathcal{T}_X(-\log D)$ is locally free were given in [8] and [3]. One of these equivalent characterisations is the regularity of the residues of logarithmic 1-forms along the divisor D.

It turns out that also in the case of pairs (X, D), where X is allowed to be singular, this notion is useful. We are always assuming that X is normal and thus there is a closed subset $Z \subset X$ of codimension at least 2 such $X \setminus Z$ is smooth and $D|_{X\setminus Z}$ is an snc divisor. Given any logarithmic 1-form σ on X, we may then define its residue by first restricting σ to $X \setminus Z$, then taking the residue along $D|_{X\setminus Z}$, which then defines a unique rational function on the normalisation $\widetilde{D}$ of D.

In general this residue will not be regular, nor does there exist a short exact residue sequence as in the case of snc pairs. In the case of dlt pairs however, we have the following result for logarithmic 1-forms:

THEOREM 3.11 (Residue sequence for dlt pairs, [10, Theorem 11.7]). *Let (X, D) be a dlt pair with $\lfloor D \rfloor \neq \emptyset$ and $D_0 \subseteq \lfloor D \rfloor$ an irreducible component. Then there is a sequence*

$$0 \longrightarrow \Omega_X^{[1]}(\log(\lfloor D \rfloor - D_0)) \longrightarrow \Omega_X^{[1]}(\log \lfloor D \rfloor) \stackrel{\mathrm{res}_{D_0}}{\longrightarrow} \mathcal{O}_{D_0} \longrightarrow 0,$$

which is exact on X outside a subset of codimension at least 3. Moreover this sequence coincides with the usual residue sequence where the pair $(X, \lfloor D \rfloor)$ is an snc pair. □

Remark 3.12. — If (X, D) is a dlt pair, $p \in D$, then by definition either the pair (X, D) is snc near p, or there is an open neighbourhood $U \subseteq X$ of p such that $(U, D|_U)$ is plt. Thus if (X, D) is not locally snc at $p \in D$, we know by [18, Proposition 5.51] that $\lfloor D \rfloor$ is normal when restricted to U and the disjoint union of its irreducible components. In particular, p is contained in only one irreducible component of $\lfloor D \rfloor$.

In the case of lc pairs the extension of logarithmic forms to resolutions also yields residues for logarithmic 1-forms:

Remark 3.13. — Let (X, D) be an lc pair. Since logarithmic 1-forms extend to logarithmic 1-forms on a log resolution of (X, D) by [10, Theorem 1.5], the residue of a logarithmic 1-form along a component of $\lfloor D \rfloor$ is a regular function on the normalisation of that component of $\lfloor D \rfloor$. Moreover, we have an exact sequence

$$0 \to \Omega_X^{[1]} \to \Omega_X^{[1]}(\log\lfloor D\rfloor) \to \bigoplus_{j=1}^{k} (\rho_j)_*(\mathcal{O}_{\tilde{D}_j}),$$

where $D_1, \ldots, D_k$ denote the irreducible components of the rounddown $\lfloor D\rfloor$ and $\rho_j : \widetilde{D}_j \to D_j$ is the normalisation of D_j. Note however that the last arrow of this sequence is in general not surjective.

Remark 3.14. — Let (X, D) be an arbitrary pair, and σ a logarithmic 1-form on X that is closed (as defined in Definition 2.6). Since $d(\mathrm{res}_{D_j}(\sigma)) = \mathrm{res}_{D_j}(d\sigma) = 0$ along each irreducible component D_j of $\lfloor D\rfloor$, the residue $\mathrm{res}(\sigma)$ is constant on each irreducible component D_j.

4. Dlt pairs with locally free sheaf of logarithmic 1-forms

If (X, D) is a dlt pair and its sheaf of logarithmic differential 1-forms is locally free, then (X, D) is necessarily snc:

THEOREM 4.1. — *Let (X, D) be a dlt pair such that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is locally free. Then X is smooth and $\lfloor D\rfloor$ is an snc divisor.*

Proof. — After shrinking X, we may assume that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is free. If $p \notin \lfloor D\rfloor$, then the pair (X, D) is klt near p and $\Omega_X^{[1]}(\log\lfloor D\rfloor) \cong \Omega_X^{[1]}$ near p. Since the Lipman–Zariski conjecture is true for klt spaces by [10, Theorem 6.1], X is smooth near p.

Assume now $p \in \lfloor D\rfloor$ and let $\pi : \widetilde{X} \to X$ be the functorial log resolution with exceptional divisor E, $\overline{D}$ the strict transform of D and $\widetilde{D} = E + \overline{D}$. Then $\Omega_{\widetilde{X}}^{[1]}(\log\lfloor \widetilde{D}\rfloor)$ is free by Corollary 3.6. Suppose that the pair (X, D) is not snc at p. Then by Remark 3.12 the point p is only contained in one irreducible component of $\lfloor D\rfloor$, and after possibly further shrinking we may thus assume that $\lfloor D\rfloor$ is irreducible. Let $\overline{D}$ denote again the strict transform of D, and $E_1, \ldots, E_m$ the exceptional divisors, $\widetilde{D} = \overline{D} + E_1 + \ldots + E_m$. Let $\sigma_1, \ldots, \sigma_n$ be logarithmic 1-forms spanning $\Omega_X^{[1]}(\log\lfloor D\rfloor)$, and let $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ denote the extensions of these to $\widetilde{X}$ (cf. Corollary 3.6 and its proof), which span $\Omega_{\widetilde{X}}^{[1]}(\log\lfloor \widetilde{D}\rfloor)$. Let $q_0 \in \pi^{-1}(p) \cap \lfloor \overline{D}\rfloor$. Since $\widetilde{D}$ is snc, there is j such that the residue $\mathrm{res}_{\lfloor \overline{D}\rfloor}(\widetilde{\sigma}_j)$ along $\lfloor \overline{D}\rfloor$ does not vanish in q_0 and hence $\mathrm{res}_{\lfloor D\rfloor}(\sigma_j)(p) = \mathrm{res}_{\lfloor \overline{D}\rfloor}(\widetilde{\sigma}_j)(q_0) \neq 0$. The divisor $\lfloor D\rfloor$, which we assume to be irreducible as argued above, is normal since (X, D) is dlt and by Theorem 3.11 the residues $\mathrm{res}_{\lfloor D\rfloor}(\sigma_i)$ are all regular functions on $\lfloor D\rfloor$.

Therefore, there exist regular functions f_i on X defined on some neighbourhood of p such that $\operatorname{res}_{\lfloor D\rfloor}(\sigma_i) = f_i|_{\lfloor D\rfloor}$ along the divisor $\lfloor D\rfloor$. As argued before we have $f_j(p) \neq 0$ and after possibly changing the numbering of the σ_i's we assume $j = 1$. Now after restricting to an appropriate neighbourhood of p and replacing $\sigma_1, \dots, \sigma_n$ by $\frac{1}{f_1}\sigma_1, \sigma_2 - \frac{f_2}{f_1}\sigma_1, \dots, \sigma_n - \frac{f_n}{f_1}\sigma_1$, we may thus assume that $\operatorname{res}_{\lfloor D\rfloor}(\sigma_1) = 1$ and $\operatorname{res}_{\lfloor D\rfloor}(\sigma_i) = 0$ for $i > 1$. Therefore, $\sigma_2, \dots, \sigma_n$ are regular 1-forms without poles. By [6, Theorem 3.1] the extensions $\widetilde{\sigma}_2, \dots, \widetilde{\sigma}_n$ of $\sigma_2, \dots, \sigma_n$ to $\widetilde{X}$ are also regular, and have in particular no poles along the exceptional divisors $E_1, \dots, E_m$, hence $\operatorname{res}_{E_i}(\widetilde{\sigma}_j) = 0$ for $j > 1$ and all i. Choose $l \in \{1, \dots, k\}$ such that E_l intersects $\lfloor \overline{D}\rfloor$, which exists since we supposed that (X, D) is not snc at $p \in \lfloor D\rfloor$. Let $q \in E_l \cap \lfloor \overline{D}\rfloor$ and η be a logarithmic 1-form on a neighbourhood of q such $\operatorname{res}_{E_l}(\eta) = 1$ and $\operatorname{res}_{\lfloor \overline{D}\rfloor}(\eta) = 0$. By Corollary 3.6 we have $\eta = \alpha_1\widetilde{\sigma}_1 + \dots + \alpha_n\widetilde{\sigma}_n$ for some regular functions α_j. This implies

$$0 = \operatorname{res}_{\lfloor \overline{D}\rfloor}(\eta) = \alpha_1|_{\lfloor \overline{D}\rfloor} \operatorname{res}_{\lfloor \overline{D}\rfloor}(\widetilde{\sigma}_1) + \dots + \alpha_n|_{\lfloor \overline{D}\rfloor} \operatorname{res}_{\lfloor \overline{D}\rfloor}(\widetilde{\sigma}_n) = \alpha_1|_{\lfloor \overline{D}\rfloor},$$

and in particular $\alpha_1(q) = 0$. But then we also have

$$\operatorname{res}_{E_l}(\eta) = \alpha_1|_{E_l} \operatorname{res}_{E_l}(\widetilde{\sigma}_1) + \dots + \alpha_n|_{E_l} \operatorname{res}_{E_l}(\widetilde{\sigma}_n) = \alpha_1|_{E_l}$$

and in particular $\operatorname{res}_{E_l}(\eta)(q) = \alpha_1(q)\operatorname{res}_{E_l}(\widetilde{\sigma}_1)(q) = 0$, which is a contradiction to $\operatorname{res}_{E_l}(\eta) = 1$. $\square$

5. Pairs with locally free sheaf of logarithmic 1-forms generated by closed forms.

If we allow slightly more general singularities for the pair (X, D) than dlt singularities, e.g. if (X, D) is lc, then the statement of Theorem 4.1 is no longer true. Even if the sheaf of logarithmic 1-forms is locally free, X could have singularities or the irreducible components of $\lfloor D\rfloor$ could be non-normal:

Example 5.1. — Let $X = \mathbb{A}^2$ and $D = \{y^2 - x^3 - x^2 = 0\}$ be the nodal curve, which is not normal. The pair (X, D) is lc and its sheaf of logarithmic 1-forms is locally free.

Example 5.2. — Let $X = \mathbb{A}^2$ and $D = \{y^2 - x^3 = 0\}$ be the cusp. In this case the pair (X, D) is not lc, D is not normal, but the sheaf of logarithmic 1-forms is locally free.

Example 5.3. — Let X be a normal toric variety. Let $T \subseteq X$ denote the open orbit of the $(\mathbb{C}^*)^n$-action and set $D = X \setminus T$. Then the pair (X, D) is lc by [15, Proposition 3.7].

Moreover, the sheaves of logarithmic vector fields on (X, D) and logarithmic 1-forms can be described rather explicitly (see e.g. [21, Section 3.1]), and in particular these are free sheaves.

In this section we consider the case of a pair (X, D) with the property that its sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ of logarithmic 1-forms is locally free and locally generated by closed 1-forms.

The main result (see Theorem 5.14) is that pairs consisting of a toric variety and boundary divisor as in Example 5.3 describe the local structure of all such pairs, i.e. if (X, D) is a pair whose sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ of logarithmic 1-forms is locally free and locally generated by closed 1-forms, then $(X, \lfloor D\rfloor)$ is toroidal. We briefly give an overview over the main steps towards the proof of Theorem 5.14 presented in this section:

The technical, though very useful Lemma 5.6 is a direct consequence of a standard formula for the calculation of the exterior derivative of a 1-form and states that if $\sigma_1, \ldots, \sigma_n$ form a local basis of logarithmic 1-form on a pair and $\xi_1, \ldots, \xi_n$ form the dual basis of logarithmic vector fields, then then logarithmic 1-forms are closed if and only if the vector fields pairwise commute.

In Proposition 5.8 the properties of the irreducible components $D_1, \ldots, D_k$ of $\lfloor D\rfloor$ of a pair (X, D) whose sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ of logarithmic 1-forms is locally free and locally generated by closed 1-forms are investigated. It is proven that in this case the intersection

$$\bigcap_{i_j \in I} D_{i_j}$$

for any subset $I \subset \{1, \ldots, k\}$ is normal. The proof of this statement heavily relies on the subsequent technical Lemma 5.9 whose proof follows from a careful study of the residues of the logarithmic 1-forms while at the same time studying the flows of a dual basis of logarithmic vector fields and it uses moreover methods from complex analytic geometry such as quotients of Stein spaces in connection with the study of attractive fixed points as investigated in [23] and globalisations of complex Stein spaces in the sense of [12, Section 1.1].

Corollary 5.11 then proves a type of extension theorem to resolutions of singularities for logarithmic 1-forms in the special setting where the sheaf

$\Omega^{[1]}_X(\log\lfloor D\rfloor)$ of logarithmic 1-forms of a pair (X, D) is locally free and locally generated by closed 1-forms.

These results with a few more technical remarks and another careful analysis of the residues of the involved logarithmic 1-forms and properties of the flows of the dual vector fields are then used to prove Theorem 5.14.

Remark 5.4. — Let X be a normal complex space. Then any closed 1-form σ on the smooth locus of X extends to any resolution of singularities of X by [13, Theorem 1.2]. As a consequence the Lipman–Zariski conjecture holds for normal complex spaces X whose sheaf $\Omega^{[1]}_X$ is locally free and locally generated by closed 1-forms, see [13, Theorem 1.1].

