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AROUND THE LIE CORRESPONDENCE FOR COMPLETE KAC–MOODY GROUPS AND GABBER–KAC SIMPLICITY

by Timothée MARQUIS (*)

ABSTRACT. — Let k be a field and A be a generalised Cartan matrix, and let $\mathfrak{G}_A(k)$ be the corresponding minimal Kac–Moody group of simply connected type over k. Consider the completion $\mathfrak{G}_A^{pma}(k)$ of $\mathfrak{G}_A(k)$ introduced by O. Mathieu and G. Rousseau, and let $\mathfrak{U}_A^{ma+}(k)$ denote the unipotent radical of the Borel subgroup of $\mathfrak{G}_A^{pma}(k)$. In this paper, we exhibit a functorial dependence of the groups $\mathfrak{U}_A^{ma+}(k)$ and $\mathfrak{G}_A^{pma}(k)$ on their Lie algebra. We also provide several contributions to fundamental questions in the general theory of maximal Kac–Moody groups: (non-)Gabber–Kac simplicity over certain finite fields, (non-)density of a minimal Kac–Moody group in its Mathieu–Rousseau completion, (non-)linearity of maximal pro-p subgroups, and the isomorphism problem.

RÉSUMÉ. — Soit $\mathfrak{G}_A(k)$ le groupe de Kac–Moody minimal simplement connexe associé à un corps k et à une matrice de Cartan généralisée A. On note $\mathfrak{G}_A^{pma}(k)$ la complétion de $\mathfrak{G}_A(k)$ introduite par O. Mathieu et G. Rousseau, et $\mathfrak{U}_A^{ma+}(k)$ le radical unipotent du sous-groupe de Borel de $\mathfrak{G}_A^{pma}(k)$. Dans cet article, nous mettons en évidence une dépendance fonctorielle des groupes $\mathfrak{U}_A^{ma+}(k)$ et $\mathfrak{G}_A^{pma}(k)$ en leur algèbre de Lie. Nous apportons en outre plusieurs contributions à certaines questions fondamentales de la théorie générale des groupes de Kac–Moody maximaux: (non-)densité du groupe de Kac–Moody minimal dans sa complétion de Mathieu–Rousseau, (non-)Gabber–Kac simplicité sur certains corps finis, (non-)linéarité des sous-groupes pro-p maximaux, et problème d'isomorphisme.

1. Introduction

The main theme of this paper is the correspondence between the properties of a complete Kac–Moody group and its Lie algebra over an arbitrary field, with a special emphasis on the case of finite ground fields.

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Let $A = (a_{ij})_{i,j\in I}$ be a generalised Cartan matrix (GCM) and let $\mathfrak{g} = \mathfrak{g}(A)$ be the associated Kac–Moody algebra ([11]). Let also $\mathfrak{G}_A$ denote the corresponding Tits functor of simply connected type, as defined by J. Tits ([29]). Given a field k, the value of $\mathfrak{G}_A$ over k is called a *minimal Kac–Moody group*. This terminology is justified by the existence of larger groups, called *maximal* or *complete Kac–Moody groups*, which can be constructed as completions $\widehat{\mathfrak{G}}_A(k)$ of $\mathfrak{G}_A(k)$ with respect to some suitable topology. For instance, the completion of the affine Kac–Moody group $\mathrm{SL}_n(k[t,t^{-1}])$ of type $\widetilde{A}_{n-1}$ is the maximal Kac–Moody group $\mathrm{SL}_n(k(\!(t)\!))$.

Roughly speaking, a minimal Kac–Moody group $\mathfrak{G}_A(k)$ is obtained by "exponentiating" the real root spaces of the Kac–Moody algebra $\mathfrak{g}$, while completions $\widehat{\mathfrak{G}}_A(k)$ of $\mathfrak{G}_A(k)$ are obtained by exponentiating both real and imaginary root spaces of $\mathfrak{g}$. As a result, it becomes easier to make computations in $\widehat{\mathfrak{G}}_A(k)$ rather than in $\mathfrak{G}_A(k)$ (see e.g. [2, Remark 2.8]). Another motivation to consider maximal Kac–Moody groups rather than minimal ones is the fact that, when k is a finite field, the groups $\widehat{\mathfrak{G}}_A(k)$ form a prominent family of simple, compactly generated totally disconnected locally compact groups. Such groups have received considerable attention in the past years (see [4] for a current state of the art).

Unlike minimal Kac–Moody groups, whose definition is somehow "canonical" (in the sense that the Tits functor $\mathfrak{G}_A$ over the category of fields is uniquely determined by a small number of axioms generalising in a natural way properties of semi-simple algebraic groups), maximal Kac–Moody groups have been constructed in the literature using different approaches. There are essentially three such constructions of completions of a minimal Kac–Moody group $\mathfrak{G}_A(k)$, which we now briefly review.

The first approach is geometric. The *Rémy–Ronan completion* $\mathfrak{G}_A^{rr}(k)$ of $\mathfrak{G}_A(k)$ ([26]) is the completion of the image of $\mathfrak{G}_A(k)$ in the automorphism group $\mathrm{Aut}(X_+)$ of its associated positive building, where $\mathrm{Aut}(X_+)$ is equipped with the topology of uniform convergence on bounded sets. A slight variant of this construction was introduced by P-E. Caprace and B. Rémy ([5, §1.2]): the resulting group $\mathfrak{G}_A^{crr}(k)$ admits $\mathfrak{G}_A(k)$ as a dense subgroup and $\mathfrak{G}_A^{rr}(k)$ as a quotient.

The second approach is representation-theoretic. The *Carbone–Garland completion* $\mathfrak{G}_A^{cg\lambda}(k)$ with dominant integral weight λ ([8]) is the completion of the image of $\mathfrak{G}_A(k)$ in the automorphism group $\mathrm{Aut}(L_k(\lambda))$ of an irreducible λ-highest-weight module $L_k(\lambda)$ over k. Again, as for $\mathfrak{G}_A^{rr}(k)$, this construction can be slightly modified to produce a group $\mathfrak{G}_A^{cgr}(k)$ containing $\mathfrak{G}_A(k)$ as a dense subgroup, rather than a quotient of $\mathfrak{G}_A(k)$ ([28, 6.2]).

The third approach is algebraic. It is closer in spirit to the construction of the Tits functor $\mathfrak{G}_A$, and produces a (topological) group functor over the category of $\mathbb{Z}$-algebras, denoted $\mathfrak{G}_A^{pma}$, such that $\mathfrak{G}_A(k)$ canonically embeds in $\mathfrak{G}_A^{pma}(k)$ for any field k. The group $\mathfrak{G}_A^{pma}(k)$ was first introduced by O. Mathieu ([18]) and further developed by G. Rousseau ([28]), and will be called the *Mathieu–Rousseau completion* of $\mathfrak{G}_A(k)$. Over $k = \mathbb{C}$, the group $\mathfrak{G}_A^{pma}(k)$ coincides with the maximal Kac–Moody group constructed by S. Kumar ([13, §6.1.6]).

The Mathieu–Rousseau completion of $\mathfrak{G}_A(k)$, which will be used in this paper, is better suited to the study of finer algebraic properties of Kac–Moody groups, as for instance illustrated in [16] and [2]. The reason for this is that the relation between $\mathfrak{G}_A^{pma}(k)$ and its Kac–Moody algebra $\mathfrak{g}$ is more transparent than for the other completions. Our first theorem further illustrates this statement.

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta(A)} \mathfrak{g}_\alpha$ be the root decomposition of $\mathfrak{g}$ with respect to its Cartan subalgebra $\mathfrak{h}$, with corresponding set of roots $\Delta(A)$ (resp. of positive roots $\Delta_+(A)$, of positive real roots $\Delta_+^{\mathrm{re}}(A)$). Let $\mathfrak{g}_\mathbb{Z}$ denote the standard $\mathbb{Z}$-form of $\mathfrak{g}$ introduced by J. Tits ([29, §4]) and set $\mathfrak{g}_k := \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$. Set also $\mathfrak{n}^+(A) := \bigoplus_{\alpha\in\Delta_+(A)} \mathfrak{g}_\alpha$ and $\mathfrak{n}_k^+(A) := (\mathfrak{n}^+(A) \cap \mathfrak{g}_\mathbb{Z}) \otimes_\mathbb{Z} k$. Finally, let $\mathfrak{U}_A^{ma+}(k)$ denote the unipotent radical of the positive Borel subgroup of $\mathfrak{G}_A^{pma}(k)$: the Lie algebra corresponding to $\mathfrak{U}_A^{ma+}(k)$ is then some completion of $\mathfrak{n}_k^+(A)$. Our first theorem exhibits a "functorial dependence" of the group $\mathfrak{U}_A^{ma+}(k)$ on its Lie algebra.

Theorem A. — *Let k be a field, and let $A = (a_{ij})_{i,j\in I}$ and $B = (b_{ij})_{i,j\in I}$ be two GCM such that $|b_{ij}| \leqslant |a_{ij}|$ for all $i,j \in I$. Then the following assertions hold:*

(1) *There exists a surjective Lie algebra homomorphism*
$$\pi\colon\ \mathfrak{n}^+(A) \to \mathfrak{n}^+(B).$$

(2) *π gives rise to a surjective, continuous and open group homomorphism*
$$\widehat{\pi}\colon\ \mathfrak{U}_A^{ma+}(k) \to \mathfrak{U}_B^{ma+}(k).$$

A more precise version of Theorem A is given in Section 3.2 below (see Theorem 3.6).

Now that we have introduced the three constructions of maximal Kac–Moody groups that can be found in the literature, a very natural question arises: how do these constructions compare to one another? Or, more optimistically stated: *do the geometric, representation-theoretic and algebraic completions of $\mathfrak{G}_A(k)$ yield isomorphic topological groups?* Surprisingly, the answer to this question is yes in many cases, and conjecturally yes in *almost* all cases. However, when the field has positive characteristic p smaller than $M_A := \max_{i\neq j} |a_{ij}|$, things become more subtle.

One obstruction to an affirmative answer in all cases is the fact that the closure $\overline{\mathfrak{G}_A}(k)$ of $\mathfrak{G}_A(k)$ in its Mathieu–Rousseau completion $\mathfrak{G}_A^{pma}(k)$ might be proper: in [16], we gave for each finite field k an infinite family of GCM A such that $\overline{\mathfrak{G}_A}(k) \neq \mathfrak{G}_A^{pma}(k)$ (see also [28, §6.10] for an example over $k = \mathbb{F}_2$). Here, we exhibit a much wider class of examples.

Proposition B. — *Let $k = \mathbb{F}_q$ be a finite field, and let $A = (a_{ij})_{i,j\in I}$ be a GCM. Assume that there exist indices $i, j \in I$ such that $|a_{ij}| \geqslant q + 1$ and $|a_{ji}| \geqslant 2$. Then $\mathfrak{G}_A(k)$ is not dense in $\mathfrak{G}_A^{pma}(k)$.*

We give two completely different proofs of this theorem. The first relies on Theorem A. The second is more constructive, and provides another perspective on this non-density phenomenon. The proof of Proposition B can be found in Section 4 below. Note that $\overline{\mathfrak{G}_A}(k) = \mathfrak{G}_A^{pma}(k)$ as soon as the characteristic of k is zero or bigger than M_A (see [28, 6.11]).

On the other hand, G. Rousseau proved that there always exist continuous group homomorphisms $\overline{\mathfrak{G}_A}(k) \to \mathfrak{G}_A^{cgr}(k)$ and $\mathfrak{G}_A^{cgr}(k) \to \mathfrak{G}_A^{crr}(k)$, which are moreover isomorphisms as soon as $\operatorname{char} k = 0$ and A is symmetrisable (see [28, 6.3 and 6.7]). When k is finite, these homomorphisms are surjective, but the question of their injectivity is open.

Assume that $\overline{\mathfrak{G}_A}(k) = \mathfrak{G}_A^{pma}(k)$ and denote by $\phi\colon \mathfrak{G}_A^{pma}(k) \to \mathfrak{G}_A^{crr}(k)$ the composition of the two above homomorphisms. The kernel of ϕ then coincides with $Z'_A \cap \mathfrak{U}_A^{ma+}(k)$, where Z'_A denotes the kernel of the $\mathfrak{G}_A^{pma}(k)$-action on its associated building X_+. The injectivity of ϕ thus amounts to $Z'_A \cap \mathfrak{U}_A^{ma+}(k)$ being trivial or, equivalently, to the statement that every normal subgroup of $\mathfrak{G}_A^{pma}(k)$ that is contained in $\mathfrak{U}_A^{ma+}(k)$ must be trivial. If this is the case, we call $\mathfrak{G}_A^{pma}(k)$ *simple in the sense of the Gabber–Kac theorem*, or simply *GK-simple*. This terminology is motivated by its Lie algebra counterpart, stating that, at least in the symmetrisable case, every (graded) ideal of the Kac–Moody algebra $\mathfrak{g}$ that is contained in $\mathfrak{n}^+(A)$ must be trivial: this is an equivalent formulation of the Gabber–Kac theorem[(1)] ([11, Theorem 9.11]). When $\operatorname{char} k = 0$ and A is symmetrisable, $\mathfrak{G}_A^{pma}(k)$ is

(1) Here, we define a Kac–Moody algebra using the Serre relations, as in [11, §5.12]

known to be GK-simple (see [28, Remarque 6.9.1]). However, in the other cases, the following problem is widely open:

Problem 1.1 (GK-simplicity problem). — *Let A be a GCM and let k be a field. Determine when $\mathfrak{G}_A^{pma}(k)$ is GK-simple.*

To give a feeling for the difficulty of Problem 1.1, note that in characteristic zero (say $k=\mathbb{C}$), the GK-simplicity of $\mathfrak{G}_A^{pma}(k)$ is equivalent to the Gabber–Kac theorem for $\mathfrak{g}_k$ (see [17, Remark 8.104(1)]); when A is not symmetrisable, this latter problem remains, decades after it was first considered, completely open. As a second application of Theorem A, we give the first (negative) contribution to Problem 1.1 over finite fields.

Proposition C. — *Let $k=\mathbb{F}_q$ be a finite field. Consider the GCM $A = \left(\begin{smallmatrix} 2 & -m \\ -n & 2\end{smallmatrix}\right)$ with $m,n\geqslant 2$ and $mn>4$. Assume that $m\equiv n\equiv 2 \pmod{q-1}$. If* $\operatorname{char} k=2$, *we moreover assume that at least one of m* and *n is odd. Then $\mathfrak{G}_A^{pma}(k)$ and $\overline{\mathfrak{G}_A}(k)$ are not GK-simple, that is, $Z'_A\cap\overline{U_A^+}(k)\neq\{1\}$.*

The proof of Proposition C is given in Section 4 (see Proposition 4.9). Note that the above counter-examples to GK-simplicity all occur for $\operatorname{char} k<M_A$; the hope is that for $\operatorname{char} k>M_A$, Problem 1.1 has a positive answer.

To illustrate why the Lie correspondence is better behaved when $\operatorname{char} k > M_A$, we make the following observations on the pro-p group $\mathfrak{U}_A^{ma+}(k)$ (for k a finite field of characteristic p) in the light of some important pro-p group concepts, such as the *Zassenhaus–Jennings–Lazard (ZJL) series* (also known as the series of *dimension subgroups*, see [9, §11.1]). Given a pro-p group G with ZJL series $(D_n)_{n\geqslant 1}$, the space $L=\bigoplus_{n\geqslant 1} D_n/D_{n+1}$ has the structure of a graded Lie algebra over $\mathbb{F}_p$, called the *ZJL Lie algebra* of G (see [9, p. 280]).

Proposition D. — *Let A be a GCM and let k be a finite field of characteristic $p>M_A$. Then the following assertions hold:*

1. *The ZJL series of $\mathfrak{U}_A^{ma+}(k)$ coincides with its lower central series.*
2. *The ZJL Lie algebra of $\mathfrak{U}_A^{ma+}(k)$ is isomorphic to $\mathfrak{n}_k^+(A)$, viewed as a Lie algebra over $\mathbb{F}_p$.*

The proof of Proposition D is given in Section 7 below.

We now present a few more functoriality results, as well as results that are either applications of Theorem A or provide motivations for the study of Problem 1.1 (or both) – besides the motivation to clarify the relations between the different completions of $\mathfrak{G}_A(k)$, and hence to provide a unified theory of complete Kac–Moody groups.

For each positive real root $\alpha \in \Delta_+^{\mathrm{re}}(A)$, we let e_α be a $\mathbb{Z}$-basis element of $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\mathbb{Z}}$, and we let $e_i = e_{\alpha_i}$, $i \in I$, be the Chevalley generators of $\mathfrak{n}^+(A)$.

Theorem E. — *Let k be a field, B a GCM, and let $\{\beta_i \mid i \in I\}$ be a linearly independent finite subset of $\Delta_+^{\mathrm{re}}(B)$ such that $\beta_i - \beta_j \notin \Delta(B)$ for all $i, j \in I$. Then the following assertions hold:*

(1) *The matrix $A := (\beta_j(\beta_i^\vee))_{i,j\in I}$ is a GCM and the map $\pi\colon \mathfrak{n}^+(A) \to \mathfrak{n}^+(B) : e_i \mapsto e_{\beta_i}$ is a Lie algebra morphism.*

(2) *π gives rise to a continuous group homomorphism*

$$\widehat{\pi}\colon \mathfrak{U}_A^{ma+}(k) \to \mathfrak{U}_B^{ma+}(k)$$

whose kernel is normal in $\mathfrak{G}_A^{pma}(k)$. In particular, if the group $\mathfrak{G}_A^{pma}(k)$ is GK-simple, then $\widehat{\pi}$ is injective.

(3) *The restriction of $\widehat{\pi}$ to $\mathfrak{U}_A^{ma+}(k) \cap \mathfrak{G}_A(k)$ extends to continuous group homomorphisms*

$$\mathfrak{G}_A(k) \to \mathfrak{G}_B(k) \quad \textit{and} \quad \overline{\mathfrak{G}_A}(k) \to \overline{\mathfrak{G}_B}(k)$$

with kernels contained in Z'_A.

Here, we view $\mathfrak{G}_A(k)$ and $\mathfrak{G}_B(k)$ as subgroups of their Mathieu–Rousseau completion (with the induced topology). Note that, in contrast to Theorem E, the surjective map $\widehat{\pi}\colon \mathfrak{U}_A^{ma+}(k) \to \mathfrak{U}_B^{ma+}(k)$ provided by Theorem A can typically *not* be extended to the whole group $\mathfrak{G}_A^{ma+}(k)$ (or even to $\mathfrak{G}_A(k)$) as soon as $A \neq B$: this is a consequence of the simplicity results for these groups (see [19], [16] and [28, §6.13]). A more precise version of Theorem E is given in Section 3.3 below (see Theorem 3.10).

As a third instance of functoriality properties of Kac–Moody groups, we also establish that every symmetrisable Kac–Moody group $\mathfrak{G}_A(k)$ can be embedded into some *simply laced* Kac–Moody group $\mathfrak{G}_B(k)$, that is, such that the off-diagonal entries of B are either 0 or -1. It is known that any symmetrisable GCM A admits a *simply laced cover*, which is a simply laced GCM B for which there is an embedding $\mathfrak{g}(A) \to \mathfrak{g}(B)$ (see [10, §2.4]).

Theorem F. — *Let k be a field and A be a GCM. Let B be a simply laced cover of A, and consider the associated embedding $\pi\colon \mathfrak{g}(A) \to \mathfrak{g}(B)$. Then π gives rise to continuous group homomorphisms*

$$\mathfrak{G}_A(k) \to \mathfrak{G}_B(k) \quad \textit{and} \quad \overline{\mathfrak{G}_A}(k) \to \overline{\mathfrak{G}_B}(k)$$

with kernels contained in Z'_A.

Note that the embeddings of the minimal Kac–Moody groups (modulo center) provided by Theorem F preserve the corresponding twin BN-pairs and hence induce embeddings of the corresponding twin buildings.

As pointed out to us by B. Mühlherr, similar embeddings can be obtained with a totally different approach (not relying on the Lie algebra), using the techniques developed in [21] (see also [22]). A more precise version of Theorem F is given in Section 3.4 below (see Theorem 3.15).

As a second motivation for the study of Problem 1.1 (besides Theorem E(2)), as well as a third application of Theorem A, we present a contribution to the linearity question of $\mathfrak{U}_A^{ma+}(k)$ for k a finite field. The long-standing question whether $\mathfrak{U}_A^{ma+}(k)$ is linear over some field k' is still open (see [2, §4.2]). Caprace and Stulemeijer [6] proved that, within the class of non-discrete, compactly generated, topologically simple totally disconnected locally compact groups G (of which the simple Kac–Moody groups $\mathfrak{G}_A^{pma}(k)/Z'_A$ for k a finite field are examples), the existence of a linear open subgroup U of G (in the sense that U has a continuous faithful finite-dimensional linear representation over a local field) is equivalent to the linearity of G itself (even more: G is in that case a simple algebraic group over a local field). The following theorem extends this result in the Kac–Moody setting, and addresses the above-mentioned linearity problem for continuous representations over local fields, provided the group $\mathfrak{G}_A^{pma}(k)$ is GK-simple (actually, an *a priori* much weaker version of the GK-simplicity of $\mathfrak{G}_A^{pma}(k)$ would be sufficient in this case, see Remark 5.3 below).

Theorem G. — *Let A be an indecomposable GCM of non-finite type and let k be a finite field. Assume that $\mathfrak{G}_A^{pma}(k)$ is GK-simple and set $G := \mathfrak{G}_A^{pma}(k)/Z'_A$. Then the following assertions are equivalent:*

(1) *Every compact open subgroup of G is just-infinite (i.e. possesses only finite proper quotients).*
(2) *$\mathfrak{U}_A^{ma+}(k)$ is linear over a local field.*
(3) *G is a simple algebraic group over a local field.*
(4) *The matrix A is of affine type.*

The proof of Theorem G relies on the paper [6] (which already contains the implications (2) $\Leftrightarrow$ (3) and (3) $\Rightarrow$ (1)), and is given in Section 5 below.

As a third motivation for the study of Problem 1.1, we also present a contribution to the isomorphism problem for complete Kac–Moody groups over finite fields. The isomorphism problem for minimal Kac–Moody groups has been addressed by P-E. Caprace ([3, Theorem A]). When k is a finite field, the group $\mathfrak{G}_A(k)$ turns out to contain, in general, very little information about A (see [3, Lemma 4.3]). The situation for $\mathfrak{G}_A^{pma}(k)$ is completely different (see [16, Theorem E]), and we expect it to be possible to recover A from $\mathfrak{G}_A^{pma}(k)$ in all cases. This difference between $\mathfrak{G}_A(k)$ and $\mathfrak{G}_A^{pma}(k)$

is in fact related to the non-density of $\mathfrak{G}_A(k)$ in $\mathfrak{G}_A^{pma}(k)$ (see the proof of Proposition B).

Given a GCM $A = (a_{ij})_{i,j\in I}$ and a subset $J \subseteq I$, we define the GCM $A|_J := (a_{ij})_{i,j\in J}$.

Proposition H. — *Let k, k' be finite fields, and let $A = (a_{ij})_{i,j\in I}$ and $B = (b_{ij})_{i,j\in J}$ be GCM. Assume that $p = \operatorname{char} k > M_A, M_B$ and that all rank 2 subgroups of $\mathfrak{G}_A^{pma}(k)$ and $\mathfrak{G}_B^{pma}(k')$ are GK-simple.*

If $\alpha\colon \mathfrak{G}_A^{pma}(k)/Z'_A \to \mathfrak{G}_B^{pma}(k')/Z'_B$ is an isomorphism of topological groups, then $k \cong k'$, and there exist an inner automorphism γ of $\mathfrak{G}_B^{pma}(k')/Z'_B$ and a bijection $\sigma\colon I \to J$ such that

(1) *$\gamma\alpha(\mathfrak{U}^{ma+}_{A|_{\{i,j\}}}(k)) = \mathfrak{U}^{ma+}_{B|_{\{\sigma(i),\sigma(j)\}}}(k')$ for all distinct $i, j \in I$.*

(2) *$B|_{\{\sigma(i),\sigma(j)\}} \in \left\{ \begin{pmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{pmatrix}, \begin{pmatrix} 2 & a_{ji} \\ a_{ij} & 2 \end{pmatrix} \right\}$ for all distinct $i, j \in I$.*

The proof of Proposition H can be found in Section 6 below. Note that if $\mathfrak{G}_A^{pma}(k)$ is of rank 2 and if α lifts to an isomorphism $\alpha\colon \mathfrak{G}_A^{pma}(k) \to \mathfrak{G}_B^{pma}(k')$, then the conclusion of the theorem holds without any GK-simplicity assumption (see Remark 6.9 below).

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2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of nonnegative integers.

2.1. Generalised Cartan matrices

An integral matrix $A = (a_{ij})_{i,j\in I}$ indexed by some finite set I is called a *generalised Cartan matrix* (GCM) if it satisfies the following conditions:

(C1) $a_{ii} = 2$ for all $i \in I$;

(C2) $a_{ij} \leqslant 0$ for all $i, j \in I$ with $i \neq j$;

(C3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Given two GCM $A = (a_{ij})_{i,j\in I}$ and $B = (b_{ij})_{i,j\in J}$, we write $B \leqslant A$ if $J \subseteq I$ and $|b_{ij}| \leqslant |a_{ij}|$ for all $i, j \in J$.

2.2. Kac–Moody algebras

The general reference for this paragraph is [11, Chapters 1–5] (see also [17, Part II]).

