ANNALES DE L'INSTITUT FOURIER

Misha GROMOV

Mean Curvature in the Light of Scalar Curvature

MEAN CURVATURE IN THE LIGHT OF SCALAR CURVATURE

by Misha GROMOV

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ABSTRACT. — We formulate several conjectures on mean convex domains in the Euclidean spaces, as well as in more general spaces with lower bounds on their scalar curvatures, and prove a few theorems motivating these conjectures

RÉSUMÉ. — Nous formulons plusieurs conjectures sur les domaines à bords de courbure moyenne positive dans l'espace euclidiens ainsi que dans des espaces plus généraux de courbure scalaire minorée. Nous prouvons quelques théorèmes qui étayent ces conjectures.

We think of *scalar curvature* as a Riemannin incarnation of *mean curvature*[1] and we search for constraints on global geometric invariants of n-spaces X with $\mathrm{Sc}(X) > 0$ that would generalise those for smooth *mean convex* domains $X \subset \mathbb{R}^n$ also called domains with *mean convex* boundaries $Y = \partial X$, i.e. with $\mathrm{mean.curv}(\partial Y) > 0$.[2]

And, as an unexpected bonus of this search, we find out that techniques developed for the study of manifolds X with $\mathrm{Sc}(X) \geqslant \sigma > 0$ yield new results for hypersurfaces $Y \subset \mathbb{R}^n$ with $\mathrm{mean.curv} \geqslant \mu > 0$.

In what follows we briefly overview of what is known and what is unknown in this regard.[3]

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Keywords: mean curvature, scalar curvature.

[1] Throughout the paper, we use the standard normalisation, where the unit sphere $S^{n-1} \subset \mathbb{R}^n$ has $\mathrm{mean.curv}(S^n) = n - 1$, the Ricci curvature of S^n is $n - 1$ and the scalar curvature of S^n is $n(n-1)$.

[2] This is similar in spirit to parallelism between spaces X with positive sectional curvatures and convex subsets in $\mathbb{R}^n$, the best instance of which is Perelman's double sided bound on the product of the n *Uryson widths* of an X by $\mathrm{const}_n^{\pm 1} \cdot \mathrm{vol}(X)$.

[3] Part of this article is a slightly edited extract from my unfinished paper [8].

1. Inscribed Balls and Distance Decreasing Maps

Let us recall classical comparison theorems between radii of balls in manifolds X with lower bounds on their Ricci curvatures by ς and on the mean curvatures of their boundaries by μ and the radii $R = R(n, \kappa, \mu)$ of the balls $B = B^n(\kappa, \mu)$ with mean.curv$(\partial B) = \mu$ in the standard (complete simply connected) n-spaces X_κ^n with sectional curvatures $\kappa = \varsigma/(n-1)$,[4] which go back to the work by Paul Levy, S. B. Myers and Richard L. Bishop.

Let X be a (metrically) complete Riemannian n-manifold with a boundary, denoted $Y = \partial X$, such that $\mathrm{Ricci}(X) \geqslant \varsigma = (n-1)\kappa$, i.e. $\mathrm{Ricci}(X) \geqslant \varsigma \cdot g_X$, and mean.curv$_\partial(Y) \geqslant \mu$.

For instance, X may be a smooth, possibly unbounded domain in the Euclidean space $\mathbb{R}^n$, that is a closed subset bounded by a smooth hypersurface $Y \subset \mathbb{R}^n$.

Inball^n-INEQUALITY. — *The in-radius of X, that is*

$$\mathrm{Rad}_{in}(X) = \sup_{x \in X} \mathrm{dist}(x, \partial X),$$

is bounded by the radius $R = R(n, \kappa, \mu)$ of the ball $B^n(\kappa, \mu)$ with

$$\mathrm{mean.curv}(\partial B^n(\kappa, \mu)) = \mu$$

in the standard n-space X_κ^n of constant sectional curvature $\kappa = \varsigma/(n-1)$.[5]

Indeed, the normal exponential map to ∂Y necessarily develops conjugate points on geodesic segments normal to ∂Y of length $> R(n, \kappa, \mu)$.

Inball^n-EQUALITY. — *If $\mathrm{Rad}_{in}(X) = R = R(n, \kappa, \mu)$, then X, assuming it is connected, is isometric to an R-ball in X_κ^n.*

This is proven by fiddling at the boundary points of the regions in ∂X, where the maximal in-ball meets ∂X.

S^{n-1}-EXTREMALITY/RIGIDITY COROLLARY. — *Let $X \subset X_\kappa^n$ be a compact connected domain with smooth connected boundary $Y = \partial X$.*

Let the mean curvature of Y be bounded from below by $\mu \geqslant 0$ and let

$$f : Y \to S^n = S^{n-1}(\kappa, \mu) = \partial B^n(\kappa, \mu)$$

be a distance non-increasing map for the distances in $Y \subset X_\kappa^n$ and in $S^{n-1}(\kappa, \mu) \subset X_\kappa^n$ induced from that in the ambient (standard) space X_κ^n.

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[4] If $\kappa \leqslant 0$, then the standard balls may only have $\mu \geqslant (n-1)\sqrt{-\kappa}$.
[5] If no such ball exists, e.g. if $\kappa \geqslant -1$ and $\mu = -(n-1)$ then, by definition, $R(n, \varsigma, \mu) = \infty$.

Extr_{mn}: EXTREMALITY OF S^{n-1}. — *If the map f is strictly distance decreasing then f is contractible.*

Rigd_{mn}: RIGIDITY OF S^{n-1}. — *If f is non-contractible, then it is an isometry.*

Proof. — Use *Kirszbraun's theorem*[6] and extend map f to a distance non-increasing map $F : X \to B = B^n(\kappa, \mu)$.

If f has non-zero degree then so does F, hence, the center of B is in the image and the pullback of this center lies within distance $\geqslant R(n, \kappa, \mu)$ from the boundary of X and the above Inball^n-inequality and Inball^n-equality apply. $\square$

Remarks.
  (a) If X is isometrically realised by a domain in the Euclidean or in the hyperbolic n-space and $\mathrm{dist}(x_0, \partial X) \geqslant R$, then the normal projection to the ball, $X \to B_{x_0}(R) \subset X$, is a distance non-increasing map of degree 1.
  (b) An essential drawback of Extr_{mn} and Rigd_{mn} is an appeal to the *extrinsic metric*, that is the Euclidean distance function restricted from $\mathbb{R}^n$ to Y and to S^{n-1}, rather than to the *intrinsic metrics* in Y and S^{n-1} associated with the *Riemannian metrics/tensors* induced from $\mathbb{R}^n$, where, observe, the intrinsic metric in Y may be incomparably greater than the extrinsic one.

Yet, we shall it in the next section, Extr_{mn} and Rigd_{mn} remain valid for the intrinsic metrics but the proof of these relies on *Dirac operators* on manifolds with *positive scalar curvatures* with no direct approach in sight. (The proof of the intrinsic Rigd_{mn} is slightly more elaborate than that of Extr_{mn}.)

The following refinements(s) of the Inball^n-inequality/equality in the standard spaces of constant curvature, e.g. in $\mathbb{R}^n$, must be also well known. I apologies to the author(s) whose paper(s) I failed to locate on the web.

Inball^{n-1}-INEQUALITY. — *Let X be a smooth domain with mean.curv$(\partial X) \geqslant \mu$ in the standard n-space X_κ^n with the sectional curvature κ and let X contain a flat R-ball B of dimension $(n-1)$, that is an R-ball in a totally geodesic hypersurface in X_κ^n.*

If $R \geqslant R(\kappa, \mu)$ then, in fact, $R = R(\kappa, \mu)$ and X is equal to an R-ball in X_κ^n.

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[6] https://en.wikipedia.org/wiki/Kirszbraun_theorem

Proof. — Let $B_\circ(r) \supset B$ be the lens-like region between two spherical caps of hight $r \leqslant R$ and with the boundaries ∂B. If $R > 1$, the mean curvatures of these caps are $< \mu$; hence they do not meet ∂X which makes the R-ball $B^n(R) = B_\circ(R)$ contained in Y and the above two "Inball^n" apply.   □

Intersection Remark. — The above argument also shows that if $X \subset \mathbb{R}^n$ with mean.curv$(\partial X) \geqslant n - 1$ contains a flat $(n-1)$-ball, $B_x^{n-1}(r) \subset X$, of radius $r < 1$, then the distance from the center of this ball to the boundary of X is bounded from below by dist$(x, \partial X) > \frac{r^2}{2}$. In words: If X is r-thin at a point $x \in X$, then the intersections of X with hyperplanes passing through x are *at most $\sqrt{2r}$-thick at this point.*

In fact, the same apply to intersections of X with arbitrary hypersurfaces $S \subset \mathbb{R}^n$ where relevant constants now depend on the bounds on the principal curvatures of S.