For the case of a pair (X, D) whose sheaf $\Omega^{[1]}_X(\log\lfloor D\rfloor)$ of logarithmic 1-forms is locally free and locally generated by closed 1-forms this directly implies that $X \setminus \lfloor D\rfloor$ is smooth.

Next, we show that the requirement to locally have a basis for $\Omega^{[1]}_X(\log \lfloor D\rfloor)$ consisting of closed forms and the requirement that $\Omega^{[1]}_X(\log\lfloor D\rfloor)$ is locally free and locally generated by closed forms are equivalent.

Lemma 5.5. — *Let (X, D) be a pair such that $\Omega^{[1]}_X(\log\lfloor D\rfloor)$ is locally free and locally generated by closed 1-forms. Then locally there exists a basis of closed 1-forms $\sigma_1, \ldots, \sigma_n$ spanning $\Omega^{[1]}_X(\log\lfloor D\rfloor)$.*

Proof. — After possibly shrinking X, let $\sigma_1, \ldots, \sigma_n$ be a basis of logarithmic 1-forms, and let $\tau_1, \ldots, \tau_m$ be closed 1-forms generating $\Omega^{[1]}_X(\log \lfloor D\rfloor)$. Since $\sigma_1, \ldots, \sigma_n$ form a basis, there is an $m \times n$-matrix A whose entries a_{ij} are regular functions on X and such that

$$\begin{pmatrix} \tau_1 \\ \vdots \\ \tau_m \end{pmatrix} = A \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}.$$

Similarly, since $\tau_1, \ldots, \tau_m$ generate $\Omega^{[1]}_X(\log\lfloor D\rfloor)$, there is an $n \times m$-matrix B whose entries are regular functions and such that

$$\begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} = B \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_m \end{pmatrix}.$$

Combining the above, we get

$$\begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix} = B \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_m \end{pmatrix} = BA \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}$$

and because $\sigma_1, \ldots, \sigma_n$ form a basis we get $BA = \mathrm{id}$. In particular, the matrix B has rank n at each point, and (after possibly reordering) $\tau_1, \ldots, \tau_n$ form a local basis for $\Omega^{[1]}_X(\log\lfloor D\rfloor)$. □

Lemma 5.6. — *Let (X, D) be a pair such that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is locally free. Let $\sigma_1, \ldots, \sigma_n$ be a local basis of the logarithmic 1-forms and let $\xi_1, \ldots, \xi_n$ be a dual local basis of logarithmic vector fields for $\mathcal{T}_X(-\log\lfloor D\rfloor)$. Then the 1-forms $\sigma_1, \ldots \sigma_n$ are closed if and only if $\xi_1, \ldots, \xi_n$ pairwise commute, i.e. $[\xi_i, \xi_j] = 0$ for all i, j.*

Proof. — On the smooth locus of any variety we have

$$d\sigma(\xi, \xi') = \xi(\sigma(\xi')) - \xi'(\sigma(\xi)) - \sigma([\xi, \xi'])$$

for any regular 1-form σ and arbitrary vector fields ξ, ξ'. Therefore, we get

$$\begin{aligned} d\sigma_i(\xi_j, \xi_k) &= \xi_j(\sigma_i(\xi_k)) - \xi_k(\sigma_i(\xi_j)) - \sigma_i([\xi_j, \xi_k]) \\ &= \xi_j(\delta_{ik}) - \xi_k(\delta_{ij}) - \sigma_i([\xi_j, \xi_k]) \\ &= -\sigma_i([\xi_j, \xi_k]) \end{aligned}$$

on the smooth locus of $X \setminus \lfloor D\rfloor$, and by continuity this holds on all of X. Since $\sigma_1, \ldots, \sigma_n$ and $\xi_1, \ldots, \xi_n$ are local bases for $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ and $\mathcal{T}_X(-\log\lfloor D\rfloor)$, we have $d\sigma_i = 0$ for any i if every commutator $[\xi_j, \xi_k]$ vanishes and vice versa. □

Let us now consider the case of a pair (X, D) with locally free sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ which is locally generated by closed forms. We start with the case where $\lfloor D\rfloor$ is irreducible.

Lemma 5.7. — *Let (X, D) be a pair such that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is locally free, $\lfloor D\rfloor$ is irreducible and assume that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is locally generated by closed 1-forms. Then X is smooth and $\lfloor D\rfloor$ is smooth.*

Proof. — By Remark 5.4 we already know that $X \setminus \lfloor D\rfloor$ is smooth. Let $p \in \lfloor D\rfloor \subset X$ be a singular point of X and shrink X such that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is free and generated by the closed forms $\sigma_1, \ldots, \sigma_n$. The residue of the closed forms σ_j along $\lfloor D\rfloor$ is constant (cf. Remark 3.14) and thus the residue of each logarithmic 1-form along $\lfloor D\rfloor$ is regular. Let $q \in \lfloor D\rfloor$ be a smooth point of X. Then locally near q, $\lfloor D\rfloor$ is given by an equation $h = 0$ for a regular function h. Moreover, $\sigma = \frac{dh}{h}$ defines a logarithmic 1-form near q and $\mathrm{res}_{\lfloor D\rfloor}(\sigma) = 1$. Thus, there is $j \in \{1, \ldots, n\}$ such that $\mathrm{res}_{\lfloor D\rfloor}(\sigma_j) \neq 0$, and hence we may assume without loss of generality that $\mathrm{res}_{\lfloor D\rfloor}(\sigma_1) = 1$ and $\mathrm{res}_{\lfloor D\rfloor}(\sigma_j) = 0$ for $j > 1$. Then $\sigma_2, \ldots, \sigma_2$ are regular.

Let $\pi : \widetilde{X} \to X$ be the functorial log resolution of the pair (X, D) as in Proposition 3.3, let E be the exceptional divisor and $\overline{D}$ the strict transform of D, $\widetilde{D} = \overline{D} + E$. Since the 1-forms $\sigma_2, \ldots, \sigma_n$ are regular and closed, they extend to regular 1-forms $\widetilde{\sigma}_2, \ldots, \widetilde{\sigma}_n$ on $\widetilde{X}$ by [13, Theorem 1.2].

Furthermore, let $\xi_1, \ldots, \xi_n$ be logarithmic vector fields which are dual to $\sigma_1, \ldots, \sigma_n$. They lift to vector fields $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ on $\widetilde{X}$ (cf. Proposition 3.3) whose flows stabilise each component of $\lfloor \widetilde{D} \rfloor = \lfloor \overline{D} \rfloor + E$ as explained in Remark 3.4. We may thus restrict these vector fields to $\lfloor \overline{D} \rfloor$. Since for any point $q_0 \in \lfloor \overline{D} \rfloor$ we have

$$\left(\widetilde{\sigma}_i|_{\lfloor \overline{D} \rfloor}\right)\left(\widetilde{\xi}_j|_{\lfloor \overline{D} \rfloor}\right)(q_0) = \widetilde{\sigma}_i\left(\widetilde{\xi}_j\right)(q_0) = \sigma_i(\xi_j)(\pi(q_0)) = \delta_{ij}$$

for any $i, j \geqslant 2$, the vector fields $\widetilde{\xi}_2|_{\lfloor \overline{D} \rfloor}, \ldots, \widetilde{\xi}_n|_{\lfloor \overline{D} \rfloor}$ are independent at each point in $\lfloor \overline{D} \rfloor$. Their flows also stabilise E and thus $E \cap \lfloor \overline{D} \rfloor$, which yields a contradiction as $n - 2 = \dim E \cap \lfloor \overline{D} \rfloor$. $\square$

If (X, D) is any pair such that $\Omega_X^{[1]}(\log\lfloor D \rfloor)$ is locally free, then it does not follow in general that the irreducible components of $\lfloor D \rfloor$ are normal as illustrated in Example 5.1. However, if we also assume that $\Omega_X^{[1]}(\log\lfloor D \rfloor)$ is locally generated by closed forms such examples cannot occur.

Proposition 5.8. — *Let (X, D) be a pair such that $\Omega_X^{[1]}(\log\lfloor D \rfloor)$ is locally free and locally generated by closed forms. Let $D_1, \ldots, D_k$ be the irreducible components of $\lfloor D \rfloor$. Then for any subset $I \subseteq \{1, \ldots, k\}$ the intersection*

$$\bigcap_{i \in I} D_i$$

is normal.

The Proposition 5.8 is a consequence of the following Lemma 5.9, which describes the local geometry of group actions induced by appropriate logarithmic vector fields.

Lemma 5.9. — *Let (X, D) be a pair such that $\Omega_X^{[1]}(\log\lfloor D \rfloor)$ is locally free and locally generated by closed forms. Let D_j be an irreducible component of $\lfloor D \rfloor$ and $p \in D_j$. Then there is a neighbourhood U of p such that the following is true:*

(1) *There is a local basis of closed logarithmic 1-forms $\sigma_1, \ldots, \sigma_n$ for $\Omega_X^{[1]}(\log\lfloor D \rfloor)$ on U such that $\operatorname{res}_{D_j}(\sigma_1) = 2\pi i$ and $\operatorname{res}_{D_j}(\sigma_k) = 0$ for all $k \neq 1$.*
(2) *Let $\xi_1, \ldots, \xi_n$ be a basis of logarithmic vector fields on U dual to $\sigma_1, \ldots, \sigma_n$. Then there is an S^1-action $\varphi : S^1 \times U \to U$ on U which induces the vector field ξ_1, i.e. $\frac{d}{dt}\big|_{t=1}(f \circ \varphi)(t, x) = \xi_1(f)(x)$ for any $x \in U$ and any holomorphic function f defined near x.*
(3) *There is an open embedding $\iota : U \hookrightarrow Y \subseteq \mathbb{C}^N$ into a normal Stein space Y such that there is a holomorphic $\mathbb{C}^*$-action $\psi : \mathbb{C}^* \times Y \to$*

Y which is induced by a linear $\mathbb{C}^*$-action on $\mathbb{C}^N$ and induces the S^1-action φ on U, i.e. $\psi|_{S^1\times U} = \varphi$, where we identify U and $\iota(U)$.

(4) *Let $A = \{y \in Y \mid \psi(t, y) = y \text{ for all } t \in \mathbb{C}^*\}$ be the fixed point set of the $\mathbb{C}^*$-action on Y, and let $\pi : Y \to Y/\!/\mathbb{C}^*$ be the categorical quotient of Y by the $\mathbb{C}^*$-action ψ. Then $A = U \cap D_j$ and the quotient space $Y/\!/\mathbb{C}^*$ is isomorphic to A.*

(5) *Let $B \subseteq U$ be a closed analytic subset which is S^1-invariant, i.e. $S^1 \cdot B = \varphi(S^1 \times B) = B$. Then $\mathbb{C}^* \cdot B = \psi(\mathbb{C}^* \times B)$ is a closed subset of Y and $(\mathbb{C}^* \cdot B) \cap U = B$. Moreover, if B is normal, then $B \cap A$ is normal.*

Before proving the Lemma 5.9, we show how the above proposition follows from the lemma.

Proof of Proposition 5.8. — Relabelling the components of $\lfloor D\rfloor$ if necessary, it is enough to show that if $D_1 \cap \ldots \cap D_{j-1}$ is normal, then $D_1 \cap \ldots \cap D_j$ is normal.

Let $p \in D_1 \cap \ldots \cap D_j$. Let $U \subseteq X$ be an open neighbourhood of p as described in the preceding lemma, $Y \subseteq \mathbb{C}^N$ a normal complex Stein space with a $\mathbb{C}^*$-action $\psi : \mathbb{C}^* \times Y \to Y$, and $\iota : U \to Y$ an embedding such that the restriction of the vector field ξ induced by the $\mathbb{C}^*$-action ψ to U is a logarithmic vector field with respect to D and such that there is a local basis $\sigma_1, \ldots, \sigma_n$ for $\Omega^{[1]}_X(\log\lfloor D\rfloor)|_U$ consisting of closed forms such that $\xi, \xi_2, \ldots, \xi_n$ is a dual basis for $\mathcal{T}_X(-\log\lfloor D\rfloor)|_U$ for appropriate logarithmic vector fields $\xi_2, \ldots, \xi_n$ on U.

Again, we identify U with its image $\iota(U) \subseteq Y$ and let $\pi : Y \to Y/\!/\mathbb{C}^*$ denote the categorical quotient. As before the quotient $Y/\!/\mathbb{C}^*$ may be identified with set the of fixed points $A = D_j \cap U$ of the $\mathbb{C}^*$-action.

Set $B = D_1 \cap \ldots \cap D_{j-1} \cap U$. Then B is a closed analytic subset of U which is S^1-invariant by construction. Moreover, the set B is normal by assumption. Then by part (5) of the preceding Lemma 5.9 the intersection $B \cap A = B \cap D_j = D_1 \cap \ldots \cap D_{j-1} \cap D_j$ is normal. □

Proof of Lemma 5.9. — By Lemma 5.7 we already know that

$$D_j \setminus \left(\bigcup_{i \neq j} D_i \right)$$

is smooth for each j. Since the question is local, we may assume that $\Omega^{[1]}_X(\log\lfloor D\rfloor)$ is free and spanned by closed forms $\sigma_1, \ldots, \sigma_n$. By the same argument as used in the proof of Lemma 5.7, we may furthermore assume that $\mathrm{res}_{D_j}(\sigma_1) = 2\pi i$ and $\mathrm{res}_{D_j}(\sigma_i) = 0$ for $i > 1$. This proves part (1).