Let $A = (a_{ij})_{i,j \in I}$ be a GCM and let $(\mathfrak{h}, \Pi = \{\alpha_i \mid i \in I\}, \Pi^\vee = \{\alpha_i^\vee \mid i \in I\})$ denote a realisation of A, as in [11, §1.1]. Define $\tilde{\mathfrak{g}}(A)$ to be the complex Lie algebra with generators e_i, f_i $(i \in I)$ and $\mathfrak{h}$, and with the following defining relations:

$$\begin{cases} [e_i, f_j] = -\delta_{ij}\alpha_i^\vee & (i, j \in I), \\ [h, h'] = 0 & (h, h' \in \mathfrak{h}), \\ [h, e_i] = \langle \alpha_i, h\rangle e_i, & (i \in I,\, h \in \mathfrak{h}), \\ [h, f_i] = -\langle \alpha_i, h\rangle f_i & (i \in I,\, h \in \mathfrak{h}). \end{cases}$$

Denote by $\tilde{\mathfrak{n}}^+ = \tilde{\mathfrak{n}}^+(A)$ (respectively, $\tilde{\mathfrak{n}}^- = \tilde{\mathfrak{n}}^-(A)$) the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by e_i, $i \in I$ (respectively, f_i, $i \in I$). Then $\tilde{\mathfrak{n}}^+$ (respectively, $\tilde{\mathfrak{n}}^-$) is freely generated by e_i, $i \in I$ (respectively, f_i, $i \in I$), and one has a decomposition

$$\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}^- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}^+ \quad \text{(direct sum of vector spaces).}$$

Moreover, there is a unique maximal ideal $\mathfrak{i}'$ of $\tilde{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially. It decomposes as

$$\mathfrak{i}' = (\mathfrak{i}' \cap \tilde{\mathfrak{n}}^-) \oplus (\mathfrak{i}' \cap \tilde{\mathfrak{n}}^+) \quad \text{(direct sum of ideals),}$$

and contains the ideal $\mathfrak{i}$ of $\tilde{\mathfrak{g}}(A)$ generated by the elements

$$x_{ij}^+ := \operatorname{ad}(e_i)^{1+|a_{ij}|} e_j \in \tilde{\mathfrak{n}}^+ \quad \text{and} \quad x_{ij}^- := \operatorname{ad}(f_i)^{1+|a_{ij}|} f_j \in \tilde{\mathfrak{n}}^-$$

for all $i, j \in I$ with $i \neq j$. The *Kac–Moody algebra* with GCM A is then the complex Lie algebra

$$\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{i}.$$

We keep the same notation for the images of $e_i, f_i, \mathfrak{h}$ in $\mathfrak{g}(A)$. The subalgebra $\mathfrak{h}$ of $\mathfrak{g}(A)$ is called its *Cartan subalgebra*. The elements e_i, f_i $(i \in I)$ are called the *Chevalley generators* of $\mathfrak{g}(A)$. They respectively generate the images $\mathfrak{n}^+ = \mathfrak{n}^+(A)$ and $\mathfrak{n}^- = \mathfrak{n}^-(A)$ of $\tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{n}}^-$ in $\mathfrak{g}(A)$. The *derived Kac–Moody algebra* $\mathfrak{g}_A := [\mathfrak{g}(A), \mathfrak{g}(A)]$ is generated by the Chevalley generators of $\mathfrak{g}(A)$.

Let $Q = Q(A) := \sum_{i \in I} \mathbb{Z}\alpha_i$ denote the free abelian group generated by the *simple roots* $\alpha_1, \dots, \alpha_n$, and set $Q_+ = Q_+(A) := \sum_{i \in I} \mathbb{N}\alpha_i$ and $Q_- = Q_-(A) := -Q_+$. Then $\mathfrak{g}(A)$ admits a Q-gradation. More precisely,

$$\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \bigoplus_{\alpha \in Q_- \setminus \{0\}} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in Q_+ \setminus \{0\}} \mathfrak{g}_\alpha,$$

where for $\alpha \in Q_+ \setminus \{0\}$ (respectively, $\alpha \in Q_- \setminus \{0\}$), the *root space* $\mathfrak{g}_\alpha$ is the linear span of all elements of the form $[e_{i_1}, \dots, e_{i_s}]$ (respectively, $[f_{i_1}, \dots, f_{i_s}]$) such that $\alpha_{i_1} + \dots + \alpha_{i_s} = \alpha$ (respectively, $= -\alpha$). Here we follow the standard notation

$$[x_1, x_2, \dots, x_s] := \mathrm{ad}(x_1)\,\mathrm{ad}(x_2) \dots \mathrm{ad}(x_{s-1})(x_s).$$

The set of *roots* of $\mathfrak{g}(A)$ is

$$\Delta = \Delta(A) := \big\{\alpha \in Q \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\big\}.$$

It decomposes as $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_\pm = \Delta_\pm(A) := \Delta \cap Q_\pm$ is the set of *positive/negative roots*. The subgroup $W = W(A)$ of $\mathrm{GL}(Q)$ generated by the reflections

$$s_i\colon\ Q \to Q : \alpha_j \mapsto \alpha_j - a_{ij}\alpha_i$$

for $i \in I$ stabilises Δ. The W-orbit $W.\{\alpha_i \mid i \in I\} \subseteq \Delta$ is called the set of *real roots* and is denoted $\Delta^{\mathrm{re}} = \Delta^{\mathrm{re}}(A)$. Its complement $\Delta^{\mathrm{im}} = \Delta^{\mathrm{im}}(A) := \Delta \setminus \Delta^{\mathrm{re}}$ is the set of *imaginary roots*. We furthermore set $\Delta^{\mathrm{re}}_\pm = \Delta^{\mathrm{re}}_\pm(A) := \Delta^{\mathrm{re}} \cap \Delta_\pm$ and $\Delta^{\mathrm{im}}_\pm = \Delta^{\mathrm{im}}_\pm(A) := \Delta^{\mathrm{im}} \cap \Delta_\pm$. Given $\alpha = \sum_{i\in I} n_i\alpha_i \in Q$, we call $\mathrm{ht}(\alpha) := \sum_{i\in I} n_i \in \mathbb{Z}$ the *height* of α. The group W also acts linearly on $Q^\vee = Q^\vee(A) := \sum_{i\in I} \mathbb{Z}\alpha_i^\vee$ by

$$s_i(\alpha_j^\vee) = \alpha_j^\vee - a_{ji}\alpha_i^\vee.$$

Given a real root $\alpha = w\alpha_i$ ($w \in W$, $i \in I$), we define the *coroot* of α as $\alpha^\vee := w\alpha_i^\vee \in Q^\vee$. Alternatively, $\alpha^\vee$ is the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\alpha(\alpha^\vee) = 2$.

2.3. Integral enveloping algebra

The general references for this paragraph are [29] and [28, Section 2] (see also [17, Chapter 7]).

Let $A = (a_{ij})_{i,j\in I}$ be a GCM, and consider the corresponding derived Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}_A$. For an element u of the enveloping algebra $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ of $\mathfrak{g}$ and an $s \in \mathbb{N}$, we write

$$u^{(s)} := \frac{u}{s!}, \quad (\mathrm{ad}\,u)^{(s)} := \frac{1}{s!}(\mathrm{ad}\,u)^s,$$

and

$$\binom{u}{s} := \frac{1}{s!}u(u-1)\dots(u-s+1).$$

Let $\mathcal{U}^+, \mathcal{U}^-$ and $\mathcal{U}^0$ be the $\mathbb{Z}$-subalgebras of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ respectively generated by the elements $e_i^{(s)}$ ($i \in I$, $s \in \mathbb{N}$), $f_i^{(s)}$ ($i \in I$, $s \in \mathbb{N}$) and $\binom{h}{s}$ ($h \in \sum_{i\in I} \mathbb{Z}\alpha_i^\vee$,

$s \in \mathbb{N}$). Then the $\mathbb{Z}$-subalgebra $\mathcal{U} = \mathcal{U}(A)$ of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ generated by $\mathcal{U}^+$, $\mathcal{U}^-$ and $\mathcal{U}^0$ is a $\mathbb{Z}$-form of $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$, called the *integral enveloping algebra* of $\mathfrak{g}$. It has the structure of a co-invertible $\mathbb{Z}$-bialgebra with respect to the coproduct ∇, co-unit ϵ, and co-inverse τ, whose restrictions to $\mathcal{U}^+ = \mathcal{U}^+(A)$ are respectively given by

$$\nabla e_i^{(m)} = \sum_{k+l=m} e_i^{(k)} \otimes e_i^{(l)}, \quad \epsilon e_i^{(m)} = 0 \text{ for } m > 0$$

and

$$\tau e_i^{(m)} = (-1)^m e_i^{(m)}.$$

The $\mathbb{Z}$-algebra $\mathcal{U}$ inherits from $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ a natural filtration, as well as a Q-gradation $\mathcal{U} = \bigoplus_{\alpha\in Q} \mathcal{U}_\alpha$. We set $\mathfrak{g}_{\mathbb{Z}} := \mathfrak{g} \cap \mathcal{U}$ and $\mathfrak{n}^+_{\mathbb{Z}} = \mathfrak{n}^+_{\mathbb{Z}}(A) := \mathfrak{n}^+ \cap \mathcal{U}$. For $\alpha \in Q_+$, we also set $(\mathfrak{n}^+_{\mathbb{Z}})_\alpha := \mathfrak{n}^+_{\mathbb{Z}} \cap \mathfrak{g}_\alpha$. For a field k, we similarly write $\mathfrak{g}_k := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$, $\mathfrak{n}^+_k = \mathfrak{n}^+_k(A) := \mathfrak{n}^+_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ and $(\mathfrak{n}^+_k)_\alpha := (\mathfrak{n}^+_{\mathbb{Z}})_\alpha \otimes_{\mathbb{Z}} k$, as well as $\mathcal{U}_k := \mathcal{U} \otimes_{\mathbb{Z}} k$.

A set of roots $\Psi \subseteq \Delta_+$ is called *closed* if for all $\alpha, \beta \in \Psi$: $\alpha + \beta \in \Delta_+ \implies \alpha + \beta \in \Psi$. For a closed set $\Psi \subseteq \Delta_+$, we let $\mathcal{U}(\Psi)$ denote the $\mathbb{Z}$-subalgebra of $\mathcal{U}^+$ generated by all $\mathcal{U}^\alpha := \mathcal{U}_{\mathbb{C}}(\oplus_{n\geqslant 1}\mathfrak{g}_{n\alpha}) \cap \mathcal{U}^+$ for $\alpha \in \Psi$. Given a field k, we define the completion $\widehat{\mathcal{U}}_k(\Psi)$ of $\mathcal{U}(\Psi)$ over k with respect to the Q_+-gradation as

$$\widehat{\mathcal{U}}_k(\Psi) = \prod_{\alpha\in Q_+} (\mathcal{U}(\Psi)_\alpha \otimes_{\mathbb{Z}} k),$$

where $\mathcal{U}(\Psi)_\alpha := \mathcal{U}(\Psi) \cap \mathcal{U}_\alpha$. For $\Psi = \Delta^+$, we also write $\widehat{\mathcal{U}}^+_k = \widehat{\mathcal{U}}^+_k(A) := \widehat{\mathcal{U}}_k(\Delta^+)$, as well as $\mathcal{U}^+_\alpha := \mathcal{U}(\Delta_+)_\alpha$.

For each $i \in I$, the element

$$s_i^* := \exp(\operatorname{ad} e_i)\exp(\operatorname{ad} f_i)\exp(\operatorname{ad} e_i) \in \operatorname{Aut}(\mathcal{U})$$

satisfies $s_i^*(\mathcal{U}_\alpha) = \mathcal{U}_{s_i(\alpha)}$ for all $\alpha \in Q$. We denote by $W^* = W^*(A)$ the subgroup of $\operatorname{Aut}(\mathcal{U})$ generated by the s_i^*, $i \in I$. There is a surjective group homomorphism

$$\pi_W\colon\ W^* \to W : s_i^* \mapsto s_i$$

such that for any $w^* \in W^*$ and any $i \in I$, the pair $E_\alpha := \{w^*e_i, -w^*e_i\}$ only depends on the root $\alpha := \pi_W(w^*)\alpha_i \in \Delta^{\mathrm{re}}$, that is, it is the same for any decomposition $\alpha = \pi_W(v^*)\alpha_j$. Moreover, for any $w \in W$ and any reduced decomposition $w = s_{i_1}s_{i_2}\dots s_{i_k}$ for w, the element $w^* := s^*_{i_1}s^*_{i_2}\dots s^*_{i_k} \in W^*$ only depends on w, and not on the choice of the reduced decomposition for w. For each $\alpha \in \Delta^{\mathrm{re}}$, we make some choice of an element $e_\alpha \in E_\alpha$ (with $e_{\alpha_i} := e_i$ and $e_{-\alpha_i} := f_i$ for $i \in I$), so that $e_\alpha = w^*e_i$ for some $w^* \in W^*$ and $i \in I$ with $\alpha = \pi_W(w^*)\alpha_i$. Then $\{e_\alpha\}$ is a $\mathbb{Z}$-basis for $\mathfrak{g}_\alpha \cap \mathcal{U}$, and we set

$$s_\alpha^* := \exp(\operatorname{ad} e_\alpha)\exp(\operatorname{ad} e_{-\alpha})\exp(\operatorname{ad} e_\alpha) = w^* s_i^* (w^*)^{-1} \in W^*.$$

Lemma 2.1. — *The group W acts on $\mathcal{U}$ by bialgebras morphisms.*

Proof. — Let $u \in \mathcal{U}$ and $i \in I$. Since the coproduct ∇ is an algebra morphism, we have

$$\begin{aligned}
\nabla(s_i^* u) &= \nabla\left(\sum_{n_1,n_2,n_3 \geqslant 0} (\operatorname{ad} e_i)^{(n_1)}(\operatorname{ad} f_i)^{(n_2)}(\operatorname{ad} e_i)^{(n_3)} u\right) \\
&= \sum_{n_1,n_2,n_3 \geqslant 0} (\operatorname{ad} e_i \otimes \mathbf{1} + \mathbf{1} \otimes \operatorname{ad} e_i)^{(n_1)}(\operatorname{ad} f_i \otimes \mathbf{1} \\
&\qquad + \mathbf{1} \otimes \operatorname{ad} f_i)^{(n_2)}(\operatorname{ad} e_i \otimes \mathbf{1} + \mathbf{1} \otimes \operatorname{ad} e_i)^{(n_3)} \nabla(u) \\
&= \sum_{\substack{r_1,r_2,r_3 \geqslant 0 \\ s_1,s_2,s_3 \geqslant 0}} \big((\operatorname{ad} e_i)^{(r_1)}(\operatorname{ad} f_i)^{(r_2)}(\operatorname{ad} e_i)^{(r_3)} \\
&\qquad \otimes (\operatorname{ad} e_i)^{(s_1)}(\operatorname{ad} f_i)^{(s_2)}(\operatorname{ad} e_i)^{(s_3)}\big)\nabla(u) \\
&= (s_i^* \otimes s_i^*)\nabla(u),
\end{aligned}$$

and hence $\nabla s_i^* = (s_i^* \otimes s_i^*)\nabla$. Since clearly $\epsilon s_i^* = \epsilon$ for all $i \in I$, the lemma follows. □

2.4. Minimal Kac–Moody groups

The general references for this paragraph are [29] and [24, Chapter 9] (see also [17, Chapter 7]).

Given a GCM $A = (a_{ij})_{i,j \in I}$, we denote by $\mathfrak{G}_A$ the corresponding *Tits functor* of simply connected type. As a group functor over the category of fields, it is characterised by a small number of properties; one of them ensures that the complex group $\mathfrak{G}_A(\mathbb{C})$ admits an adjoint action by automorphisms on the corresponding derived Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}_A$. *Minimal Kac–Moody groups* are by definition the groups obtained by evaluating such Tits functors over a field k.

The minimal Kac–Moody group $\mathfrak{G}_A(k)$ can be constructed by generators and relations, as follows. For each real root $\alpha \in \Delta^{\mathrm{re}}$, we let U_α denote the affine group scheme over $\mathbb{Z}$ with Lie algebra $\mathfrak{g}_\alpha \cap \mathfrak{g}_{\mathbb{Z}} = \mathbb{Z}e_\alpha$, and we denote by $x_\alpha \colon \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ the isomorphism from the additive group scheme $\mathbb{G}_a$ to

U_α determined by the choice of $e_\alpha \in E_\alpha$ as a $\mathbb{Z}$-basis element, that is,

$$x_\alpha\colon \mathbb{G}_a(k) = (k,+) \xrightarrow{\sim} U_\alpha(k) : r \mapsto \exp(re_\alpha) \quad \text{for any field } k.$$

A pair of roots $\{\alpha,\beta\} \subseteq \Delta^{\mathrm{re}}$ is called *prenilpotent* if there exist $w, w' \in W$ such that $\{w\alpha, w\beta\} \subseteq \Delta^{\mathrm{re}}_+$ and $\{w'\alpha, w'\beta\} \subseteq \Delta^{\mathrm{re}}_-$. In this case, the interval

$$[\alpha,\beta]_{\mathbb{N}} := (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta^{\mathrm{re}}$$

is finite. One then defines a group functor $\mathfrak{St}_A$, called the *Steinberg functor* associated to A, such that for any field k, the group $\mathfrak{St}_A(k)$ is the quotient of the free product of the *real root groups* $U_\gamma(k)$, $\gamma \in \Delta^{\mathrm{re}}$, by the relations

$$[x_\alpha(r), x_\beta(s)] = \prod_\gamma x_\gamma(C^{\alpha\beta}_{ij} r^i s^j) \qquad \text{for any } r,s \in k \text{ and any prenilpotent pair } \{\alpha,\beta\} \subseteq \Delta^{\mathrm{re}}, \tag{2.1}$$

where $\gamma = i\alpha + j\beta$ runs through $]\alpha,\beta[_{\mathbb{N}} := [\alpha,\beta]_{\mathbb{N}} \backslash \{\alpha,\beta\}$ in some prescribed order, and where the $C^{\alpha\beta}_{ij}$ are integral constants (that can be computed) depending on α,β and on the chosen order on $]\alpha,\beta[_{\mathbb{N}}$ (see [24, 9.2.2]). The canonical homomorphisms $U_\gamma(k) \to \mathfrak{St}_A(k)$ turn out to be injective, and we may thus identify each $U_\gamma(k)$ with its image in $\mathfrak{St}_A(k)$. There is a W^*-action on $\mathfrak{St}_A(k)$, defined for any $w^* \in W^*$, $r \in k$ and $\gamma \in \Delta^{\mathrm{re}}$ by

$$w^*(x_\gamma(r)) = w^*(\exp(re_\gamma)) := \exp(rw^*e_\gamma) = x_{w\gamma}(\epsilon r),$$

where $w := \pi_W(w^*) \in W$ and where $\epsilon \in \{\pm 1\}$ corresponds to the choice $e_{w\gamma} = \epsilon w^* e_\gamma \in E_{w\gamma}$. For any $i \in I$ and $r \in k^\times$, we define the element

$$\widetilde{s}_i(r) := x_{\alpha_i}(r) x_{-\alpha_i}(r^{-1}) x_{\alpha_i}(r)$$

of $\mathfrak{St}_A(k)$ and we set $\widetilde{s}_i := \widetilde{s}_i(1)$.

The second step of the construction is to define the *split torus scheme* $\mathfrak{T} = \mathfrak{T}_A$. Let Λ be the free $\mathbb{Z}$-module whose $\mathbb{Z}$-dual $\Lambda^\vee$ is freely generated by $\{\alpha_i^\vee \mid i \in I\}$. In particular, $\{\alpha_i \mid i \in I\} \subseteq \Lambda$, where we view each simple root α_i as a linear functional on $\sum_{i\in I} \mathbb{C}\alpha_i^\vee$. For any field k, we set

$$\mathfrak{T}(k) := \mathrm{Hom}_{\mathrm{gr}}(\Lambda, k^\times) \cong (k^\times)^{|I|}.$$

The torus $\mathfrak{T}(k)$ is then generated by the elements

$$r^{\alpha_i^\vee}\colon\ \Lambda \to k^\times : \lambda \mapsto r^{\alpha_i^\vee}(\lambda) := r^{\langle \lambda, \alpha_i^\vee\rangle}$$

for $r \in k^\times$ and $i \in I$. There is a W-action on $\mathfrak{T}(k)$, defined for any $i,j \in I$ and $r \in k^\times$ by

$$s_i(r^{\alpha_j^\vee}) = r^{s_i(\alpha_j^\vee)} = r^{\alpha_j^\vee - a_{ji}\alpha_i^\vee}.$$

For any field k, the *minimal Kac–Moody group* $\mathfrak{G}_A(k)$ *of simply connected type* is now defined as the quotient of the free product $\mathfrak{St}_A(k) * \mathfrak{T}(k)$ by the following relations, where $i \in I$, $r \in k$ and $t \in \mathfrak{T}(k)$:

$$t \cdot x_{\alpha_i}(r) \cdot t^{-1} = x_{\alpha_i}(t(\alpha_i)r), \tag{2.2}$$

$$\widetilde{s}_i \cdot t \cdot \widetilde{s}_i^{-1} = s_i(t), \tag{2.3}$$

$$\widetilde{s}_i(r^{-1}) = \widetilde{s}_i \cdot r^{\alpha_i^\vee} \qquad \text{for } r \neq 0, \tag{2.4}$$

$$\widetilde{s}_i \cdot u \cdot \widetilde{s}_i^{-1} = s_i^*(u) \qquad \text{for } u \in U_\gamma(k), \quad \gamma \in \Delta^{\mathrm{re}}. \tag{2.5}$$

We let $U^+(k) = U_A^+(k)$ denote the subgroup of $\mathfrak{G}_A(k)$ generated by all $U_\alpha(k)$ with $\alpha \in \Delta_+^{\mathrm{re}}$. The normaliser of $U^+(k)$ in $\mathfrak{G}_A(k)$ is the *standard Borel subgroup* $\mathfrak{B}^+(k) = \mathfrak{T}(k) \ltimes U^+(k)$. The center $\mathcal{Z}_A(k)$ of $\mathfrak{G}_A(k)$ is given by

$$\mathcal{Z}_A(k) = \bigcap_{g \in \mathfrak{G}_A(k)} g\mathfrak{B}^+(k)g^{-1} = \{t \in \mathfrak{T}(k) \mid t(\alpha_i) = 1 \ \ \forall\, i \in I\}. \tag{2.6}$$

We let $\mathfrak{N}(k) = \mathfrak{N}_A(k)$ denote the subgroup of $\mathfrak{G}_A(k)$ generated by $\mathfrak{T}(k)$ and by the elements $\tilde{s}_i$ for $i \in I$. Then the assignment $\widetilde{s}_i \mapsto s_i$ for $i \in I$ induces an isomorphism $\mathfrak{N}(k)/\mathfrak{T}(k) \cong W$, and $(\mathfrak{B}^+(k), \mathfrak{N}(k))$ is a BN-pair for $\mathfrak{G}_A(k)$.

2.5. Mathieu–Rousseau completions

The general reference for this paragraph is [28] (see also [17, §8.5]).

Let $A = (a_{ij})_{i,j \in I}$ be a GCM. For each closed set $\Psi \subseteq \Delta_+(A)$ of positive roots, we let $\mathfrak{U}_\Psi^{ma}$ denote the affine group scheme (viewed as a group functor) whose algebra is the restricted dual $\mathbb{Z}[\mathfrak{U}_\Psi^{ma}] := \bigoplus_{\alpha \in \mathbb{N}\Psi} \mathcal{U}(\Psi)_\alpha^*$ of $\mathcal{U}(\Psi)$. One can then define *real* and *imaginary root groups* $\mathfrak{U}_{(\alpha)} = \mathfrak{U}_{(\alpha)}^A$ in

$$\mathfrak{U}_A^{ma+} := \mathfrak{U}_{\Delta_+}^{ma}$$

by setting $\mathfrak{U}_{(\alpha)} := \mathfrak{U}_{\{\alpha\}}^{ma}$ for $\alpha \in \Delta_+^{\mathrm{re}}$ and $\mathfrak{U}_{(\alpha)} := \mathfrak{U}_{\mathbb{Z}_{>0}\alpha}^{ma}$ for $\alpha \in \Delta_+^{\mathrm{im}}$.

The *Mathieu–Rousseau completion* $\mathfrak{G}_A^{pma}$ of the Tits functor $\mathfrak{G}_A$ is a group functor, with the following properties. It contains the split torus scheme $\mathfrak{T}_A$, as well as the group functors $\mathfrak{U}_A^{ma+}$ and $\mathfrak{N}_A$ as subfunctors. Over a field k, the identification of the real root groups $U_\alpha(k)$ ($\alpha \in \Delta_+^{\mathrm{re}}$) of $\mathfrak{G}_A(k)$ with the corresponding real root groups $\mathfrak{U}_{(\alpha)}(k)$ in $\mathfrak{G}_A^{pma}(k)$ produces injections of $U_A^+(k)$ in $\mathfrak{U}_A^{ma+}(k)$ and of $\mathfrak{G}_A(k)$ in $\mathfrak{G}_A^{pma}(k)$. Again, the normaliser of $\mathfrak{U}_A^{ma+}(k)$ in $\mathfrak{G}_A^{pma}(k)$ is the *standard Borel subgroup* $\mathfrak{B}^{ma+}(k) = \mathfrak{T}(k) \ltimes \mathfrak{U}_A^{ma+}(k)$, and $(\mathfrak{B}^{ma+}(k), \mathfrak{N}(k))$ is a BN-pair for $\mathfrak{G}_A^{pma}(k)$.