An instance of such an S, which we shall meet later in Section 3, is that of the sphere $S = S_x^{n-1}(1 + \delta)$, $x \in X$, where the existence of an $(n-1)$-disc of radius $r > 20\sqrt{\delta}$ in $S \cap X$ for this S and small $\delta > 0$ necessities the existence of a ball

$$B_{x'}^n(2\delta) \subset X \cap B_x^n(1 + \delta)$$

such that

$$\text{dist}(x', x) = 1 - \delta.$$

Inball^{n-1} MAPPING COROLLARY. — *Let $X \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $Y = \partial X$, such that* mean.curv$(\partial X) \geqslant n - 1$ *and let $Y_+ \subset Y$ be the intersection of Y with the half subspace $\mathbb{R}_+^n = \{x_1, \ldots, x_i, \ldots, x_n\}_{x_1 \geqslant 0} \subset \mathbb{R}^n$. Then*

Y_+ admits no proper distance decreasing map with non-zero degree to the unit ball $B^{n-1}(1) \subset \mathbb{R}^{n-1}$, where "distance decreasing" refers to the Euclidean distance on $Y_+ \subset \mathbb{R}^n$ and "proper" signifies that $\partial Y_+ \to S^{n-2} = \partial B^{n-1}(1)$.

MAXIMAL PRINCIPLE FOR PRINCIPAL CURVATURES. — *The maximum principle argument, which was used for the proof of the strict inball inequalities (but not their extremal equality cases!) and their corollaries for hypersurfaces Y in $\mathbb{R}^n$ with* mean.curv$(Y) \geqslant n - 1$, *trivially extends to hypersurfaces $Y \subset \mathbb{R}^n$, where the maxima of principal curvatures are $\geqslant 1$ at all points $y \in Y$, and the same applies to hypersurfaces in the standard spaces X_κ^n with constant sectional curvatures κ for positive and negative κ as well.*

What is more interesting is the following, probably known, simple generalisation of the above Inball^{n-1}-mapping corollary derived with this "principle".

SHARP BOUND ON THE FILLING RADIUS IN CODIMENSION 2. — *Let $Y_0^{n-1} \subset \mathbb{R}^n$ be a smooth cooriented hypersurface with connected boundary $Z^{n-2} = \partial Y_0$, such that the maxima of the principal curvatures of Y_0 satisfy*

$$\text{max.curv}(Y_0, y) \geqslant 1 \text{ for all } y \in Y_0.$$

Then Z^{n-2} bounds a submanifold[7] $Y_1^{n-1} \subset \mathbb{R}^n$, i.e. $\partial Y_1^{n-1} = Z^{n-2}$, such that

$$\text{dist}_{\mathbb{R}^n}(y, Z^{n-2}) \leqslant 1 \text{ for all } y \in Y_1^{n-1}.$$

Consequently, Z^{n-2} admits no distance decreasing map with non-zero degree to the unit sphere $S^{n-2}(1) \subset \mathbb{R}^{n-2}$ with non-zero degree, where "distance decreasing" refers to the Euclidean distances restricted to $Z^{n-2} \subset \mathbb{R}^n$ and to $S^{n-2} \subset \mathbb{R}^{n-1}$.

Proof. — If Z^{n-2} doesn't bound in its ρ-neighbourhood in $\mathbb{R}^n$ then, by the *Alexander duality* there exists a simple curve $C \subset \mathbb{R}^n$, with both ends going to infinity in $\mathbb{R}^n$, which is *non-trivially linked with Z* and such that

$$\text{dist}_X(Z, C) > \rho.$$

We claim that there is a point $c \in C$ such that the ρ-sphere with the center c, say $S_c^{n-1}(\rho)$ is tangent to Y_0 at some point $y_0 \in Y_0^{n-1}$, where, this sphere is locally positioned "inside" Y_0 relative to the given coorientation of Y_0.

To see this let $\tau : S^{n-1}(\rho) \times C \to \mathbb{R}^n$ be the map (tautologically) defined via the identification of the sphere $S^{n-1}(\rho) = S_0^{n-2}(\rho) \subset \mathbb{R}^{n-1}$ with all $S_c^{n-1}(\rho)$ by parallel translations and let us assume this map is transversal to Y_0.

Then the τ-pullback of Y_0,

$$\Sigma = \tau^{-1}(Y_0) \subset S^{n-1}(\rho) \times C$$

is a smooth hypersurface in $S^{n-1}(\rho) \times C$, the projection of which to $S^{n-1}(\rho)$ has non-zero degree, namely the degree equal the linking number between Z^{n-2} and C.

It follows, that there is a connected component, say $Z_0 \subset Z$ which separates the two ends of the cylinder $S^{n-1} \times C$, where one of the ends is regarded as "internal" with respect to the coorientation of Y_0 and the other one as "external".

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[7] A priori, this Y_0 may be singular, but it can be made smooth for $\text{codim}(Y_0) = 1$.

Then we let $(s, c)_0$ be the minimum point of the function $(s, c) \mapsto c \in C = \mathbb{R}$ restricted to the "external" connected component of the complement to Σ_0 in the cylinder $S^{n-1}(\rho) \times C$ and observe that

$$\tau((s, c)_0) \in Y_0 \cap S_c^{n-1}(\rho)$$

serves as the desired "internal kissing point" of the sphere $S_c^{n-1}(\rho)$ with Y_0.

Finally, by the maximal principal, all principal curvatures of Y_0 at this "kissing point" are bounded by $\frac{1}{\rho}$ and the proof is concluded.   □

Remark. — The above automatically generalises to the standard spaces with constant curvatures and to many spaces with variable curvatures.

Also there is the following (rather trivial) version of this for submanifolds $Z^{n-k} \subset \mathbb{R}^n$ of codimensions $k \geqslant 3$: If Z^{n-k} doesn't bound in its ρ-neighbourhood, and if $Y_0^{n-k+1} \subset \mathbb{R}^n$ has $\partial Y_0^{n-k+1} = Z^{n-k}$, then there exists a normal vector ν_0 to Y_0 at some point, such that all principal curvatures in the direction of ν at this point are bounded by $\frac{1}{\rho}$.

2. Manifolds with Lower Bounds on their Scalar curvatures and on the Mean Curvatures of their Boundaries

An essential link between positive mean and positive scalar curvatures is furnished by an elementary observation [6] that the natural *continuous* Riemannin metric on the double $X \cup_Y X$ of a domain $\subset \mathbb{R}^n$ with boundary $Y = \partial X$ with *positive mean curvature* can be approximated by smooth metrics with *positive scalar curvatures*

This leads (see [12, Section 3.6 and 4.3]) to the following "intrinsic" improvement of the above "extrinsic extremality" of the Euclidean spheres.

$\odot_{mn}^{\to S^{n-1}}$ SHARP BOUND ON $\mathrm{Rad}_{S^{n-1}}(Y \subset \mathbb{R}^n)$. — *Let $Y \subset \mathbb{R}^n$ be a closed hypersurface and let f be a Lipschitz map of Y to the unit sphere,*

$$f : Y \to S^{n-1} = S^{n-1}(1) \subset \mathbb{R}^n.$$

If mean.curv$(Y) \geqslant n-1$ *and if f strictly decreases the lengths of the curves in Y, then f is contractible.*

In fact, this is a corollary to the following result derived in [11] from Goette–Semmelmann's estimates [2] for *twisted Dirac operators.*

$\odot_{\mathrm{Sc} \geqslant 0}^{\to S^{n-1}}$ SHARP BOUND ON $\mathrm{Rad}_{S^{n-1}}(Y = \partial X)_{\mathrm{Sc}(X) \geqslant 0}$. — *If a Riemannin manifold Y serves as a boundary with mean curvature $\geqslant n-1$ of a*

compact Riemannian spin n-manifold X with non-negative scalar curvature, then Y admits no non-contractible map $f : Y \to S^{n-1}$ which strictly decreases the lengths of curves in Y.