In order to prove part (2), let $\xi_1, \dots, \xi_n$ be a basis of logarithmic vector fields dual to $\sigma_1, \dots, \sigma_n$. Let $\chi : \Omega \to X$ be the flow map of ξ_1 (cf. Theorem 3.2), $\Omega \subseteq \mathbb{C} \times X$. Let $q \in D_j \setminus (\bigcup_{i \neq j} D_i)$. Then X and D_j are smooth near q by Lemma 5.7. We may now define local coordinates on a suitable neighbourhood of q by setting

$$z_1(x) = \exp\left(\int_{q_0}^{x} \sigma_1\right)$$

and

$$z_i(x) = \int_{q_0}^{x} \sigma_i$$

for $i > 1$ and a fixed point $q_0 \in X \setminus \lfloor D \rfloor$ near q. Note that the integrals are independent of the chosen path since $\sigma_1, \dots, \sigma_n$ are closed, $\operatorname{res}_{D_j}(\sigma_1) = 2\pi i$ and $\sigma_2, \dots, \sigma_n$ are holomorphic near q. With respect to these coordinates we have $D_j = \{z_1 = 0\}$ and

$$\sigma_1 = d\log(z_1) = \frac{dz_1}{z_1}, \sigma_2 = dz_2, \dots, \sigma_n = dz_n,$$

and the dual vector fields $\xi_1, \dots, \xi_n$ are thus necessarily of the form

$$\xi_1 = z_1 \frac{\partial}{\partial z_1}, \xi_2 = \frac{\partial}{\partial z_2}, \dots, \xi_n = \frac{\partial}{\partial z_n}.$$

Therefore ξ_1 vanishes along $\{z_1 = 0\}$ and by the identity principle along all of D_j. Since each point in D_j is a fixed point of the flow $\chi : \Omega \to X$ of the vector field ξ_1, there is an open connected neighbourhood V of the point $p \in D_j$ (as in the statement of the Lemma 5.9) such that the domain Ω of definition of χ can be chosen such that

$$((-1, 1) \times (-4\pi, 4\pi)) \times V \subset \Omega \subseteq \mathbb{C} \times X,$$

and such that $V \cap D_j$ is connected and contains p and q.

The flow of ξ_1 with respect to the local coordinates $z_1, \dots, z_n$ is given by

$$\chi\left(t, \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}\right) = \begin{pmatrix} e^t z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

Thus $\chi(2\pi i, x) = x$ for all x near q and by the identity principle we get $\chi(2\pi i, x) = x$ for all $x \in V$. Let V' be a relatively compact open subset of V such that $V' \cap D_j$ is also connected and contains p and such that $\chi(\{0\} \times (-4\pi, 4\pi) \times V') \subseteq V$. Consequently, $\chi(i(s+t), x)$ and $\chi(is, \chi(it, x))$ are defined for all $x \in V'$, $s, t \in (-4\pi, 4\pi)$ with $s + t \in (-4\pi, 4\pi)$, and we have

$$\chi(i(s+t), x) = \chi(is, \chi(it, x)).$$

Set $U = \chi(\{0\} \times (-4\pi, 4\pi) \times V')$. Then U is an open neighbourhood of p, $U \subseteq V$, and we can define a map $\varphi : S^1 \times U \to U$ by setting

$$\varphi(e^{it}, x) = \chi(it, x).$$

This is well-defined since $\chi(2\pi i, x) = x$ for all $x \in V$, $\chi(i(s+t), x) = \chi(is, \chi(it, x))$ implies $\varphi(S^1 \times U) \subseteq U$ and that φ is a group action. Moreover, this S^1-action φ induces ξ_1 by construction, and thus we proved (2).

By standard arguments (see for example [4, Proposition 2.3]), there is an open neighbourhood $U' \subseteq U$ of p and an embedding $\iota : U' \to \mathbb{C}^N$, and moreover we can choose N minimal in the sense that $N = \dim T_p(U') = \dim T_p(X)$. We may assume $\iota(p) = 0$. Consider now the set

$$U'' = \bigcap_{s \in S^1} \varphi(\{s\} \times U') \subseteq U'.$$

This set U'' is open and contains p since S^1 is compact and p a fixed point because $p \in D_j$ and $\xi_1|_{D_j} = 0$. Moreover, U'' is S^1-invariant, i.e. $\varphi(S^1 \times U'') = U''$. After shrinking, we may thus assume that $U = U' = U''$. Moreover, we will always identify U and $\iota(U) \subseteq \mathbb{C}^N$ in the following and also denote the inclusion map $U = \iota(U) \hookrightarrow \mathbb{C}^N$ by ι.

Since p is a fixed point of the S^1-action φ, we get a linear S^1-action on $T_pX \cong \mathbb{C}^N$ by differentiation, which we denote by $\rho : S^1 \times \mathbb{C}^N \to \mathbb{C}^N$. Next, we want to average the embedding $\iota : U \hookrightarrow \mathbb{C}^N$ in order to obtain an embedding $U \hookrightarrow \mathbb{C}^N$ which is equivariant with respect to the S^1-action φ on U and the linear S^1-action ρ on $\mathbb{C}^N$. Let μ denote the normalised Haar measure on S^1 and set

$$\widetilde{\iota} : U \to \mathbb{C}^N, \quad \widetilde{\iota}(u) = \int_{S^1} \rho(s)\iota\left(\varphi\left(s^{-1}, u\right)\right) d\mu(s).$$

Then $\widetilde{\iota}(p) = 0$ and $\widetilde{\iota}$ is equivariant by construction. Identifying $T_pU \cong \mathbb{C}^N$ and $T_0(\iota(U)) = T_0\mathbb{C}^N \cong \mathbb{C}^N$ appropriately, we have $D\iota(p) = \mathrm{id}_{\mathbb{C}^N}$ and thus

$$\begin{aligned} D\widetilde{\iota}(p) &= \int_{S^1} D\rho(s) D\iota(p) D\varphi\left(s^{-1}, p\right) d\mu(s) \\ &= \int_{S^1} \rho(s) \circ \mathrm{id} \circ \rho\left(s^{-1}\right) d\mu(s) = \mathrm{id} \end{aligned}$$

and consequently $\widetilde{\iota}$ is an immersion at p. Thus we can shrink U and get an equivariant embedding $\widetilde{\iota} : U \hookrightarrow \mathbb{C}^N$, and identifying U and $\widetilde{\iota}(U) \subseteq \mathbb{C}^N$, the S^1-action φ on U is induced by a linear S^1-action on $\mathbb{C}^N$. After possibly further shrinking U and rescaling, we may assume that U is a closed subset of the open unit ball $B_N = \{z \in \mathbb{C}^N \,|\, \langle z, z\rangle < 1\}$ with respect to an S^1-invariant hermitian inner product $\langle\, ,\rangle$.

The linear S^1-action on $\mathbb{C}^N$ extends to a linear $\mathbb{C}^*$-action $\psi : \mathbb{C}^* \times \mathbb{C}^N \to \mathbb{C}^N$ on $\mathbb{C}^N$, and the restriction of the induced vector field $\frac{d}{dt}\big|_{t=1} \psi(t, -)$ to U is precisely the vector field ξ_1.

The function $\alpha : \mathbb{C}^N \to \mathbb{R}$, $z \mapsto \langle z, z\rangle$, is S^1-invariant and plurisubharmonic. Therefore, the open unit ball $B_N = \{z \in \mathbb{C}^N \,|\, \alpha(z) < 1\}$ is orbit-convex (cf. [12, Section 3.4 Proposition]), i.e. for every $z \in B_N$ and $v \in \mathbb{R} = i\,\mathrm{Lie}(S^1)$ such that $\exp(v) \cdot z = \psi(\exp(v), z) \in B_N$ we also have $\exp(tv) \cdot z = \psi(\exp(tv), z) \in B_N$ for all $t \in [0, 1]$.

Define now $Y = \mathbb{C}^* \cdot U = \psi(\mathbb{C}^* \times U) \subseteq \mathbb{C}^N$. Then Y is an irreducible normal complex space since U is normal and $U \subseteq Y$ is an open subset. By [12, Section 3.3], the complex space Y is the S^1-complexification (in the sense of [12, Section 1.1]) of the S^1-invariant analytic subset U of the open unit ball B_N and consequently, Y is a Stein space by [12, Section 6.6]. This finishes the proof of part (3).

The categorical quotient $\pi : Y \to Y/\!/\mathbb{C}^*$ of Y by the $\mathbb{C}^*$-action ψ exists and is a complex Stein space by [23, Theorem 5.3]. Furthermore, $Y/\!/\mathbb{C}^*$ is normal since Y is normal (see [23, Lemma 3.2 and the subsequent remark]).

By definition of Y as $Y = \mathbb{C}^* \cdot U = \psi(\mathbb{C}^* \times U)$, the fixed point set

$$A = \{y \in Y \,|\, \psi(t, y) = y \text{ for all } t \in \mathbb{C}^*\}$$

is contained in U. For elements $u \in U$ we know that $u \in A$ precisely if $\xi_1(u) = 0$, and thus we get $D_j \cap U \subseteq A$.

Let $q \in D_j \cap U$, $q \notin \bigcup_{i \neq j} D_i$, such that q is a smooth point of U and D_j. As argued before, there are local coordinates $z_1, \ldots, z_n$ for U near q such that locally $D_j = \{z_1 = 0\}$ and $\xi_1 = z_1 \frac{\partial}{\partial z_1}$, and locally near q the set of fixed points A and D_j coincide.

Moreover, q is an attractive fixed point of the $\mathbb{C}^*$-action, i.e. there is a neighbourhood $W \subseteq Y$ of q such that for any $y \in W$ the closure of the orbit $\mathbb{C}^* \cdot y$ through y contains a fixed point. Then the set of fixed points A is a closed irreducible subspace of Y by [23, Theorem 6.2] and hence $A = D_j \cap U$.

Since every fibre of π contains precisely one closed orbit, and the set of fixed points A is the set of orbits of minimal dimension, $\pi(A)$ is closed. Moreover, $\pi(A)$ is open since there is an attractive fixed point. Therefore, we get that A is isomorphic to $Y/\!/\mathbb{C}^*$ and every fixed point is an attractive fixed point, see also [23, Theorem 6.2].

In order to prove part (5) of the lemma, let $B \subseteq U$ be a closed analytic S^1-invariant subset of U. The results of [12, Section 3.3] now directly imply that $\mathbb{C}^* \cdot B = \psi(\mathbb{C}^* \times B)$ is a closed analytic subset of Y, and $(\mathbb{C}^* \cdot B) \cap U = B$.

In particular, $\mathbb{C}^* \cdot B$ is a complex Stein space, and normal if B is normal. Let A denote again the set of fixed point of the $\mathbb{C}^*$-action on Y. By similar arguments as used before, it follows that the categorical quotient $(\mathbb{C}^* \cdot B)//\mathbb{C}^*$, which is normal if $\mathbb{C}^* \cdot B$ is normal, can be identified with the set of fixed points A' in $\mathbb{C}^* \cdot B$, and we have

$$A' = A' \cap U = A \cap (\mathbb{C}^* \cdot B) \cap U = A \cap B.$$

This shows in particular that $A \cap B$ is normal if B is normal. $\square$

Lemma 5.10. — *Let (X, D) be a pair. Let $\sigma_1, \ldots, \sigma_k$ be closed 1-forms on $X \setminus \lfloor D \rfloor$ and $\sigma_{k+1}, \ldots, \sigma_n$ closed 1-forms on X such that the sheaf $\Omega_X^{[1]}|_{X \setminus \lfloor D \rfloor}$ is spanned by $\sigma_1, \ldots, \sigma_n$. Let $\xi_1, \ldots, \xi_n$ be vector fields on X which are logarithmic with respect to $\lfloor D \rfloor$ and dual to $\sigma_1, \ldots, \sigma_n$ (on $X \setminus \lfloor D \rfloor$), i.e. $\sigma_i(\xi_j) = \delta_{ij}$. Assume that the vector fields $\xi_1, \ldots, \xi_k$ are induced by S^1-actions, i.e. there are actions $\psi_j : S^1 \times X \to X$ of the Lie group S^1 by holomorphic transformations such that the induced vector field $\frac{d}{ds}\big|_{s=1} \psi_j(s, \cdot)$ coincides with ξ_j.*

Then the 1-forms $\sigma_1, \ldots, \sigma_k$ extend to logarithmic 1-forms $\sigma_1, \ldots, \sigma_k \in \Omega_X^{[1]}(\log\lfloor D \rfloor)(X)$.