The group $\mathfrak{G}_A^{pma}(k)$ is a Hausdorff topological group, with basis of neighbourhoods of the identity the normal subgroups $\mathfrak{U}_n^{ma}(k)$ ($n \in \mathbb{N}$) of $\mathfrak{U}_A^{ma+}(k)$ defined by

$$\mathfrak{U}_n^{ma} = \mathfrak{U}_{A,n}^{ma} := \mathfrak{U}_{\Psi(n)}^{ma} \quad \text{where } \Psi(n) = \{\alpha \in \Delta^+ \mid \operatorname{ht}(\alpha) \geqslant n\}.$$

It is topologically generated by $\mathfrak{G}_A(k)$, together with the imaginary root groups $\mathfrak{U}_{(\alpha)}(k)$, $\alpha \in \Delta_+^{\mathrm{im}}$. Unlike the minimal Kac–Moody group $\mathfrak{G}_A(k)$, the Mathieu–Rousseau completion $\mathfrak{G}_A^{pma}(k)$ of $\mathfrak{G}_A(k)$ is thus obtained by not only "exponentiating" the real root spaces of the derived Kac–Moody algebra $\mathfrak{g}$, but also the imaginary root spaces.

The group functor $\mathfrak{U}_A^{ma+}$ admits a more tractable description in terms of root groups, which we now briefly review. We call an element $x \in \mathfrak{n}_{\mathbb{Z}}^+$ *homogeneous* if $x \in (\mathfrak{n}_{\mathbb{Z}}^+)_\alpha$ for some $\alpha \in \Delta_+$. In this case, we call $\deg(x) := \alpha$ the *degree* of x. Given an homogeneous element $x \in \mathfrak{n}_{\mathbb{Z}}^+$ with $\deg(x) = \alpha$, we call a sequence $(x^{[n]})_{n\in\mathbb{N}}$ an *exponential sequence* for x if it satisfies the following conditions:

(ES1) $x^{[0]} = 1$, $x^{[1]} = x$, and $x^{[n]} \in \mathcal{U}_{n\alpha}$ for all $n \in \mathbb{N}$.
(ES2) $x^{[n]} - x^{(n)}$ has filtration less than n in $\mathcal{U}_{\mathbb{C}}(\mathfrak{g}(A))$ for all $n > 0$.
(ES3) $\nabla(x^{[n]}) = \sum_{k+l=n} x^{[k]} \otimes x^{[l]}$ and $\epsilon(x^{[n]}) = 0$ for all $n > 0$.

For a field k and an element $\lambda \in k$, one can then define the *twisted exponential*

$$[\exp]\lambda x := \sum_{n\geqslant 0} \lambda^n x^{[n]} \in \widehat{\mathcal{U}}_k^+.$$

Note that an exponential sequence $(x^{[n]})_{n\in\mathbb{N}}$ for x always exists and is essentially unique, in the following sense (see [28, §2.9] or [17, Proposition 8.50]): if $(x^{\{n\}})_{n\in\mathbb{N}}$ is another exponential sequence for x with associated twisted exponential $\{\exp\}x$, then for any given choice of exponential sequences $(y^{[m]})_{m\in\mathbb{N}}$ for the homogeneous elements of $\bigoplus_{r\geqslant 2} \mathfrak{g}_{r\alpha\mathbb{Z}}$, there exist (uniquely determined) elements $x_m \in \mathfrak{g}_{m\alpha\mathbb{Z}}$ ($m \geqslant 2$) such that

$$\{\exp\}x = [\exp]x \cdot \prod_{m\geqslant 2} [\exp]x_m.$$

In particular, when $\alpha \in \Delta_+^{\mathrm{re}}$, one has $x^{[n]} = x^{(n)} = x^n/n!$ for all $n \in \mathbb{N}$. The element $[\exp]\lambda x$ satisfies $\epsilon([\exp]\lambda x) = 1$ and is group-like, that is, $\nabla[\exp]\lambda x = [\exp]\lambda x \widehat{\otimes} [\exp]\lambda x$. It is moreover invertible in $\widehat{\mathcal{U}}_k^+$, with inverse $\tau[\exp]\lambda x = \sum_{n\geqslant 0} \lambda^n \tau x^{[n]}$.

Remark 2.2. — As we will see, it is also convenient to allow $x \in \mathfrak{g}_{\alpha\mathbb{Z}}$ in the definition of an exponential sequence to be zero, in which case one has to specify α, so as to make sense of (ES1). In other words, the sequence

$(x^{[n]})_{n\in\mathbb{N}}$ is an exponential sequence for $x=0$ *viewed as an element of* $\mathfrak{g}_{\alpha\mathbb{Z}}$ if it satisfies the conditions (ES1)–(ES3) for that α. Of course, in that case, one should rather call $\sum_{n\geqslant 0} x^{[n]}\otimes r^n$ for r in some ring k the *twisted exponential of x associated to $(x^{[n]})_{n\in\mathbb{N}}$ and to r*, and one should replace the notation $[\exp](rx)$ by some other notation, such as $[\exp](r,x)$ (keeping the notation $[\exp](x)$ for $\sum_{n\geqslant 0} x^{[n]}$).

By the above uniqueness statement, the exponential sequences $(x^{[n]})_{n\in\mathbb{N}}$ for $x=0\in\mathfrak{g}_{\alpha\mathbb{Z}}$ ($\alpha\in\Delta_+$) can be described as follows. Fix a choice of exponential sequences for the homogeneous elements of $\bigoplus_{r\geqslant 2}\mathfrak{g}_{r\alpha\mathbb{Z}}$; in particular, for $y=0$ viewed as a homogeneous element of $\mathfrak{g}_{r\alpha\mathbb{Z}}$ ($r\geqslant 2$), one could take $[\exp]y:=1$. Then there exist (uniquely determined) $x_r\in\mathfrak{g}_{r\alpha\mathbb{Z}}$ ($r\geqslant 2$) such that $[\exp](x):=\sum_{n\geqslant 0}x^{[n]}=\prod_{r\geqslant 2}[\exp](x_r)$; conversely, any such product defines an exponential sequence for $x=0\in\mathfrak{g}_{\alpha\mathbb{Z}}$.

For each $\alpha\in\Delta_+$, let $\mathcal{B}_\alpha$ be a $\mathbb{Z}$-basis of $(\mathfrak{n}^+_{\mathbb{Z}})_\alpha$. For $\alpha\in\Delta^{\mathrm{re}}_+$, we choose $\mathcal{B}_\alpha=\{e_\alpha\}$. For a closed subset $\Psi\subseteq\Delta_+$, we then call $\mathcal{B}_\Psi=\mathcal{B}_\Psi(A):=\bigcup_{\alpha\in\Psi}\mathcal{B}_\alpha$ a *standard $\mathbb{Z}$-basis* of $\mathfrak{n}^+_{\mathbb{Z}}\cap\mathcal{U}(\Psi)$. The announced description of $\mathfrak{U}^{ma+}_A$ is provided by the following proposition.

Proposition 2.3 ([28, Proposition 3.2]). — *Let $\Psi\subseteq\Delta_+$ be closed and let k be a field. Then the following hold:*

1. *$\mathfrak{U}^{ma}_\Psi(k)$ can be identified to the multiplicative subgroup of $\widehat{\mathcal{U}}_k(\Psi)$ consisting of all group-like elements of $\widehat{\mathcal{U}}_k(\Psi)$ of constant term 1.*
2. *Let $\mathcal{B}_\Psi$ be a standard $\mathbb{Z}$-basis of $\mathfrak{n}^+_{\mathbb{Z}}\cap\mathcal{U}(\Psi)$, and choose for each $x\in\mathcal{B}_\Psi$ an exponential sequence. Then $\mathfrak{U}^{ma}_\Psi(k)\subseteq\widehat{\mathcal{U}}_k(\Psi)$ consists of the products*
$$\prod_{x\in\mathcal{B}_\Psi}[\exp]\lambda_x x$$
for $\lambda_x\in k$, where the product is taken in any (arbitrary) chosen order on $\mathcal{B}_\Psi$. The expression of an element of $\mathfrak{U}^{ma}_\Psi(k)$ in the form of such a product is unique.

In this paper, we will always identify $\mathfrak{U}^{ma+}_A(k)$ with a subset of $\widehat{\mathcal{U}}^+_k$, as in Proposition 2.3(1). The conjugation action of the torus $\mathfrak{T}(k)$ on $\mathfrak{U}^{ma+}_A(k)$ is then given by

$$t([\exp]x)t^{-1}=[\exp]t(\alpha)x \tag{2.7}$$

for all $t\in\mathfrak{T}(k)$ and $x\in(\mathfrak{n}^+_k)_\alpha$, $\alpha\in\Delta_+$. Given $i\in I$, $\lambda\in k$ and $\alpha\in\{\pm\alpha_i\}$, we also have a conjugation action of $\exp\lambda e_\alpha$ on $\mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}(k)$ given by

$$\exp(\lambda e_\alpha)u\exp(-\lambda e_\alpha)=\sum_{n\geqslant 0}(\mathrm{ad}\,\lambda e_\alpha)^{(n)}u \tag{2.8}$$

for all $u \in \mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha\}}(k)$.

Lemma 2.4. — *Let $i \in I$, and let $x \in \mathfrak{n}^+_{\mathbb{Z}}$ be an homogeneous element of degree $\alpha \in \Delta_+\setminus\{\alpha_i\}$. Then for any choice of exponential sequence $(x^{[n]})_{n\in\mathbb{N}}$ for x, the sequence $(s_i^* x^{[n]})_{n\in\mathbb{N}}$ is an exponential sequence for $s_i^* x$, and we have*

$$\tilde{s}_i([\exp]x)\tilde{s}_i^{-1} = [\exp](s_i^* x) \in \mathfrak{U}^{ma+}_A(k)$$

for the corresponding twisted exponentials.

Proof. — We first prove that $(s_i^* x^{[n]})_{n\in\mathbb{N}}$ is an exponential sequence for $s_i^* x$. Since s_i^* preserves the natural gradation and filtration on $\mathcal{U} = \mathcal{U}(A)$ and maps $\mathcal{U}_{n\alpha}$ to $\mathcal{U}_{ns_i(\alpha)}$ ($n \in \mathbb{N}$), the axioms (ES1) and (ES2) are clearly satisfied. Since moreover s_i^* acts on $\mathcal{U}$ by bialgebra morphisms by Lemma 2.1, the axiom (ES3) is also satisfied, as desired. The second statement of the lemma follows from (2.8). □

2.6. Gabber–Kac kernel and non-density

The general reference for this paragraph is [28, Section 6] (see also [17, §8.5–§8.6]).

Let $A = (a_{ij})_{i,j\in I}$ be a GCM and k be a field. The minimal Kac–Moody group $\mathfrak{G}_A(k)$ acts strongly transitively by simplicial automorphisms on its positive building X_+, associated to the BN-pair $(\mathfrak{B}^+(k), \mathfrak{N}(k))$ of $\mathfrak{G}_A(k)$. (For general background on buildings and BN-pairs, we refer the reader to [1, Chapter 6]).

The *Rémy–Ronan completion* $\mathfrak{G}^{rr}_A(k)$ of $\mathfrak{G}_A(k)$ (see [26]) is the completion of the image of $\mathfrak{G}_A(k)$ in the automorphism group $\mathrm{Aut}(X_+)$ of X_+, where $\mathrm{Aut}(X_+)$ is equipped with the topology of uniform convergence on bounded sets. The BN-pair $(\mathfrak{B}^{ma+}(k), \mathfrak{N}(k))$ of the Mathieu–Rousseau completion $\mathfrak{G}^{pma}_A(k)$ of $\mathfrak{G}_A(k)$ yields the same building X_+ (possibly with a larger apartment system). The kernel of the action of $\mathfrak{G}^{pma}_A(k)$ on X_+ is given by

$$Z'_A := \bigcap_{g\in\mathfrak{G}^{pma}_A(k)} g\mathfrak{B}^{ma+}(k)g^{-1}$$

and decomposes as $Z'_A = Z_A \cdot (Z'_A \cap \mathfrak{U}^{ma+}_A(k))$, where $Z_A = \mathcal{Z}_A(k)$ is the center of $\mathfrak{G}_A(k)$. We call the intersection $Z'_A \cap \mathfrak{U}^{ma+}_A(k)$ the *Gabber-Kac kernel* of $\mathfrak{G}^{pma}_A(k)$, for reasons that will become clear in Section 2.7 below. It can also be described as

$$Z'_A \cap \mathfrak{U}^{ma+}_A(k) = \bigcap_{u\in\mathfrak{U}^{ma+}_A(k)} uU^{im+}u^{-1},$$

where $U^{im+} := \mathfrak{U}^{ma}_{\Delta^{\mathrm{im}}_+}(k)$ is the *imaginary subgroup* of $\mathfrak{U}^{ma+}_A(k)$. Note that if A is of indefinite type and k is of characteristic zero or is finite, the quotient $\mathfrak{G}^{pma}_A(k)/Z'_A$ is simple (see [16] and [28, Theorem 6.19]).

Unlike the Rémy–Ronan completion, the Mathieu–Rousseau completion $\mathfrak{G}^{pma}_A(k)$ of $\mathfrak{G}_A(k)$ is, in general, not the completion of $\mathfrak{G}_A(k)$ (in its own topology). Note however that $\mathfrak{G}_A(k)$ is dense in $\mathfrak{G}^{pma}_A(k)$ as soon as the characteristic of k is either zero or bigger than

$$M_A := \max_{i\neq j} |a_{ij}|.$$

We denote by $\overline{U^+_A}(k)$ (respectively, $\overline{\mathfrak{G}_A}(k)$) the completion of $U^+_A(k)$ (respectively, $\mathfrak{G}_A(k)$) in $\mathfrak{G}^{pma}_A(k)$. The completions $\overline{\mathfrak{G}_A}(k)$ and $\mathfrak{G}^{rr}_A(k)$ of $\mathfrak{G}_A(k)$ are strongly related: there is a continuous homomorphism

$$\varphi_A \colon \overline{\mathfrak{G}_A}(k) \to \mathfrak{G}^{rr}_A(k)$$

with kernel $Z'_A \cap \overline{\mathfrak{G}_A}(k) = Z_A \cdot (Z'_A \cap \overline{U^+_A}(k))$, which is moreover surjective if k is finite.

2.7. GK-simplicity

The general reference for this paragraph is [28, 6.5] (see also [17, §8.6]).

Let $A = (a_{ij})_{i,j\in I}$ be a GCM and k be a field. By a theorem of Gabber–Kac (see [11, Proposition 1.7 and Theorem 9.11]), every ideal of the derived Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}_A$ intersecting the Cartan subalgebra $\mathfrak{h}$ trivially is reduced to $\{0\}$ (at least when A is symmetrisable). Equivalently, every graded sub-$\mathfrak{g}$-module of $\mathfrak{g}$ that is contained in $\mathfrak{n}^+$ is reduced to $\{0\}$. The Lie algebra $\mathfrak{g}_k$ is called *simple in the sense of the Gabber–Kac theorem*, or simply *GK-simple* if every graded sub-$\mathcal{U}_k$-module of $\mathfrak{g}_k$ that is contained in $\mathfrak{n}^+_k$ is reduced to $\{0\}$. Similarly, the Kac–Moody group $\mathfrak{G}^{pma}_A(k)$ is called *GK-simple* if every normal subgroup of $\mathfrak{G}^{pma}_A(k)$ that is contained in $\mathfrak{U}^{ma+}_A(k)$ is reduced to $\{1\}$.

It is easy to see that the Lie algebra $\mathfrak{g}_k$ is GK-simple if and only if for all $\delta \in \Delta^{\mathrm{im}}_+$, any homogeneous element $x \in \mathfrak{g}_k$ of degree δ such that $(\operatorname{ad} f_i)^{(s)}x = 0$ for all $i \in I$ and $s \in \mathbb{N}$ must be zero. By the Gabber–Kac theorem, $\mathfrak{g}_k$ is GK-simple when A is symmetrisable and $\operatorname{char} k = 0$. When $\operatorname{char} k = p > 0$, this is not true anymore: for instance, the affine Kac–Moody algebra $\mathfrak{g}_k = \mathfrak{sl}_m(k) \otimes_k k[t, t^{-1}]$ is not GK-simple as soon as p divides m. Note, however, that the corresponding Kac–Moody group $\mathfrak{G}^{pma}_A(k) = \mathrm{SL}_m(k(\!(t)\!))$ is GK-simple (see [28, Exemple 6.8]).

Note that $\mathfrak{G}_A^{pma}(k)$ is GK-simple if and only if its Gabber–Kac kernel $Z'_A \cap \mathfrak{U}_A^{ma+}(k)$ is trivial, that is, if and only if $Z'_A = Z_A$. If $\mathfrak{g}_k$ is GK-simple and k is infinite, then $\mathfrak{G}_A^{pma}(k)$ is GK-simple by [28, Remarque 6.9.1]. In particular, $\mathfrak{G}_A^{pma}(k)$ is GK-simple as soon as A is symmetrisable and $\operatorname{char} k = 0$.

3. Functoriality

In this section, given two GCM A and B, we define a family of Lie algebra maps $\mathfrak{n}^+(A) \to \mathfrak{n}^+(B)$, which we call $\mathbb{Z}$-*regular*, and which give rise to continuous group homomorphisms $\mathfrak{U}_A^{ma+}(k) \to \mathfrak{U}_B^{ma+}(k)$ over any field k. We then give concrete examples of such maps, respectively yielding surjective and injective exponentials, as in Theorems A and E. Finally, we show how Theorem F can be deduced using the same lines of proof.

3.1. The exponential of a $\mathbb{Z}$-regular map

Definition 3.1. — *Let $A = (a_{ij})_{i,j\in I}$ and B be two GCM. We call a map $\pi\colon \mathfrak{n}^+(A) \to \mathfrak{n}^+(B)$* $\mathbb{Z}$-regular *if it is a Lie algebra morphism such that for each $i \in I$, there is some $\beta_i \in \Delta_+^{\mathrm{re}}(B)$ with $\pi(e_i) \in (\mathfrak{n}_{\mathbb{Z}}^+(B))_{\beta_i}$. In this case, we denote by $\overline{\pi}\colon Q(A) \to Q(B)$ the $\mathbb{Z}$-linear map defined by*

$$\overline{\pi}(\alpha_i) = \beta_i \quad \forall\, i \in I.$$

Theorem 3.2. — *Let k be a field, and let $A = (a_{ij})_{i,j\in I}$ and B be two GCM. Let $\pi\colon \mathfrak{n}^+(A) \to \mathfrak{n}^+(B)$ be $\mathbb{Z}$-regular. Then there is a continuous group homomorphism*

$$\widehat{\pi}\colon \mathfrak{U}_A^{ma+}(k) \to \mathfrak{U}_B^{ma+}(k)$$

such that for any nonzero homogeneous $x \in \mathfrak{n}_{\mathbb{Z}}^+(A)$ and any choice of exponential sequence for x, there is a choice of exponential sequence $(\pi(x)^{[n]})_{n\in\mathbb{N}}$ for $\pi(x)$ such that

$$\widehat{\pi}([\exp]\lambda x) = \sum_{n\in\mathbb{N}} \lambda^n \pi(x)^{[n]} \quad \textit{for all } \lambda \in k. \tag{3.1}$$

Proof. — By assumption, there exist for each $i \in I$ some real root $\beta_i \in \Delta_+^{\mathrm{re}}(B)$ and some $\lambda_i \in \mathbb{Z}$ such that

$$\pi(e_i) = \lambda_i e_{\beta_i} \quad \text{for all } i \in I.$$

Since $e_{\beta_i}^{(n)} \in \mathcal{U}^+(B)$ for all $i \in I$ and $n \in \mathbb{N}$, the map $\mathcal{U}_{\mathbb{C}}(\mathfrak{n}^+(A)) \to \mathcal{U}_{\mathbb{C}}(\mathfrak{n}^+(B))$ lifting π at the level of the corresponding enveloping algebras restricts to an algebra morphism

$$\pi_1 \colon \mathcal{U}^+(A) \to \mathcal{U}^+(B).$$

Since $W(B)$ acts on $\mathcal{U}^+(B)$ by bialgebra morphisms (see Lemma 2.1), we get

$$\begin{aligned}\nabla_B \pi_1(e_i^{(m)}) = \lambda_i^m \nabla_B e_{\beta_i}^{(m)} &= \lambda_i^m \sum_{r+s=m} e_{\beta_i}^{(r)} \otimes e_{\beta_i}^{(s)} \\ &= (\pi_1 \otimes \pi_1) \sum_{r+s=m} e_i^{(r)} \otimes e_i^{(s)} = (\pi_1 \otimes \pi_1) \nabla_A e_i^{(m)}\end{aligned}$$

for all $i \in I$ and $m \in \mathbb{N}$, where ∇_X denotes the coproduct on $\mathcal{U}^+(X)$, $X = A, B$. Hence $\nabla_B \pi_1 = (\pi_1 \otimes \pi_1)\nabla_A$. Similarly, denoting by ϵ_X the co-unit on $\mathcal{U}^+(X)$, we have $\epsilon_B \pi_1 = \epsilon_A$, and hence π_1 is a bialgebra morphism.

Note also that π_1 preserves the natural gradations on $\mathcal{U}^+(A)$ and $\mathcal{U}^+(B)$, in the sense that

$$\pi_1(\mathcal{U}^+_\alpha(A)) \subseteq \mathcal{U}^+_{\overline{\pi}(\alpha)}(B) \quad \text{for all } \alpha \in Q_+(A). \tag{3.2}$$

In particular, the map

$$\mathcal{U}^+(A) \otimes_{\mathbb{Z}} k \to \mathcal{U}^+(B) \otimes_{\mathbb{Z}} k$$

obtained from π_1 by extension of scalars can be further extended to a bialgebra morphism

$$\pi_2 \colon \widehat{\mathcal{U}}^+_k(A) \to \widehat{\mathcal{U}}^+_k(B)$$

between the corresponding completions. Finally, since π_2 preserves the group-like elements of constant term 1, it restricts to a group homomorphism

$$\widehat{\pi} \colon \mathfrak{U}^{ma+}_A(k) \to \mathfrak{U}^{ma+}_B(k)$$

by Proposition 2.3 (1).

Let now $x \in \mathfrak{n}^+_{\mathbb{Z}}(A)$ be homogeneous of degree $\alpha \in \Delta_+(A)$, and choose an exponential sequence $(x^{[n]})_{n\in\mathbb{N}}$ for x. Then $y := \pi(x) \in \mathfrak{n}^+_{\mathbb{Z}}(B)$ is homogeneous of degree $\overline{\pi}(\alpha) \in Q_+(B)$. We claim that the sequence $(y^{[n]})_{n\in\mathbb{N}}$ defined by

$$y^{[n]} := \pi_1(x^{[n]}) \quad \text{for all } n \in \mathbb{N} \tag{3.3}$$

is an exponential sequence for y (viewed as an element of degree $\overline{\pi}(\alpha)$ if $y = 0$, cf. Remark 2.2), so that

$$\widehat{\pi}([\exp]\lambda x) = \sum_{n\in\mathbb{N}} \lambda^n \pi(x)^{[n]} \quad \text{for all } \lambda \in k. \tag{3.4}$$

Indeed, $y^{[0]} = 1$ and $y^{[1]} = y$ by the corresponding properties for x. Since $\pi_1(\mathcal{U}^+_{n\alpha}(A)) \subseteq \mathcal{U}^+_{n\overline{\pi}(\alpha)}(B)$ for all $n \in \mathbb{N}$, we also have $y^{[n]} \in \mathcal{U}^+_{n\overline{\pi}(\alpha)}(B)$ for all n, so that the condition (ES1) is satisfied. Similarly,

$$y^{[n]} - y^{(n)} = \pi_1(x^{[n]}) - \pi_1(x)^{(n)} = \pi_1(x^{[n]} - x^{(n)})$$

has filtration less than n in $\mathcal{U}_{\mathbb{C}}(\mathfrak{g}(B))$, because π_1 preserves the natural filtrations, yielding (ES2). Finally, (ES3) readily follows from the corresponding property for x and the fact that π_1 is a bialgebra morphism.