RIGIDITY OF SPHERES AND BALLS. — By using (a slightly generalised) Alexandrov's theorem on sphericity of closed hypersurfaces with constant mean curvature in $\mathbb{R}^n$, one can show that non-contractible maps of smooth closed hypersurfaces $Y \subset \mathbb{R}^n$ with mean.curv$(Y) \geqslant n-1$ to the unit sphere $S^n = S^n(1) \subset \mathbb{R}^{n+1}$,

$$f : Y \to S^{n-1},$$

which do not increase the length of curves, are isometric.

Moreover, compact Riemannian spin manifolds X with $\mathrm{Sc}(X) \geqslant 0$, the boundaries Y of which admit non-contractible length non-increasing maps to S^{n-1}, are isometric to the unit balls $B^n(1)\mathbb{R}^n$, but the proof of this needs an additional bit of reasoning.

However, the following remains problematic.

QUESTION 2.1. — Is there a direct elementary proof of $\odot^{\to S^{n-1}}_{mn}$?

QUESTION 2.2. — Does $\odot^{\to S^{n-1}}_{mn}$ generalise to hypersurfaces Y in the standard spaces X^n_κ with sectional curvatures $\kappa \neq 0$?

For instance, do minimal hypersurfaces $Y \subset S^n$ with their intrinsic (induces Riemannin) metrics have $Rad_{S^{n-1}}(Y) \leqslant 1$?

QUESTION 2.3. — Is the spin condition in $\odot^{\to S^{n-1}}_{\mathrm{Sc}\geqslant 0}$ essential?

QUESTION 2.4. — Is there a sharp version of the Inball^{n-1}-mapping corollary to the intrinsic metric in Y?

Namely, let $Y \subset \mathbb{R}^n$ be a smooth closed hypersurface with mean curvature $\geqslant n - 1$ and let $Z \subset Y$ be a hypersurface, $\dim(Z) = n - 2$, which divides Y in two halves, say $Y_- \subset Y$ and $Y_+ \subset Y$.

Does this $Z = \partial Y_- = \partial Y_+$ admit a non-contractible distance decreasing map to $S^{n-2} = \partial S^{n-1}_- = \partial S^{n-1}_+$, where the distance in Z is induced from the Riemannian distance in Y and the distance in S^{n-2} is the intrinsic spherical one (which is equal to the distance coming from the ambient sphere $S^{n-1} \supset S^{n-2}$)?

Notice in this regard that $\odot^{\to S^{n-1}}_{mn}$ implies a non-sharp version of this, due to the following simple corollary of the extension property of Lipschitz maps to $\mathbb{R}^n$.

[Lip$_{\sqrt{n}}$]. — Recall that *1-Lipschitz*, i.e. distance non-increasing, maps from subsets in metric spaces to the Euclidean space $\mathbb{R}^k$ with the *sup-metric*,

$$f : A \to (\mathbb{R}^k, \text{dist}_{\text{sup}})$$

for

$$\text{dist}_{\text{sup}}((x_1, \ldots, x_i, \ldots, x_k), (x'_1, \ldots, x'_i, \ldots, x'_k)) = \sup_i |x_i - x'_i|$$

extend to *1-Lipschitz* maps of the ambient spaces $B \supset A$,

$$F : B \to (\mathbb{R}^k, \text{dist}_{\text{sup}}).$$

Since the Euclidean (Pythagorean) metric in $\mathbb{R}^k$ satisfies,

$$\text{dist}_{\text{sup}} \leqslant \text{dist}_{\text{Eucl}} \leqslant \sqrt{k} \cdot \text{dist}_{\text{sup}},$$

1-Lipschitz maps

$$f : A \to \mathbb{R}^k = (\mathbb{R}^k, \text{dist}_{\text{Eucl}})$$

extend to $\sqrt{k}$-Lipschitz maps to $\mathbb{R}^,$ and the same applies to maps to convex subsets, e.g. balls in $\mathbb{R}^k$.

Finally we notice that the unit Euclidean ball $B^{n-1} \subset \mathbb{R}^{n-1}$ admits a $\frac{\pi}{2}$-Lipschitz homeomorphism onto the hemisphere S^{n-1}_+ which keeps the boundary sphere $S^{n-2} = \partial S^{n-1}_+ = \partial B^{n-1}$ fixed.

Thus we conclude that *1-Lipschitz* maps $Z = \partial Y_\mp \to \partial S^{n-1}_\mp \subset S^{n-1}$ of non-zero degree extend to λ-Lipschitz maps $Y \to S^{n-1}$ for $\lambda = \frac{\pi}{2}\sqrt{n-1}$ which also have non-zero degrees.

In fact, this extension property together with $\odot^{\to S^{n-1}}_{\text{Sc} \geqslant 0}$ yield the following (alas, non-sharp) bound on the *(hyper)spherical radius* $\text{Rad}_{S^{n-2}}(Z \subset Y)$.

$\odot^{\to S^{n-2}}_{\text{Sc} \geqslant 0}$. — *Let X be a compact n-dimensional Riemannian spin manifold with boundary $Y = \partial X$, such that $\text{Sc}(X) \geqslant 0$ and $\text{mean.curv}(Y) \geqslant n - 1$.*

If a closed hypersurface $Z \subset Y$ homologous to zero in Y admits an ε-Lipschitz map $Z \to S^{n-2} = \partial S^{n-1}_\pm$ of non-zero degree, where "Lipschitz" is understood for the distance associated with the Riemannian metric in Y induced from X, then

$$\varepsilon \geqslant \frac{2}{\pi\sqrt{n-1}}.$$

Remark/Example. — The "homologous to zero" condition is essential: non-contractible curves in 2-tori $Y \subset \mathbb{R}^3$ with $\text{mean.curv}(Y) \geqslant 2$ may be uncontrollably long.

Now let us show that a combination of the above argument with that used in the proof of the *sharp bound in codimension 2* in Section 1 yields the following.

FILLING RADIUS BOUND FOR $Z^{n-2} \subset \mathbb{R}^n$. — *Let $Y \subset \mathbb{R}^n$ be a smooth closed (embedded without self-intersection!) connected hypersurface with* mean.curv$(Y) \geqslant n-1$ *and let $Z = Z^{n-2} \subset X$ be a closed (embedded without self-intersection) hypersurface in Y.*

If Z is homologous to zero in Y then it is homologous to zero in its ρ-neighbourhood in $\mathbb{R}^n \supset Z$ for

$$\rho \leqslant \frac{\pi}{2} \frac{n}{n-1}.$$

Proof. — If Z doesn't bound in its ρ-neighbourhood in $\mathbb{R}^n$ then, by the *Alexander duality* there exists a closed curve $S \subset \mathbb{R}^n$, which is *non-trivially linked* with Z and such that

$$\mathrm{dist}_X(Z, S) > \rho.$$

Then the radial projections of Z to the ρ-spheres with the centers $s \in S$, say

$$f_s : Z \to S_s^{n-1}(\rho),$$

are *distance decreasing*, for the *Euclidean* distances restricted to Z and to S_s^{n-1} while the resulting map from $Z \times S$ to the ρ-sphere, call it

$$f : Z \times S \to S^{n-1}(\rho) = S_{\mathbf{0}}^{n-1}(\rho),$$

has non-zero degree, since this degree is equal to the linking number between Z and S.

Then, using Kirszbraun theorem and an obvious $\frac{\pi}{2}$-Lipschitz homeomorphisms from the unit ball onto the unit hemisphere, $B^n \to S_\pm^n$, one extends this f to a $\frac{\pi}{2}$-Lipschitz map

$$F : Y \times S \to S^n(\rho),$$

such that $\deg(F) = \deg(f) \neq 0$, where, by using (the corresponding version of) $\mathrm{Lip}_{\sqrt{n}}$ and by making the curve S longer if necessary, one gets such an F with

$$\mathrm{Lip}(F) \leqslant \lambda = \frac{\pi}{2}.$$

Finally observe that $Y \times S$ serves as the boundary of the manifold X^{n+1}, that is the domain in $\mathbb{R}^n$ bounded by Y times S, where $\mathrm{Sc}(X) = 0$ and mean.curv$(Y \times S) = $ mean.curv$(Y) \geqslant n-1$ and that $\odot_{\mathrm{Sc} \geqslant 0}^{\to S^n}$ implies that

$$\rho \leqslant \frac{\lambda n}{n-1},$$

by which the proof is concluded. $\square$

Remarks and Conjectures.