Proof. — Since the pair $(X, \lfloor D \rfloor)$ is snc outside a set of codimension at least 2 and $\Omega_X^{[1]}(\log\lfloor D \rfloor)$ is reflexive, it is enough to consider the case where X is smooth and $\lfloor D \rfloor$ an snc divisor. By Lemma 5.6 the vector fields $\xi_1, \ldots, \xi_n$ pairwise commute since the dual forms $\sigma_1, \ldots, \sigma_n$ are closed. Hence, the S^1-actions ψ_j all commute and thus induce an $(S^1)^k$-action $\psi : (S^1)^k \times X \to X$ by setting

$$\psi\left(\begin{pmatrix} s_1 \\ \vdots \\ s_k \end{pmatrix}, p\right) = \psi_1(s_1, \psi_2(s_2, \ldots \psi_k(s_k, p) \ldots))$$

for

$$\begin{pmatrix} s_1 \\ \vdots \\ s_k \end{pmatrix} \in (S^1)^k, p \in X.$$

Let $p_0 \in \lfloor D \rfloor$. First, we consider the case where $k = n$ and p_0 is a fixed point of the $(S^1)^n$-action ψ, or equivalently $\xi_1(p_0) = \ldots = \xi_n(p_0) = 0$. Then the action can locally be linearised, i.e. there are local coordinates $z_1, \ldots, z_n$ near p_0 such that $p_0 = 0$ and the action ψ is linear in these coordinates. Moreover, we may assume that $z_1, \ldots, z_n$ are chosen such that there are constants a_{ij} for $i, j = 1, \ldots, n$ such that

$$\psi\left(\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}, \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}\right) = \begin{pmatrix} s_1^{a_{11}}\cdot\ldots\cdot s_n^{a_{n1}} & & \\ & \ddots & \\ & & s_1^{a_{1n}}\cdot\ldots\cdot s_n^{a_{nn}} \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$
$$= \begin{pmatrix} s_1^{a_{11}}\cdot\ldots\cdot s_n^{a_{n1}}\cdot z_1 \\ \vdots \\ s_1^{a_{1n}}\cdot\ldots\cdot s_n^{a_{nn}} z_n \end{pmatrix}$$

and then

$$\xi_i(z) = \sum_{j=1}^{n} a_{ij} z_j \frac{\partial}{\partial z_j}.$$

Since $\sigma_1, \ldots, \sigma_n$ and hence $\xi_1, \ldots, \xi_n$ are linearly independent on $X \setminus \lfloor D \rfloor$ we get that the matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ has to be invertible. We may thus replace the vector fields $\xi_1, \ldots, \xi_n$ (and also $\sigma_1, \ldots, \sigma_n$) by an invertible linear combination of them and then get $\xi_j = z_j \frac{\partial}{\partial z_j}$ for all $j = 1, \ldots n$. This implies $\sigma_j = \frac{1}{z_j} dz_j$, and therefore $\sigma_1, \ldots, \sigma_n$ are logarithmic 1-forms on X.

Let now $p_0 \in \lfloor D \rfloor$ be any point and k be arbitrary. Let $G = ((S^1)^k)_{p_0} = \{s \in (S^1)^k \mid \psi(s, p_0) = p_0\}$ denote the isotropy group in p_0. Since G is a closed subgroup of $(S^1)^k$, we have $G \cong (S^1)^l$ for some $l \leqslant k$, and since $(S^1)^k$ is abelian, there is a Lie subgroup $H \cong (S^1)^{k-l}$ of $(S^1)^k$ such that $(S^1)^k \cong G \times H$. After possibly again replacing the vector fields $\xi_1, \ldots, \xi_k$ by an invertible linear combination of them we may assume that the Lie algebra of G is spanned by $\xi_1, \ldots, \xi_l$ and the Lie algebra of H by $\xi_{l+1}, \ldots, \xi_k$. Since the isotropy group of H in p_0 is trivial by construction and the vector fields ξ_j all satisfy $\sigma_i(\xi_j) = \delta_{ij}$ on X for $i > k$, we get that $\xi_{l+1}, \ldots, \xi_n$ are independent at each point in a neighbourhood of p_0.

Moreover, $\xi_{l+1}, \ldots, \xi_n$ are commuting and thus span an involutive distribution (of rank $n - l$) locally near p_0. Therefore, by Frobenius' theorem there are local coordinates $w_1, \ldots, w_n$ such that

$$\xi_{l+1} = \frac{\partial}{\partial w_{l+1}}, \ldots, \xi_n = \frac{\partial}{\partial w_n}$$

and we may assume also $w_1(p_0) = \ldots = w_n(p_0) = 0$. Potentially, we could have $\xi_i(w_j) \neq 0$ for some $i \leqslant l$ and $j > l$.

We now want to average $w_{l+1}, \ldots, w_n$ in order to obtain G-invariant coordinates $z_1, \ldots, z_n$ with $\xi_i(z_j) = 0$ for all $i \leqslant l$ and $j > l$. We define

$$z_j(p) = \int_G w_j(\psi(s, p)) d\mu(s)$$

for $j > l$ and where μ denotes the normalised Haar measure on $G \cong (S^1)^l$. Setting $z_1 = w_1, \ldots, z_l = w_l$ we get new coordinates $z_1, \ldots, z_n$ such that $z_1(p_0) = \ldots = z_n(p_0) = 0$, $\xi_{l+1} = \frac{\partial}{\partial z_{l+1}}, \ldots, \xi_n = \frac{\partial}{\partial z_n}$ and such that $z_{l+1}, \ldots, z_n$ are G-invariant. Since the vector fields $\xi_1, \ldots, \xi_l$ are induced by the G-action and $z_{l+1}, \ldots, z_n$ are G-invariant, we now get $\xi_i(z_j) = 0$ for all $i \leqslant l$ and $j > l$.

Moreover, the subset $S = \{z_{l+1} = \ldots = z_n = 0\}$ is G-invariant, a smooth submanifold and parametrised by the coordinates $z_1, \ldots, z_l$. We have $p_0 \in S$ by construction and may now apply the argument from the beginning of the proof to S and the $G \cong (S^1)^l$-action on S. This then yields that (after possibly changing the coordinates $z_1, \ldots, z_l$) the vector fields $\xi_1, \ldots, \xi_l$ locally have the form $\xi_i = z_i \frac{\partial}{\partial z_i}$, $i = 1, \ldots, l$, on S. Since $\xi_i(z_j) = 0$ for all $i \leqslant l$ and $j > l$ when the ξ_i's are considered as vector fields on X, we also get that locally on X with respect to the coordinates $z_1, \ldots, z_n$ the vector fields $\xi_1, \ldots, \xi_l$ have the form $\xi_i = z_i \frac{\partial}{\partial z_i}$, $i = 1, \ldots, l$. This implies $\sigma_i = \frac{1}{z_i} dz_i$, $i = 1, \ldots l$ for the dual 1-forms and hence yields the desired result. □

Corollary 5.11. — *Let (X, D) be a pair such that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is locally free and locally generated by closed forms. Let $\pi : \widetilde{X} \to X$ be any log resolution of the pair (X, D), let E be the exceptional divisor of π and $\overline{D}$ the strict transform of D.*

Then every logarithmic 1-from on (X, D) extends to a logarithmic 1-from on $(\widetilde{X}, \widetilde{D})$, where $\widetilde{D} = E + \overline{D}$.

Proof. — As explained in [9, Lemma 2.13] (and keeping in mind their notation/definition as in [9, Definition 2.8]), the statement of this Corollory 5.11 holds for one specific log resolution of the pair (X, D) if and only if it holds for all log resulations of the pair. Therefore, we will assume in the following that $\pi : \widetilde{X} \to X$ is the functionral resolution of the pair as needed for Proposition 3.3.

Let $D_1, \ldots, D_k$ denote the irreducible components of $\lfloor D\rfloor$, $\lfloor D\rfloor = D_1 + \ldots + D_k$. Since the statement is local, it is enough to prove the statement for a neighbourhood of a point $p \in D_1 \cap \ldots \cap D_k$. By Lemma 5.5 there exist closed logarithmic 1-forms $\sigma_1, \ldots, \sigma_n$ which span $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ in a neighbourhood of p.

Let

$$A = (\mathrm{res}_{D_i}(\sigma_j))_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n}$$

be the $k \times n$-matrix whose entry at the position (i, j) is the residue of σ_j along the divisor D_i. Note that by Remark 3.14 all entries of A are complex numbers. After relabelling the indices of the D_i's and passing to a linear combination of $\sigma_1, \ldots, \sigma_n$ we may assume that there is $l \leqslant n$ such that $\operatorname{res}_{D_i}(\sigma_i) = 2\pi i$ and $\operatorname{res}_{D_i}(\sigma_j) = 0$ for all $i \leqslant l$ and all $j \neq i$, and $\operatorname{res}_{D_i}(\sigma_j) = 0$ for all $j > l$, i.e. $\sigma_{l+1}, \ldots, \sigma_n$ are regular. By [13, Theorem 1.2] we already know that $\sigma_{l+1}, \ldots, \sigma_n$ extend to regular 1-forms on $\widetilde{X}$.

Let $\xi_1, \ldots, \xi_n$ be logarithmic vector fields dual to $\sigma_1, \ldots, \sigma_n$. For any $i \leqslant l$ we have $\operatorname{res}_{D_i}(\sigma_i) = 2\pi i$ and $\operatorname{res}_{D_i}(\sigma_j) = 0$ for $j \neq i$, and thus by Lemma 5.9 (2) we get that for $i \leqslant l$ there are open neighbourhoods U_i of p and S^1-actions $\varphi_i : S^1 \times U_i \to U_i$ which induce ξ_i.

Let $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ denote the lifts of $\xi_1, \ldots, \xi_n$ to $\widetilde{X}$ (cf. Proposition 3.3). The S^1-actions φ_i also lift to $\widetilde{X}$ and induce the vector fields $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_l$. Moreover, the vector fields $\widetilde{\xi}_{l+1}, \ldots, \widetilde{\xi}_n$ are independent at each point since $\sigma_{l+1}, \ldots, \sigma_n$ extend to regular 1-forms on $\widetilde{X}$. An application of Lemma 5.10 now yields that $\sigma_1, \ldots, \sigma_l$ extend to logarithmic 1-forms on $\widetilde{X}$. □

Lemma 5.12. — *Let (X, D) be a pair such that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is free and generated by closed logarithmic 1-forms $\sigma_1, \ldots, \sigma_n$ such that $\sigma_{l+1}, \ldots, \sigma_n$ are regular. Let $D_1, \ldots, D_k$ be the irreducible components of $\lfloor D\rfloor$.*

Then $\sigma_{l+1}, \ldots, \sigma_n$ can be restricted to any intersection $D_{i_1} \cap \ldots \cap D_{i_j}$ for $i_1, \ldots, i_j \in \{1, \ldots, k\}$, i.e. there are regular 1-forms $\eta_{l+1}, \ldots, \eta_n$ on $D_{i_1} \cap \ldots \cap D_{i_j}$ such that $\iota^*(\sigma_i) = \eta_i$ if $\iota : D_{i_1} \cap \ldots \cap D_{i_j} \hookrightarrow X$ denotes the inclusion map.

Moreover, we have $\dim(D_1 \cap \ldots \cap D_k) \geqslant n - l$ if $D_1 \cap \ldots \cap D_k \neq \emptyset$, and $D_1 \cap \ldots \cap D_k$ is smooth.

Proof. — Without loss of generality, let $D_{i_1} \cap \ldots \cap D_{i_j} = D_1 \cap \ldots \cap D_j$, and suppose $D_1 \cap \ldots \cap D_j \neq \emptyset$. Recall that by Proposition 5.8 this intersection $D_1 \cap \ldots \cap D_j$ is normal.

Let $\xi_1, \ldots, \xi_n$ be a basis of logarithmic vector fields dual to $\sigma_1, \ldots, \sigma_n$. Since the flows of $\xi_1, \ldots, \xi_n$ stabilise each irreducible component of $\lfloor D\rfloor$, the vector fields $\xi_1, \ldots, \xi_n$ induce vector fields on $D_1 \cap \ldots \cap D_j$ for any $j \leqslant k$.

Let $\pi : \widetilde{X} \to X$ be a log resolution of (X, D) with exceptional divisor E and let $\overline{D}_i$ be the strict transform of D_i. By Corollary 5.11, $\sigma_1, \ldots, \sigma_n$ extend to logarithmic 1-forms $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ on $\widetilde{X}$.

We first want to restrict to D_1. We may assume that $\operatorname{res}_{D_1}(\sigma_1) = 1$ and $\operatorname{res}_{D_1}(\sigma_i) = 0$ for $i > 1$. Then also $\operatorname{res}_{\overline{D}_1}(\widetilde{\sigma}_i) = 0$ for $i > 1$ and $\widetilde{\sigma}_i$ is thus regular along $\overline{D}_1 \backslash (E \cup \overline{D}_2 \cup \ldots \cup \overline{D}_k)$. Therefore the restriction of $\widetilde{\sigma}_2, \ldots, \widetilde{\sigma}_n$ to $\overline{D}_1$ yields logarithmic 1-forms with respect to $(E + \overline{D}_2 + \ldots + \overline{D}_k)|_{\overline{D}_1}$. Since D_1 is normal, D_1 is isomorphic to an open subset of $\overline{D}_1$ outside a closed subset of codimension at least 2. Consequently, the logarithmic 1-forms $\widetilde{\sigma}_2|_{\overline{D}_1}, \dots, \widetilde{\sigma}_n|_{\overline{D}_1}$ on the strict transform $\overline{D}_1$ induce logarithmic 1-forms on $(D_1, (D_2+\dots+D_k)|_{D_1})$ since $\Omega^{[1]}_{D_1}(\log(D_2+\dots+D_k)|_{D_1})$ is reflexive, and these give the desired restrictions $\sigma_2|_{D_1}, \dots, \sigma_n|_{D_1}$ of $\sigma_2, \dots, \sigma_n$ to D_1. Moreover, the restricted vector fields $\xi_2|_{D_1}, \dots, \xi_n|_{D_1}$ are dual to these and hence $\sigma_2|_{D_1}, \dots, \sigma_n|_{D_1}$ yield a basis of logarithmic 1-forms on D_1. The vector fields $\xi_2|_{D_1}, \dots, \xi_n|_{D_1}$ commute and $\sigma_2|_{D_1}, \dots, \sigma_n|_{D_1}$ are closed.