Note that (3.2) and (3.4), together with Proposition 2.3 (2), imply that

$$\widehat{\pi}(\mathfrak{U}^A_{(\alpha)}(k)) \subseteq \mathfrak{U}^B_{(\overline{\pi}(\alpha))}(k) \quad \text{for all } \alpha \in \Delta_+(A), \tag{3.5}$$

where $\mathfrak{U}^B_{(\overline{\pi}(\alpha))}(k) := \{1\}$ if $\overline{\pi}(\alpha) \notin \Delta_+(B)$ (see also Remark 2.2). Since $\mathrm{ht}(\overline{\pi}(\alpha)) \to \infty$ as $\mathrm{ht}(\alpha) \to \infty$, $\alpha \in \Delta_+(A)$, we deduce in particular that $\widehat{\pi}$ is continuous. This concludes the proof of the theorem. $\square$

Definition 3.3. — *For a $\mathbb{Z}$-regular map $\pi\colon \mathfrak{n}^+(A) \to \mathfrak{n}^+(B)$, we have just proved that the unique continuous map*

$$\widehat{\pi}\colon \mathfrak{U}^{ma+}_A(k) \to \mathfrak{U}^{ma+}_B(k)$$

defined on the (topological) generators $[\exp]\lambda x$ of $\mathfrak{U}^{ma+}_A(k)$ by the formulas (3.3) and (3.4) (where $\lambda \in k$, $x \in \mathfrak{n}^+_{\mathbb{Z}}(A)$ is a homogeneous element, and $[\exp]\lambda x = \sum_{n\geqslant 0} \lambda^n x^{[n]}$ a twisted exponential) is a group homomorphism, which we call the exponential *of π.*

3.2. Surjective $\mathbb{Z}$-regular maps

Lemma 3.4. — *Let $A = (a_{ij})_{i,j\in I}$ be a GCM, and let $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{i}$ be the associated Kac–Moody algebra. Then $\mathfrak{i}$ decomposes as a direct sum of ideals $\mathfrak{i} = \mathfrak{i}^+ \oplus \mathfrak{i}^-$, where $\mathfrak{i}^\pm \subseteq \tilde{\mathfrak{n}}^\pm$ is generated, as an ideal of the Lie algebra $\tilde{\mathfrak{n}}^\pm$, by the elements $x^\pm_{ij}$, $i, j \in I$.*

Proof. — Let $\mathfrak{i}^\pm$ denote the ideal of $\tilde{\mathfrak{n}}^\pm$ generated by the elements $x^\pm_{ij}$, $i, j \in I$. We claim that $[f_k, x^+_{ij}] = 0$ for all $i, j, k \in I$. If $k \neq i$, this is clear. For $k = i$, this follows from the formula

$$[f_i, (\mathrm{ad}\, e_i)^m e_j] = m(m - 1 - |a_{ij}|)(\mathrm{ad}\, e_i)^{m-1} e_j,$$

obtained by an easy induction on $m \geqslant 0$. This implies that $\mathfrak{i}^+$ is in fact an ideal in $\tilde{\mathfrak{g}}(A)$, and similarly for $\mathfrak{i}^-$. In particular, $\mathfrak{i} = \mathfrak{i}^+ \oplus \mathfrak{i}^-$, as desired. $\square$

Lemma 3.5. — *Let $A = (a_{ij})_{i,j\in I}$ and $B = (b_{ij})_{i,j\in J}$ be two GCM such that $B \leqslant A$. Then the assignment $e_i \mapsto e_i$ if $i \in J$ and $e_i \mapsto 0$ otherwise defines a surjective Lie algebra morphism*

$$\pi_{AB}\colon\ \mathfrak{n}_+(A) \to \mathfrak{n}_+(B)$$

such that $\pi(\mathfrak{g}(A)_\alpha) = \mathfrak{g}(B)_\alpha$ for all $\alpha \in Q_+(B) = \sum_{i\in J}\mathbb{N}\alpha_i \subseteq Q_+(A) = \sum_{i\in I}\mathbb{N}\alpha_i$. In particular, π_{AB} is $\mathbb{Z}$-regular and

$$\Delta_+(B) \subseteq \Delta_+(A).$$

Proof. — The assignment $e_i \mapsto e_i$ if $i \in J$ and $e_i \mapsto 0$ otherwise defines a surjective Lie algebra morphism $\tilde{\pi}_{AB}\colon \tilde{\mathfrak{n}}_+(A) \to \tilde{\mathfrak{n}}_+(B)$, which by hypothesis maps the ideal $\mathfrak{i}^+(A)$ inside the ideal $\mathfrak{i}^+(B)$. In particular, $\tilde{\pi}_{AB}$ factors through a surjective Lie algebra morphism $\pi_{AB}\colon \mathfrak{n}_+(A) \to \mathfrak{n}_+(B)$ by Lemma 3.4. Since $\mathfrak{g}(A)_\alpha$ is spanned by all iterated brackets $[e_{i_1}, \dots, e_{i_s}]$ with $\alpha_{i_1} + \cdots + \alpha_{i_s} = \alpha$ ($\alpha \in Q_+(A)$) and similarly for $\mathfrak{g}(B)_\alpha$, the other claims follow. □

Theorem 3.6. — *Let k be a field. Let A and B be two GCM such that $B \leqslant A$. Let $\pi_{AB}\colon \mathfrak{n}_+(A) \to \mathfrak{n}_+(B)$ be the corresponding $\mathbb{Z}$-regular map, as in Lemma 3.5. Then the following hold:*

(1) *The exponential $\widehat{\pi}_{AB}\colon \mathfrak{U}^{ma+}_A(k) \to \mathfrak{U}^{ma+}_B(k)$ of π_{AB} is surjective, continuous and open.*
(2) *For any closed set of roots $\Psi_A \subseteq \Delta_+(A)$,*

$$\widehat{\pi}(\mathfrak{U}^{ma}_{\Psi_A}(k)) = \mathfrak{U}^{ma}_{\Psi_A\cap\Delta_+(B)}(k).$$

In particular, $\widehat{\pi}(\mathfrak{U}^A_{(\alpha)}(k)) = \mathfrak{U}^B_{(\alpha)}(k)$ for all $\alpha \in \Delta^+(A)$, where $\mathfrak{U}^B_{(\alpha)}(k) := \{1\}$ if $\alpha \notin \Delta^+(B)$.

Proof. — Recall that, for any closed set of roots $\Psi_A \subseteq \Delta_+(A)$, the subgroup $\mathfrak{U}^{ma}_{\Psi_A}(k)$ of $\mathfrak{U}^{ma+}_A(k)$ is topologically generated by the twisted exponentials $[\exp]\lambda x$ for $\lambda \in k$ and $x \in \mathfrak{n}^+_{\mathbb{Z}}(A)$ a homogeneous element of degree in Ψ_A (and similarly for subgroups of $\mathfrak{U}^{ma+}_B(k)$). The surjectivity of $\widehat{\pi}_{AB}$ as well as the second statement of the theorem thus readily follow from (3.1). In particular, $\widehat{\pi}_{AB}(\mathfrak{U}^{ma}_{A,n}(k)) = \mathfrak{U}^{ma}_{B,n}(k)$ for all $n \in \mathbb{N}$, and hence π_{AB} is also open, as desired. □

Corollary 3.7. — *Let k be a field. Let A and B be two GCM such that $B \leqslant A$. Then the map $\widehat{\pi}_{AB}\colon \mathfrak{U}^{ma+}_A(k) \to \mathfrak{U}^{ma+}_B(k)$ restricts to group homomorphisms*

$$U^+_A(k) \to U^+_B(k) \quad \text{and} \quad \overline{U^+_A}(k) \to \overline{U^+_B}(k).$$

Proof. — By Lemma 3.5, we may identify $\Delta_+(B)$ with a subset of $\Delta_+(A)$. Since a root α is real if and only if 2α is not a root by [11, Propositions 5.1 and 5.5], we deduce that $\Delta_+^{\mathrm{re}}(A) \cap \Delta_+(B) \subseteq \Delta_+^{\mathrm{re}}(B)$. It then follows from Theorem 3.6 that $\widehat{\pi}_{AB}(\mathfrak{U}^A_{(\alpha)}(k)) \subseteq U_B^+(k)$ for any $\alpha \in \Delta_+^{\mathrm{re}}(A)$, and hence that $\widehat{\pi}_{AB}$ restricts to a map $U_A^+(k) \to U_B^+(k)$. The corresponding statement for the completions follows from the continuity of $\widehat{\pi}_{AB}$. $\square$

Remark 3.8. — Let $A = (a_{ij})_{i,j\in I}$ and $B = (b_{ij})_{i,j\in I}$ be two GCM such that $B \leqslant A$. Then the restriction $U_A^+(k) \to U_B^+(k)$ of $\widehat{\pi}_{AB}$ provided by Corollary 3.7 is, in general, not surjective anymore. Indeed, assume for instance that the matrices A and B are symmetric, and for $X \in \{A, B\}$, let $(\cdot,\cdot)_X$ denote the bilinear form on $Q_+(X)$ introduced in [11, §2.1]. Thus, given $\alpha = \sum_{i\in I} n_i\alpha_i \in Q_+(A) = Q_+(B)$ with support $J := \{i \in I \mid n_i \neq 0\}$, we have

$$\begin{aligned}(\alpha,\alpha)_A &= \sum_{i,j\in I} n_in_ja_{ij} = 2\sum_{i\in I} n_i^2 - \sum_{i\neq j} n_in_j|a_{ij}| \\ &\leqslant 2\sum_{i\in I} n_i^2 - \sum_{i\neq j} n_in_j|b_{ij}| = (\alpha,\alpha)_B.\end{aligned} \tag{3.6}$$

Moreover, if $\alpha \in \Delta_+(X)$, then $\alpha \in \Delta_+^{\mathrm{re}}(X)$ if and only if $(\alpha,\alpha)_X = 2$, while $\alpha \in \Delta_+^{\mathrm{im}}(X)$ if and only if $(\alpha,\alpha)_X \leqslant 0$ (see [11, Propositions 3.9 and 5.2]). In particular, if $a_{ij} \neq b_{ij}$ for some i,j in the support J of the real root $\alpha \in \Delta_+^{\mathrm{re}}(A)$, then $\alpha \notin \Delta_+(B)$, because $(\alpha,\alpha)_A < (\alpha,\alpha)_B$ by (3.6). Hence in that case the real root group $\mathfrak{U}^A_{(\alpha)}(k)$ is in the kernel of $\widehat{\pi}_{AB}$. For instance, if $A = \left(\begin{smallmatrix} 2 & -a \\ -a & 2\end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 2 & -b \\ -b & 2\end{smallmatrix}\right)$ with $b < a$, then $\widehat{\pi}_{AB}(\mathfrak{U}^A_\alpha(k)) = \{1\}$ for all $\alpha \in \Delta_+^{\mathrm{re}}(A) \setminus \{\alpha_1, \alpha_2\}$. Thus, in that case, $\widehat{\pi}_{AB}(U_A^+(k))$ is the subgroup of $U_B^+(k)$ generated by the real root groups $\mathfrak{U}^B_{\alpha_1}(k)$ and $\mathfrak{U}^B_{\alpha_2}(k)$ associated to the simple roots.

3.3. Injective $\mathbb{Z}$-regular maps

Lemma 3.9. — *Let B be a GCM, and let $\{\beta_i \mid i \in I\}$ be a finite subset of $\Delta_+^{\mathrm{re}}(B)$ such that $\beta_i - \beta_j \notin \Delta(B)$ for all $i,j \in I$. Then the matrix $A := (\beta_j(\beta_i^\vee))_{i,j\in I}$ is a GCM. Moreover, the assignment $e_i \mapsto e_{\beta_i}$, $f_i \mapsto e_{-\beta_i}$ for all $i \in I$ defines a Lie algebra morphism $\pi\colon \mathfrak{g}_A \to \mathfrak{g}_B$.*

Proof. — Let $\widetilde{\pi}$ be the Lie algebra morphism from the free complex Lie algebra on the generators $\{e_i, f_i \mid i \in I\}$ to $\mathfrak{g}_B$ defined by the assignment $e_i \mapsto e_{\beta_i}$, $f_i \mapsto e_{-\beta_i}$ for all $i \in I$. Since $\beta_j - \beta_i \notin \Delta(B)$ for all $i,j \in I$, we deduce from [11, Corollary 3.6] that $\beta_j(\beta_i^\vee) \leqslant 0$, so that A is indeed a GCM. Moreover,

$$[e_{\beta_i}, e_{-\beta_j}] = 0 \quad \text{for all } i,j \in I \text{ with } i \neq j.$$

Similarly, since $s_i(\beta_j - \beta_i) = (|\beta_j(\beta_i^\vee)| + 1)\beta_i + \beta_j \notin \Delta(B)$, we have

$$(\operatorname{ad} e_{\pm\beta_i})^{|\beta_j(\beta_i^\vee)|+1} e_{\pm\beta_j} = 0 \quad \text{for all } i,j \in I \text{ with } i \neq j.$$

Finally, the elements $\beta_i^\vee = [e_{-\beta_i}, e_{\beta_i}]$ of $\mathfrak{g}_B$ $(i \in I)$ satisfy

$$[\beta_i^\vee, \beta_j^\vee] = 0 \quad \text{and} \quad [\beta_i^\vee, e_{\pm\beta_j}] = \pm\beta_j(\beta_i^\vee) e_{\pm\beta_j}$$

for all $i,j \in I$. Hence all the defining relations of $\mathfrak{g}_A = [\mathfrak{g}(A), \mathfrak{g}(A)]$ (see Section 2.2) lie in the kernel of $\widetilde{\pi}$, so that $\widetilde{\pi}$ factors through a Lie algebra morphism $\pi\colon\, \mathfrak{g}_A \to \mathfrak{g}_B$. □

Theorem 3.10. — *Let k be a field, B a GCM, and let $\{\beta_i \mid i \in I\}$ be a linearly independent finite subset of $\Delta_+^{\mathrm{re}}(B)$ such that $\beta_i - \beta_j \notin \Delta(B)$ for all $i,j \in I$. Let $A := (\beta_j(\beta_i^\vee))_{i,j\in I}$ be the corresponding GCM, and consider the $\mathbb{Z}$-regular map $\pi\colon\, \mathfrak{n}^+(A) \to \mathfrak{n}^+(B) : e_i \mapsto e_{\beta_i}$. Then the following holds:*

1. *The kernel of the exponential $\widehat{\pi}\colon\, \mathfrak{U}_A^{ma+}(k) \to \mathfrak{U}_B^{ma+}(k)$ of π is a normal subgroup of $\mathfrak{G}_A^{pma}(k)$. In particular, if $\mathfrak{G}_A^{pma}(k)$ is GK-simple, then $\widehat{\pi}$ is injective.*
2. *The restriction of $\widehat{\pi}$ to $U_A^+(k)$ extends to continuous group homomorphisms*

$$\mathfrak{G}_A(k) \to \mathfrak{G}_B(k) \quad \text{and} \quad \overline{\mathfrak{G}_A}(k) \to \overline{\mathfrak{G}_B}(k)$$

with kernels respectively contained in $\mathcal{Z}_A(k)$ and $\mathcal{Z}_A(k) \cdot (Z'_A \cap \overline{U_A^+}(k))$. Here, we view $\mathfrak{G}_A(k)$ and $\mathfrak{G}_B(k)$ as subgroups of $\mathfrak{G}_A^{pma}(k)$ and $\mathfrak{G}_B^{pma}(k)$ respectively, with the induced topology.

Proof. — By Lemma 3.9, the $\mathbb{Z}$-regular map π extends to a Lie algebra morphism

$$\mathfrak{g}_A \to \mathfrak{g}_B : e_i \mapsto e_{\beta_i},\ f_i \mapsto e_{-\beta_i}.$$

Since $e_{\pm\beta_i}^{(n)} \in \mathcal{U}(B)$ for all $i \in I$ and $n \in \mathbb{N}$, this then yields a $\mathbb{Z}$-algebra morphism $\mathcal{U}(A) \to \mathcal{U}(B)$, which in turn extends to a k-algebra morphism

$$\pi_1\colon\, \mathcal{U}_k(A) \to \mathcal{U}_k(B).$$

Note that the $\mathbb{Z}$-linear map

$$\overline{\pi}\colon\, Q(A) \to Q(B) : \alpha_i \mapsto \beta_i$$

induced by π is injective because the β_i are linearly independent. Set $K := \ker\widehat{\pi}$, where $\widehat{\pi}$ is the exponential of π (see Definition 3.3).

Choose a $\mathbb{Z}$-basis $\mathcal{B}$ of $\mathfrak{n}_{\mathbb{Z}}^{+}(A)$, as well as exponential sequences for the elements of $\mathcal{B}$. Choose also exponential sequences for the elements $\pi(x) \in \mathfrak{n}_{\mathbb{Z}}^{+}(B)$, $x \in \mathcal{B}$, as in Theorem 3.2, so that for any $y = \prod_{x\in\mathcal{B}} [\exp]\lambda_x x \in \mathfrak{U}_A^{ma+}(k)$ we have

$$\widehat{\pi}(y) = \prod_{x\in\mathcal{B}} [\exp]\lambda_x \pi(x) \in \mathfrak{U}_B^{ma+}(k).$$

Thus, if $y \in K$, then for any $i \in I$ the component of degree β_i of $\prod_{x\in\mathcal{B}} [\exp]\lambda_x \pi(x) \in \widehat{\mathcal{U}}_k^{+}$ must be zero. Since $\overline{\pi}$ is injective and $\pi(e_i) = e_{\beta_i} \neq 0$, this implies that $\lambda_{e_i} = 0$. Hence $K \subseteq \mathfrak{U}^{ma}_{\Delta_+\setminus\{\alpha_i \mid i\in I\}}(k)$.

For each real root α and each $r \in k^{\times}$, we set

$$s_\alpha^*(r) := \exp(\operatorname{ad} re_\alpha)\exp(\operatorname{ad} r^{-1}e_{-\alpha})\exp(\operatorname{ad} re_\alpha) \in \operatorname{Aut}(\mathcal{U}_k),$$

so that $s_\alpha^* = s_\alpha^*(1)$ (cf. Section 2.3). For any $i \in I$ and $r \in k^{\times}$, any homogeneous $x \in \mathfrak{n}_k^{+}(A)$ of degree $\alpha \neq \alpha_i$ and any choice of exponential sequence $(x^{[n]})_{n\in\mathbb{N}}$ for x, we deduce from (2.8) that

$$\begin{aligned}\widehat{\pi}\big(\widetilde{s}_i(r)([\exp]x)\widetilde{s}_i(r)^{-1}\big) &= \widehat{\pi}\left(\sum_{n\geqslant 0} s^*_{\alpha_i}(r)x^{[n]}\right) = \sum_{n\geqslant 0}\pi_1\big(s^*_{\alpha_i}(r)x^{[n]}\big)\\ &= \sum_{n\geqslant 0} s^*_{\beta_i}(r)\pi_1\big(x^{[n]}\big) = s^*_{\beta_i}(r)\big(\widehat{\pi}([\exp]x)\big).\end{aligned}$$

In particular, $\widetilde{s}_i(r)K\,\widetilde{s}_i(r)^{-1} \subseteq K$ for any $i \in I$ and $r \in k^{\times}$. Since the torus $\mathfrak{T}(k)$ is generated by

$$\big\{\widetilde{s}_i^{\,-1}\widetilde{s}_i(r) \mid i \in I,\ r \in k^{\times}\big\}$$

(see Section 2.4), we deduce that $\mathfrak{N}_A(k) \subseteq \mathfrak{G}_A^{pma}(k)$ normalises K. As $\mathfrak{G}_A^{pma}(k)$ is generated by $\mathfrak{U}_A^{ma+}(k)$ and $\mathfrak{N}_A(k)$, we conclude that K is a normal subgroup of $G_A^{pma}(k)$, proving (1).

We now turn to the proof of (2). Let $X \in \{A, B\}$, and let I_X denote the indexing set of X. Given $w \in W(X)$ and a reduced decomposition $w = s_{i_1}s_{i_2}\dots s_{i_k}$ for w, we write

$$w^* := s^*_{i_1}s^*_{i_2}\dots s^*_{i_k} \in W^*$$

and

$$\widetilde{w} := \widetilde{s}_{i_1}\widetilde{s}_{i_2}\dots\widetilde{s}_{i_k} \in \mathfrak{N}_X(k) \subseteq \mathfrak{St}_X(k).$$

We recall that w^* depends only on w. Similarly, the coset $\widetilde{w}\,\mathfrak{T}_X(k)$ is uniquely determined by w. The relations (2.3) and (2.5) in $\mathfrak{G}_X(k)$ respectively imply that

$$\widetilde{w}\cdot t\cdot\widetilde{w}^{-1} = w(t) \quad \text{for any } t \in \mathfrak{T}_X(k) \tag{3.7}$$

and

$$\widetilde{w}\cdot u\cdot\widetilde{w}^{-1}=w^*(u)\quad\text{for any } u\in\mathfrak{St}_X(k). \tag{3.8}$$

Moreover, in view of the relations (2.4), the torus $\mathfrak{T}_X(k)$ is generated by the elements

$$r^{\alpha_i^\vee}=\widetilde{s}_i^{-1}\widetilde{s}_i(r^{-1})\quad\text{for all } r\in k^\times \text{ and } i\in I_X. \tag{3.9}$$

For each positive real root $\gamma\in\Delta_+^{\mathrm{re}}(X)$, we fix some $w_\gamma\in W(X)$ and some $i_\gamma\in I_X$ such that $\gamma=w_\gamma\alpha_{i_\gamma}$ (with the choice $w_\gamma=1$ if $\gamma=\alpha_i$), and we choose the basis elements $e_\gamma\in E_\gamma$ and $e_{-\gamma}\in E_{-\gamma}$ so that $e_\gamma=w_\gamma^*e_{i_\gamma}$ and $e_{-\gamma}=w_\gamma^*f_{i_\gamma}$. To lighten the notation, we will also write $w_j:=w_{\beta_j}\in W(B)$ and $\sigma_j:=i_{\beta_j}$ for all $j\in I_A$, so that

$$\beta_i=w_i\alpha_{\sigma_i}\quad\text{and}\quad e_{\pm\beta_i}=w_i^*e_{\pm\alpha_{\sigma_i}}\quad\text{for all } i\in I_A.$$

Defining for all $\gamma\in\Delta_+^{\mathrm{re}}(X)$ the reflection

$$s_\gamma: Q(X)\to Q(X):\lambda\mapsto\lambda-\langle\lambda,\gamma^\vee\rangle\,\gamma,$$

we then have $s_\gamma=w_\gamma s_{i_\gamma}w_\gamma^{-1}\in W(X)$. We will also view s_γ as acting on the coroot lattice $Q^\vee(X)=\sum_{i\in I_X}\mathbb{Z}\alpha_i^\vee$ by

$$s_\gamma: Q^\vee(X)\to Q^\vee(X): h\mapsto h-\langle\gamma,h\rangle\,\gamma^\vee.$$

We define the map

$$\widetilde{\pi}\colon\ \mathfrak{T}_A(k)*\Big(\mathop{\ast}_{\gamma\in\Delta^{\mathrm{re}}(A)}U_\gamma(k)\Big)\to\mathfrak{G}_B(k):$$
$$x_{\pm\alpha_i}(r)\mapsto x_{\pm\beta_i}(r),\begin{cases} r^{\alpha_i^\vee} & \mapsto\widetilde{\pi}(\widetilde{s}_i^{-1}\widetilde{s}_i(r^{-1}))\\ x_{\pm\gamma}(r) & \mapsto\widetilde{\pi}(\widetilde{w}_\gamma x_{\pm\alpha_{i_\gamma}}(r)\widetilde{w}_\gamma^{-1})\end{cases}$$

on the free product of $\mathfrak{T}_A(k)$ with all real root groups $U_\gamma(k)$, and we prove that $\widetilde{\pi}$ factors through a group homomorphism $\mathfrak{G}_A(k)\to\mathfrak{G}_B(k)$. Note first that

$$\begin{aligned}\widetilde{\pi}(\widetilde{s}_i(r))&=\widetilde{\pi}(x_{\alpha_i}(r)x_{-\alpha_i}(r^{-1})x_{\alpha_i}(r))\\ &=x_{\beta_i}(r)x_{-\beta_i}(r^{-1})x_{\beta_i}(r)=w_i^*(\widetilde{s}_{\sigma_i}(r))\end{aligned} \tag{3.10}$$

for all $r\in k^\times$ and $i\in I_A$. In particular, we deduce from (3.8) that

$$\widetilde{\pi}(\widetilde{s}_i)=w_i^*(\widetilde{s}_{\sigma_i})=\widetilde{w}_i\widetilde{s}_{\sigma_i}\widetilde{w}_i^{-1}\in\mathfrak{N}_B(k)\quad\text{for all } i\in I_A. \tag{3.11}$$

Hence for any $\gamma\in\Delta_+^{\mathrm{re}}$ and $r\in k$, we have

$$\begin{aligned}\widetilde{\pi}(x_{\pm\gamma}(r))&=\widetilde{\pi}(\widetilde{w}_\gamma x_{\pm\alpha_{i_\gamma}}(r)\widetilde{w}_\gamma^{-1})\\ &=\widetilde{w}_\gamma^\pi x_{\pm\beta_{i_\gamma}}(r)(\widetilde{w}_\gamma^\pi)^{-1}=w_\gamma^{\pi*}(x_{\pm\beta_{i_\gamma}}(r)),\end{aligned} \tag{3.12}$$

where

$$\widetilde{w}_\gamma^\pi := w_{i_1}^*(\widetilde{s}_{\sigma_{i_1}})\dots w_{i_k}^*(\widetilde{s}_{\sigma_{i_k}}) \in \mathfrak{N}_B(k)$$

and

$$w_\gamma^{\pi *} := s_{\beta_{i_1}}^* \dots s_{\beta_{i_k}}^* \in W^*(B)$$

for some prescribed reduced decomposition $w_\gamma = s_{i_1}\dots s_{i_k}$ of $w_\gamma \in W(A)$. Finally, using (3.7), (3.8), (3.9) and (3.10), we see that the restriction of $\widetilde{\pi}$ to $\mathfrak{T}_A(k)$ is given for all $r \in k^\times$ and $i \in I_A$ by

$$\begin{aligned}\widetilde{\pi}(r^{\alpha_i^\vee}) &= \widetilde{\pi}(\widetilde{s}_i^{-1}\widetilde{s}_i(r^{-1})) = w_i^*(\widetilde{s}_{\sigma_i}^{-1}\widetilde{s}_{\sigma_i}(r^{-1}))\\ &= \widetilde{w}_i \cdot r^{\alpha_{\sigma_i}^\vee} \cdot \widetilde{w}_i^{-1} = w_i(r^{\alpha_{\sigma_i}^\vee}) = r^{w_i\alpha_{\sigma_i}^\vee} = r^{\beta_i^\vee}.\end{aligned} \tag{3.13}$$

We are now ready to prove that the image by $\widetilde{\pi}$ of the relations (2.1), (2.2), (2.3), (2.4) and (2.5) defining $\mathfrak{G}_A(k)$ are still satisfied in $\mathfrak{G}_B(k)$. Observe first that $\widetilde{\pi}$ and $\widehat{\pi}$ coincide on $U_A^+(k)$. Indeed, this follows from (3.12) and the fact that for any $\gamma \in \Delta_+^{\mathrm{re}}(A)$ and any $r \in k$,

$$\begin{aligned}\widehat{\pi}(x_\gamma(r)) &= \widehat{\pi}(\exp re_\gamma) = \exp r\pi(e_\gamma)\\ &= \exp r\pi_1(w_\gamma^* e_{i_\gamma}) = \exp rw_\gamma^{\pi *} e_{\beta_{i_\gamma}}\\ &= w_\gamma^{\pi *}(x_{\beta_{i_\gamma}}(r)).\end{aligned}$$

In particular, the image by $\widetilde{\pi}$ of the relations (2.1) are satisfied in $\mathfrak{G}_B(k)$ for any prenilpotent pair $\{\alpha,\beta\} \subseteq \Delta_+^{\mathrm{re}}(A)$ of positive real roots (and hence also of negative real roots by symmetry). Let now $\{\alpha,\beta\} \subseteq \Delta^{\mathrm{re}}(A)$ be a prenilpotent pair of roots of opposite sign, say $\alpha \in \Delta_+^{\mathrm{re}}(A)$ and $\beta \in \Delta_-^{\mathrm{re}}(A)$. Then there exists some $w \in W$ such that $\{w\alpha, w\beta\} \subseteq \Delta_+^{\mathrm{re}}(A)$. Up to modifying $e_{w\alpha}$ and $e_{w\beta}$ by their opposite, we may then assume that $we_\alpha = e_{w\alpha}$ and $we_\beta = e_{w\beta}$ (note that $\{\alpha,\beta\} \neq \{w\alpha, w\beta\} \subseteq \Delta_+^{\mathrm{re}}(A)$). Hence $ww_\alpha e_{i_\alpha} = e_{w\alpha}$ and we may thus assume, up to modifying $w_{w\alpha}$, that $w_{w\alpha}w_\alpha^{-1} = w$. Set

$$w^{\pi *} := w_{w\alpha}^{\pi *}(w_\alpha^{\pi *})^{-1}.$$

Consider the relation

$$[x_\alpha(r), x_\beta(s)] = \prod_\gamma x_\gamma(C_{ij}^{\alpha\beta} r^i s^j)$$

in $\mathfrak{G}_A(k)$ for some $r, s \in k$, where $\gamma = i\alpha + j\beta$ runs, as in (2.1), through the interval $]\alpha,\beta[_{\mathbb{N}}$. For each $\gamma \in\,]\alpha,\beta[_{\mathbb{N}}$, let $\epsilon_\gamma \in \{\pm 1\}$ be such that $e_{w\gamma} = \epsilon_\gamma w^* e_\gamma$. Note that $w\big(]\alpha,\beta[_{\mathbb{N}}\big) =]w\alpha, w\beta[_{\mathbb{N}}$. We have

$$\begin{aligned}[x_{w\alpha}(r), x_{w\beta}(s)] &= w^*([x_\alpha(r), x_\beta(s)])\\ &= w^*\left(\prod_\gamma x_\gamma(C_{ij}^{\alpha\beta} r^i s^j)\right) = \prod_\gamma x_{w\gamma}(\epsilon_\gamma C_{ij}^{\alpha\beta} r^i s^j),\end{aligned}$$

so that $C_{ij}^{w\alpha,w\beta}=\epsilon_{i\alpha+j\beta}C_{ij}^{\alpha\beta}$ for all i,j.