(a). — Our inequality with const $= \frac{\pi}{2} \frac{n}{n-1} > 1$ is not especially exciting in view of the sharp estimate (where const $= 1$) available with the maximum principle. But the above argument applies to a class of spaces quite different from $\mathbb{R}^n$, where the scalar curvatures are bounded from below, and which admit (exactly or approximately) contracting projections to the balls, such, for instance as manifolds with $\mathrm{Sc} \geqslant -\varepsilon$ which are bi-Lipschitz equivalent to $\mathbb{R}^n$.

However, *sharp* filling inequalities in this kind of spaces remain conjectural.

(b). — Let a hypersurface $Z \subset Y = \partial X$, where X is a compact Riemannin manifold, divide Y into two halves, say Y_+ and Y_-, such that
$[\mathbf{MN}_{\pm(\mathbf{n-1})}]$ mean.curv$(Y_+) \geqslant n-1$ and mean.curv$(Y_-) \geqslant -(n-1)$.
Then one knows that Z bounds a *stable hypersurface* $Y_{\min} \subset X$ with *constant mean curvature* $n-1$ (called "μ-bubble" in [3]).[8]

If $\mathrm{Sc}(X) \geqslant 0$, this yields — we show it somewhere else — alternative proofs of non-sharp bounds on the (hyper)spherical radius and on the filling radius of Z provided the mean curvature of $Y \supset Z$ satisfy the condition $[\mathbf{MN}_{\pm(\mathbf{n-1})}]$, which is weaker than mean.curv$(Y) \geqslant n-1$.

(c). — Let X^+ be a complete Riemannian n-manifold, which, for simplicity's sake, we assume having uniformly bounded local geometry, such as $X^+ = \mathbb{R}^n$, for instance.

Let $Z = Z^{n-2} \subset X^+$ be a smooth closed oriented submanifold which bounds a cooriented submanifold $Y_0 = Y_0^{n-1} \subset X^+$ with mean.curv$(Y_0) \geqslant n-1$. Does then Z bound a stable hypersurface $Y_{\min} \subset X^+$ with mean.curv$(Y_{\min}) = n-1$?

(Intuitively, this $Y_{\min}$ could be obtained by an "inward deformation" of Y_0, where, in general, even if Y_0 is compact, the bubble $Y_{\min}$ may somewhere go to infinity; yet, this is ruled out by the "bounded geometry" condition as in [13, 11.6]).

If this works, we would obtain bounds on filling (hyperspherical) radii and filling volumes of Z, in manifolds X^+ with $\mathrm{Sc}(X^+) \geqslant \sigma$, similar to the bound in $\mathbb{R}^n$ obtained with the maximum principle.

But since minimal bubbles $Y_{\min}$ can intersect Y_0 away from $\partial Y_0 = \partial Y_{\min}$, the existence of these $V_{\min}$ seems problematic.

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[8] If $n \geqslant 9$, this $Y_{\min}$ may, a priori, have "stable singularities" but these can be rendered harmless with the recent Schoen–Yau theorem in [19] and/or, possibly, using the work by Lohkamp [16].

(d). — Let $Z = Z^{n-2} \subset X^+$, e.g. for $X^+ = \mathbb{R}^n$ be as above.

Can our bound on fill.rad(Z) be upgraded to such a bound for an $(n-1)$-volume minimizing filling of Z that is a volume minimizing hypersurface, say $Y_* \subset X^+$ with $\partial Y_* = Z$?

Namely, is there a bound on $\sup_{y \in Y_*} \text{dist}(y, Z)$ for our Z? in terms of the lower bounds on the scalar curvature of X^+ and on the mean curvature of some $Y \supset Z$?

(Ideally we would like this "dist" to be associated with the Riemannin metric in Y_* rather than with that in X^+.)

3. Back to Mean Convex Domains in $\mathbb{R}^n$

The following may be known but I couldn't find a reference.

CONJECTURE 3.1. — *Let X be an (infinite!) mean convex domain $X \subset \mathbb{R}^n$, i.e. mean.curv$(\partial X) \geqslant 0$.*

If the boundary ∂X is disconnected, then none of the connected component of ∂X may have its mean curvature separated away from 0, i.e. infima of the mean curvatures of all components are zero.

This is known in the case $n = 3$, where the following better result is available.

STRONG HALF-SPACE/SLAB THEOREM ([17]). — *The only mean convex domains, i.e. with mean.curv $\geqslant 0$, in $\mathbb{R}^3$ with disconnected boundaries are slabs between parallel planes.*

In fact, this easily follows from

FISCHER COLBRIE–SCHOEN PLANARITY THEOREM. — *Complete stable minimal surfaces in $\mathbb{R}^3$ are flat.*

Warning. — The Euclidean spaces $\mathbb{R}^n$ for $n \geqslant 4$, contain mean convex domains bounded by pairs of n-dimensional catenoids [14].

Mean convex domains $X \subset \mathbb{R}^3$, even with connected boundaries, are subjects to strong geometric constrains.

◀-*Example.* — If a mean convex $X \subset \mathbb{R}^3$ contains a plane, then, assuming ∂X is non-empty and connected, Y is equal to a half-space.

This follows from *the half-catenoid maximal principle* that was originally used by Hoffman and Meeks [14] to show that *properly embedded minimal surfaces $Y \subset \mathbb{R}^3_+$ are flat.*

It may be unclear what to expect of general mean convex domains $X \subset \mathbb{R}^n$ but if the mean curvature is separated away from zero, then one expects the following.

CONJECTURE 3.2. — *If a domain $X \subset \mathbb{R}^n$ has mean.curv$(\partial X) \geqslant n - 1$, then there exists a continuous self-map $R : X \to X$, such that*

- *the image $R(X) \subset X$ has topological dimension $n - 2$;*
- dist$(x, R(x)) \leqslant \text{const}_n$ *for all $x \in X$, with the best expected* $\text{const}_n = 1$.

Comments. — This is a baby version of the corresponding *conjectural* bound on the *Uryson width* of complete (also non-complete?) n-manifolds X with $\text{Sc}(X) \geqslant n(n-1)$: There exists a continuous map from X to an $(n-2)$ dimensional, say polyhedral, space, say $\rho : X \to P^{n-2}$, such that the diameters (also areas?) of the pullbacks of all points $p \in P$ are bounded by

$$\text{diam}(\rho^{-1}(p)) \leqslant \text{const}_n,$$

possibly with $\text{const}_n = 1$ (and $\text{const}_n = 4\pi$ in the case of areas).[9]

Ideally, one would like to have the above map R as the end result of a homotopy, a kind of mean curvature flow, which would collapse X to P and blow up the mean curvature at all points in the process.

Similarly, ρ may result from a Ricci kind of flow which would shrink X to P^{n-2} in finite time with a simultaneous blow up of the scalar curvature.

Exercices.

(a). — Let $n = 3$ and show that there exists a map $f : \partial X \to \mathbb{R}^3$, such that the image $R(X) \subset X$ has topological dimension 1 and dist$(x, R(x)) \leqslant 100$ for all $x \in \partial X$.

Hint. — Argue as in the proof of Corollary 10.11 in [7].

(b). — Let X be a complete n-manifold with *disconnected* boundary, where mean.curv$(\partial X) \geqslant \mu \geqslant 0$. If $n \leqslant 7$. Then, for all positive $\nu \leqslant \mu$, X contains an n-submanifold X_ν with a *disconnected* boundary which have *constant* mean curvature mean.curv$(\partial X_\nu) = \nu$.

Remark. — This may be only of "negative" use for $X \subset \mathbb{R}^n$, where it may help to settle Conjecture 3.2, but it may be more relevant for domains in the hyperbolic space $\mathbf{H}^n$ with the sectional curvature -1, where mean convex domains of all kinds are abundant and where the counterpart of Conjecture 3.2 refers to $X \subset \mathbf{H}^n$ with mean.curv$(\partial X) > n - 1$. (Complete

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[9] See [7] and [18] for the case $n = 3$.

non-compact hypersurfaces with mean.curv $= n - 1$ in $\mathbf{H}^n$, which have no Euclidean analogues, look especially intriguing.)