If $k > 1$ and if there is D_i, say $D_2 = D_i$, with $\operatorname{codim}_{D_1}(D_1 \cap D_i) = 1$, we apply the procedure again. It might now happen that $D_i \cap (D_1 \cap D_2) = D_1 \cap D_2$ for some $i > 2$. Assume $(D_1 \cap D_2) \cap \dots \cap D_i = D_1 \cap D_2$ and $(D_1 \cap D_2) \cap D_{i'} \subsetneq D_1 \cap D_2$ for all $i' > i$. In this case, $\sigma_3, \dots, \sigma_n$ restrict to logarithmic 1-forms of the pair $(D_1 \cap \dots \cap D_i, (D_{i+1}+\dots+D_k)|_{D_1 \cap \dots \cap D_i})$, and $\xi_3, \dots, \xi_n$ induce dual logarithmic vector fields on $D_1 \cap \dots \cap D_i$.

We continue then iteratively. At each step either the boundary $(D_{i+1} + \dots + D_k)|_{D_1 \cap \dots \cap D_{i'}}$ of the pair $(D', D_0) = (D_1 \cap \dots \cap D_i, (D_{i+1} + \dots + D_k)|_{D_1 \cap \dots \cap D_i})$ is empty or otherwise there is $i' > i$ such that $D_{i'} \cap (D_1 \cap \dots \cap D_i)$ has codimension 1 in $D_1 \cap \dots \cap D_i$ as explained in the following:

By construction we have $D_{i'} \cap (D_1 \cap \dots \cap D_i) \neq D' = D_1 \cap \dots \cap D_i$ and thus the codimension is at least 1. On $D' = D_1 \cap \dots \cap D_i$ we have the restricted logarithmic 1-forms $\sigma_r|_{D'}, \dots, \sigma_n|_{D'}$ and dual logarithmic vector fields $\xi_r|_{D'}, \dots, \xi_n|_{D'}$, where $r \leqslant i+1$, $r-1 = \operatorname{codim}_X(D')$. If the codimension of $D_{i'} \cap D'$ in D' is at least 2 for all $i' > i$, then the logarithmic 1-forms $\sigma_r|_{D'}, \dots, \sigma_n|_{D'}$ are regular since D' is normal. But this is in contradiction to the fact that the vector fields $\xi_r|_{D'}, \dots, \xi_n|_{D'}$ stabilise each $D_{i'}$.

This procedure eventually gives rise to regular 1-forms $\sigma_r|_{D_1 \cap \dots \cap D_k}, \dots, \sigma_n|_{D_1 \cap \dots \cap D_k}$ on $D_1 \cap \dots \cap D_k$, $r \leqslant l+1$ and also their dual vector fields $\xi_r|_{D_1 \cap \dots \cap D_k}, \dots \xi_n|_{D_1 \cap \dots \cap D_k}$. Hence, we have $\dim(D_1 \cap \dots \cap D_k) = n - r + 1 \geqslant n - (l+1) - 1 = n - l$. Furthermore, $D_1 \cap \dots \cap D_k$ is smooth by [13, Theorem 1.1] since $\sigma_r|_{D_1 \cap \dots \cap D_k}, \dots, \sigma_n|_{D_1 \cap \dots \cap D_k}$ are a basis for $\Omega^{[1]}_{D_1 \cap \dots \cap D_k}$ and each $\sigma_j|_{D_1 \cap \dots \cap D_k}$ is closed. $\square$

Remark 5.13. — In the setting of the previous Lemma 5.12 and its proof, we also get that $r = \operatorname{rk}(A) + 1$ and hence

$$\dim(D_1 \cap \dots \cap D_k) = n - r + 1 = n - \operatorname{rk}(A),$$

where

$$A = (\operatorname{res}_{D_i}(\sigma_j))_{1 \leqslant i \leqslant k,\, 1 \leqslant j \leqslant n}$$

is the matrix of residues as before.

Theorem 5.14. — *Let (X, D) be a pair such that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is locally free and locally generated by closed logarithmic* 1*-forms. Then $(X, \lfloor D\rfloor)$ is toroidal, i.e. for any point there is a neighbourhood $U \subseteq X$ which is isomorphic to an open subset of a toric variety Y with open $(\mathbb{C}^*)^n$-orbit T, and the divisor $\lfloor D\rfloor$ corresponds to the complement $Y \setminus T$ of T in Y.*

Proof. — Let $D_1, \dots, D_k$ denote the irreducible components of $\lfloor D\rfloor$, and let $p \in X$. Since the statement of the theorem is local, we may assume that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is free and generated by the closed logarithmic 1-forms $\sigma_1, \dots, \sigma_n$, and that $p \in D_1 \cap \dots \cap D_k$.

We first consider the case where $D_1 \cap \dots \cap D_k = \{p\}$. Let

$$A = (\operatorname{res}_{D_i}(\sigma_j))_{1 \leqslant i \leqslant k,\, 1 \leqslant j \leqslant n}$$

denote again the residue matrix of the forms $\sigma_1, \dots, \sigma_n$. Since we assumed $\dim(D_1 \cap \dots \cap D_k) = 0$, we have $\operatorname{rk}(A) = n$ by Remark 5.13. In particular, there are at least $n = \dim X$ irreducible components of $\lfloor D\rfloor$ containing the point p, and without loss of generality we may assume that A is of the form

$$A = (\operatorname{res}_{D_i}(\sigma_j))_{1 \leqslant i \leqslant k,\, 1 \leqslant j \leqslant n} = \left(\begin{array}{ccc} 2\pi i & & \\ & \ddots & \\ & & 2\pi i \\ \hline & B & \end{array}\right),$$

where B is an arbitrary $(k - n) \times n$-matrix. Let $\xi_1, \dots, \xi_n$ be a basis of logarithmic vector fields dual to $\sigma_1, \dots, \sigma_n$. By Lemma 5.9 (2) there is an open neighbourhood U_j of p for any $j = 1, \dots, n$ and an S^1-action $\varphi_j : S^1 \times U_j \to U_j$ which induces the vector field ξ_j on U_j and such that p is a fixed point of this S^1-action.

There is a neighbourhood U' of p such that $\varphi_1, \dots, \varphi_n$ define a map $\varphi : (S^1)^n \times U' \to X$ by setting

$$\varphi\left(\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}, q\right) = \varphi_1(s_1, \varphi_2(s_2, \dots \varphi_n(s_n, q) \dots)$$

for

$$\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \in (S^1)^n \text{ and } q \in U'.$$

Moreover, the vector fields $\xi_1, \dots, \xi_n$ all commute by Lemma 5.6 and hence the S^1-actions $\varphi_1, \dots, \varphi_n$ all commute. Thus, there is an open neighbourhood U of p such that $\varphi : (S^1)^n \times U \to U$ is an $(S^1)^n$-action; set e.g.

$$U = \bigcap_{t \in (S^1)^n} \varphi(\{t\} \times U')$$

and note that U is open since $(S^1)^n$ is compact and U contains p since $\varphi((S^1)^n \times \{p\}) = \{p\}$.

By the same arguments as used in the proof of Lemma 5.9 (3), we can shrink U such that there is a normal Stein space $Y \subseteq \mathbb{C}^N$ with a holomorphic $(\mathbb{C}^*)^n$-action $\psi : (\mathbb{C}^*)^n \times Y \to Y$ which is induced by a linear $(\mathbb{C}^*)^n$-action on $\mathbb{C}^N$ and such that there is an open equivariant embedding $\iota : U \hookrightarrow Y$ with $Y = \psi((\mathbb{C}^*)^n \times \iota(U))$, and we identify again U and $\iota(U)$. Moreover, we may assume that U is a closed analytic subset of the open unit ball $B_N = \{z \in \mathbb{C}^N \mid \langle z, z\rangle < 1\}$ with respect to an $(S^1)^n$-invariant hermitian inner product $\langle\, ,\rangle$.

Let $q \in U \setminus \lfloor D \rfloor$ and consider the orbit $(\mathbb{C}^*)^n \cdot q = \psi((\mathbb{C}^*)^n \times \{q\})$, which is open and dense in Y. The unique closed orbit in its closure $\overline{(\mathbb{C}^*)^n \cdot q}$ in the ambient space $\mathbb{C}^N$ is 0. Thus every orbit $(\mathbb{C}^*)^n \cdot x$ with $x \in \overline{(\mathbb{C}^*)^n \cdot q}$ contains 0 in its closure and there is $x' \in B_N$ with $(\mathbb{C}^*)^n \cdot x = (\mathbb{C}^*)^n \cdot x'$. Since $U \subset B_N$ is analytic and $(S^1)^n$-invariant and B_N is orbit-convex, we have $((\mathbb{C}^*)^n \cdot U) \cap B_N = U$ (cf. [12, Section 3.3 Corollary]) and then

$$((\mathbb{C}^*)^n \cdot q) \cap B_N \subset ((\mathbb{C}^*)^n \cdot U) \cap B_N = U.$$

This implies $x' \in U$ and hence $\overline{(\mathbb{C}^*)^n \cdot q} = Y$. Consequently, Y is an affine toric variety.

Now, let $\dim(D_1 \cap \ldots \cap D_k)$ be arbitrary. The intersection $D_1 \cap \ldots \cap D_k$ is smooth by Lemma 5.12 and we have $\dim(D_1 \cap \ldots \cap D_k) = n - \mathrm{rk}(A)$ by Remark 5.13. Thus we may assume that $\sigma_{l+1}, \dots, \sigma_n$, where $l = \mathrm{rk}(A)$, are regular 1-forms and $\mathrm{res}_{D_i}(\sigma_j) = 2\pi i \delta_{ij}$ for $i, j \leqslant l$. Let $p \in D_1 \cap \ldots \cap D_k$ and let $\xi_1, \dots, \xi_n$ denote again the logarithmic vector fields dual to $\sigma_1, \dots, \sigma_n$. Applying Lemma 5.9 (2) to the vector fields $\xi_1, \dots, \xi_l$, we get commuting S^1-actions $\varphi_j : S^1 \times U_j \to U_j$ on some neighbourhood U_j of p, $j = 1, \dots, l$, which induce the vector fields ξ_j. This gives now rise to an $(S^1)^l$-action $\varphi : (S^1)^l \times U \to U$ on some neighbourhood U of p. As before (cf. Lemma 5.9 (3)) we may globalise the corresponding local$(\mathbb{C}^*)^l$-action and get that there are a normal complex Stein space $Y \subseteq \mathbb{C}^N$ with a linear $(\mathbb{C}^*)^l$-action $\psi : (\mathbb{C}^*)^l \times Y \to Y$ and an equivariant open embedding $\iota : U \hookrightarrow Y$. Identifying U and its image $\iota(U)$ we have that the set A of fixed points of the $(\mathbb{C}^*)^l$-action ψ in Y is precisely $A = U \cap (D_1 \cap \ldots \cap D_k)$

and moreover A is isomorphic to $Y/\!/(\mathbb{C}^*)^l$ if $\pi : Y \to Y/\!/(\mathbb{C}^*)^l$ denotes the categorical quotient of Y by the action ψ. The vector fields $\xi_{l+1}, \ldots, \xi_n$ induce commuting and independent vector fields on $D_1 \cap \ldots \cap D_k$, and since they also commute with $\xi_1, \ldots, \xi_l$, they induce vector fields $\widehat{\xi}_{l+1}, \ldots, \widehat{\xi}_n$ on the quotient $Y/\!/(\mathbb{C}^*)^l$ with $\xi_j \circ \pi^* = \pi^* \circ \widehat{\xi}_j$ for $j = l+1, \ldots, n$. The fibre $\pi^{-1}(p)$ of $p \in U \cap D_1 \cap \ldots \cap D_k \cong Y/\!/(\mathbb{C}^*)^l$ is l-dimensional and the flows of $\xi_1, \ldots, \xi_l$ stabilise $\pi^{-1}(p)$ by construction such that $\xi_1, \ldots, \xi_l$ induce commuting vector fields on $\pi^{-1}(p)$. The flows of $\xi_{l+1}, \ldots, \xi_n$ and $\widehat{\xi}_{l+1}, \ldots, \widehat{\xi}_n$ now induce a local isomorphism $\chi : S \times X' \to X$ onto its image, where S is an open neighbourhood of p in $U \cap D_1 \cap \ldots \cap D_k \cong Y/\!/(\mathbb{C}^*)^l$ and X' an open neighbourhood of p in $\pi^{-1}(p)$. Since $U \cap D_1 \cap \ldots \cap D_k$ is smooth, S is also smooth and the divisors $D_1, \ldots, D_k$ induce divisors $D_1|_{X'}, \ldots, D_k|_{X'}$ on X'. Moreover, we may restrict $\sigma_1, \ldots, \sigma_l$ to X', they are dual to $\xi_1|_{X'}, \ldots, \xi_l|_{X'}$ and thus give rise to a basis of closed logarithmic 1-forms $\sigma_1|_{X'}, \ldots, \sigma_l|_{X'}$ of $\Omega_{X'}(\log\lfloor D\rfloor|_{X'})$. The intersection of the divisors $D_j|_{X'}$ is now $D_1|_{X'} \cap \ldots \cap D_k|_{X'} = \{p\}$ and applying the above arguments to X' we conclude that X' is toroidal. Consequently, $S \times X'$, which is isormorphic to a neighbourhood of p in X, is toroidal. □

6. Lc pairs with (locally) free sheaf of logarithmic 1-forms

In this section the case of an lc pair (X, D) with (locally) free sheaf of logarithmic 1-forms is considered.