It then follows from (3.12) that

$$\begin{aligned}
&\widetilde{\pi}([x_\alpha(r),x_\beta(s)])\\
&\quad=\widetilde{\pi}((w^*)^{-1}([x_{w\alpha}(r),x_{w\beta}(s)]))=(w^{\pi*})^{-1}\widetilde{\pi}([x_{w\alpha}(r),x_{w\beta}(s)])\\
&\quad=(w^{\pi*})^{-1}\widehat{\pi}([x_{w\alpha}(r),x_{w\beta}(s)])\\
&\quad=(w^{\pi*})^{-1}\widehat{\pi}\Bigg(\prod_\gamma x_{w\gamma}(C_{ij}^{w\alpha,w\beta}r^is^j)\Bigg)\\
&\quad=(w^{\pi*})^{-1}\widetilde{\pi}\Bigg(\prod_\gamma x_{w\gamma}(\epsilon_\gamma C_{ij}^{\alpha\beta}r^is^j)\Bigg)\\
&\quad=(w^{\pi*})^{-1}\widetilde{\pi}w^*\Bigg(\prod_\gamma x_{\gamma}(C_{ij}^{\alpha\beta}r^is^j)\Bigg)=\widetilde{\pi}\Bigg(\prod_\gamma x_{\gamma}(C_{ij}^{\alpha\beta}r^is^j)\Bigg),
\end{aligned}$$

so that the relations (2.1) are indeed satisfied.

We next check (2.2). Let $t=r^{\alpha_j^\vee}\in\mathfrak{T}_A(k)$ for some $r\in k^\times$ and some $j\in I_A$, and let $s\in k$ and $i\in I_A$. We then deduce from (3.13) and the relations (2.2), (3.7) and (3.8) in $\mathfrak{G}_B(k)$ that

$$\begin{aligned}
\widetilde{\pi}(t\cdot x_{\alpha_i}(s)\cdot t^{-1})&=r^{\beta_j^\vee}w_i^*(x_{\alpha_{\sigma_i}}(s))r^{-\beta_j^\vee}\\
&=w_i^*\big(r^{w_i^{-1}\beta_j^\vee}x_{\alpha_{\sigma_i}}(s)r^{-w_i^{-1}\beta_j^\vee}\big)\\
&=w_i^*\big(x_{\alpha_{\sigma_i}}(r^{\langle w_i\alpha_{\sigma_i},\beta_j^\vee\rangle}s)\big)=\widetilde{\pi}\big(x_{\alpha_i}(r^{\langle\beta_i,\beta_j^\vee\rangle}s)\big)\\
&=\widetilde{\pi}(x_{\alpha_i}(t(\alpha_i)s)).
\end{aligned}$$

To check (2.3), let again $t=r^{\alpha_j^\vee}\in\mathfrak{T}_A(k)$ for some $r\in k^\times$ and some $j\in I_A$, and let $i\in I_A$. We then deduce from (3.11), (3.13) and the relations (2.3) in $\mathfrak{G}_B(k)$ that

$$\begin{aligned}
\widetilde{\pi}(\widetilde{s}_it\widetilde{s}_i^{-1})&=w_i^*\big(\widetilde{s}_{\sigma_i}r^{w_i^{-1}\beta_j^\vee}\widetilde{s}_{\sigma_i}^{-1}\big)=w_i^*\big(r^{s_{\sigma_i}w_i^{-1}\beta_j^\vee}\big)\\
&=r^{w_is_{\sigma_i}w_i^{-1}\beta_j^\vee}=r^{s_{\beta_i}\beta_j^\vee}=r^{\beta_j^\vee-\langle\beta_i,\beta_j^\vee\rangle\beta_i^\vee}\\
&=\widetilde{\pi}\big(r^{\alpha_j^\vee-\langle\beta_i,\beta_j^\vee\rangle\alpha_i^\vee}\big)=\widetilde{\pi}\big(r^{s_i\alpha_j^\vee}\big)\\
&=\widetilde{\pi}(s_i(t)).
\end{aligned}$$

Since (2.4) and (2.5) are an immediate consequence of the definition of $\widetilde{\pi}$, we conclude that $\widetilde{\pi}$ factors through a group homomorphism

$$\widetilde{\pi}\colon\ \mathfrak{G}_A(k)\to\mathfrak{G}_B(k),$$

which is continuous because it coincides with the continuous group homomorphism $\widehat{\pi}$ on $U_A^+(k)$. In particular, it extends to a continuous group homomorphism $\overline{\widetilde{\pi}}\colon \overline{\mathfrak{G}_A}(k) \to \overline{\mathfrak{G}_B}(k)$ coinciding with $\widehat{\pi}$ on $\overline{U_A^+}(k)$. It thus remains to show that $\ker \widetilde{\pi} \subseteq \mathcal{Z}_A(k)$ and $\ker \overline{\widetilde{\pi}} \subseteq Z'_A \cap \overline{\mathfrak{G}_A}(k) = \mathcal{Z}_A(k) \cdot (Z'_A \cap \overline{U_A^+}(k))$.

Note that $\widetilde{\pi}(U_A^+(k)) = \widehat{\pi}(U_A^+(k)) \subseteq U_B^+(k)$. Similarly, (3.11) and (3.13) respectively imply that

$$\widetilde{\pi}(\mathfrak{N}_A(k)) \subseteq \mathfrak{N}_B(k) \quad \text{and} \quad \widetilde{\pi}(\mathfrak{T}_A(k)) \subseteq \mathfrak{T}_B(k).$$

Let $g \in \ker \widetilde{\pi}$. The Bruhat decomposition

$$\mathfrak{G}_A(k) = \dot{\bigcup}_{w \in W(A)} \mathfrak{B}^+(k)\widetilde{w}\mathfrak{B}^+(k)$$

for $\mathfrak{G}_A(k)$ implies that $g = b_1\widetilde{w}b_2$ for some $w \in W(A)$ and some $b_1, b_2 \in \mathfrak{B}^+(k)$. Hence

$$\widetilde{\pi}(g) = \widetilde{\pi}(b_1)\widetilde{\pi}(\widetilde{w})\widetilde{\pi}(b_2) = 1,$$

so that the Bruhat decomposition for $\mathfrak{G}_B(k)$ implies that $\widetilde{\pi}(\widetilde{w}) = 1$. We claim that for any reduced decomposition $w = s_{i_1} \dots s_{i_k}$ with $k \geqslant 1$, the element $w^\pi := s_{\beta_{i_1}} \dots s_{\beta_{i_k}} \in W(B)$ is nontrivial. Indeed, for any $i \in I_A$ and $\lambda \in Q(A)$, we have

$$\overline{\pi}(s_i(\lambda)) = \overline{\pi}(\lambda) - \langle \lambda, \alpha_i^\vee \rangle \beta_i = \overline{\pi}(\lambda) - \langle \overline{\pi}(\lambda), \beta_i^\vee \rangle \beta_i = s_{\beta_i}(\overline{\pi}(\lambda)).$$

In particular, $\overline{\pi}(w(\lambda)) = w^\pi(\overline{\pi}(\lambda))$ for all $\lambda \in Q(A)$. Since $\overline{\pi}$ is injective, the claim follows.

This shows that $\widetilde{w} \in \mathfrak{T}_A(k)$, and hence that $\ker \widetilde{\pi} \subseteq \mathfrak{B}^+(k)$. Therefore,

$$\ker \widetilde{\pi} \subseteq \bigcap_{h \in \mathfrak{G}_A(k)} h\mathfrak{B}^+(k)h^{-1} = \mathcal{Z}_A(k).$$

The same argument (using the Bruhat decompositions in $\mathfrak{G}_A^{pma}(k)$ and $\mathfrak{G}_B^{pma}(k)$) yields $\ker \overline{\widetilde{\pi}} \subseteq Z'_A$, as desired. This concludes the proof of the theorem. $\square$

Remark 3.11. — Note that the map $\widetilde{\pi}\colon \mathfrak{G}_A(k) \to \mathfrak{G}_B(k)$ provided by Theorem 3.10 maps $\mathcal{Z}_A(k)$ into $\mathcal{Z}_B(k)$. Indeed, recall from (2.6) that $\mathcal{Z}_A(k) = \{t \in \mathfrak{T}_A(k) \mid t(\alpha_j) = 1 \ \ \forall\, j \in I\}$ (and similarly for $\mathcal{Z}_B(k)$). Hence, if we write $t \in \mathfrak{T}_A(k)$ as a product $t = \prod_{i \in I} r_i^{\alpha_i^\vee}$ for some $r_i \in k^\times$, then $t \in \mathcal{Z}_A(k)$ if and only if $\prod_{i \in I} r_i^{\langle \alpha_j, \alpha_i^\vee \rangle} = 1$ for all $j \in I$ (and similarly for $t \in \mathfrak{T}_B(k)$, with α_i replaced by β_i). Since $\langle \alpha_j, \alpha_i^\vee \rangle = \langle \beta_j, \beta_i^\vee \rangle$ for all $i, j \in I$, the claim then follows from (3.13).

In particular, $\widetilde{\pi}$ induces a continuous injective group homomorphism

$$\mathfrak{G}_A(k)/\mathcal{Z}_A(k) \to \mathfrak{G}_B(k)/\mathcal{Z}_B(k).$$

Example 3.12. — Let k be a field and let $a \in \mathbb{N}$ with $a \geqslant 2$. We define recursively the sequence $(a_n)_{n\in\mathbb{N}}$ by $a_0 := a$ and $a_{n+1} := a_n(a_n^2 - 3)$. For each $n \in \mathbb{N}$, consider the GCM $A_n = \left(\begin{smallmatrix} 2 & -a_n \\ -a_n & 2 \end{smallmatrix}\right)$. By Theorem 3.6, the assignment $e_i \mapsto e_i$, $i = 1, 2$, defines surjective group homomorphisms

$$\pi_n \colon \mathfrak{U}^{ma+}_{A_{n+1}}(k) \to \mathfrak{U}^{ma+}_{A_n}(k).$$

Similarly, by Theorem 3.10, the assignment $e_i \mapsto e_{\beta_i}$, $i = 1, 2$, where $\beta_1 = s_1\alpha_2$ and $\beta_2 = s_2\alpha_1$, defines group homomorphisms

$$\iota_n \colon \mathfrak{U}^{ma+}_{A_{n+1}}(k) \to \mathfrak{U}^{ma+}_{A_n}(k),$$

which are moreover injective if the corresponding Kac–Moody groups are GK-simple. Indeed, this follows from the fact that

$$\begin{aligned}\beta_1(\beta_2^\vee) &= \langle s_1\alpha_2, s_2\alpha_1^\vee\rangle = \langle a_n\alpha_1 + \alpha_2, \alpha_1^\vee + a_n\alpha_2^\vee\rangle \\ &= 3a_n - a_n^3 = -a_{n+1},\end{aligned}$$

and similarly for $\beta_2(\beta_1^\vee)$. Thus, we get two projective systems

$$\begin{gathered}\cdots \xrightarrow{\pi_{n+1}} \mathfrak{U}^{ma+}_{A_{n+1}}(k) \xrightarrow{\pi_n} \mathfrak{U}^{ma+}_{A_n}(k) \cdots \xrightarrow{\pi_1} \mathfrak{U}^{ma+}_{A_1}(k) \xrightarrow{\pi_0} \mathfrak{U}^{ma+}_{A_0}(k) \\ \cdots \xrightarrow{\iota_{n+1}} \mathfrak{U}^{ma+}_{A_{n+1}}(k) \xrightarrow{\iota_n} \mathfrak{U}^{ma+}_{A_n}(k) \cdots \xrightarrow{\iota_1} \mathfrak{U}^{ma+}_{A_1}(k) \xrightarrow{\iota_0} \mathfrak{U}^{ma+}_{A_0}(k).\end{gathered}$$

The projective limit of the first system should be, in some sense to be made precise, the group $\mathfrak{U}^{ma+}_{A_\infty}(k)$ associated to the matrix $A_\infty = \left(\begin{smallmatrix} 2 & -\infty \\ -\infty & 2 \end{smallmatrix}\right)$ and with corresponding Lie algebra $\mathfrak{n}^+(A_\infty) = \tilde{\mathfrak{n}}^+$ freely generated by e_1, e_2 (see also [12, Remark on page 55]). The projective limit of the second system is trivial.

Remark 3.13. — If B is a GCM of affine type, then every subsystem $\{\beta_i \mid i \in I\} \subseteq \Delta(B)$ as in Theorem 3.10 yields a GCM $A = (\beta_i(\beta_j^\vee))_{i,j\in I}$ all whose factors are of finite or affine type. For instance, the case $a = 2$ in Example 3.12 together with Theorem 3.10 show that for the affine matrix $B = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$, the Kac–Moody group $\mathfrak{G}_B(k)/\mathcal{Z}_B(k)$ embeds properly into itself. Note that, at the algebraic level, $\mathfrak{G}_B(k) = \mathrm{SL}_2(k[t, t^{-1}])$ and the maps $k[t, t^{-1}] \to k[t, t^{-1}] : t \mapsto t^m$ $(m \geqslant 2)$ provide examples of such embeddings.

By constrast, as soon as B is of indefinite type, Example 3.12 shows that there exist GCM $A = \left(\begin{smallmatrix} 2 & -m \\ -n & 2 \end{smallmatrix}\right)$ with m, n arbitrarily large such that $\mathfrak{G}_A(k)/\mathcal{Z}_A(k)$ embeds into $\mathfrak{G}_B(k)/\mathcal{Z}_B(k)$.

3.4. Simply laced covers

A GCM A is called *simply laced* if every off-diagonal entry of A is either 0 or -1. Equivalently, A is simply laced if its Dynkin diagram $D(A)$ is a graph with only simple (unoriented, unlabelled) edges (see [11, §4.7]).

Let $A = (a_{ij})_{i,j\in I}$ be a symmetrisable GCM. A *simply laced cover* of A is a simply-laced GCM B whose Dynkin diagram $D(B)$ has n_i vertices $\alpha_{(i,1)},\dots,\alpha_{(i,n_i)}$ for each simple root $\alpha_i \in \Delta(A)$ (where the n_i are some positive integers), and such that each $\alpha_{(i,r)}$ is connected in $D(B)$ to exactly $|a_{ji}|$ of the vertices $\alpha_{(j,1)},\dots,\alpha_{(j,n_j)}$ for $j \neq i$, and to none of the other vertices $\alpha_{(i,s)}$. Such simply laced covers B of A always exist, but are in general non-unique (if one restricts to those of minimal rank). For more details about simply laced covers, we refer to [10, §2.4].

Given a simply laced cover B of A as above, we write the indexing set J of B as the set of couples

$$J = \{(i,j) \mid i \in I,\ 1 \leqslant j \leqslant n_i\}.$$

In particular, we denote by $e_{(i,j)}$ and $e_{-(i,j)} := f_{(i,j)}$ the Chevalley generators of $\mathfrak{g}_B$, by $s_{(i,j)}$ the simple reflections generating $W(B)$, and so on. For a field k, and elements $i \in I$ and $r \in k$, we also set for short

$$x_{\pm(i,\cdot)}(r) := \prod_{j=1}^{n_i} x_{\pm\alpha_{(i,j)}}(r) \in \mathfrak{G}_B(k),$$

$$\widetilde{s}_{(i,\cdot)}(r) := x_{(i,\cdot)}(r)x_{-(i,\cdot)}(r^{-1})x_{(i,\cdot)}(r) = \prod_{j=1}^{n_i} \widetilde{s}_{(i,j)}(r) \in \mathfrak{G}_B(k),$$

as well as

$$s_{(i,\cdot)} := \prod_{j=1}^{n_i} s_{(i,j)} \in W(B), \quad \widetilde{s}_{(i,\cdot)} := \widetilde{s}_{(i,\cdot)}(1) \in \mathfrak{N}_B(k)$$

and

$$s^*_{(i,\cdot)} := \prod_{j=1}^{n_i} s^*_{(i,j)} \in W^*(B).$$

Note that each of the above four products (indexed by j) consists of pairwise commuting factors. For $i \in I$, we also set

$$e_{\pm(i,\cdot)} := \sum_{j=1}^{n_i} e_{\pm(i,j)} \in \mathfrak{g}_B, \quad \alpha_{(i,\cdot)} := \sum_{j=1}^{n_i} \alpha_{(i,j)} \in Q(B)$$

and

$$\alpha^\vee_{(i,\cdot)} := \sum_{j=1}^{n_i} \alpha^\vee_{(i,j)} \in Q^\vee(B).$$

Then for all $i,j \in I$ and $m \in \{1,\dots,n_j\}$,

$$\langle \alpha_{(j,m)}, \alpha^\vee_{(i,\cdot)} \rangle = a_{ij}.$$

The following lemma is extracted from [10, §2.4]; we give here a more detailed proof.

LEMMA 3.14. — *Let $A=(a_{ij})_{i,j\in I}$ be a symmetrisable GCM, and let B be a simply laced cover of A as above. Then the assignment $e_{\pm\alpha_i} \mapsto e_{\pm(i,\cdot)}$ for $i \in I$ defines an injective Lie algebra morphism $\pi\colon \mathfrak{g}_A \to \mathfrak{g}_B$.*

Proof. — We proceed as in the proof of Lemma 3.9. Let $\widetilde{\pi}$ be the Lie algebra morphism from the free complex Lie algebra on the generators $\{e_{\pm\alpha_i} \;|i \in I\}$ to $\mathfrak{g}_B$ defined by the assignment $e_{\pm\alpha_i} \mapsto e_{\pm(i,\cdot)}$ for $i \in I$. Since $\alpha_{(j,\cdot)} - \alpha_{(i,\cdot)} \notin \Delta(B)$ for all $i,j \in I$, we have

$$[e_{(i,\cdot)}, e_{-(j,\cdot)}] = 0 \quad \text{for all } i,j \in I \text{ with } i \neq j.$$

Similarly,

$$\begin{aligned}(\operatorname{ad} e_{\pm(i,\cdot)})^{|a_{ij}|+1} e_{\pm(j,\cdot)} \\ = \sum_{r_1+\dots+r_{n_i}=|a_{ij}|+1} \binom{|a_{ij}|+1}{r_1,\dots,r_{n_i}} (\operatorname{ad} e_{\pm(i,1)})^{r_1} \dots \\ (\operatorname{ad} e_{\pm(i,n_i)})^{r_{n_i}} e_{\pm(j,\cdot)} = 0\end{aligned}$$

for all $i,j \in I$ with $i \neq j$. Indeed, each homogeneous component of $(\operatorname{ad} e_{\pm(i,\cdot)})^{|a_{ij}|+1} e_{\pm(j,\cdot)}$ has degree of the form $\alpha := \pm(\alpha_{j,m} + \sum_{s=1}^{n_i} r_s \alpha_{(i,s)})$ for some $m \in \{1,\dots,n_j\}$ and some $r_s \in \mathbb{N}$ with $\sum_{s=1}^{n_i} r_s = |a_{ij}|+1$. On the other hand, since

$$s_{(i,\cdot)}\alpha_{(i,s)} = -\alpha_{(i,s)}$$

and

$$s_{(i,\cdot)}\alpha_{(j,m)} = \alpha_{(j,m)} - \sum_{s=1}^{n_i} \langle \alpha_{(j,m)}, \alpha^\vee_{(i,s)} \rangle \alpha_{(i,s)} = \alpha_{(j,m)} + \alpha_{[i]}$$

for some $\alpha_{[i]} \in \sum_{s=1}^{n_i} \mathbb{N}\alpha_{(i,s)}$ of height $|a_{ij}|$, we have

$$s_{(i,\cdot)}\alpha = \pm(\alpha_{j,m} + \alpha_{[i]} - \sum_{s=1}^{n_i} r_s \alpha_{(i,s)}) = \pm(\alpha_{j,m} + \alpha'_{[i]})$$

for some $\alpha'_{[i]} \in \sum_{s=1}^{n_i} \mathbb{Z}\alpha_{(i,s)}$ of height -1. Hence $s_{(i,\cdot)}\alpha$ (and thus also α) cannot be a root, yielding the claim.

Finally, the elements $\alpha^\vee_{(i,\cdot)} = [e_{-(i,\cdot)}, e_{(i,\cdot)}]$ of $\mathfrak{g}_B$ $(i \in I)$ satisfy

$$[\alpha^\vee_{(i,\cdot)}, \alpha^\vee_{(j,\cdot)}] = 0$$

and

$$\begin{aligned}[\alpha^\vee_{(i,\cdot)}, e_{\pm(j,\cdot)}] &= \sum_{m=1}^{n_j} [\alpha^\vee_{(i,\cdot)}, e_{\pm(j,m)}] = \sum_{m=1}^{n_j} \pm a_{ij} e_{\pm(j,m)} \\ &= \pm a_{ij} e_{\pm(j,\cdot)}\end{aligned}$$

for all $i, j \in I$. Hence all the defining relations of $\mathfrak{g}_A = [\mathfrak{g}(A), \mathfrak{g}(A)]$ (see Section 2.2) lie in the kernel of $\widetilde{\pi}$, so that $\widetilde{\pi}$ factors through a Lie algebra morphism $\pi\colon\ \mathfrak{g}_A \to \mathfrak{g}_B$.

For the injectivity, note that $\ker \pi$ intersects the Cartan subalgebra of $\mathfrak{g}_A$ trivially. Hence $\ker \pi = \{0\}$ by the Gabber–Kac theorem (see Section 2.7), as desired. □

The proof of the following theorem follows the lines of the proof of Theorems 3.2 and 3.10. We prefer, however, to repeat the arguments, as a common treatment of these results would necessitate very cumbersome notation.