Also it may be amusing to look at the Conjecture 3.2 from the position of manifolds with Ricci $\geqslant 0$, where the natural guess is as follows.[10]

CONJECTURE 3.3. — *If an X with $\mathrm{Ricci}(X) \geqslant 0$ has disconnected boundary with the mean curvature $\geqslant -\mu$, then no connected component of this boundary can have mean curvature $\geqslant \mu + \varepsilon$.*

(This is, of course, obvious for compact X.)

Symmetrization.

Intersections of mean convex subsets X in Riemannin manifolds are mean convex with a properly defined generalised mean curvature and, more generally, the inequality mean.curv$(\partial X) \geqslant \mu$, is stable under finite and infinite intersections of such domains for all $-\infty < \mu < \infty$.

This allows *G-symmetrization* of X's under actions of isometry groups G acting on the ambient manifold $X^* \supset X$, e.g. for $X^* = \mathbb{R}^n \supset X$

$$X \rightsquigarrow X_{\mathrm{sym}} = \bigcap_{g \in G} g(X).$$

This suggests, for instance, the proof of the above ◖ in some (all?) cases by symmetrization $X \rightsquigarrow X_{\mathrm{sym}} \subset \mathbb{R}^3$ where X_{sym} is equal to the intersection of the copies of X obtained by rotations of X around the axis normal to the hyperplane $\mathbb{R}^2 \subset X$.

Similarly, one can apply symmetrizations to mean convex domains $X \subset \mathbb{R}^n$ for $n \geqslant 4$ which contain hyperplanes and where definite results need fast decay conditions on the mean curvatures of ∂X. For instance, one shows with symmetrization — this must be classically known — that the hyperplane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ admits no mean convex perturbations with *compact supports,* which also can be derived, from non-existence of $\mathbb{Z}^n$-invariant metrics with Sc > 0 on $\mathbb{R}^n$.

In fact, such perturbations admit $\mathbb{Z}^{n-1}$-invariant extensions, more precisely invariant under the action of the group (isomorphic to $\mathbb{Z}^{n-1}$) generated by sufficiently large mutually normal translations. and at the same time the metric in X can be perturbed to have the scalar curvature > 0.

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[10] Another natural guess is that the answer to this, along with Conjecture 3.4, must be known to right people.

Then the metric on the double $X \cup_\partial X$ of X can be similarly made $\mathbb{Z}^n$-invariant keeping $\mathrm{Sc} > 0$.

We mentioned all this to motivate, albeit not very convincingly, the following.

CONJECTURE 3.4. — *The only $\mathbb{Z}^{n-3}$-invariant mean convex domains in $\mathbb{R}^n$ with disconnected boundaries are slabs between parallel hyperplanes.*

VAGUE QUESTION. — *What could be a counterpart of the mean curvature symmetrization for metrics with $\mathrm{Sc} \geqslant \sigma$?*

STABILITY OF THE Inball^n-INEQUALITY. — *Let the boundary of a smooth domain $X \subset \mathbb{R}^n$ have $\mathrm{mean.curv}(\partial X) \geqslant n-1-\varepsilon$ for $\varepsilon > 0$ and let $x_1, x_2 \in X$ satisfy $\mathrm{dist}(x_1, \partial X) \geqslant 1$, and $\mathrm{dist}(x_2, \partial X) \geqslant r$ for $0 < r \leqslant 1$. Then*

$$\text{either } \mathrm{dist}(x_1, x_2) \leqslant 1 - r + \delta \text{ or } \mathrm{dist}(x_1, x_2) \geqslant 1 + r - \delta,$$

where $\delta \leqslant \delta(\varepsilon) \to 0$ for $\varepsilon \to 0$.

Proof. — Symmetrize around the axes between x_1 and x_2.   □

COROLLARY. — *If ε is small, then the unit balls which are contained in X can't be continuously moved δ-far in X from their original positions.*

Moreover, the part of $Y = \partial X$ which lies δ close to the ball $B_{x_1}^n(1) \subset X$ has only small holes, about $\sqrt{\delta}$ in size, which can be sealed by a small perturbation of Y and thus lock $B_{x_1}^n(1)$ in the concentric ball of radius $1 + \delta$. This means, more precisely, the following.

There exists continuous self-mappings $f = f_\varepsilon : X \to X$, $\varepsilon > 0$, such that

(1) *f is supported (i.e. $\neq \mathrm{Id}$) in a given arbiltrarilly small neighbourhood of the boundary $Y = \partial X$,*

(2) *f moves all points by a small amount for small ε,*

$$\mathrm{dist}(f(x), x) \leqslant \delta(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

(3) *the image $f(Y) \subset X$ "locks" all unit balls in X, that is every point $x \in X$ with $\mathrm{dist}(x, \partial X) \geqslant 1$ is contained in a connected component of the complement $\mathbb{R}^n \setminus f(Y)$, where this components itself is contained in the ball $B_x(1 + \delta)$, such that this δ is also bounded by $\delta(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0$.*

Proof. — Confront the above with the intersection corollary from Section 1 and conclude that if the unit ball $B_x(1)$ is contained in X, then the intersection of X with the δ-greater concentric sphere $S = S^{n-1}(1+\delta) \subset \mathbb{R}^n$ contains no disc of radius $> 20\sqrt{\delta}$

Then collapse this intersection, call it

$$V = X \cap S \subset S = S^{n-1}(1 + \delta),$$

to its cut locus with respect to the boundary, denoted $\Sigma \subset V$, and then extend the map $V \to \Sigma$ to the required map $X \to X$.   □

Stability in the Limit. — Given a domain $X \subset \mathbb{R}^n$ let Δ_X be the distance function to the boundary inside X extended by zero outside X,

$$\Delta_X(x) = \mathrm{dist}(x, \partial X) \text{ for } x \in X \quad \text{and} \quad \Delta_X(x) = 0 \text{ for } x \in \mathbb{R}^n \setminus X.$$

Say that a sequence of subsets $X_i \subset \mathbb{R}^n$ *regularly converges* if the functions Δ_{X_i} uniformly converge on compact subsets in $\mathbb{R}^n$. Then the *regular limit* $X_{\infty\circ}$ of X_i is defined as the (open!) subset, where the limit function $\lim_{i\to\infty} \Delta_{X_i}$ is *strictly positive*.

(Observe that every sequence X_i contains a regularly convergent subsequence.)

Now, the existence of the above $f : X \to X$ (obviously) implies the following.

If $\mathrm{mean.curv}(\partial X_i) \geqslant \mu_i \xrightarrow[i\to\infty]{} n - 1$, and X_i regularly converge to $X_{\infty\circ}$, *then the connected components in $X_{\infty\circ}$ of the points $x \in X_{\infty\circ}$ where $\mathrm{dist}(x, \partial X_{\infty\circ}) = 1$ are exactly open unit balls with the centers at these points x.*

Remark. — The non-inclusion $\mathrm{dist}(x_1, x_2) \notin [1 - r + \delta, 1 + r - \delta]$ is sharp: if $\mathrm{dist}(x_1, x_2) > 2$, then, for all $\varepsilon > 0$, there are smooth domains X in $\mathbb{R}^n$ with $\mathrm{mean.curv}(\partial X) \geqslant n - 1 - \varepsilon$, which contains both unit balls $B_{x_1}(1)$ and $B_{x_1}(1)$, as we shall explain in Section 5.

CONJECTURE 3.5. — [11] *Let $X_i \subset \mathbb{R}^n$ be a sequence of smooth domains, such that all of them contain the unit ball $B = B_0^n(1)$ and such that*

$$\mathrm{mean.curv}(X_i) \geqslant \mu_i \to n - 1 \text{ for } i \to \infty.$$

Then there exists a sequence of compact domains B_i' with smooth boundaries $S_i' = \partial B_i'$, which approximate B from above, i.e.

$$\bigcap_i B_i' = B,$$

and such that

$$\mathrm{vol}_{n-1}(S_i' \cap X_i) \to 0 \text{ for } i \to \infty.$$

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[11] This is motivated by an aspect of the Penrose conjecture explained to me by Christina Sormani.

Remark. — Similar stability/limit problem for Riemannin manifolds X_i with $\mathrm{Sc}(X_i) \geqslant \sigma$ is discussed in [12, Section 4.5] in the spirit of the *intrinsic flat convergence of metrics* [20].

Exercises.

(a). — Decide what are possibilities for the values of the distance between x_1 and x_2, in the case mean.curv$(\partial X) \geqslant 1$ and dist$(x_i, \partial X) = \frac{n-2}{n-1} + \varepsilon_i$, $i = 1, 2$.