As already noted in Examples 5.1 and 5.3 we cannot expect that X is smooth in this case. However, the singularities in these examples are contained in the support of $\lfloor D\rfloor$, and this is true in general. Since the Lipman–Zariski conjecture holds for lc pairs (see [6, Corollary 1.3] or [2, Theorem 1.1]), we have the following:

Remark 6.1. — If (X, D) is lc and $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is locally free, then $X \setminus \lfloor D\rfloor$ is smooth since the sheaf of 1-forms and the sheaf of logarithmic 1-forms agree on $X \setminus \lfloor D\rfloor$.

In the following, we first consider the case of an lc pair (X, D) where X is projective and the sheaf of logarithmic 1-forms is free. Then we deal with the case of a (not necessarily projective) lc pair (X, D) with locally free sheaf of logarithmic 1-forms. The goal is to prove that (X, D) is toroidal by reducing to the case where the sheaf of logarithmic 1-forms is spanned by closed forms as in the previous section.

6.1. Lc pairs with free sheaf of logarithmic 1-forms

Let us consider the case of an lc pair (X, D) such that its sheaf of logarithmic 1-forms is free and assume additionally that X is projective.

In the case of a smooth compact Kähler (or weakly Kähler) manifold X and an snc divisor D, Winkelmann described precisely under which conditions the logarithmic tangent bundle is trivial. In particular, the following result for smooth projective varieties is obtained.

Theorem 6.2 ([24, Corollary 1]). — *Let X be a smooth projective variety and D a reduced snc divisor on X. Then $\mathcal{T}_X(-\log D)$ is a free sheaf if and only if there is a semi-abelian variety T acting on X with $X \setminus D$ as an open orbit.* □

Recall that a semi-abelian variety is an algebraic group which is a quotient of $(\mathbb{C}^*)^n$ by a lattice Γ which contains a $\mathbb{C}$-basis of $\mathbb{C}^n$.

As a consequence of this result, we get an explicit description of projective lc pairs (X, D) with free sheaf of logarithmic 1-forms.

Corollary 6.3. — *Let (X, D) be an lc pair such that X is projective. Then the logarithmic tangent sheaf $\mathcal{T}_X(-\log\lfloor D\rfloor)$ is free if and only of there is a semi-abelian variety T which acts on X with $X \setminus \lfloor D\rfloor$ as an open orbit.*

Proof. — Let $\pi : \widetilde{X} \to X$ be a resolution of the pair (X, D) as in Proposition 3.3 and denote $\widetilde{D} = E + \overline{D}$, where E is the exceptional divisor and $\overline{D}$ the strict transform of D. Then by Proposition 3.3 and Corollary 3.6, the sheaf $\mathcal{T}_X(-\log\lfloor D\rfloor)$ is free if and only if $\mathcal{T}_{\tilde{X}}(-\log\lfloor \widetilde{D}\rfloor)$ is free.

Consequently, if $\mathcal{T}_X(-\log\lfloor D\rfloor)$ is free, then Theorem 6.2 implies that there is a semi-abelian variety T acting on $\widetilde{X}$ with $\widetilde{X} \setminus \lfloor\widetilde{D}\rfloor$ as an open orbit. Each component of $\lfloor\widetilde{D}\rfloor$ and thus in particular the exceptional divisors are T-invariant. This T-action on $\widetilde{X}$ hence induces n commuting vector fields $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ on $\widetilde{X}$ which are logarithmic with respect to $\lfloor\widetilde{D}\rfloor$ and such that their flow maps give rise to the action of T. By Remark 3.5 these vector fields induce vector fields $\xi_1, \ldots, \xi_n$ on X which are logarithmic with respect to $\lfloor D\rfloor$ and which satisfy $\widetilde{\xi}_j \circ \pi^* = \pi^* \circ \xi_j$. Since π induces an isomorphism $\widetilde{X} \setminus \pi^{-1}(Z) \to X \setminus Z$ outside the singular locus $Z = (X, D)_{\mathrm{sing}}$ of the pair (X, D), the identity principles implies that the vector fields $\xi_1, \ldots, \xi_n$ pairwise commute on all of X. Moreover, since X is projective the flows of $\xi_1, \ldots, \xi_n$ are all global and their combination thus gives rise to a $\mathbb{C}^n$-action on X. Using again that π induces an isomorphism

$\widetilde{X} \setminus \pi^{-1}(Z) \to X \setminus Z$ outside the singular locus $Z = (X, D)_{\mathrm{sing}}$ and the identity principle, we get that this $\mathbb{C}^n$-action actually descends to a T-action on X. Moreover, by construction it follows that π is equivariant with respect to the T-actions on $\widetilde{X}$ and X. Since $\widetilde{X} \setminus \lfloor \widetilde{D} \rfloor$ and $X \setminus \lfloor D \rfloor$ are isomorphic via π, $X \setminus \lfloor D \rfloor$ has to be an open orbit of T.

Conversely, if there is an action of a semi-abelian variety T on X with $X \setminus \lfloor D \rfloor$ as an open orbit, then this action lifts to $\widetilde{X}$ by [16, Proposition 3.9.1] with $\widetilde{X} \setminus \lfloor \widetilde{D} \rfloor$ as an open orbit. Consequently, $\mathcal{T}_{\tilde{X}}(-\log \lfloor \widetilde{D} \rfloor)$ and hence $\mathcal{T}_X(-\log \lfloor D \rfloor)$ are free. □

6.2. Lc pairs with locally free sheaf of logarithmic 1-forms

We now consider the case of an arbitrary lc pair (X, D) whose logarithmic tangent sheaf $\mathcal{T}_X(-\log \lfloor D \rfloor)$ is locally free.

First, we deal with the isolated case in the sense that there is a point at which every logarithmic vector field vanishes. Then the general case is considered and reduced to isolated case by an inductive argument via hyperplane sections.

Lemma 6.4. — *Let (X, D) be an lc pair with locally free tangent sheaf $\mathcal{T}_X(-\log \lfloor D \rfloor)$. Suppose that there is $p \in X$ such that $\xi(p) = 0$ for all logarithmic vector fields ξ defined on some neighbourhood of p.*

Then there exists a log resolution $\pi : \widetilde{X} \to X$ of the pair (X, D) with exceptional divisor E with the following properties:

(1) *Each irreducible component of $\pi^{-1}(p)$ is a toric variety.*
(2) *There is a point $q \in \pi^{-1}(p)$ such that $\xi(q) = 0$ for any logarithmic vector field $\xi \in \mathcal{T}_{\tilde{X}}(-\log \lfloor \widetilde{D} \rfloor)(U)$ defined on some open neighbourhood $U \subseteq \widetilde{X}$ of q, where $\widetilde{D} = \overline{D} + E$ for the strict transform $\overline{D}$ of D.*

Proof of Lemma 6.4(1). — Shrink X such $\mathcal{T}_X(-\log \lfloor D \rfloor)$ is free and let $\sigma_1, \ldots, \sigma_n$ denote a basis of logarithmic 1-forms and $\xi_1, \ldots, \xi_n$ the dual logarithmic vector fields.

Let $\pi' : X' \to X$ be the blow-up of X in p, and let D' be the sum of the exceptional divisor E_p and the strict transform of D. Since each vector field ξ_j fixes the the point p, these vector fields lift to logarithmic vector fields $\xi_1', \ldots, \xi_n'$ of the pair (X', D'), which can be proven by the same argument as used for Proposition 3.3.

Let $\widetilde{\pi} : \widetilde{X} \to X'$ be the functorial log resolution of the pair (X', D'), and let $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ denote the lifts of $\xi_1', \ldots, \xi_n'$ to $\widetilde{X}$. The composition

$\pi = \pi' \circ \widetilde{\pi} : \widetilde{X} \to X$ is also a log resolution of (X, D), and thus $\sigma_1, \ldots, \sigma_n$ extend to logarithmic 1-forms $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ on $\widetilde{X}$.

Let E denote the exceptional divisor of π, $\overline{D}$ the strict transform of D, and set $\widetilde{D} = E + \overline{D}$. Since $\pi^{-1}(p) = \widetilde{\pi}^{-1}(E_p)$, the fibre $\pi^{-1}(p)$ has pure codimension 1, and each irreducible component of $\pi^{-1}(p)$ is a component of the exceptional divisor E. Furthermore, the flows of $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ all stabilise $\pi^{-1}(p)$ since $\xi_1, \ldots, \xi_n$ vanish at p and hence $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ induce vector fields on $\pi^{-1}(p)$. Let E_1 be an irreducible component of $\pi^{-1}(p)$ and let $q \in E_1$ be a point which is not contained in any other irreducible component of E and also not contained in $\lfloor \widetilde{D} \rfloor$. Since $E_1 \subseteq \pi^{-1}(p)$ is projective, we may assume that the residues of $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n$ satisfy

$$\operatorname{res}_{E_1}(\widetilde{\sigma}_1) = 1$$

and

$$\operatorname{res}_{E_1}(\widetilde{\sigma}_j) = 0$$

if $j > 1$.

Therefore $\widetilde{\sigma}_2, \ldots, \widetilde{\sigma}_n$ induce logarithmic 1-forms on E_1 with respect to the divisor $B = (E_2 + \ldots + E_k + \overline{D})|_{E_1}$ if $E_1, \ldots, E_k$ denote the irreducible components of E. The restrictions of $\widetilde{\xi}_2, \ldots, \widetilde{\xi}_n$ to E_1 are dual to these forms. Therefore, the sheaves $\Omega^{[1]}_{E_1}(\log\lfloor B \rfloor)$ and $\mathcal{T}_{E_1}(-\log\lfloor B \rfloor)$ are free. By [24, Corollary 1] there is a semi-abelian variety T acting on E_1 with $E_1 \setminus \lfloor B \rfloor$ as an open orbit, where T admits a short exact sequence of algebraic groups

$$0 \to (\mathbb{C}^*)^d \to T \to \operatorname{Alb}(E_1) \to 0$$

for some d and where $\operatorname{Alb}(E_1)$ denotes the Albanese variety of E_1. Furthermore, the Lie algebra of T is spanned by the vector fields $\widetilde{\xi}_2|_{E_1}, \ldots, \widetilde{\xi}_n|_{E_1}$.

The flow of each vector field $\widetilde{\xi}_j|_{E_1}$ is global, i.e. we can take $\mathbb{C} \times E_1$ as its domain of definition, and for every relatively compact open subset $U \subset \mathbb{C}$, there is a neighbourhood $V \subset \widetilde{X}$ of E_1 such that the flow φ^j of $\widetilde{\xi}_j$ is defined on $U \times V$, $\varphi^j : U \times V \to \widetilde{X}$, $(t, x) \mapsto \varphi^j(t, x) = \varphi^j_t(x)$.

Let $\mathcal{L} = \mathcal{O}(-E_1)$ denote the line bundle associated with the divisor E_1. We have $\varphi^j_t(E_1) = E_1$ and thus get $(\varphi^j_t)^*(\mathcal{L}|_W) = \mathcal{L}|_V$ for any $t \in \mathbb{C}$ and appropriate neighbourhoods V, W of E_1 in $\widetilde{X}$ with $\varphi^j_t : V \to W$. Consequently, we have $(\varphi^j_t)^*(\mathcal{L}|_{E_1}) = \mathcal{L}|_{E_1}$ for all $t \in \mathbb{C}$ and hence $g^*(\mathcal{L}|_{E_1}) = \mathcal{L}|_{E_1}$ for any $g \in T$.