Theorem 3.15. — *Let k be a field and $A = (a_{ij})_{i,j\in I}$ be a symmetrisable GCM. Let B be a simply laced cover of A, and let $\pi\colon\ \mathfrak{g}_A \to \mathfrak{g}_B$ be the embedding provided by Lemma 3.14. Then the following holds:*

(1) *There is a continuous group morphism $\widehat{\pi}\colon\ \mathfrak{U}^{ma+}_A(k) \to \mathfrak{U}^{ma+}_B(k)$ such that for all $r \in k$, $i \in I$ and $\gamma \in \Delta^{\mathrm{re}}_+(A) \setminus \{\alpha_i\}$,*

$$\widehat{\pi}(x_{\alpha_i}(r)) = x_{(i,\cdot)}(r)$$

and

$$\widehat{\pi}(\widetilde{s}_i \cdot x_\gamma(r) \cdot \widetilde{s}_i^{-1}) = \widetilde{s}_{(i,\cdot)} \cdot \widehat{\pi}(x_\gamma(r)) \cdot \widetilde{s}_{(i,\cdot)}^{-1}.$$

(2) *The restriction of $\widehat{\pi}$ to $U^+_A(k)$ extends to continuous group homomorphisms*

$$\mathfrak{G}_A(k) \to \mathfrak{G}_B(k) \quad \text{and} \quad \overline{\mathfrak{G}_A}(k) \to \overline{\mathfrak{G}_B}(k)$$

with kernels respectively contained in $\mathcal{Z}_A(k)$ and $\mathcal{Z}_A(k) \cdot (Z'_A \cap \overline{U^+_A}(k))$. Here, we view $\mathfrak{G}_A(k)$ and $\mathfrak{G}_B(k)$ as subgroups of $\mathfrak{G}^{pma}_A(k)$ and $\mathfrak{G}^{pma}_B(k)$ respectively, with the induced topology.

Proof. — For $i \in I$ and a multi-index $\boldsymbol{m} = (m_1, \ldots, m_{n_i}) \in \mathbb{N}^{n_i}$, we write

$$|\boldsymbol{m}| := \sum_{j=1}^{n_i} m_j \quad \text{and} \quad e^{(\boldsymbol{m})}_{\pm(i,\cdot)} := \prod_{j=1}^{n_i} e^{(m_j)}_{\pm(i,j)} \in \mathcal{U}(B).$$

Note that the $e_{\pm(i,j)}$ pairwise commute (for i fixed). Since for any $i \in I$ and $n \in \mathbb{N}$,

$$e^{(n)}_{\pm(i,\cdot)} = \Big(\sum_{j=1}^{n_i} e_{\pm(i,j)}\Big)^{(n)} = \sum_{|\boldsymbol{m}|=n} e^{(\boldsymbol{m})}_{\pm(i,\cdot)} \in \mathcal{U}(B)$$

and since $\pi(\sum_{i\in I}\mathbb{Z}\alpha_i^\vee) \subseteq \sum_{i\in I}\sum_{j=1}^{n_i}\mathbb{Z}\alpha^\vee_{(i,j)}$, the map $\mathcal{U}_{\mathbb{C}}(\mathfrak{g}_A) \to \mathcal{U}_{\mathbb{C}}(\mathfrak{g}_B)$ lifting π at the level of the corresponding enveloping algebras restricts to an algebra morphism

$$\pi_1\colon \mathcal{U}(A) \to \mathcal{U}(B).$$

Moreover, for any $i \in I$ and $n \in \mathbb{N}$,

$$\begin{aligned}\nabla_B\pi_1\big(e_i^{(n)}\big) &= \nabla_B e^{(n)}_{(i,\cdot)} = \sum_{|\boldsymbol{m}|=n} \nabla_B e^{(\boldsymbol{m})}_{(i,\cdot)}\\ &= \sum_{r+s=n}\sum_{|\boldsymbol{r}|=r}\sum_{|\boldsymbol{s}|=s} e^{(\boldsymbol{r})}_{(i,\cdot)} \otimes e^{(\boldsymbol{s})}_{(i,\cdot)} = \sum_{r+s=n} e^{(r)}_{(i,\cdot)} \otimes e^{(s)}_{(i,\cdot)}\\ &= (\pi_1\otimes\pi_1)\nabla_A e_i^{(n)}.\end{aligned}$$

Since clearly $\epsilon_B\pi_1 = \epsilon_A$, we deduce that the restriction of π_1 to $\mathcal{U}^+(A)$ is a bialgebra morphism.

Note also that π_1 preserves the $\mathbb{N}$-gradations on $\mathcal{U}^+(A)$ and $\mathcal{U}^+(B)$ induced by $\mathrm{ht}\colon Q_+ \to \mathbb{N}$. In particular, the map

$$\mathcal{U}^+(A)\otimes_{\mathbb{Z}} k \to \mathcal{U}^+(B)\otimes_{\mathbb{Z}} k$$

obtained from π_1 by extension of scalars can be further extended to a bialgebra morphism

$$\pi_2\colon \widehat{\mathcal{U}}^+_k(A) \to \widehat{\mathcal{U}}^+_k(B)$$

between the corresponding completions. Finally, since π_2 preserves the group-like elements of constant term 1, it restricts to a group homomorphism

$$\widehat{\pi}\colon \mathfrak{U}^{ma+}_A(k) \to \mathfrak{U}^{ma+}_B(k)$$

by Proposition 2.3(1), which is moreover continuous because π_2 preserves the $\mathbb{N}$-gradations on $\widehat{\mathcal{U}}^+_k(A)$ and $\widehat{\mathcal{U}}^+_k(B)$.

Let now $i \in I$ and $r \in k$. By definition,

$$\begin{aligned}\widehat{\pi}(x_{\alpha_i}(r)) &= \pi_2\Big(\sum_{n\geqslant 0} r^n e_i^{(n)}\Big) = \sum_{n\geqslant 0} r^n e^{(n)}_{(i,\cdot)}\\ &= \sum_{n\geqslant 0}\sum_{|\boldsymbol{m}|=n} r^n e^{(\boldsymbol{m})}_{(i,\cdot)} = \prod_{j=1}^{n_i} \exp re_{(i,j)} = x_{(i,\cdot)}(r).\end{aligned}$$

Moreover, since for any $u \in \mathcal{U}(A)$,

$$\begin{aligned}\pi_1\big((\exp\operatorname{ad} e_{\pm\alpha_i})(u)\big) &= \pi_1\left(\sum_{n\geqslant 0}\sum_{r+s=n}(-1)^r e^{(r)}_{\pm\alpha_i} u e^{(s)}_{\pm\alpha_i}\right)\\ &= \sum_{n\geqslant 0}\sum_{r+s=n}\sum_{|\boldsymbol{r}|=r}\sum_{|\boldsymbol{s}|=s}(-1)^r e^{(\boldsymbol{r})}_{\pm(i,\cdot)}\pi_1(u)e^{(\boldsymbol{s})}_{\pm(i,\cdot)}\\ &= \left(\prod_{j=1}^{n_i}\exp\operatorname{ad} e_{\pm(i,j)}\right)(\pi_1(u)),\end{aligned}$$

so that

$$\pi_1(s_i^* u) = s^*_{(i,\cdot)}\pi_1(u),$$

we deduce from the relations (2.5) that for any $\gamma \in \Delta^{\mathrm{re}}_+(A)\setminus\{\alpha_i\}$,

$$\begin{aligned}\widehat{\pi}(\widetilde{s}_i\cdot x_\gamma(r)\cdot\widetilde{s}_i^{-1}) &= \widehat{\pi}\left(\sum_{n\geqslant 0} r^n s_i^* e^{(n)}_\gamma\right) = \sum_{n\geqslant 0} r^n \pi_1\big(s_i^* e^{(n)}_\gamma\big)\\ &= \sum_{n\geqslant 0} r^n s^*_{(i,\cdot)}\pi_1\big(e^{(n)}_\gamma\big) = s^*_{(i,\cdot)}\widehat{\pi}(x_\gamma(r))\\ &= \widetilde{s}_{(i,\cdot)}\cdot\widehat{\pi}(x_\gamma(r))\cdot\widetilde{s}^{-1}_{(i,\cdot)}.\end{aligned}$$

This concludes the proof of (1).

We now turn to the proof of (2). Let $X \in \{A, B\}$, and let I_X denote the indexing set of X. Given $w \in W(X)$ and a reduced decomposition $w = s_{i_1}s_{i_2}\ldots s_{i_k}$ for w, we write

$$\begin{aligned}&w^* := s^*_{i_1}s^*_{i_2}\ldots s^*_{i_k} \in W^* \quad\text{and}\\ &\widetilde{w} := \widetilde{s}_{i_1}\widetilde{s}_{i_2}\ldots\widetilde{s}_{i_k} \in \mathfrak{N}_X(k) \subseteq \mathfrak{St}_X(k).\end{aligned} \tag{3.14}$$

For each positive real root $\gamma \in \Delta^{\mathrm{re}}_+(X)$, we fix some $w_\gamma \in W(X)$ and some $i_\gamma \in I_X$ such that $\gamma = w_\gamma\alpha_{i_\gamma}$ (with the choice $w_\gamma = 1$ if $\gamma = \alpha_i$), and we choose the basis elements $e_\gamma \in E_\gamma$ and $e_{-\gamma} \in E_{-\gamma}$ so that $e_\gamma = w^*_\gamma e_{i_\gamma}$ and $e_{-\gamma} = w^*_\gamma f_{i_\gamma}$.

We define the map

$$\begin{aligned}\widetilde{\pi}\colon\ \mathfrak{T}_A(k) * \left(\mathop{\Large *}_{\gamma\in\Delta^{\mathrm{re}}(A)} U_\gamma(k)\right) &\to \mathfrak{G}_B(k):\\ x_{\pm\alpha_i}(r) \mapsto x_{\pm(i,\cdot)}(r),\quad &\begin{cases} r^{\alpha_i^\vee} & \mapsto \widetilde{\pi}(\widetilde{s}_i^{-1}\widetilde{s}_i(r^{-1}))\\ x_{\pm\gamma}(r) & \mapsto \widetilde{\pi}(\widetilde{w}_\gamma x_{\pm\alpha_{i_\gamma}}(r)\widetilde{w}_\gamma^{-1})\end{cases}\end{aligned}$$

on the free product of $\mathfrak{T}_A(k)$ with all real root groups $U_\gamma(k)$, and we prove that $\widetilde{\pi}$ factors through a group homomorphism $\mathfrak{G}_A(k) \to \mathfrak{G}_B(k)$. Note first that

$$\begin{aligned}\widetilde{\pi}(\widetilde{s}_i(r)) &= \widetilde{\pi}(x_{\alpha_i}(r)x_{-\alpha_i}(r^{-1})x_{\alpha_i}(r)) \\ &= x_{(i,\cdot)}(r)x_{-(i,\cdot)}(r^{-1})x_{(i,\cdot)}(r) = \widetilde{s}_{(i,\cdot)}(r)\end{aligned} \tag{3.15}$$

for all $r \in k^\times$ and $i \in I_A$. In particular,

$$\widetilde{\pi}(\widetilde{s}_i) = \widetilde{s}_{(i,\cdot)} \in \mathfrak{N}_B(k) \quad \text{for all } i \in I_A. \tag{3.16}$$

Hence for any $\gamma \in \Delta_+^{\mathrm{re}}$ and $r \in k$, we have

$$\begin{aligned}\widetilde{\pi}(x_{\pm\gamma}(r)) = \widetilde{\pi}(\widetilde{w}_\gamma x_{\pm\alpha_{i_\gamma}}(r)\widetilde{w}_\gamma^{-1}) &= \widetilde{w}_\gamma^\pi x_{\pm(i_\gamma,\cdot)}(r)(\widetilde{w}_\gamma^\pi)^{-1} \\ &= w_\gamma^{\pi*}(x_{\pm(i_\gamma,\cdot)}(r)),\end{aligned} \tag{3.17}$$

where

$$\widetilde{w}_\gamma^\pi := \widetilde{s}_{(i_1,\cdot)} \dots \widetilde{s}_{(i_k,\cdot)} \in \mathfrak{N}_B(k)$$

and

$$w_\gamma^{\pi*} := s^*_{(i_1,\cdot)} \dots s^*_{(i_k,\cdot)} \in W^*(B)$$

for some prescribed reduced decomposition $w_\gamma = s_{i_1} \dots s_{i_k}$ of $w_\gamma \in W(A)$. Finally, using (3.15) and the relations (2.4) in $\mathfrak{G}_B(k)$, we see that the restriction of $\widetilde{\pi}$ to $\mathfrak{T}_A(k)$ is given for all $r \in k^\times$ and $i \in I_A$ by

$$\begin{aligned}\widetilde{\pi}(r^{\alpha_i^\vee}) &= \widetilde{\pi}(\widetilde{s}_i^{-1}\widetilde{s}_i(r^{-1})) = \widetilde{s}_{(i,\cdot)}^{-1}\widetilde{s}_{(i,\cdot)}(r^{-1}) \\ &= \prod_{j=1}^{n_i} \left(\widetilde{s}_{(i,j)}^{-1}\widetilde{s}_{(i,j)}(r^{-1})\right) = \prod_{j=1}^{n_i} r^{\alpha^\vee_{(i,j)}} = r^{\alpha^\vee_{(i,\cdot)}}.\end{aligned} \tag{3.18}$$

We are now ready to prove that the image by $\widetilde{\pi}$ of the relations (2.1), (2.2), (2.3), (2.4) and (2.5) defining $\mathfrak{G}_A(k)$ are still satisfied in $\mathfrak{G}_B(k)$. Observe first that $\widetilde{\pi}$ and $\widehat{\pi}$ coincide on $U_A^+(k)$ by (3.16) and the first statement of the theorem. In particular, the image by $\widetilde{\pi}$ of the relations (2.1) are satisfied in $\mathfrak{G}_B(k)$ for any prenilpotent pair $\{\alpha,\beta\} \subseteq \Delta_+^{\mathrm{re}}(A)$ of positive real roots (and hence also of negative real roots by symmetry). Let now $\{\alpha,\beta\} \subseteq \Delta^{\mathrm{re}}(A)$ be a prenilpotent pair of roots of opposite sign, say $\alpha \in \Delta_+^{\mathrm{re}}(A)$ and $\beta \in \Delta_-^{\mathrm{re}}(A)$. Then there exists some $w \in W$ such that $\{w\alpha, w\beta\} \subseteq \Delta_+^{\mathrm{re}}(A)$. Up to modifying $e_{w\alpha}$ and $e_{w\beta}$ by their opposite, we may then assume that $we_\alpha = e_{w\alpha}$ and $we_\beta = e_{w\beta}$ (note that $\{\alpha,\beta\} \neq \{w\alpha, w\beta\} \subseteq \Delta_+^{\mathrm{re}}(A)$). Hence $ww_\alpha e_{i_\alpha} = e_{w\alpha}$ and we may thus assume, up to modifying $w_{w\alpha}$, that $w_{w\alpha}w_\alpha^{-1} = w$. Set

$$w^{\pi*} := w_{w\alpha}^{\pi*}(w_\alpha^{\pi*})^{-1}.$$

Consider the relation

$$[x_\alpha(r), x_\beta(s)] = \prod_\gamma x_\gamma(C_{ij}^{\alpha\beta} r^i s^j)$$

in $\mathfrak{G}_A(k)$ for some $r, s \in k$, where $\gamma = i\alpha + j\beta$ runs, as in (2.1), through the interval $]\alpha, \beta[_{\mathbb{N}}$. For each $\gamma \in\ ]\alpha, \beta[_{\mathbb{N}}$, let $\epsilon_\gamma \in \{\pm 1\}$ be such that $e_{w\gamma} = \epsilon_\gamma w^* e_\gamma$. Note that $w\big(]\alpha, \beta[_{\mathbb{N}}\big) =]w\alpha, w\beta[_{\mathbb{N}}$. We have

$$\begin{aligned}[x_{w\alpha}(r), x_{w\beta}(s)] = w^*([x_\alpha(r), x_\beta(s)]) &= w^*\left(\prod_\gamma x_\gamma(C^{\alpha\beta}_{ij} r^i s^j)\right)\\ &= \prod_\gamma x_{w\gamma}(\epsilon_\gamma C^{\alpha\beta}_{ij} r^i s^j),\end{aligned}$$

so that $C^{w\alpha,w\beta}_{ij} = \epsilon_{i\alpha+j\beta} C^{\alpha\beta}_{ij}$ for all i, j. It then follows from (3.17) that

$$\begin{aligned}&\widetilde{\pi}([x_\alpha(r), x_\beta(s)])\\ &\quad = \widetilde{\pi}((w^*)^{-1}([x_{w\alpha}(r), x_{w\beta}(s)])) = (w^{\pi *})^{-1}\widetilde{\pi}([x_{w\alpha}(r), x_{w\beta}(s)])\\ &\quad = (w^{\pi *})^{-1}\widehat{\pi}([x_{w\alpha}(r), x_{w\beta}(s)])\\ &\quad = (w^{\pi *})^{-1}\widehat{\pi}\left(\prod_\gamma x_{w\gamma}(C^{w\alpha,w\beta}_{ij} r^i s^j)\right)\\ &\quad = (w^{\pi *})^{-1}\widetilde{\pi}\left(\prod_\gamma x_{w\gamma}(\epsilon_\gamma C^{\alpha\beta}_{ij} r^i s^j)\right)\\ &\quad = (w^{\pi *})^{-1}\widetilde{\pi} w^*\left(\prod_\gamma x_\gamma(C^{\alpha\beta}_{ij} r^i s^j)\right) = \widetilde{\pi}\left(\prod_\gamma x_\gamma(C^{\alpha\beta}_{ij} r^i s^j)\right),\end{aligned}$$

so that the relations (2.1) are indeed satisfied.

We next check (2.2). Let $t = r^{\alpha_j^\vee} \in \mathfrak{T}_A(k)$ for some $r \in k^\times$ and some $j \in I_A$, and let $s \in k$ and $i \in I_A$. We then deduce from (3.18) and the relations (2.2) in $\mathfrak{G}_B(k)$ that

$$\begin{aligned}\widetilde{\pi}(t \cdot x_{\alpha_i}(s) \cdot t^{-1}) = r^{\alpha^\vee_{(j,\cdot)}} x_{(i,\cdot)}(s) r^{-\alpha^\vee_{(j,\cdot)}} &= \prod_{m=1}^{n_i} \left(r^{\alpha^\vee_{(j,\cdot)}} x_{(i,m)}(s) r^{-\alpha^\vee_{(j,\cdot)}}\right)\\ &= \prod_{m=1}^{n_i} x_{(i,m)}\big(r^{\langle \alpha_{(i,m)}, \alpha^\vee_{(j,\cdot)}\rangle} s\big) = \prod_{m=1}^{n_i} x_{(i,m)}(r^{a_{ji}} s)\\ &= x_{(i,\cdot)}(t(\alpha_i)s) = \widetilde{\pi}(x_{\alpha_i}(t(\alpha_i)s)).\end{aligned}$$

To check (2.3), let again $t = r^{\alpha_j^\vee} \in \mathfrak{T}_A(k)$ for some $r \in k^\times$ and some $j \in I_A$, and let $i \in I_A$. We then deduce from (3.16), (3.18) and the relations (2.3) in $\mathfrak{G}_B(k)$ that

$$\begin{aligned}\widetilde{\pi}(\widetilde{s}_i t \widetilde{s}_i^{-1}) &= \widetilde{s}_{(i,\cdot)} r^{\alpha^\vee_{(j,\cdot)}} \widetilde{s}_{(i,\cdot)}^{-1} = s_{(i,\cdot)}\big(r^{\alpha^\vee_{(j,\cdot)}}\big) = r^{s_{(i,\cdot)}(\alpha^\vee_{(j,\cdot)})} \\ &= r^{\alpha^\vee_{(j,\cdot)} - \sum_{m=1}^{n_i} \langle \alpha_{(i,m)}, \alpha^\vee_{(j,\cdot)}\rangle \alpha^\vee_{(i,m)}} = r^{\alpha^\vee_{(j,\cdot)} - a_{ji}\alpha^\vee_{(i,\cdot)}} \\ &= \widetilde{\pi}\Big(r^{\alpha_j^\vee - a_{ji}\alpha_i^\vee}\Big) = \widetilde{\pi}\big(r^{s_i(\alpha_j^\vee)}\big) \\ &= \widetilde{\pi}(s_i(t)).\end{aligned}$$

Since (2.4) and (2.5) are an immediate consequence of the definition of $\widetilde{\pi}$, we conclude that $\widetilde{\pi}$ factors through a group homomorphism

$$\widetilde{\pi}\colon\ \mathfrak{G}_A(k) \to \mathfrak{G}_B(k),$$

which is continuous because it coincides with the continuous group homomorphism $\widehat{\pi}$ on $U_A^+(k)$. In particular, it extends to a continuous group homomorphism $\overline{\widetilde{\pi}}\colon\ \overline{\mathfrak{G}_A}(k) \to \overline{\mathfrak{G}_B}(k)$ coinciding with $\widehat{\pi}$ on $\overline{U_A^+}(k)$. It thus remains to show that $\ker\widetilde{\pi} \subseteq \mathcal{Z}_A(k)$ and $\ker\overline{\widetilde{\pi}} \subseteq Z'_A \cap \overline{\mathfrak{G}_A}(k) = \mathcal{Z}_A(k) \cdot (Z'_A \cap \overline{U_A^+}(k))$.

Note that $\widetilde{\pi}(U_A^+(k)) = \widehat{\pi}(U_A^+(k)) \subseteq U_B^+(k)$. Similarly, (3.16) and (3.18) respectively imply that

$$\widetilde{\pi}(\mathfrak{N}_A(k)) \subseteq \mathfrak{N}_B(k) \quad \text{and} \quad \widetilde{\pi}(\mathfrak{T}_A(k)) \subseteq \mathfrak{T}_B(k).$$

Let $g \in \ker\widetilde{\pi}$. The Bruhat decomposition

$$\mathfrak{G}_A(k) = \dot{\bigcup}_{w\in W(A)} \mathfrak{B}^+(k)\widetilde{w}\mathfrak{B}^+(k)$$

for $\mathfrak{G}_A(k)$ implies that $g = b_1\widetilde{w}b_2$ for some $w \in W(A)$ and some $b_1, b_2 \in \mathfrak{B}^+(k)$. Hence

$$\widetilde{\pi}(g) = \widetilde{\pi}(b_1)\widetilde{\pi}(\widetilde{w})\widetilde{\pi}(b_2) = 1,$$

so that the Bruhat decomposition for $\mathfrak{G}_B(k)$ implies that $\widetilde{\pi}(\widetilde{w}) = 1$. We claim that for any reduced decomposition $w = s_{i_1}\dots s_{i_k}$ with $k \geqslant 1$, the element $w^\pi := s_{(i_1,\cdot)}\dots s_{(i_k,\cdot)} \in W(B)$ is nontrivial. Indeed, define the $\mathbb{Z}$-linear map

$$\overline{\pi}\colon\ Q^\vee(A) \to Q^\vee(B) : \alpha_i^\vee \mapsto \alpha^\vee_{(i,\cdot)}.$$

Then for any $i, j \in I_A$, we have

$$\begin{aligned}s_{(i,\cdot)}(\overline{\pi}(\alpha_j^\vee)) = s_{(i,\cdot)}(\alpha^\vee_{(j,\cdot)}) &= \alpha^\vee_{(j,\cdot)} - \sum_{m=1}^{n_i} \langle \alpha_{(i,m)}, \alpha^\vee_{(j,\cdot)}\rangle \alpha^\vee_{(i,m)} \\ &= \alpha^\vee_{(j,\cdot)} - a_{ji}\alpha^\vee_{(i,\cdot)} = \overline{\pi}(s_i(\alpha_j^\vee))\end{aligned}$$

and hence $\overline{\pi}(s_i(h)) = s_{(i,\cdot)}(\overline{\pi}(h))$ for any $i \in I$ and $h \in Q^\vee(A)$. In particular, $\overline{\pi}(w(h)) = w^\pi(\overline{\pi}(h))$ for all $h \in Q^\vee(A)$. Since $\overline{\pi}$ is injective, the claim follows.

This shows that $\widetilde{w} \in \mathfrak{T}_A(k)$, and hence that $\ker \widetilde{\pi} \subseteq \mathfrak{B}^+(k)$. Therefore,

$$\ker \widetilde{\pi} \subseteq \bigcap_{h \in \mathfrak{G}_A(k)} h\mathfrak{B}^+(k)h^{-1} = \mathcal{Z}_A(k).$$

The same argument (using the Bruhat decompositions in $\mathfrak{G}_A^{pma}(k)$ and $\mathfrak{G}_B^{pma}(k)$) yields $\ker \overline{\widetilde{\pi}} \subseteq Z'_A$, as desired. This concludes the proof of the theorem. □

Remark 3.16. — Proceeding exactly as in Remark 3.11, we see that the map $\widetilde{\pi}\colon \mathfrak{G}_A(k) \to \mathfrak{G}_B(k)$ provided by Theorem 3.15 maps $\mathcal{Z}_A(k)$ into $\mathcal{Z}_B(k)$ by (3.18), and hence induces a continuous injective group homomorphism

$$\mathfrak{G}_A(k)/\mathcal{Z}_A(k) \to \mathfrak{G}_B(k)/\mathcal{Z}_B(k).$$

4. Non-density and Gabber–Kac simplicity

This section is devoted to the proof of Propositions B and C.

Proposition 4.1. — *Let k be a field and let B be a GCM. Assume that $U_B^+(k)$ is not dense in $\mathfrak{U}_B^{ma+}(k)$. Then $U_A^+(k)$ is not dense in $\mathfrak{U}_A^{ma+}(k)$ for all GCM A such that $B \leqslant A$.*

Proof. — By Corollary 3.7, the surjective group homomorphism $\widehat{\pi}_{AB}\colon \mathfrak{U}_A^{ma+}(k) \to \mathfrak{U}_B^{ma+}(k)$ provided by Theorem 3.6 restricts to a group homomorphism $\overline{U_A^+}(k) \to \overline{U_B^+}(k)$. Thus, if $U_A^+(k)$ were dense in $\mathfrak{U}_A^{ma+}(k)$, we would conclude that

$$\overline{U_B^+}(k) \supseteq \widehat{\pi}_{AB}(\overline{U_A^+}(k)) = \widehat{\pi}_{AB}(\mathfrak{U}_A^{ma+}(k)) = \mathfrak{U}_B^{ma+}(k),$$

and hence that $\overline{U_B^+}(k) = \mathfrak{U}_B^{ma+}(k)$, yielding the desired contradiction. □

Lemma 4.2. — *Let k be a field and A be a GCM. Then $U_A^+(k)$ is dense in $\mathfrak{U}_A^{ma+}(k)$ if and only if the minimal Kac–Moody group $\mathfrak{G}_A(k)$ is dense in its Mathieu–Rousseau completion $\mathfrak{G}_A^{pma}(k)$.*

Proof. — This follows from the fact that $\mathfrak{G}_A^{pma}(k)$ is generated by $\mathfrak{U}_A^{ma+}(k)$ and $\mathfrak{G}_A(k)$ and that $U_A^+(k) = \mathfrak{U}_A^{ma+}(k) \cap \mathfrak{G}_A(k)$ (see [28, 3.16]). □

The following lemma is a slight generalisation of [16, Lemma 5.4].