(c). — [12] Let

$$X_0 = B^{n_1}(r_1) \times B^{n_2}(r_2) \subset \mathbb{R}^{n_1 + n_2}$$

and evaluate the maximal μ, for which there exists $X \supset X_0$ with mean.curv$(X) \geqslant \mu$.

Hint/Remark. — G-Symmetrization where G is product of two orthogonal groups, $G = O(n_1) \times O(n_2)$, renders the problem 1-dimensional, the analysis of which — I haven't tried it myself — seems easy. But finding this maximal μ for products of k balls,

$$X_0 = B^{n_1}(r_1) \times \cdots \times B^{n_k}(r_n) \subset \mathbb{R}^{n_1 + \cdots + n_k},$$

where $k \geqslant 3$, may be more difficult.

(d) (δ, μ)-regularisation. — Given a closed (i.e the closure of open) domain X in a complete Riemannin manifold X^*, let $X_{\pm\delta} \subset X$ be the δ-neighbourhood of the δ-sub-level $\Delta_X^{-1}[,\delta] \subset X$, that is the union of the δ-balls from X^* which are contains in X.

Then let $X_{\pm\delta,\mu}$ be the intersection of all subsets in X^* which contain $X_{\pm\delta}$ and have the mean curvatures of their boundaries $\geqslant \mu$. Show that:

(1) the operation $X \mapsto X_{\mp\delta}$ is *idempotent*, moreover,

$$(X_{\mp\delta})_{\mp\delta'} = X_{\mp\delta} \text{ for } \delta' \leqslant \delta;$$

(2) if X is compact mean convex with a *piecewise smooth* boundary, then the boundary $\partial X_{\mp\delta}$ is C^1-*smooth* for small $\delta > 0$ and

$$X_{\pm\delta,\mu} = X_{\pm\delta}.$$

for $\mu \ll \frac{1}{\delta}$.

Remark. — The operation $X \mapsto X_{\mp\delta}$ with (relatively) large δ doesn't seem to preserve mean convexity even in $\mathbb{R}^n$, for $n \geqslant 3$, where the apparent example can be obtained — I didn't check it all 100% — by a C^∞-small

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[12] This is a replacement to the *erroneous*-inequality from my "101-Questions" paper.

perturbation of $X_0 = \blacktriangle \times \mathbb{R}^{n-2} \subset \mathbb{R}^n$, where one of the angles of the triangle $\blacktriangle \subset \mathbb{R}^2$ is very small.

This raises the problem of evaluating the maximal distance from $X_{\pm\delta,\mu}$ to $X_{\pm\delta} \subset X_{\pm\delta,\mu}$ for μ-convex domains X, where this "maximal distance" is

$$\sup_{x \in X_{\pm\delta,\mu}} \text{dist}(x, X_{\pm\delta}).$$

4. Being Thick in Many Directions

Start by observing that the products of the unit balls $B^{n_1} = B^{n_1}(1) \subset \mathbb{R}^{n_1}$ by Euclidean spaces,

$$B^{n_1} \times \mathbb{R}^{n_2} \subset \mathbb{R}^{n_1+n_2}$$

are *mean curvature (weakly) extremal* in $\mathbb{R}^{n_1+n_2}$ if a domain $X \subset \mathbb{R}^{n_1+n_2}$, $X \neq \mathbb{R}^n$, with mean.curv$(\partial X) \geqslant n_1 - 1$ contains $B^{n_1}(r) \times \mathbb{R}^{n_2}$ then $r \leqslant 1$.

Proof. — Assume $r > 1$ and let $0 \leqslant \phi(x) < 1 - r$, $x \in \mathbb{R}^{n_1}$, be a smooth function with compact support, such that let $\phi(0) = r - 1$, while the norms of the first and the second differential $d\phi$ and $d^2\phi$ are much smaller then $r - 1$.

Let $x_\varepsilon \in \mathbb{R}^{n-1} \subset \mathbb{R}^{n_1+n_2}$ be a point where the distance function from $x \in \mathbb{R}^{n_1} \subset \mathbb{R}^{n_1+n_2}$ to ∂X is ε-close to the minimum and let us parallel transform ϕ to make $x_\varepsilon = 0$.

Let $U_t \supset B^{n_1}(1) \times \mathbb{R}^{n_2}$, $t \geqslant 0$ be neighbourhoods of $B^{n_1}(1) \times \mathbb{R}^{n_2} \subset \mathbb{R}^{n_1+n_2}$ defined by

$$U_t = \bigcup_{x \in \mathbb{R}^{n_2}} B^{n_1}((t+1)(\phi(x)+1)) \subset \mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$$

and observe that, for small $\|d\phi\|$ and $\|d^2\phi\|$ their boundaries have mean curvatures < 1.

If $\varepsilon = \varepsilon(\phi) > 0$ is taken sufficiently small, then there exists a $t > 0$, such that $U_t \subset X$ and such that the boundary ∂U_t meets ∂X; this yields the proof by an (obvious in this case) maximum principle argument.   □

Remark. — In an earlier version of this paper we claimed *strong extremality* of $B^{n_1}(r) \times \mathbb{R}^{n_2}$ if $X \supset B^{n_1}(r) \times \mathbb{R}^{n_2}$ and mean.curv$(\partial X) \geqslant n_1 - 1$ then $X = B^{n_1}(r) \times \mathbb{R}^{n_2}$.

My (two different) arguments suggested for this purpose were erroneous, and I am uncertain if the strong extremality holds for $n_1 \geqslant 2$.

The (weak) extremality of $B^{n_1} \times \mathbb{R}^{n_2}$ motivates the following conjectural bound on the *macroscopic dimensions* of the sets of (the centers of) large

balls in X, where, recall, the macroscopic dimension of a metric space M is the minimal dimension of a polyhedral space P, for which M admits a continuous map $\Delta : M \to P$, such that $\mathrm{diam}_M(\Delta^{-1}(p)) \leqslant d$ for some constant $d = d(M) < \infty$.

CONJECTURE 4.1. — *Let a domain $X \subset \mathbb{R}^n$ satisfy*

$$\mathrm{mean.curv}(\partial X) \geqslant n_1 - 1 + \varepsilon$$

for $n_1 < n$ and $\varepsilon > 0$. Then the macroscopic dimension of the subset $X_{-1} \subset X$ of the points $x \in X$, such that $\mathrm{dist}(x, \partial X) \geqslant 1$, satisfies:

$$\mathrm{macr.dim}(X) < n - n_1,$$

where the equality $\mathrm{macr.dim}(X) = n - n_1$ *with $\varepsilon = 0$ is achieved* only *for*

$$X_0 = B^{n_1} \times \mathbb{R}^{n-n_1}.$$

Remarks.

(a) *The case of $n_1 = n - 1$.* — Conjecture 4.1 in this case says that if $\mathrm{mean.curv}(\partial X) \geqslant n - 1 - \alpha$ for $\alpha < 1$, then the diameters of all connected components of the above subset $X_{-1} \subset X$ are bounded by $\beta \leqslant \beta(\alpha) < \infty$.

This can be shown for $\alpha \leqslant \alpha_n \sim \frac{1}{n}$ by the argument used for proving stability of the Inball^n-inequality, where slightly better estimates on α_n can be, probably, obtained with symmetrization around k-planes containing centers of suitable $(k+1)$-tuples of unit balls in X.

(b). — The macroscopic dimension of $B^{n_1} \times \mathbb{R}^{n_2}$ is equal to n_2 by

LEBESGUE'S LEMMA. — *A continuous map $[0,1]^n \to \mathbb{R}^n$ necessary brings a pair of points from opposite faces in the cube $[0,1]^n$ to a single point in $\mathbb{R}^n$.*

There is a counterpart to Conjecture 6 in the context of Riemannian manifolds (with and without boundaries) which express the idea that if the scalar curvature of an n-dimensional manifold X is bounded from below by $n(n-1)$ then the space of "large" balls $B_x(r) \subset X$, say of radii $r \approx \pi$, must be small, where the size of a ball is evaluated in comparison with geodesic balls in the unit sphere S^n.