By a version of the Negativity Lemma as in [5, Proposition 1.6], the line bundle $\mathcal{L}|_{E_1}$ is big. Let $(\mathcal{L}|_{E_1})^{\otimes r}$ be a multiple of the line bundle $\mathcal{L}|_{E_1}$ such that there is an effective divisor F in E_1 with $(\mathcal{L}|_{E_1})^{\otimes r} = \mathcal{O}(F)$. By [20, Proposition 5.5.28] the connected component $\operatorname{St}(F)^0$ of the stabiliser $\operatorname{St}(F)$

is contained in the maximal linear subgroup $(\mathbb{C}^*)^d$ of T. Moreover, the connected components of $\mathrm{St}((\mathcal{L}|_{E_1})^{\otimes r}) = \{t \in T \mid t^*((\mathcal{L}|_{E_1})^{\otimes r}) = (\mathcal{L}|_{E_1})^{\otimes r}\}$ and $\mathrm{St}(F)$ coincide by [20, Lemma 5.5.8]. Therefore, the stabiliser

$$T = \mathrm{St}(\mathcal{L}|_{E_1})^0 = \mathrm{St}(\mathcal{L}|_{E_1}) = \{t \in T \mid t^*(\mathcal{L}|_{E_1}) = \mathcal{L}|_{E_1}\} \subseteq \mathrm{St}\left((\mathcal{L}|_{E_1})^{\otimes r}\right)^0$$

of $\mathcal{L}|_{E_1}$ is contained in the maximal connected linear subgroup $(\mathbb{C}^*)^d$ of T. Thus we conclude that $T = (\mathbb{C}^*)^d$ with $d = n-1$, $\mathrm{Alb}(E_1) = 0$, and E_1 is a toric variety. □

Proof of Lemma 6.4(2). — Let E_1 be any irreducible component of $\pi^{-1}(p)$. Since E_1 is smooth and projective, the action of the torus $T = (\mathbb{C}^*)^{n-1}$ on E_1 has a fixed point $q \in E_1$. The Lie algebra of T is spanned by $\widetilde{\xi}_2|_{E_1}, \dots, \widetilde{\xi}_n|_{E_1}$ and thus we have $\widetilde{\xi}_j(q) = 0$ for all $j > 1$. By construction only $\widetilde{\sigma}_1$ has a pole along E_1 and therefore we necessarily have $\widetilde{\xi}_1|_{E_1} = 0$ for the dual vector field. Moreover, $\widetilde{\xi}_1, \dots, \widetilde{\xi}_n$ span the sheaf $\mathcal{T}_{\tilde{X}}(-\log\lfloor\widetilde{D}\rfloor)$ on some neighbourhood of E_1 and the statement follows. □

Proposition 6.5. — *Let (X, D) be an lc pair whose logarithmic tangent sheaf $\mathcal{T}_X(-\log\lfloor D\rfloor)$ is locally free. Suppose that there is $p \in X$ such that $\xi(p) = 0$ for all logarithmic vector fields ξ defined on some neighbourhood of p.*

Then there exist closed logarithmic 1-forms $\sigma_1, \dots, \sigma_n$ which span the sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ in a neighbourhood of p. In particular, the pair (X, D) is toroidal in a neighbourhood of p by Theorem 5.14.

The proof consists of two main steps. First, we consider a local basis of logarithmic vector fields $\xi_1, \dots, \xi_n$, and consider their lifts $\widetilde{\xi}_1, \dots, \widetilde{\xi}_n$ to a log resolution $(\widetilde{X}, \widetilde{D})$. The statement of the preceding lemma is then used to study their behaviour near a point q where $\widetilde{\xi}_1(q) = \ldots = \widetilde{\xi}_n(q) = 0$ and a version of Poincaré's theorem (see e.g. [1, p. 190]) on the normal form of holomorphic vector fields allows us to modify $\xi_1, \dots, \xi_n$ in such a way that these vector fields are induced by local $\mathbb{C}^*$-actions, or equivalently by S^1-actions, on a neighbourhood of p.

Then, averaging by an appropriate S^1-action yields commuting vector fields $\eta_1, \dots, \eta_n$, which can be shown to still span the sheaf of logarithmic vector fields locally near p. The logarithmic 1-forms dual to $\eta_1, \dots, \eta_n$ are then closed by Lemma 5.6 and yield the desired local basis of closed logarithmic 1-forms.

Proof. — Let $\pi : \widetilde{X} \to X$ be a log resolution of the pair (X, D) as in Lemma 6.4. As before, let E denote the exceptional divisor and $\overline{D}$ the strict transform of D, $\widetilde{D} = E + \overline{D}$.

Since the question is local we may assume again that $\mathcal{T}_X(-\log\lfloor D\rfloor)$ and $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ are free. Let $\xi_1, \ldots, \xi_n$ be a basis of logarithmic vector fields and let $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ denote their lifts to $\widetilde{X}$. By Lemma 6.4, there is a point $q \in \pi^{-1}(p)$ with $\widetilde{\xi}_1(q) = \ldots = \widetilde{\xi}_n(q) = 0$. Since $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ form a basis for $\mathcal{T}_{\tilde{X}}(-\log\lfloor\widetilde{D}\rfloor)$, n irreducible components $\widetilde{D}_{i_1}, \ldots, \widetilde{D}_{i_n}$ of $\lfloor\widetilde{D}\rfloor$ have to meet in q. There exist local coordinates $z_1, \ldots, z_n$ near q such that $q = 0$ and locally $D_{i_l} = \{z_l = 0\}$ for $l = 1, \ldots, n$, and there are local holomorphic functions $a_{kl}(z)$ such that locally

$$\begin{pmatrix} \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_n \end{pmatrix} = A(z) \begin{pmatrix} z_1 \frac{\partial}{\partial z_1} \\ \vdots \\ z_n \frac{\partial}{\partial z_n} \end{pmatrix}$$

for $A(z) = (a_{kl}(z))_{1 \leqslant k,l \leqslant n}$.

We now want to prescribe the linear part of the vector field $\widetilde{\xi}_1$ at the point q such that ξ_1 is conjugated to its linear part and its flow induces a local $\mathbb{C}^*$-action. For this, we substitute $\xi_1, \ldots, \xi_n$ by a invertible linear combination of them such that $A(0)$ is of the form

$$A(0) = \left(\begin{array}{c|ccc} n+1 & n+2 & \cdots & 2n \\ \hline 0 & & & \\ \vdots & & A_0 & \\ 0 & & & \end{array}\right)$$

where A_0 is an invertible $(n-1) \times (n-1)$-matrix.

The linear part of $\widetilde{\xi}_1$ at q is then given by

$$(n+1) z_1 \frac{\partial}{\partial z_1} + \cdots + 2n z_n \frac{\partial}{\partial z_n},$$

and the important point is that its n-tuple of eigenvalues $(n+1, \ldots, 2n)$ is non-resonant (in the sense of [1, Section 22]) and the convex hull of the eigenvalues does not contain 0. Hence we may apply Poincaré's theorem (see e.g. [1, p. 190]) and get that there is a neighbourhood of q on which $\widetilde{\xi}_1$ is biholomorphically conjugated to its linear part $(n+1) z_1 \frac{\partial}{\partial z_1} + \cdots + 2n z_n \frac{\partial}{\partial z_n}$. Moreover, the eigenvalues are all different, which we will need later on.

In a neighbourhood of q the flow $\widetilde{\varphi}^1$ of $\widetilde{\xi}_1$ is given by

$$\left(t, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right) \mapsto \begin{pmatrix} e^{(n+1)t} w_1 \\ \vdots \\ e^{2nt} w_n \end{pmatrix}$$

in appropriate local coordinates $w_1, \ldots, w_n$. Since $\pi^{-1}(p)$ is compact, the vector field $\widetilde{\xi}_1$ induces a global flow on each irreducible component of

$\pi^{-1}(p)$, and there is an open connected neighbourhood $V \subseteq X$ of p such that $\widetilde{\varphi}^1$ can be defined on $U \times \pi^{-1}(V)$, where

$$U = \{t \in \mathbb{C} \,|\, |\operatorname{Re}(t)| < 1,\ |\operatorname{Im}(t)| < 4\pi\},$$

and the flow φ^1 of ξ_1 is defined on $U \times V$, and we have a commutative diagram:

$$\begin{tikzcd} U \times \pi^{-1}(V) \arrow[r, "\widetilde{\varphi}^1"] \arrow[d, "\mathrm{id} \times \pi"'] & \widetilde{X} \arrow[d, "\pi"] \\ U \times V \arrow[r, "\varphi^1"] & X \end{tikzcd}$$

Locally near q we have $\widetilde{\varphi}^1(2\pi i, w) = w$ and by the identity principle we thus get $\widetilde{\varphi}^1(2\pi i, y) = y$ for any $y \in \pi^{-1}(V)$ and moreover $\varphi^1(2\pi i, x) = x$ for any $x \in V$. Hence the flow map φ^1 induces a local $\mathbb{C}^*$-action and we may define an S^1-action $\chi^1 : S^1 \times V \to V$ on V (after possibly shrinking V) by setting $\chi^1(e^{is}, x) = \varphi^1(is, x)$ (as explained in the proof of Lemma 5.9) which induces ξ_1. Moreover, $\widetilde{\varphi}^1$ gives rise to an S^1-action $\widetilde{\chi}^1 : S^1 \times \pi^{-1}(V) \to \pi^{-1}(V)$ which induces the vector field $\widetilde{\xi}_1$.

We now want to use the S^1-action $\chi^1 : U \times V \to V$ to average the other vector fields ξ_j and obtain commuting vector fields. For this purpose we define vector fields

$$\xi_j' = \int_{S^1} (\chi_s^1)_*(\xi_j)\, d\mu(s)$$

for $j \geqslant 2$, where μ denotes the unique normalised Haar measure on S^1, we write χ_s^1 for $\chi^1(s, \cdot)$, and the push-forward $(\chi_s^1)_*(\xi_j)$ of the vector field ξ_j is as usually defined by

$$\left(\chi_s^1\right)_* (\xi_j)(f)(x) = \xi_j \left(f \circ \chi_s^1\right) \left(\chi_{s^{-1}}^1(x)\right)$$

for any $x \in V$ and local holomorphic function f. The vector fields ξ_j' are all logarithmic with respect to D since the S^1-action χ^1 stabilises each irreducible component D_i of $\lfloor D \rfloor$. Moreover, for any $t \in S^1$ we have

$$\begin{aligned} \left(\chi_t^1\right)_* \left(\xi_j'\right) &= \left(\chi_t^1\right)_* \int_{S^1} \left(\chi_s^1\right)_* (\xi_j)\, d\mu(s) = \int_{S^1} \left(\chi_t^1\right)_* \left(\chi_s^1\right)_* (\xi_j)\, d\mu(s) \\ &= \int_{S^1} \left(\chi_{st}^1\right)_* (\xi_j)\, d\mu(s) = \xi_j' \end{aligned}$$

due to the invariance of the Haar measure. This implies $(\varphi_t^1)_*(\xi_j') = \xi_j'$ for any $t \in \mathbb{C}$ in a neighbourhood of 0. Consequently, the vector fields ξ_1 and ξ_j' commute:

$$\left[\xi_1, \xi_j'\right] = -\left.\frac{d}{dt}\right|_{t=0} \left(\varphi_t^1\right)_* (\xi_j') = -\left.\frac{d}{dt}\right|_{t=0} \xi_j' = 0$$

Next, we prove that $\xi_1', \ldots, \xi_n'$ (where we set $\xi_1' = \xi_1$) still form a local basis for the logarithmic tangent sheaf $\mathcal{T}_X(-\log\lfloor D\rfloor)$ near p and that $\xi_1', \ldots, \xi_n'$ are pairwise commuting. In order to do so, we consider the lifts of ξ_j' to $\pi^{-1}(V)$, which are given by

$$\widetilde{\xi}_j' = \int_{S^1} \left(\widetilde{\chi}_s^1\right)_* \left(\widetilde{\xi}_j\right) \, d\mu(s),$$

and analyse them near the point q. Recall that in appropriate coordinates $w_1, \ldots, w_n$ with $w_j(q) = 0$ we have $\widetilde{\xi}_1' = \widetilde{\xi}_1 = (n{+}1)w_1\frac{\partial}{\partial w_1}+\ldots+2nw_n\frac{\partial}{\partial w_n}$ near q. Let $b_{jkl}(w)$ be holomorphic functions defined locally near $q = 0$ such that

$$\widetilde{\xi}_j = \sum_{k,l=1}^{n} b_{jkl}(w) w_k \frac{\partial}{\partial w_l}$$

for $j \geqslant 2$.