Lemma 4.3. — *Let k be a field, and let $A = \left(\begin{smallmatrix} 2 & -m \\ -n & 2 \end{smallmatrix}\right)$ be a GCM such that $mn > 4$. If $\mathrm{char}\, k = 2$, we moreover assume that at least one of m and n is odd and $\geqslant 3$. Then the imaginary subgroup U^{im+} of $\mathfrak{U}^{ma+}_A(k)$ is not contained in Z'_A.*

Proof. — Assume for a contradiction that U^{im+} is contained in Z'_A.

Note first that

$$U^{im+} = \bigcap_{w \in W} \widetilde{w}\mathfrak{U}^{ma+}_A(k)\widetilde{w}^{-1}, \tag{4.1}$$

where $\widetilde{w}$ is as in (3.14) (see also [17, Definition 7.58]): indeed, the inclusion $\subseteq$ readily follows from Lemma 2.4 and the fact that W stabilises Δ^{im}_+ (see [11, Theorem 5.4]). Conversely, if $g \in \mathfrak{U}^{ma+}_A(k) \setminus U^{im+}$, then by Proposition 2.3(2) we can write g as a product $g = \prod_{x \in \mathcal{B}_{\Delta_+}} [\exp]\lambda_x x$ for some $\lambda_x \in k$ such that $\lambda_y \neq 0$ for some y with $\deg(y) \in \Delta^{\mathrm{re}}_+$. In particular, by Lemma 2.4, we find some $v \in W$ such that $\widetilde{v}g\widetilde{v}^{-1} = x_{\alpha_i}(r)h$ for some $i \in I$, some nonzero $r \in k$, and some $h \in \mathfrak{U}^{ma}_2(k)$. Hence $\widetilde{w}g\widetilde{w}^{-1} \notin \mathfrak{U}^{ma+}_A(k)$ for $w := s_i v \in W$, proving the reverse inclusion.

As $Z'_A = Z_A \cdot (Z'_A \cap \mathfrak{U}^{ma+}_A(k))$ and as $Z'_A \cap \mathfrak{U}^{ma+}_A(k)$ is normal in $\mathfrak{G}^{pma}_A(k)$ by [28, Proposition 6.4], we deduce that $U^{im+} = Z'_A \cap \mathfrak{U}^{ma+}_A(k)$ is a normal subgroup of $\mathfrak{G}^{pma}_A(k)$. We now exhibit some imaginary root $\delta \in \Delta^{\mathrm{im}}_+$, some simple root α_i, and some element $x \in (\mathfrak{n}^+_k)_\delta$ such that $\delta - \alpha_i \in \Delta^{\mathrm{re}}_+$ and such that $\mathrm{ad}(f_i)x$ is nonzero in $\mathfrak{n}^+_k$. This will show that the element $\exp(f_i) \in \mathfrak{U}_{(-\alpha_i)}(k) \subseteq \mathfrak{G}^{pma}_A(k)$ conjugates the element $[\exp]x \in \mathfrak{U}_{(\delta)}(k) \subseteq U^{im+}$ outside U^{im+} (see (2.8)), yielding the desired contradiction.

Set $p = \mathrm{char}\, k$. By hypothesis, $mn > 4$. Up to interchanging m and n, we may then assume that $n \geqslant 3$. If $p = 2$, we may moreover assume that n is odd. Set $\beta := s_1(\alpha_2) = \alpha_2 + m\alpha_1 \in \Delta^{\mathrm{re}}_+$ and $\gamma := s_2(\alpha_1) = \alpha_1 + n\alpha_2 \in \Delta^{\mathrm{re}}_+$, so that

$$\langle \gamma, \alpha_1^\vee \rangle = \langle \beta, \alpha_2^\vee \rangle = 2 - mn, \quad \langle \gamma, \alpha_2^\vee \rangle = n, \quad \langle \beta, \alpha_1^\vee \rangle = m,$$

and

$$\langle \gamma, \beta^\vee \rangle = n(3 - mn).$$

Assume first that p does not divide $2 - mn$ or that $p = 0$. Set $\delta := \alpha_1 + \gamma$. Then $\delta \in \Delta^{\mathrm{im}}_+$ because $\delta(\alpha_1^\vee) = 4 - mn < 0$ and $\delta(\alpha_2^\vee) = 0$ (see [11, Lemma 5.3]). Set also $x := [e_1, e_\gamma] \in \mathfrak{n}^+_k$. Since $\gamma - \alpha_1 = n\alpha_2 \notin \Delta$, we deduce that

$$[f_1, x] = \langle \gamma, \alpha_1^\vee \rangle \cdot e_\gamma = (2 - mn) \cdot e_\gamma \neq 0 \quad \text{in } \mathfrak{n}^+_k,$$

as desired.

Assume next that p divides $2 - mn$. Since n is odd if $p = 2$, this implies that p does not divide $n(3 - mn)$. Set $\delta := s_1(\beta + \gamma) = \alpha_2 + s_1(\gamma)$. Note that if $m \geqslant 2$, then

$$\langle s_1(\delta), \alpha_1^\vee \rangle = m + 2 - mn \leqslant 2 - 2m < 0$$

and

$$\langle s_1(\delta), \alpha_2^\vee \rangle = 2 - mn + n \leqslant 2 - n < 0,$$

while if $m = 1$, so that $n \geqslant 5$, then

$$\langle s_2 s_1(\delta), \alpha_1^\vee \rangle = \langle s_1(\delta), \alpha_1^\vee + \alpha_2^\vee \rangle = 5 - n \leqslant 0$$

and

$$\langle s_2 s_1(\delta), \alpha_2^\vee \rangle = -\langle s_1(\delta), \alpha_2^\vee \rangle = -2 < 0.$$

Hence $\delta \in \Delta_+^{\mathrm{im}}$ by [11, Theorem 5.4]. Set $x := [e_2, e_{\gamma'}] \in \mathfrak{n}_k^+$, where $\gamma' = s_1(\gamma) \in \Delta_+^{\mathrm{re}}$. Since $\gamma - \beta = -(m-1)\alpha_1 + (n-1)\alpha_2 \notin \Delta$ and hence also $\gamma' - \alpha_2 = s_1(\gamma - \beta) \notin \Delta$, we deduce that

$$[f_2, x] = \langle \gamma', \alpha_2^\vee \rangle \cdot e_{\gamma'} = \langle \gamma, \beta^\vee \rangle \cdot e_{\gamma'} = n(3 - mn) \cdot e_{\gamma'} \neq 0 \quad \text{in } \mathfrak{n}_k^+,$$

as desired. □

We record the following more precise version of [16, Theorem E].

Proposition 4.4. — *Let $k = \mathbb{F}_q$ be a finite field. Consider the GCM $A_1 = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 2 & -m \\ -n & 2 \end{smallmatrix}\right)$ with $m, n \geqslant 2$ and $mn > 4$. Assume that $m \equiv n \equiv 2 \pmod{q-1}$. If $\operatorname{char} k = 2$, we moreover assume that at least one of m and n is odd. Then the minimal Kac–Moody groups $\mathfrak{G}_{A_1}(\mathbb{F}_q)$ and $\mathfrak{G}_{A_2}(\mathbb{F}_q)$ are isomorphic as abstract groups, but the simple quotients $\mathfrak{G}_{A_1}^{pma}(\mathbb{F}_q)/Z'_{A_1}$ and $\mathfrak{G}_{A_2}^{pma}(\mathbb{F}_q)/Z'_{A_2}$ of the corresponding Mathieu–Rousseau completions are not isomorphic as topological groups.*

Proof. — The proof of [16, Theorem E] on p. 725 of *loc. cit.* applies *verbatim*, with the same notation [note: that proof uses Lemma 5.3 in [16]; for the convenience of the reader, we provide below (see Lemma 4.5) a more detailed proof of that lemma]. The only difference is that [16, Lemma 5.4], which is used to conclude the proof, must be replaced by its generalisation, Lemma 4.3 above (hence the extra assumption in characteristic 2). □

Lemma 4.5. — *Let $k = \mathbb{F}_q$ be a finite field. Consider the GCM $A = \left(\begin{smallmatrix} 2 & -m \\ -n & 2 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 2 & -m' \\ -n' & 2 \end{smallmatrix}\right)$ with $m, m', n, n' \geqslant 2$. Assume moreover that $m \equiv m' \pmod{q-1}$ and $n \equiv n' \pmod{q-1}$. Then the minimal Kac–Moody groups $\mathfrak{G}_A(k)$ and $\mathfrak{G}_B(k)$ are isomorphic as abstract groups, and the corresponding Rémy–Ronan completions $\mathfrak{G}_A^{rr}(\mathbb{F}_q)$ and $\mathfrak{G}_B^{rr}(\mathbb{F}_q)$ are isomorphic as topological groups.*

Proof. — We can identify the Weyl groups $W(A)$ and $W(B)$ (both isomorphic to the infinite dihedral group), and hence also the corresponding sets of real roots $\Delta^{\mathrm{re}}(A)$ and $\Delta^{re}(B)$. Moreover, since A and B do not have any -1 entry, it follows from [20, §3] that the commutation relations (2.1) are all trivial, and hence one can identify the Steinberg functors $\mathfrak{St}_A$ and $\mathfrak{St}_B$. Let us fix this identification $\mathfrak{St}_A \xrightarrow{\sim} \mathfrak{St}_B$ as follows (we add a superscript A or B to the usual notations, to distinguish between the objects related to the GCM A or B). For $X \in \{A, B\}$, each real root $\alpha \in \Delta^{\mathrm{re}}_+(X)$ can be uniquely written as $\alpha = w_\alpha \alpha_i$ for some $w_\alpha \in W(X)$ and $i \in I = \{1, 2\}$. We then choose the sign of $e^X_{\pm\alpha}$ in the double basis E^X_α by setting

$$e^X_\alpha := w^*_\alpha e^X_i \quad \text{and} \quad e^X_{-\alpha} := w^*_\alpha f^X_i,$$

where w^*_α is as in (3.14) (see also [17, Definition 7.58]), and we define the corresponding parametrisations $x^X_{\pm\alpha} \colon\ k \to U^X_{\pm\alpha} : r \mapsto \exp(re_{\pm\alpha})$ of the real root groups accordingly. The identification $\mathfrak{St}_A \xrightarrow{\sim} \mathfrak{St}_B$ is now obtained by mapping x^A_α to x^B_α for each $\alpha \in \Delta^{\mathrm{re}}(A) = \Delta^{\mathrm{re}}(B)$.

Similarly, identifying the coroots associated to A and B, we obtain an identification of the tori $\mathfrak{T}_A(k) \xrightarrow{\sim} \mathfrak{T}_B(k)$ mapping $r^{\alpha_i^\vee} \in \mathfrak{T}_A(k)$ ($r \in k^\times$, $i \in I$) to the corresponding element of $\mathfrak{T}_B(k)$. This yields an isomorphism $\varphi\colon\ \mathfrak{St}_A(k) * \mathfrak{T}_A(k) \to \mathfrak{St}_B(k) * \mathfrak{T}_B(k)$, and to see it induces an isomorphism $\mathfrak{G}_A(k) \to \mathfrak{G}_B(k)$, we only have to show that the relations (2.2)–(2.5) are the same in $\mathfrak{St}_A(k) * \mathfrak{T}_A(k)$ and $\mathfrak{St}_B(k) * \mathfrak{T}_B(k)$.

For the relations (2.4), this is clear by construction. For the relations (2.2) and (2.3), this follows from the fact that

$$r^m = r^{m'} \quad \text{and} \quad r^n = r^{n'} \quad \text{for all } r \in k. \tag{4.2}$$

Finally, for the relations (2.5), let $i \in I$, $\alpha \in \Delta^{\mathrm{re}}$ and $r \in k$, and let us check that $\widetilde{s}_i \cdot x^A_\alpha(r) \cdot \widetilde{s}_i^{-1} \cdot (s^*_i x^A_\alpha(r))^{-1}$ is mapped to $\widetilde{s}_i \cdot x^B_\alpha(r) \cdot \widetilde{s}_i^{-1} \cdot (s^*_i x^B_\alpha(r))^{-1}$ under φ, or else that

$$\varphi(s^*_i x^A_\alpha(r)) = s^*_i x^B_\alpha(r). \tag{4.3}$$

We may assume that $\alpha \in \Delta^{\mathrm{re}}_+$ (the case $\alpha \in \Delta^{\mathrm{re}}_-$ being symmetric). Let $X \in \{A, B\}$. If $\alpha = \alpha_i$, then $s^*_i x^X_\alpha(r) = x^X_{-\alpha_i}(r)$, yielding (4.3) in that case. Assume now that $\alpha \in \Delta^{\mathrm{re}}_+ \setminus \{\alpha_i\}$. By definition, $x^X_\alpha(r) = \exp(re^X_\alpha) = \exp(r \cdot w^*_\alpha e^X_j) = w^*_\alpha x^X_{\alpha_j}(r)$ for some $j \in I$ (determined by α). If $\ell(s_i w_\alpha) = \ell(w_\alpha) + 1$ (where $\ell\colon\ W \to \mathbb{N}$ is the word length on $W = W(X)$ with respect to the generating set $\{s_1, s_2\}$), then $s^*_i w^*_\alpha = (s_i w_\alpha)^* = w^*_{s_i\alpha}$, and hence $s^*_i x^X_\alpha(r) = x^X_{s_i\alpha}(r)$, yielding (4.3) in that case. Finally, suppose $\ell(s_i w_\alpha) = \ell(w_\alpha) - 1$. Then $w^*_\alpha = s^*_i \cdot (s_i w_\alpha)^*$, and hence $s^*_i x^X_\alpha(r) = (s^*_i)^2 \cdot x^X_{s_i\alpha}(r)$. On the other hand, by [17, Proposition 4.18 (6)], we have

$$(s_i^*)^2 \cdot x^X_{s_i\alpha}(r) = x^X_{s_i\alpha}((-1)^{\langle s_i\alpha,\alpha_i^\vee\rangle}r) = x^X_{s_i\alpha}((-1)^{\langle \alpha,\alpha_i^\vee\rangle}r).$$

It thus remains to check that $(-1)^{\langle\alpha,\alpha_i^\vee\rangle}$ yields the same element of k, regardless of whether α is viewed as a root of $\Delta_+^{\mathrm{re}}(A)$ or of $\Delta_+^{\mathrm{re}}(B)$. But if $\alpha = \alpha_j$ is a simple root, this follows from (4.2), and in general, this follows from an easy induction on $\ell(w_\alpha)$ using the fact that $\langle s_j\alpha, \alpha_i^\vee\rangle = \langle\alpha,\alpha_i^\vee\rangle - \langle\alpha,\alpha_j^\vee\rangle\cdot\langle\alpha_j,\alpha_i^\vee\rangle$.

We have thus shown that the map φ induces an isomorphism $\phi\colon\ \mathfrak{G}_A(k)\to\mathfrak{G}_B(k)$. On the other hand, note that ϕ identifies the (positive) BN-pairs of $\mathfrak{G}_A(k)$ and $\mathfrak{G}_B(k)$ (see Section 2.6), and hence also induces an isomorphism of topological groups between the corresponding Rémy–Ronan completions, yielding the lemma. □

Finally, we prove a slight generalisation of [16, Corollary F].

Lemma 4.6. — *Let $k=\mathbb{F}_q$ be a finite field. Consider the GCM $A = \left(\begin{smallmatrix} 2 & -m \\ -n & 2\end{smallmatrix}\right)$ with $m,n\geqslant 2$ and $mn>4$. Assume that $m\equiv n\equiv 2 \pmod{q-1}$. If $\operatorname{char} k = 2$, we moreover assume that at least one of m and n is odd. Then $U_A^+(\mathbb{F}_q)$ is not dense in $\mathfrak{U}_A^{ma+}(\mathbb{F}_q)$.*

Proof. — Set $A_1 = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2\end{smallmatrix}\right)$ and $A_2 = A$. For $i=1,2$, we also set $G_i := \mathfrak{G}_{A_i}(\mathbb{F}_q)$, $\widehat{G}_i := \mathfrak{G}^{pma}_{A_i}(\mathbb{F}_q)$, and $Z_i' := Z'_{A_i}$. It follows from Proposition 4.4 that G_1 and G_2 are isomorphic as abstract groups, whereas $\widehat{G}_1/Z_1'$ and $\widehat{G}_2/Z_2'$ are not isomorphic as topological groups. Note also that the Rémy–Ronan completions $\mathfrak{G}^{rr}_{A_1}(\mathbb{F}_q)$ of G_1 and $\mathfrak{G}^{rr}_{A_2}(\mathbb{F}_q)$ of G_2 are isomorphic as topological groups by Lemma 4.5. Finally, we may assume without loss of generality that G_1 is dense in $\widehat{G}_1$, for otherwise $U^+_{A_1}(\mathbb{F}_q)$ would not be dense in $\mathfrak{U}^{ma+}_{A_1}(\mathbb{F}_q)$ by Lemma 4.2, so that the conclusion of the lemma would immediately follow from Proposition 4.1.

Assume for a contradiction that $U_A^+(\mathbb{F}_q)$ is dense in $\mathfrak{U}_A^{ma+}(\mathbb{F}_q)$. Then G_2 is dense in $\widehat{G}_2$ by Lemma 4.2. Hence the continuous surjective group homomorphisms $\varphi_{A_i}\colon\ \widehat{G}_i\to\mathfrak{G}^{rr}_{A_i}(\mathbb{F}_q)$, $i=1,2$, induce isomorphisms

$$\widehat{G}_1/Z_1' \cong \mathfrak{G}^{rr}_{A_1}(\mathbb{F}_q) \cong \mathfrak{G}^{rr}_{A_2}(\mathbb{F}_q) \cong \widehat{G}_2/Z_2'$$

of topological groups, yielding the desired contradiction. □

Theorem 4.7. — *Let $k=\mathbb{F}_q$ be a finite field, and let $A=(a_{ij})_{i,j\in I}$ be a GCM. Assume that there exist indices $i,j\in I$ such that $|a_{ij}|\geqslant q+1$ and $|a_{ji}|\geqslant 2$. Then $U_A^+(k)$ is not dense in $\mathfrak{U}_A^{ma+}(k)$.*

Proof. — Consider the GCM $B = \left(\begin{smallmatrix} 2 & -m \\ -n & 2 \end{smallmatrix}\right)$ with $m = q+1$ and $n = 2$. Then $U_B^+(k)$ is not dense in $\mathfrak{U}_B^{ma+}(k)$ by Lemma 4.6. Since $B \leqslant \left(\begin{smallmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{smallmatrix}\right)$ or $B \leqslant \left(\begin{smallmatrix} 2 & a_{ji} \\ a_{ij} & 2 \end{smallmatrix}\right)$, the conclusion then follows from Proposition 4.1. □

We now give a completely different proof of Theorem 4.7, which provides another perspective on this non-density phenomenon.

Proposition 4.8. — *Let $k = \mathbb{F}_q$ be a finite field, and let $A = (a_{ij})_{i,j \in I}$ be a GCM. Fix distinct $i, j \in I$, and let $g \in \mathfrak{U}_A^{ma+}(k) \subseteq \widehat{\mathcal{U}}_k^+$ be one of the twisted exponentials $[\exp][e_i, e_j]$ or $[\exp](\operatorname{ad} e_i)^{(q)} e_j$.*

(1) *If $|a_{ij}| \geqslant q$, then $g \notin \overline{[\mathfrak{U}_A^{ma+}(k), \mathfrak{U}_A^{ma+}(k)]}$.*
(2) *If moreover $|a_{ij}| \geqslant q+1$ and $|a_{ji}| \geqslant 2$, then $g \notin \overline{U_A^+}(k)$.*

Proof. — As usual, we realise $\mathfrak{U}_A^{ma+}(k)$ inside $\widehat{\mathcal{U}}_k^+$. Assume that $|a_{ij}| \geqslant q$. We claim that for any element $h = \sum_{\alpha \in Q_+} h_\alpha \in V := \overline{[\mathfrak{U}_A^{ma+}(k), \mathfrak{U}_A^{ma+}(k)]}$, where $h_\alpha \in \mathcal{U}_\alpha^+ \otimes_{\mathbb{Z}} k$ for all $\alpha \in Q_+$, the homogeneous components $h_{\alpha_i + \alpha_j}$ and $h_{q\alpha_i + \alpha_j}$ are either both zero or both nonzero.

Set

$$\Psi = Q_+ \setminus \{m\alpha_i + n\alpha_j \in Q_+ \mid 0 \leqslant m \leqslant q,\ 0 \leqslant n \leqslant 1\}$$

and

$$\widehat{\mathcal{U}}_\Psi^+ := \prod_{\alpha \in \Psi} (\mathcal{U}_\alpha^+ \otimes_{\mathbb{Z}} k) \subseteq \widehat{\mathcal{U}}_k^+.$$

Note that $\widehat{\mathcal{U}}_\Psi^+$ is an ideal of the k-algebra $\widehat{\mathcal{U}}_k^+$. To prove the claim, we will compute modulo $\widehat{\mathcal{U}}_\Psi^+$.

Any element of $\mathfrak{U}_A^{ma+}(k)$ is congruent modulo $\widehat{\mathcal{U}}_\Psi^+$ to an element of the form

$$\begin{aligned} g_{\underline{\lambda}} &:= \exp \lambda e_i \cdot \prod_{s=0}^{q} [\exp] \lambda_s (\operatorname{ad} e_i)^{(s)} e_j \\ &\equiv \exp \lambda e_i \cdot (1 + x(\underline{\lambda})) \mod \widehat{\mathcal{U}}_\Psi^+ \end{aligned}$$

for some tuple $\underline{\lambda} := (\lambda, \lambda_0, \dots, \lambda_q) \in k^{q+2}$, where

$$x(\underline{\lambda}) := \sum_{s=0}^{q} \lambda_s (\operatorname{ad} e_i)^{(s)} e_j.$$

Using the identity (see for instance [15, Lemma 4.9])

$$\begin{aligned} \exp \mu e_i \cdot x(\underline{\lambda}) \cdot \exp(-\mu e_i) &= (\exp \operatorname{ad} \mu e_i) x(\underline{\lambda}) \\ &= \sum_{s=0}^{q} \sum_{t \geqslant 0} \lambda_s \mu^t \binom{s+t}{t} (\operatorname{ad} e_i)^{(s+t)} e_j \end{aligned}$$

and the fact that

$$\big([\exp]\lambda_s(\operatorname{ad} e_i)^{(s)}e_j\big)^{-1} \equiv 1 - \lambda_s(\operatorname{ad} e_i)^{(s)}e_j \mod \widehat{\mathcal{U}}^+_\Psi,$$

we may now compute, for two tuples $\underline{\lambda}$ and $\underline{\mu}$ in k^{q+2} as above, that

$$\begin{aligned}[g_{\underline{\lambda}}, g_{\underline{\mu}}] &\equiv \exp\lambda e_i \cdot (1 + x(\underline{\lambda})) \cdot \exp\mu e_i \cdot (1 + x(\underline{\mu})) \cdot (1 - x(\underline{\lambda})) \\ &\qquad\qquad\qquad \cdot \exp(-\lambda e_i) \cdot (1 - x(\underline{\mu})) \cdot \exp(-\mu e_i) \\ &\equiv 1 + (\exp\operatorname{ad}\lambda e_i)x(\underline{\lambda}) + (\exp\operatorname{ad}(\lambda+\mu)e_i)(x(\underline{\mu}) - x(\underline{\lambda})) \\ &\qquad\qquad\qquad - (\exp\operatorname{ad}\mu e_i)x(\underline{\mu}) \\ &\equiv 1 + \sum_{s=1}^{q} C_s(\underline{\lambda}, \underline{\mu}) \cdot (\operatorname{ad} e_i)^{(s)}e_j \mod \widehat{\mathcal{U}}^+_\Psi\end{aligned}$$

for some polynomials $C_s \in k[\lambda, \lambda_0, \dots, \lambda_q, \mu, \mu_0, \dots, \mu_q]$ satisfying

$$C_1(\underline{\lambda}, \underline{\mu}) = \lambda\mu_0 - \mu\lambda_0 = \lambda^q\mu_0 - \mu^q\lambda_0 = C_q(\underline{\lambda}, \underline{\mu}).$$

Here we used the fact that $\binom{q}{t} = 0$ in k unless $t = 0$ or $t = q$.