A conceptually simple instance of this concerns maps of closed n-manifolds X to the unit sphere S^n, namely the space of distance decreasing maps of non-zero degrees $X \to S^n$, which we denote $\mathrm{Lip}_1(X^{\to \circ}_{\neq 0})$, and which we endow with the metric associated in the usual way to the natural *length structure* in the space of maps to S^n, where the length of a curve in this space, that is a family of maps $f_t : X \to S^n$, is defined as the supremum

of the lengths of the t-curves in S^n drawn by individual points $x \in X$,

$$\text{length}(f_t) = \sup_{x \in X} \text{length}(f_t(x)).$$

CONJECTURE 4.2. — *If* $\text{Sc}(X) \geqslant n_1(n_1 - 1) + \varepsilon$, $\varepsilon > 0$, *then*

$$\text{macr.dim}(\text{Lip}_1(X_{\neq 0}^{\to \circ})) < n - n_1.$$

The simplest instance of this conjecture concerns manifolds X with $\text{Sc}(X) \geqslant n(n-1) - \varepsilon$, where it says that all connected components of the space $\text{Lip}_1(X_{\neq 0}^{\to \circ})$ are bounded. In fact, when $\varepsilon \to 0$, one expects more of this space, which we formulate as follows.

CONJECTURE 4.3 (Stability of S^n). — *Let X be a closed orientable Riemannin n-manifold, such that $\text{Sc}(X) \geqslant n(n-1) - \varepsilon$. Then the diameters of the connected components of the quotient space of $\text{Lip}_1(X_{\neq 0}^{\to \circ})$ under the orthogonal group action on S^n satisfy:*

$$\text{diam}(\text{conn}(\text{Lip}_1(X_{\neq 0}^{\to \circ})/O(n))) \leqslant \delta \leqslant \delta(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0.$$

Digression

To get a feeling for our metric in $\text{Lip}_1(X_{\neq 0}^{\to \circ})$ observe that such length metrics are defined for spaces maps to all length metric spaces S and look at a few examples.

(i) Continuous Maps. — If S is a compact locally contractible space and B^n is the topological n-ball, then the space $C(B^n \to S)$ of continuous maps $B^n \to S$, $n \geqslant 1$, has finite diameter if and only if S has finite fundamental group.

For instance

$$\text{diam}(C(B^n \to S^m(1))) = \text{diam}(S^m(1)) = \pi \text{ for } m > n.$$

Somewhat less obviously,

$$\text{diam}(C(B^n \to S^n(1))) \leqslant 3\pi,$$

which implies that the (*infinite* cyclic) universal covering of the space $C(S^{m-1} \to S^m(1))$ also has diameter $\leqslant 3\pi$.

More generally,

$$\text{diam}(C(B^n \to S)) \leqslant n \cdot \text{const}(S),$$

for all compact, say cellular, spaces S with finite fundamental groups and, probably, the universal coverings of all connected components of the spaces $C(X \to S)$ are similarly bounded by $\dim(X) \cdot \text{const}(S)$.

The above linear bound on $\operatorname{diam}(C(B^n \to S))$ is asymptotically matched by a lower bound for most (all?) compact non-contractible spaces S.

For instance, it follows from 1.4 in [4] that

$$\operatorname{diam}(C(B^n \to S)) \geqslant n \cdot \operatorname{const}(S) \text{ with } \operatorname{const}(S) > 0$$

if S is the m-sphere $S^m(1)$, $m \geqslant 2$, or, more generally, if the iterated loop space $\Omega^k(S)$ for some $k \geqslant 1$ has non-zero rational homology groups $H_i(\Omega^k(S); \mathbb{Q})$ for i from a subset of positive density in $\mathbb{Z}_+$.

(ii) Lipschitz maps. — Let X and S be metric spaces and $\operatorname{Lip}_\lambda(X \to S)$ be the space of λ-Lipschitz maps with the above length metric which we now denote $\operatorname{dist}_\lambda$ and where we observe that the inclusions

$$\operatorname{Lip}_{\lambda_1}((X \to S), \operatorname{dist}_{\lambda_1}) \subset \operatorname{Lip}_{\lambda_2}((X \to S), \operatorname{dist}_{\lambda_2}), \lambda_1 \leqslant \lambda_2,$$

are distance decreasing.

The simplest space here, as earlier is where X is a ball, but now the geometry of the ball is essential. For instance,

$$\operatorname{diam}_\lambda \operatorname{Lip}_\lambda(B^n(R) \to S)) \leqslant \lambda R + \operatorname{diam}(S),$$

where $B^n(R)$ is the Euclidean R-ball, where $\operatorname{diam}_\lambda$ is the diameter measured with $\operatorname{dist}_\lambda$ and where, observe,

$$\operatorname{Lip}_{\lambda_1}(B^n(R_1) \to S) = \operatorname{Lip}_{\lambda_2}(B^n(R_2) \to S) \text{ for } \lambda_1 R_1 = \lambda_2 R_2.$$

More interesting is the lower bound

$$\operatorname{diam}_{c\lambda}(\operatorname{Lip}_\lambda(B^n(R) \to S)) \geqslant \operatorname{const}(S, c)\lambda R,$$

which holds whenever *the real homology $H_i(S, \mathbb{R})$ does not vanish for some* $i \leqslant n$.

In fact,

$$\operatorname{diam}_{c\lambda}(\operatorname{Lip}_\lambda(B^n(R) \to S)) \geqslant \operatorname{diam}_{c\lambda}(\operatorname{Lip}_\lambda(B^i(R) \to S)) \text{ for } n \geqslant i,$$

while evaluation $f \mapsto h(f)$ of a non-cohomologous to zero real i-cocycle h at 1-Lipschitz maps $f : B^i(R) \to S$ [13] defines a $C \cdot R^{n-1}$-*Lipschitz map* from $\operatorname{Lip}_1(B^i(R) \to S))$ to $\mathbb{R}$ for some $C = C(S, c))$. It follows that the 1-Lipschitz maps f with $h(f) \approx \operatorname{vol}_i(B^i(R)) \approx R^n$ – these exist by the Hurewicz–Serre theorem for the *minimal i where $H_i(S, \mathbb{R}) \neq 0$* – are within distance $\gtrsim R$ from the constant maps.

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[13] Such an h in our case is a differential i-form on S and its "evaluation" is the integral of this form over B^i mapped to S by f.

THREE QUESTIONS.

(1) *Are the diameters*

$$\mathrm{diam}_c(\mathrm{Lip}_1(B^n(R) \to S)))$$

bounded for a large fixed c and $R \to \infty$ if $H_i(S, \mathbb{R}) = 0$ for $i = 1, 2, \ldots, n$.

(It is not hard to show that $\mathrm{diam}_1(\mathrm{Lip}_\lambda(B^1(R) \to S^m(1))) \leqslant \mathrm{const} \cdot \log(R)$.)

(2) *What is the asymptotics of the diameters*

$$\mathrm{diam}_c(\mathrm{Lip}_1(B^n_H(R) \to S)))$$

for the hyperbolic balls $B^n_H(R)$ and $R \to \infty$?

(3) *Let S be a Riemannian manifold homeomorphic to the connected sum of twenty copies of $S^2 \times S^2$. Are there 1-Lipschitz maps $f_R : B^4(R) \to S$, $R \to \infty$, such that $h(f_R) \geqslant \mathrm{const} \cdot R^4$ for a cocycle h (e.g. a closed 4-form) which represents the fundamental cohomology class $[S] \in H^4(S; \mathbb{R})$, and some $\mathrm{const} = \mathrm{const}(S) > 0$?*

5. Thin Mean Convex

Start by observing that smooth domains $V \subset \mathbb{R}^n$ with mean.curv$(\partial V) \geqslant 0$ are diffeotopic to regular neighbourhoods of subpolyhedra $P \subset U$ with codim$(P) \geqslant 2$.

In fact, if such a V is bounded, this follows by Morse theory applied to a linear function on V,[14] while an unbounded V can be exhausted by bounded $V_i \subset Y$ with min.curv$(\partial V_i) > 0$.

Conversely, smooth submanifolds and, more general, piecewise smooth polyhedral subsets of codimension $\geqslant 2$ in $\mathbb{R}^n$ possess arbitrary thin mean convex neighbourhoods.

In fact, the "staircase" surgery construction for manifolds with positive scalar curvature (see [1, 5]) applied to mean curvature allows an attachment of such thin domains to thick ones, as follows.[15]

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[14] Since the second fundamental form is *nowhere strictly negative definite*, local minima of a generic linear function *restricted to ∂V* locally minimize this function *on $V \supset \partial V$*; therefore, all Morse cells/handles have dimensions $\leqslant n - 2$.