The definition of the S^1-action $\widetilde{\chi}_s^1$ via the flow map $\widetilde{\varphi}^1$ of ξ_1 yields that with respect to the local coordinates $w_1, \ldots, w_n$ the map $\widetilde{\chi}_s^1$ is given by

$$\widetilde{\chi}_s^1(w) = \widetilde{\chi}_s^1 \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} s^{(n+1)} w_1 \\ \vdots \\ s^{2n} w_n \end{pmatrix}$$

Therefore, we have

$$\begin{aligned}\left((\widetilde{\chi}_s^1)_*\widetilde{\xi}_j\right)(w_l) &= \widetilde{\xi}_j\left(w_l \circ \widetilde{\chi}_s^1\right) \circ \chi_{s^{-1}}^1 = \widetilde{\xi}_j\left(s^{n+l} w_l\right) \circ \chi_{s^{-1}}^1 \\ &= \left(\sum_{k=1}^{n} b_{jkl} \begin{pmatrix} s^{-(n+1)} w_1 \\ \vdots \\ s^{-2n} w_n \end{pmatrix}\right) w_l,\end{aligned}$$

where we used the specific form of $\widetilde{\xi}_j$ in the local coordinates. Since the exponents of s in the expression

$$\begin{pmatrix} s^{-(n+1)} w_1 \\ \vdots \\ s^{-2n} w_n \end{pmatrix}$$

are all negative integers and remembering that

$$\int_{S^1} s^r d\mu(s) = \int_0^{2\pi} e^{i\theta r} \frac{d\theta}{2\pi} = 0$$

for all integers $r \neq 0$, we get

$$\widetilde{\xi}_j'(w_l) = \int_{S^1} \left((\widetilde{\chi}_s^1)_*\widetilde{\xi}_j\right)(w_l) d\mu(s) = \sum_{k=1}^{n} b_{jkl}(0) w_l,$$

also keeping in mind that the b_{jkl} are holomorphic functions in a neighbourhood of $q = 0$. These local calculations thus yield that $\widetilde{\xi}'_j$ is linear (with respect to the coordinates $w_1, \ldots, w_n$) and of the form

$$\widetilde{\xi}'_j = \sum_{k,l=1}^{n} b_{jkl}(0) w_k \frac{\partial}{\partial w_l}.$$

Moreover, since all eigenvalues of $\widetilde{\xi}_1 = \widetilde{\xi}'_1$ at $q = 0$ are different and $\widetilde{\xi}'_j$ and $\widetilde{\xi}'_1$ commute we get that $b_{jkl}(0) = 0$ if $k \neq l$ and hence $\widetilde{\xi}'_j$ is of the form

$$\widetilde{\xi}'_j = \sum_{k=1}^{n} b_{jk} w_k \frac{\partial}{\partial w_k}$$

for some constants b_{jk}. In particular, we see now that the vector fields $\widetilde{\xi}'_j$ are all pairwise commuting near q, thus by the identity principle on all of $\pi^{-1}(V)$ and consequently $\xi'_1, \ldots, \xi'_n$ are also pairwise commuting vector fields.

Moreover, we have

$$\begin{pmatrix} \tilde{\xi}'_1 \\ \vdots \\ \tilde{\xi}'_n \end{pmatrix} = \widetilde{C}(w) \begin{pmatrix} \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_n \end{pmatrix}$$

for some matrix $\widetilde{C}(w)$ whose entries $\widetilde{c}_{jk}(w)$ are local holomorphic functions and which satisfies $\widetilde{C}(q) = \widetilde{C}(0) = E_n$ since we have

$$\widetilde{\xi}'_1 = \widetilde{\xi}_1 \quad \text{and} \quad \widetilde{\xi}'_j(0) = \sum_{k=1}^{n} b_{jk} w_k \frac{\partial}{\partial w_k} = \sum_{k,l=1}^{n} b_{jkl}(0) w_k \frac{\partial}{\partial w_l} = \widetilde{\xi}_j(0)$$

for $j \geqslant 2$ at the point $q = 0$.

Since $\xi_1, \ldots, \xi_n$ form a basis of logarithmic vector fields on X, we have

$$\begin{pmatrix} \xi'_1 \\ \vdots \\ \xi'_n \end{pmatrix} = C(x) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

for a matrix $C(x)$ whose entries $c_{jk}(x)$ are holomorphic functions on a neighbourhood of p. Using that $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ are the lifts of $\xi_1, \ldots, \xi_n$ and $\widetilde{\xi}'_1, \ldots, \widetilde{\xi}'_n$ the lifts of $\xi'_1, \ldots, \xi'_n$, we get $C(\pi(y)) = \widetilde{C}(y)$ on a neighbourhood of $q \in \widetilde{X}$ and in particular $C(p) = C(\pi(q)) = \widetilde{C}(q) = E_n$ is invertible. Hence $\xi'_1, \ldots, \xi'_n$ also form a local basis of logarithmic vector fields on X near q. These vector fields $\xi'_1, \ldots, \xi'_n$ commute and their dual logarithmic 1-forms $\sigma_1, \ldots, \sigma_n$ are thus closed (cf. Lemma 5.6). □

The statement of the next Lemma 6.6 on hyperplane sections will be useful when reducing the case of an lc pair (X, D) with locally free sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ to the isolated case as in Proposition 6.5. For more details on hyperplane sections and their properties relevant to our setting the reader is referred to [10, Section 2.E].

LEMMA 6.6. — *Let (X, D) be an lc pair, $D_1, \dots D_k$ the irreducible components of D, $D = \sum_i a_i D_i$, and let H be a general member of an ample basepoint free linear system on X and assume that the sheaf $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is locally free. Then $\Omega_H^{[1]}(\log\lfloor D\rfloor|_H)$ is locally free.*

Moreover, for any point $p \in H$ there is a logarithmic vector field of (X, D) defined on a neighbourhood U of p in X which does not vanish on this neighbourhood and is transversal to the hyperplane H all points $q \in U \cap H$.

Remark 6.7. — By [10, Lemma 2.23] the divisor H is normal and irreducible, and the intersections $D_j \cap H$ are all distinct. Therefore, $(H, D|_H)$ with $D|_H = a_1(D_1 \cap H) + \ldots + a_k(D_k \cap H)$ is a pair, and $(H, D|_H)$ is lc if (X, D) is lc; see [10, Lemma 2.25].

Proof of Lemma 6.6. — Since the question is local, we may assume that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is free and H is given by the reduced equation $h = 0$ for a regular function h on X.

Let $\pi : \widetilde{X} \to X$ be a log resolution of (X, D) and let $\widetilde{H} = \pi^{-1}(H)$. By [10, Lemma 2.24] the restricted morphism $\pi|_{\tilde{H}} : \widetilde{H} \to H$ is a log resolution of the pair $(H, D|_H)$, and the exceptional sets $\mathrm{Exc}(\pi)$ of π and $\mathrm{Exc}(\pi|_{\tilde{H}})$ of $\pi|_{\tilde{H}}$ satisfy $\mathrm{Exc}(\pi|_{\tilde{H}}) = \mathrm{Exc}(\pi) \cap \widetilde{H}$.

Let $\sigma_1, \dots, \sigma_n$ be a basis of logarithmic 1-forms on (X, D), and let $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_n$ denote their lifts to $\widetilde{X}$. The hyperplane $\widetilde{H} \subset \widetilde{X}$ is given by the reduced equation $\widetilde{h} = h \circ \pi = 0$. Let $\alpha_1, \dots, \alpha_n$ be regular functions on X such that

$$dh = \sum_{j=1}^{n} \alpha_j \sigma_j \quad \text{and} \quad d\widetilde{h} = \sum_{j=1}^{n} (\alpha_j \circ \pi) \widetilde{\sigma}_j.$$

After possibly shrinking X, the 1-form $d\widetilde{h}$ has no zeroes and there is j, say $j = 1$, with $\alpha_j(\pi(y)) = \alpha_1(\pi(y)) \neq 0$ for all $y \in \widetilde{X}$. Consequently, we may assume $d\widetilde{h} = \widetilde{\sigma}_1$ and $dh = \sigma_1$ without loss of generality.

Let H° be largest open subset of H such that $(H^\circ, D|_{H^\circ})$ is snc. Then (X, D) is snc along H° and the restrictions of $\sigma_2, \dots, \sigma_n$ to logarithmic forms $\sigma_2|_H, \dots, \sigma_n|_H$ in $\Omega_H^{[1]}(\log\lfloor D\rfloor|_H)$ are well-defined. On H^0 we have an exact sequence

$$0 \to \mathcal{O}_{H^\circ}\langle h\rangle \to \Omega_X^{[1]}(\log\lfloor D\rfloor)|_{H^\circ} \to \Omega_H^{[1]}(\log\lfloor D\rfloor|_H)|_{H^\circ} \to 0,$$

where the kernel of the morphism $\Omega_X^{[1]}(\log\lfloor D\rfloor)|_{H^\circ} \to \Omega_H^{[1]}(\log\lfloor D\rfloor|_H)|_{H^\circ}$ is generated by $dh = \sigma_1$. Consequently, the forms $\sigma_2|_{H^\circ}, \ldots, \sigma_n|_{H^\circ}$ form a basis of $\Omega_H^{[1]}(\log\lfloor D\rfloor|_H)|_{H^\circ}$ and hence $\sigma_2|_H, \ldots, \sigma_n|_H$ also form a basis of $\Omega_H^{[1]}(\log\lfloor D\rfloor|_H)$ since $H \setminus H^\circ$ has at least codimension 2. Therefore, $\Omega_H^{[1]}(\log\lfloor D\rfloor|_H)$ is locally free.

Let $\xi_1, \ldots, \xi_n$ be the dual vector fields to the logarithmic 1-forms $\sigma_1, \ldots, \sigma_n$. By construction we have

$$1 = \sigma_1(\xi_1) = dh(\xi_1) = \xi_1(h).$$

Hence $\xi_1(q) \neq 0$ for all q on a neighbourhood of H. Moreover, ξ_1 is transversal to H in every point since $H = \{h = 0\}$ and $\xi_1(h) = 1 \neq 0$ and ξ_1 thus is a vector field as required in the statement of the Lemma 6.6. □

Theorem 6.8. — *Let (X, D) be an lc pair such that $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ is locally free. Then $(X, \lfloor D\rfloor)$ is toroidal.*

Proof. — Let $Z \subset X$ be the smallest closed analytic subset such that the pair $(X \setminus Z, \lfloor D\rfloor|_{X\setminus Z})$ is toroidal. We shrink X such that $\mathcal{T}_X(-\log\lfloor D\rfloor)$ and $\Omega_X^{[1]}(\log\lfloor D\rfloor)$ are free and Z is connected. We now want to do induction on the dimension of Z.

If Z is 0-dimensional, then Z consists of a single point $Z = \{p\}$, and the statement of the theorem is the content of Proposition 6.5. Assume now that $m = \dim Z$ and that the theorem is proven for those pairs (X', D') such that the non-toroidal locus Z' of the pair $(X', \lfloor D'\rfloor)$ has $\dim Z' < m$. Let H be a general hyperplane section of an ample basepoint free linear system on X as described in Lemma 6.6. Then $(H, D|_H)$ is lc, $\Omega_H^{[1]}(\log\lfloor D|_H\rfloor)$ is locally free and $\dim(Z \cap H) = \dim Z - 1 = m - 1 < m$. Hence $(H, \lfloor D|_H\rfloor)$ is toroidal by the induction hypothesis.

Let $p \in Z$ such that $p \in H \cap Z$ and let ξ be a vector field on a neighbourhood of p that is transversal to H, which exists by the second part of the statement of Lemma 6.6. We may assume that $X \subset \mathbb{A}^n$ and H is the intersection of a smooth divisor $\widehat{H}$ and X. The vector field ξ extends to a holomorphic vector field $\widehat{\xi}$ on an open neighbourhood of $p \in X \subset \mathbb{A}^n$ in $\mathbb{A}^n$. Let $\widehat{\varphi} : \Omega \to \mathbb{A}^n$, $\Omega \subseteq \mathbb{C} \times \mathbb{A}^n$, denote the flow map of $\widehat{\xi}$. Since ξ is not tangent to H at p, $\widehat{\xi}$ is not tangent to $\widehat{H}$ at p and the flow $\widehat{\varphi}$ induces a morphism $\chi : U \times \widehat{H} \to \mathbb{A}^n$, $(t, q) \mapsto \widehat{\varphi}(t, q)$, where U is an open subset of $\mathbb{C}$ with $0 \in U$, such that χ is biholomorphic near p. Moreover, we have $\chi(U \times H) \subseteq X$ by construction, and thus we get that $U \times H$ and X

are biholomorphic near p. In particular, it follows that X is toroidal in a neighbourhood of p, which is a contradiction to our assumption that p is contained in the non-toroidal subset Z of X. □

A version of Theorem 6.8 for Du Bois pairs can now directly be deduced by applying the results of [7]. For definitions and a detailed discussion of Du Bois pairs the reader is referred to [17, Chapter 6].

Corollary 6.9. — *Let X be a normal quasi-projective variety and $\Sigma \subsetneq X$ a reduced closed subscheme such that (X, Σ) is a Du Bois pair. Let Σ_{div} denote the largest reduced divisor whose support is contained in Σ. Assume that $\Omega_X^{[1]}(\log \Sigma_{\mathrm{div}})$ is locally free. Then $(X, \Sigma_{\mathrm{div}})$ is toroidal.*

Proof. — It is again enough to consider the case where $\mathcal{T}_X(-\log \Sigma_{\mathrm{div}})$ and $\Omega_X^{[1]}(\log \Sigma_{\mathrm{div}})$ are free. Then the twisted canonical sheaf $\omega_X(\Sigma_{\mathrm{div}}) \cong \Omega_X^{[n]}(\log \Sigma_{\mathrm{div}})$, $n = \dim X$, is also free, which implies that the divisor $K_X + \Sigma_{\mathrm{div}}$ is linearly equivalent to 0, where K_X denote a canonical divisor of X. In particular, $K_X + \Sigma_{\mathrm{div}}$ is Cartier. Therefore, the pair $(X, \Sigma_{\mathrm{div}})$ is lc by [7, Theorem 1.4.2] and $(X, \Sigma_{\mathrm{div}})$ is toroidal by Theorem 6.8. □

Remark 6.10. — Alternatively, the statement of Corollary 6.9 could be proven along the lines as the statement for lc pairs noting that extension of logarithmic forms to log resolutions and the cutting down procedure via hyperplanes also work for Du Bois pairs by [7, Theorem 4.1 and Lemma 4.4].

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