[15] It is worth remembering that the natural continuous Riemannian metrics on the doubles of a smooth domain $V \subset \mathbb{R}^n$ with mean.curv$(\partial V) > 0$ admits arbitrarily fine approximations by smooth metrics with Sc > 0. Thus all shapes and constructions you encounter with mean.curv > 0 are also seen with Sc > 0.

[●⪬]. — *Let X be a Riemannin manifold, $V \subset X$ a smooth domain and $P \subset X$ a piecewise smooth polyhedral subset. Let $\phi(x)$, $x \in X$, be a continuous function such that*

- $\phi(x) \leqslant$ mean.curv$(\partial V, x)$ *for all* $x \in \partial X$;
- $\phi(x) <$ mean.curv$(\partial V, x)$ *for all* $x \in \partial X \cap P$.

Given a neighbourhood $U \subset X$ of the closure of the difference $P \setminus V$, there exists a smooth domain V' in X, such that

- ∗ $U \supset V' \supset V \cup P$;
- ∗ $V' \setminus U = V \setminus U$;
- ∗ mean.curv$(\partial V', x) \geqslant \phi(x)$ *for all* $x \in \partial V'$.

Moreover, if P is transversal to the boundary ∂V then there is such a V', whose intersection with U serves as a regular neighbourhood of $V \cap P$ intersected with U. In particular, V' homotopy retracts to $V \cup P$.

This shows that, unlike to what happens to "thick" mean convex domains, there are few (if at all) *global* restrictions on the shapes of the "thin" ones but there are plenty of local ones, where an essential point is to understand which domains should be qualified as "very thin". Below are some definitions and observations which may clarify this point.

Thin$_{\text{mean}>0}$. — Given a closed subset Y in a Riemannian n-manifold X, define vol$_\partial(Y)$ as the infimum of the $(n-1)$-volumes of the boundaries of arbitrarily small neighbourhoods $U \supset Y$ of Y in X, i.e.

$$\text{vol}_\partial(Y) = \liminf_{U \to Y} \text{vol}_{n-1}(\partial U).$$

Thus, *vol$_\partial(Y) < \alpha$ if and only there exist arbitrary small neighbourhoods $U \supset Y$ with vol$_{n-1}(\partial U) < \alpha$.*

Observe that the so defined vol$_\partial$ is bounded by the $(n-1)$-dimensional Hausdorff measure,

$$\text{vol}_\partial(Y) \leqslant \text{mes}_{n-1}(Y).$$

In particular, *closed subsets $Y \subset X$ with vanishing $(n-1)$-dimensional Hausdorff measure have vol$_\partial(Y) = 0$* (but the converse, probably, is not true). Next, write

$$\text{mean.curv}_\partial(Y) \geqslant \mu_0$$

if for all $\varepsilon > 0$ all neighbourhoods $U' \supset Y$ contain smaller smooth neighbourhoods $U \supset Y$ such that

$$\text{mean.curv}(\partial U) \geqslant \mu_0 - \varepsilon$$

Implication $\mathrm{vol}_\partial(Y) = 0 \Rightarrow \mathrm{mean.curv}_\partial(Y) = \infty$. — To show this let $\mu = \mu(x)$ be a continuous function on $U' \setminus Y$ which is $\geqslant \mu_0$ for a given μ_0 and which may blow up at Y.

Let U_0 be a $\mu - bubble$ pinched between Y and U', i.e. U_0 minimises the following functional

$$U \mapsto \mathrm{vol}_{n-1}(\partial U) - \int_U \mu(x)\,\mathrm{d}x.$$

If Y is compact and μ is sufficiently large near X such a μ-bubble U_0 exists and

(1) the boundary ∂U_0 is smooth away from a possible singular subset $\Sigma \subset \partial U_0$ of codimension $\geqslant 7$;
(2) $\mathrm{mean.curv}(\partial U_0, x) = \mu(x)$ at the regular points $x \in \partial U_0$;
(3) U_0 can be approximated by domains U with *smooth* boundaries ∂U such that $\mathrm{mean.curv}(\partial U) \geqslant \mu - \varepsilon$ for a given $\varepsilon > 0$.

(These (1) and (2) are standard results of the geometric measure theory and (3) is an elementary exercise, see [9, 10] for details.[16])

Probably, the implication $\mathrm{vol}_\partial(\mathbf{Y}) = \mathbf{0} \Rightarrow \mathrm{mean.curv}_\partial(\mathbf{Y}) = \infty$ remains valid for all closed subsets in X, but the above argument, as it stands, delivers the following weaker property in the non-compact case.

If X has uniformly bounded geometry[17] *then every closed subsets $Y \subset X$ with $\mathrm{vol}_\partial(Y) = 0$ is equal to the intersection of a decreasing family of domains $U_\mu \subset X$, $\mu \to \infty$, where $\mathrm{mean.curv}(\partial U_\mu) \geqslant \mu$.*

Remark. — The role of bounded geometry is to ensure a lower bound on the volumes of balls in the μ-bubble away from Y where μ is small and, thus, keep domains U which minimise the function $U \mapsto \mathrm{vol}_{n-1}(\partial U) - \int_U \mu(x)\,\mathrm{d}x$ within an ε-neighbourhood of Y.

Exercises.

(a). — Let X be a Riemannian manifold isometrically acted upon by a group G and let $P \subset X$ be a G-invariant piecewise smooth polyhedral subset of codimension 2.

Show that P admits a G-invariant arbitrarily small *regular* neighbourhoods with arbitrarily large mean curvatures of their boundaries.

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[16] I apologise for referring to my own articles, but I could not find what is needed on the web except for a 1987-paper by F. H. Lin [15], where only a special case is treated.
[17] This means that there exist $\rho > 0$ and $\lambda > 0$, such that all ρ-balls $B_x(\rho) \subset X$ are λ-bi-Lipschitz homeomorphic to a Euclidean ball.

(b). — Show that every bounded smooth domain $U_0 \subset \mathbb{R}^n$, $n \geqslant 3$ contains arbitrarily long simple curves (arcs) $C \subset U$, such that curvature$(C) \leqslant$ const $=$ const(U).

Construct such curves with regular neighbourhoods $U \subset U_0$ which fill almost all of U_0 and such that the mean curvatures of the boundaries ∂U tend to infinity.

Apply this successively to $C_i \subset U_{i-1}$ and obtain

$$U_0 \supset U_1 \supset \cdots \supset U_{i-1} \supset U_i \supset \dots.$$

such that

$$\text{mean.curv}(\partial U_i) \to \infty \quad \text{and} \quad \text{vol}(U_i) \geqslant \text{const} > 0.$$

Show that the intersection $Y_\infty = \cap_i U_i$ is a compact set such that

- Y has positive Lebesgue measure, $\text{vol}_n(Y) > 0$;
- $\text{vol}_\partial(Y_\infty) = \infty$;
- $\text{mean.curv}_\partial(Y) = \infty$;
- the topological dimension of Y is one.

(c). — Construct similar $Y \subset \mathbb{R}^n$ with $\dim_{\text{top}} = m$ for all $m \leqslant n - 2$.

(d). — Show that the function $\lambda|\sin t|$, can be uniformly approximated, for all $\lambda > 0$, by C^2-functions $\varphi(t) > 0$, such that the hypersurface $H_\varphi \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, obtained by rotating the graphs of $\varphi(t)$ around the t-axis in $\mathbb{R}^{n+1}$ has mean.curv$(H_\varphi) > 0$ and, moreover, $\text{Sc}(H_\varphi) > 0$ if $n \geqslant 3$.

(e). — Construct Cantor (compact 0-dimensional) subsets Y in the plane which *are not* intersections of locally convex subsets, i.e. disjoint unions of convex ones.

(f). — Define "random" Cantor sets in $\mathbb{R}^n$ of positive measure, and show for $n \geqslant 2$ that they are not intersections of smooth mean convex domains. Then, do the same for Cantor sets in $\mathbb{R}^n$ with the Hausdorff dimensions $> n - 1$.

Admission. — Frankly, I am not 100% certain as I haven't seriously tried to solve this exercise.

Acknowledgement

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Misha GROMOV
Courant Institute of Mathematical Sciences,
NYU,

New York,
(U.S.A)
and
IHES,
Bures-Sur-Yvette (France)
gromov@ihes.fr