Let now $h = \sum_{\alpha\in Q_+} h_\alpha \in V$. Then h is congruent modulo $\widehat{\mathcal{U}}^+_\Psi$ to a (finite) product of elements of the form $[g_{\underline{\lambda}}, g_{\underline{\mu}}]$ as above, say

$$h \equiv \prod_r [g_{\underline{\lambda}^r}, g_{\underline{\mu}^r}] \equiv 1 + \sum_r ([g_{\underline{\lambda}^r}, g_{\underline{\mu}^r}] - 1) \mod \widehat{\mathcal{U}}^+_\Psi$$

for some tuples $\underline{\lambda}^r$, $\underline{\mu}^r$ in k^{q+2}. The above discussion then implies that there is some $c \in k$ such that

$$h_{\alpha_i+\alpha_j} = c[e_i, e_j] \quad \text{and} \quad h_{q\alpha_i+\alpha_j} = c(\operatorname{ad} e_i)^{(q)}e_j,$$

proving our claim. This shows in particular that $g \notin V + \widehat{\mathcal{U}}^+_\Psi$. Since $1+\widehat{\mathcal{U}}^+_\Psi$ contains the open subgroup $\mathfrak{U}^{ma}_{q+2}(k)$, so that in particular $V\mathfrak{U}^{ma}_{q+2}(k) \subseteq V + \widehat{\mathcal{U}}^+_\Psi$, we deduce that $g \notin \overline{V}$, proving (1).

Assume now that $|a_{ij}| \geqslant q+1$ and $|a_{ji}| \geqslant 2$. In particular, the only real roots not in Ψ are the simple roots α_i and α_j (see [11, Chapter 5]). Assume for a contradiction that $g \in \overline{U^+_A}(k)$. Then

$$g \equiv \exp(\lambda e_i)\exp(\mu e_j) \mod V + \widehat{\mathcal{U}}^+_\Psi$$

for some $\lambda, \mu \in k$. Since $V + \widehat{\mathcal{U}}^+_\Psi \subseteq 1 + \widehat{\mathcal{U}}^+_{\geqslant 2}$, where

$$\widehat{\mathcal{U}}^+_{\geqslant 2} := \prod_{\operatorname{ht}(\alpha)\geqslant 2} (\mathcal{U}^+_\alpha \otimes_{\mathbb{Z}} k) \subseteq \widehat{\mathcal{U}}^+_k,$$

the components of degree α_i and α_j of $\exp(\lambda e_i)\exp(\mu e_j)$ must be zero, so that $\lambda = \mu = 0$. Hence $g \in V + \widehat{\mathcal{U}}^+_\Psi$. But this contradicts the first part of the proof, yielding (2). □

As pointed out to us by Pierre-Emmanuel Caprace, the methods of this section can also be used to show that Kac–Moody groups $G_A^{pma}(k)$ (or even $\overline{\mathfrak{G}_A}(k)$) are in general not GK-simple if $\operatorname{char} k \leqslant M_A$.

Proposition 4.9. — *Let $k = \mathbb{F}_q$ be a finite field. Consider the GCM $A = \left(\begin{smallmatrix} 2 & -m \\ -n & 2 \end{smallmatrix}\right)$ with $m, n \geqslant 2$ and $mn > 4$. Assume that $m \equiv n \equiv 2 \pmod{q-1}$. If $\operatorname{char} k = 2$, we moreover assume that at least one of m and n is odd. Then $\mathfrak{G}_A^{pma}(k)$ and $\overline{\mathfrak{G}_A}(k)$ are not GK-simple, that is, $Z'_A \cap \overline{U_A^+}(k) \neq \{1\}$.*

Proof. — Consider the (affine) GCM $B = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Note first that the hypotheses of Lemma 4.5 are satisfied. Hence, as noted in the proof of this lemma, there is an isomorphism of the Rémy–Ronan completions $\mathfrak{G}_A^{rr}(k)$ of $\mathfrak{G}_A(k)$ and $\mathfrak{G}_B^{rr}(k)$ of $\mathfrak{G}_B(k)$ that preserves the corresponding BN-pair structures. In particular, the Rémy–Ronan completions $U_A^{rr+}(k) \subseteq \mathfrak{G}_A^{rr}(k)$ of $U_A^+(k)$ and $U_B^{rr+}(k) \subseteq \mathfrak{G}_B^{rr}(k)$ of $U_B^+(k)$ are isomorphic.

Assume for a contradiction that $Z'_A \cap \overline{U_A^+}(k) = \{1\}$. Then the surjective homomorphism $\varphi_A \colon \overline{U_A^+}(k) \to U_A^{rr+}(k)$ (see Section 2.6) is an isomorphism, so that $\overline{U_A^+}(k) \cong U_A^{rr+}(k) \cong U_B^{rr+}(k)$. On the other hand, it follows from [27] (and the fact that $\mathfrak{G}_B^{rr}(k) \cong \mathrm{PSL}_2(k(\!(t)\!))$) that $U_B^{rr+}(k)$ is just-infinite: every proper quotient of $U_B^{rr+}(k)$ is finite. But Corollary 3.7 provides a map $\pi_{AB} \colon \overline{U_A^+}(k) \to \overline{U_B^+}(k)$ with nontrivial kernel: in fact, $\ker \pi_{AB}$ is even infinite, as it contains all real root groups in $U_A^+(k)$ associated to positive real roots $\alpha = x\alpha_1 + y\alpha_2$ with $x, y \geqslant 2$ (i.e., by [11, Exercises 5.25–5.27], the element α is a positive real root in both $\Delta_+^{\mathrm{re}}(A)$ and $\Delta_+^{\mathrm{re}}(B)$ if and only if $nx^2 - mnxy + my^2 \in \{m, n\}$ and $|x - y| = 1$, which is easily seen to have no positive integral solutions (x, y) other than $(x, y) = (1, 2)$ if $n = 2$ and $(x, y) = (2, 1)$ if $m = 2$. One then concludes as in Remark 3.8). Moreover, π_{AB} has infinite image, as $\pi_{AB}(U_A^+(k))$ contains the subgroup of $U_B^+(k)$ generated by the simple root groups. Hence $\overline{U_A^+}(k)$ cannot be just-infinite, a contradiction. □

5. Non-linearity

This section is devoted to the proof of Theorem G. For earlier contributions to the linearity problem for the group $\mathfrak{U}_A^{ma+}(k)$ over a finite field k, we refer to [2, §4.2] (see also [6]).

We recall that a GCM $A = (a_{ij})_{i,j \in I}$ is called *indecomposable* if, up to a permutation of the index set I, it does not admit any nontrivial block-diagonal decomposition $A = \left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right)$. Indecomposable GCM are either of finite, affine or indefinite type (see [11, Chapter 4]). If A is of indefinite type and all proper submatrices of A (corresponding to proper subdiagrams of the Dynkin diagram of A) are of finite type, then A is moreover said to be of *compact hyperbolic type*.

Lemma 5.1. — *Let A be a GCM of compact hyperbolic type. Then there exists some $B \leqslant A$ such that B is of affine type.*

Proof. — We use the notation of [11, §4.8] for the parametrisation of affine GCM. If A is of rank 2, then one can take for B the GCM of affine type $A_1^{(1)}$ or $A_2^{(2)}$. If the Dynkin diagram of A is a cycle of length $\ell+1$ for some $\ell \geqslant 2$, then one can take for B the GCM of affine type $A_\ell^{(1)}$. Assume now that the Dynkin diagram of A is not a cycle and that A is of rank at least 3. Then A must correspond to one of the 7 Dynkin diagrams $H_{100}^{(3)}$, $H_{106}^{(3)}$, $H_{101}^{(3)}$, $H_{105}^{(3)}$, $H_{114}^{(3)}$, $H_{115}^{(3)}$ and $H_{116}^{(3)}$ from [7, Section 7]. One can then respectively choose B to be affine of type $D_4^{(3)}$, $G_2^{(1)}$, $D_4^{(3)}$, $G_2^{(1)}$, $D_4^{(3)}$, $D_4^{(3)}$ and $G_2^{(1)}$. $\square$

Using the results of [6], we can now prove our non-linearity theorem.

Theorem 5.2. — *Let A be an indecomposable GCM of non-finite type and let k be a finite field. Assume that $\mathfrak{G}_A^{pma}(k)$ is GK-simple and set $G := \mathfrak{G}_A^{pma}(k)/Z'_A$. Then the following assertions are equivalent:*

1. *Every compact open subgroup of G is just-infinite (i.e. possesses only finite proper quotients).*
2. *$\mathfrak{U}_A^{ma+}(k)$ is linear over a local field.*
3. *G is a simple algebraic group over a local field.*
4. *The matrix A is of affine type.*

Proof. — Note that the GK-simplicity assumption on $\mathfrak{G}_A^{pma}(k)$ allows to view $\mathfrak{U}_A^{ma+}(k)$ (rather than a quotient of $\mathfrak{U}_A^{ma+}(k)$) as a subgroup of the simple group G.

The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear. Since G is a non-discrete, compactly generated, topologically simple, totally disconnected locally compact group (see, for instance, [4, Appendix A]) and since $\mathfrak{U}_A^{ma+}(k)$ is an open compact subgroup of G, the implication (2) $\Rightarrow$ (3) follows from [6, Corollary 1.4], while the implication (3) $\Rightarrow$ (1) follows from [6, Theorem 2.6]. We are thus left with the proof of (1) $\Rightarrow$ (4).

Assume thus that $\mathfrak{U}_A^{ma+}(k)$ is just-infinite, and suppose for a contradiction that A is of indefinite type. Assume first that $A = (a_{ij})_{i,j\in I}$ has a proper submatrix $(a_{ij})_{i,j\in J}$ of non-finite type. Consider the closed sets of positive roots

$$\Psi_J := \Delta_+(A) \cap \bigoplus_{j\in J} \mathbb{N}\alpha_j \quad \text{and} \quad \Psi_{I\setminus J} := \Delta_+(A) \setminus \Psi_J.$$

Note that $\Psi_{I\setminus J}$ is an ideal in $\Delta_+(A)$, in the sense that $\alpha+\beta \in \Psi_{I\setminus J}$ for all $\alpha \in \Delta_+(A)$ and $\beta \in \Psi_{I\setminus J}$ such that $\alpha+\beta \in \Delta_+(A)$. It then follows from [28, Lemme 3.3 (c)] that $\mathfrak{U}^{ma}_{\Psi_{I\setminus J}}(k)$ is normal in $\mathfrak{U}^{ma+}_A(k)$ and that

$$\mathfrak{U}^{ma+}_A(k)/\mathfrak{U}^{ma}_{\Psi_{I\setminus J}}(k) \cong \mathfrak{U}^{ma}_{\Psi_J}(k)$$

is infinite, contradicting (1).

We may thus assume that A is of compact hyperbolic type. By Lemma 5.1, there exists a matrix B of affine type such that $B \leqslant A$. It then follows from Theorem 3.6 that there is a surjective map $\widehat{\pi}_{AB}\colon \mathfrak{U}^{ma+}_A(k) \to \mathfrak{U}^{ma+}_B(k)$. This again yields an infinite quotient

$$\mathfrak{U}^{ma+}_A(k)/K \cong \mathfrak{U}^{ma+}_B(k)$$

of $\mathfrak{U}^{ma+}_A(k)$ for $K := \ker \widehat{\pi}_{AB}$, in contradiction with (1). This concludes the proof of the theorem. □

Remark 5.3. — Note that, up to replacing $\mathfrak{U}^{ma+}_A(k)$ by the quotient $\mathfrak{U}^{ma+}_A(k)/Z$ where $Z := Z'_A \cap \mathfrak{U}^{ma+}_A(k)$ in the statement of Theorem 5.2, the GK-simplicity assumption on $\mathfrak{G}^{pma}_A(k)$ can be substantially weakened. Indeed, the only issue that may arise in the above proof of Theorem 5.2 if we replace $\mathfrak{U}^{ma+}_A(k)$ by its quotient $\mathfrak{U}^{ma+}_A(k)/Z$ is that for A of compact hyperbolic type, the implication (1) ⇒ (4) would require to ensure that the map

$$\mathfrak{U}^{ma+}_A(k)/Z \to \mathfrak{U}^{ma+}_B(k)/\widehat{\pi}_{AB}(Z)$$

induced by $\widehat{\pi}_{AB}$ has still infinite image. In other words, we need to know that KZ is not open in $\mathfrak{U}^{ma+}_A(k)$ where $K := \ker \widehat{\pi}_{AB}$, which is *a priori* much weaker than the GK-simplicity assumption $Z = \{1\}$.

6. On the isomorphism problem

This section is devoted to the proof of Proposition H. Let $A = (a_{ij})_{i,j\in I}$ be a GCM, and let k be a field. Set $\gamma_1(\mathfrak{U}^{ma+}_A(k)) := \mathfrak{U}^{ma+}_A(k)$, and for each $n \geqslant 1$, define recursively

$$\gamma_{n+1}(\mathfrak{U}^{ma+}_A(k)) := \overline{[\mathfrak{U}^{ma+}_A(k), \gamma_n(\mathfrak{U}^{ma+}_A(k))]},$$

that is, $\gamma_{n+1}(\mathfrak{U}^{ma+}_A(k))$ is the closure in $\mathfrak{U}^{ma+}_A(k)$ of the commutator subgroup $[\mathfrak{U}^{ma+}_A(k), \gamma_n(\mathfrak{U}^{ma+}_A(k))]$.

Remark 6.1. — If k is a finite field of characteristic $p > M_A$, then $\mathfrak{U}_A^{ma+}(k)$ is a finitely generated pro-p group by [2, §2.2]. It then follows from [9, Exercise 1.17] that $\big(\gamma_n(\mathfrak{U}_A^{ma+}(k))\big)_{n\geqslant 1}$ coincides with the lower central series of $\mathfrak{U}_A^{ma+}(k)$.

The proof of the following proposition is an adaptation of the proof of [28, Proposition 6.11] (see also [2, §2.2]).

Proposition 6.2. — *Let $A = (a_{ij})_{i,j\in I}$ be a GCM and let k be a field. Assume that* $\operatorname{char} k = 0$ *or that* $\operatorname{char} k > M_A$. *Then $\gamma_n(\mathfrak{U}_A^{ma+}(k)) = \mathfrak{U}_{A,n}^{ma}(k)$ for all $n \geqslant 1$.*

Proof. — To lighten the notation, we set $U^{ma+} = \mathfrak{U}_A^{ma+}(k)$ and $U_n^{ma} = \mathfrak{U}_{A,n}^{ma}(k)$. Given some $n \geqslant 1$, it follows from [28, proof of Proposition 6.11] that

$$U_m^{ma} \subseteq [U^{ma+}, U_n^{ma}] \cdot U_{m+1}^{ma} \quad \text{for all } m \geqslant n+1.$$

Indeed, in the notation of *loc. cit.*, G. Rousseau proves that for any given $g \in U_m^{ma}$, there exists some $i \in I$ and some $h \in U_{m-1}^{ma}$ such that $g \equiv [\exp e_i, h] \mod U_{m+1}^{ma}$, yielding the claim. By definition of the topology on U^{ma+}, we deduce that $U_{n+1}^{ma} \subseteq \overline{[U^{ma+}, U_n^{ma}]}$ for all $n \geqslant 1$. Since the reverse inclusion holds as well by [28, Lemme 3.3], so that

$$U_{n+1}^{ma} = \overline{[U^{ma+}, U_n^{ma}]} \quad \text{for all } n \geqslant 1,$$

the proposition follows from an easy induction on n. $\square$

Remark 6.3. — If $k = \mathbb{F}_q$ is finite and such that $|a_{ij}| \geqslant q$ for some $i, j \in I$, Proposition 4.8 shows that the conclusion of Proposition 6.2 does not hold anymore, i.e. $\gamma_2(\mathfrak{U}_A^{ma+}(k))$ is properly contained in $\mathfrak{U}_{A,2}^{ma}(k)$.

We now apply the above observations to the study of the isomorphism problem for Mathieu–Rousseau completions of Kac–Moody groups over finite fields. We first record some known facts about complete Kac–Moody groups allowing to recognise specific subgroups from the topological group structure.

For this, we will need to define Kac–Moody groups in a slightly more general context, by considering arbitrary Kac–Moody root data (see for instance [28, §1.1] or [17, §7.3]).

To simplify the notation, we have so far considered Kac–Moody root data $\mathcal{D}$ of simply connected type, as we are mainly interested in the structure of the subgroup $\mathfrak{U}_A^{ma+}$, which only depends on the GCM A and not on a specific choice of $\mathcal{D}$. For $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ arbitrary with associated GCM $A = (a_{ij})_{i,j\in I}$, we denote by $\mathfrak{G}_{\mathcal{D}}^{pma}$ the Mathieu–Rousseau completion of the Tits functor $\mathfrak{G}_{\mathcal{D}}$ of type $\mathcal{D}$ (see [28, §3.19]), and by $Z'_{\mathcal{D}}$ the kernel of the action of $\mathfrak{G}^{pma}_{\mathcal{D}}$ on its associated building.

The additional information provided by $\mathcal{D}$ is encoded in the torus scheme $\mathfrak{T}_{\mathcal{D}}$. We denote as before by $\mathfrak{B}^{ma+}_{\mathcal{D}} = \mathfrak{T}_{\mathcal{D}} \ltimes \mathfrak{U}^{ma+}_A$ the standard Borel subgroup of $\mathfrak{G}^{pma}_{\mathcal{D}}$. Given a subset $J \subseteq I$, we let $\mathfrak{P}^{ma+}_{\mathcal{D}}(J)$ denote the standard parabolic subgroup of $\mathfrak{G}^{pma}_{\mathcal{D}}$ of type J (see [28, §3.10]). We also set

$$\mathcal{D}(J) := (J, A|_J, \Lambda, (c_i)_{i\in J}, (h_i)_{i\in J})$$

where $A|_J = (a_{ij})_{i,j\in J}$ and $\Delta_+(J) := \Delta_+ \cap \bigoplus_{j\in J} \mathbb{Z}\alpha_j$.

Lemma 6.4. — *Let $\mathcal{D}$ be a Kac–Moody root datum with associated GCM $A = (a_{ij})_{i,j\in I}$ and let k be a finite field of characteristic p. Then the following hold:*

1. *If $\mathfrak{G}^{pma}_{\mathcal{D}}(k)$ contains an open pro-q subgroup for some prime q, then $q = p$.*
2. *Every maximal pro-p subgroup of $\mathfrak{G}^{pma}_{\mathcal{D}}(k)$ is conjugate to $\mathfrak{U}^{ma+}_A(k)$.*
3. *The normaliser of $\mathfrak{U}^{ma+}_A(k)$ in $\mathfrak{G}^{pma}_{\mathcal{D}}(k)$ is the standard Borel subgroup $\mathfrak{B}^{ma+}_{\mathcal{D}}(k)$.*
4. *The subgroups of $\mathfrak{G}^{pma}_{\mathcal{D}}(k)$ containing $\mathfrak{B}^{ma+}_{\mathcal{D}}(k)$ are precisely the standard parabolic subgroups of $\mathfrak{G}^{pma}_{\mathcal{D}}(k)$.*
5. *For any subset $J \subset I$, one has a Levi decomposition $\mathfrak{P}^{ma+}_{\mathcal{D}}(J) = \mathfrak{G}^{pma}_{\mathcal{D}(J)} \ltimes \mathfrak{U}^{ma}_{\Delta_+\setminus\Delta_+(J)}$. Moreover,*

$$\bigcap_{g\in\mathfrak{P}^{ma+}_{\mathcal{D}}(J)} g\mathfrak{U}^{ma+}_A(k)g^{-1} = (Z'_{\mathcal{D}(J)} \cap \mathfrak{U}^{ma}_{\Delta_+(J)}(k)) \ltimes \mathfrak{U}^{ma}_{\Delta_+\setminus\Delta_+(J)}(k).$$

Proof. — To prove (1), let V be an open pro-q subgroup of $\mathfrak{G}^{pma}_{\mathcal{D}}(k)$. Then $V' := V \cap \mathfrak{U}^{ma+}_A(k)$ is open in V, hence an open pro-q subgroup of $\mathfrak{U}^{ma+}_A(k)$ (see e.g. [9, Proposition 1.11 (i)]). Since $\mathfrak{U}^{ma+}_A(k)$ is pro-p, the same argument implies that V' is pro-p, and hence $q = p$.

The second statement follows from [25, 1.B.2] (see also [2, Section 2]). The statements (3) and (4) are standard (see e.g. [1, Theorem 6.43]). The Levi decomposition in (5) follows from [28, 3.10].

Let us now prove the identity in (5). Since $\mathfrak{U}^{ma}_{\Delta_+\setminus\Delta_+(J)}(k)$ is normal in $\mathfrak{P}^{ma+}_{\mathcal{D}}(J)$ (see the above Levi decomposition) and since $Z'_{\mathcal{D}(J)} \cap \mathfrak{U}^{ma}_{\Delta_+(J)}(k)$ is the Gabber–Kac kernel of $\mathfrak{G}^{pma}_{\mathcal{D}(J)}$ (hence is conjugate under any element of $\mathfrak{P}^{ma+}_{\mathcal{D}}(J) = \mathfrak{G}^{pma}_{\mathcal{D}(J)} \ltimes \mathfrak{U}^{ma}_{\Delta_+\setminus\Delta_+(J)}$ to an element of $\mathfrak{U}^{ma+}_A(k)$), the inclusion from right to left is clear. Conversely, since the Gabber–Kac kernel $Z'_{\mathcal{D}(J)} \cap \mathfrak{U}^{ma}_{\Delta_+(J)}(k)$ is the largest normal subgroup of $\mathfrak{G}^{pma}_{\mathcal{D}(J)}$ that is contained in

$\mathfrak{U}^{ma}_{\Delta_+(J)}(k)$, the image of $\bigcap_{g\in\mathfrak{P}^{ma+}_{\mathcal{D}}(J)} g\mathfrak{U}^{ma+}_A(k)g^{-1}$ under the quotient map

$$\mathfrak{U}^{ma+}_A(k)=\mathfrak{U}^{ma}_{\Delta_+(J)}(k)\ltimes\mathfrak{U}^{ma}_{\Delta_+\setminus\Delta_+(J)}(k)\to\mathfrak{U}^{ma}_{\Delta_+(J)}(k)$$

(see [28, Lemme 3.3 (c)]) is contained in $Z'_{\mathcal{D}(J)}\cap\mathfrak{U}^{ma}_{\Delta_+(J)}(k)$, as desired. $\square$

To lighten the notation, we will write $H/Z'_A := H/(H\cap Z'_A)$ for any subgroup H of $\mathfrak{G}^{pma}_A(k)$.

Lemma 6.5. — *Let $\mathcal{D},\mathcal{D}'$ be Kac–Moody root data with associated GCM $A=(a_{ij})_{i,j\in I}$ and $A'=(a'_{ij})_{i,j\in I'}$, respectively. Let also k,k' be finite fields. If $\alpha\colon \mathfrak{G}^{pma}_{\mathcal{D}}(k)/Z'_{\mathcal{D}}\to\mathfrak{G}^{pma}_{\mathcal{D}'}(k')/Z'_{\mathcal{D}'}$ is an isomorphism of topological groups, then $k\cong k'$ and there exist an inner automorphism γ of $\mathfrak{G}^{pma}_{\mathcal{D}'}(k')/Z'_{\mathcal{D}'}$ and a bijection $\sigma\colon I\to I'$ such that*

$$\gamma\alpha(\mathfrak{U}^{ma+}_{A|_{\{i,j\}}}(k)/Z'_{\mathcal{D}})=\mathfrak{U}^{ma+}_{A'|_{\{\sigma(i),\sigma(j)\}}}(k')/Z'_{\mathcal{D}'}$$

for all distinct $i,j\in I$.

Proof. — By Lemma 6.4 (1) and (2), there exists an inner automorphism γ of $\mathfrak{G}^{pma}_{\mathcal{D}'}(k')/Z'_{\mathcal{D}'}$ such that $\gamma\alpha$ maps $\mathfrak{U}^{ma+}_A(k)/Z'_{\mathcal{D}}$ to $\mathfrak{U}^{ma+}_{A'}(k')/Z'_{\mathcal{D}'}$. Then $\gamma\alpha$ maps $\mathfrak{B}^{ma+}_{\mathcal{D}}(k)/Z'_{\mathcal{D}}$ to $\mathfrak{B}^{ma+}_{\mathcal{D}'}(k')/Z'_{\mathcal{D}'}$ by Lemma 6.4 (3). Hence Lemma 6.4 (4) implies that $\gamma\alpha$ maps maximal chains of standard parabolic subgroups in $\mathfrak{G}^{pma}_{\mathcal{D}}(k)/Z'_{\mathcal{D}}$ to maximal chains of standard parabolic subgroups in $\mathfrak{G}^{pma}_{\mathcal{D}'}(k')/Z'_{\mathcal{D}'}$. In particular, $|I|=|I'|$ and there exists a bijection $\sigma\colon I\to I'$ such that

$$\gamma\alpha(\mathfrak{P}^{ma+}_{\mathcal{D}}(\{i\})/Z'_{\mathcal{D}})=\mathfrak{P}^{ma+}_{\mathcal{D}'}(\{\sigma(i)\})/Z'_{\mathcal{D}'}\quad\text{for all } i\in I.$$

Hence

$$\gamma\alpha(\mathfrak{P}^{ma+}_{\mathcal{D}}(\{i,j\})/Z'_{\mathcal{D}})=\mathfrak{P}^{ma+}_{\mathcal{D}'}(\{\sigma(i),\sigma(j)\})/Z'_{\mathcal{D}'}$$

for all $i,j\in I$. It then follows from Lemma 6.4 (5) that

$$\gamma\alpha(\mathfrak{U}^{ma}_{\Delta_+\setminus\Delta_+(\{i,j\})}(k)/Z'_{\mathcal{D}})=\mathfrak{U}^{ma}_{\Delta_+\setminus\Delta_+(\{\sigma(i),\sigma(j)\})}(k')/Z'_{\mathcal{D}'}$$

and hence that

$$\gamma\alpha(\mathfrak{U}^{ma}_{\Delta_+(\{i,j\})}(k)/Z'_{\mathcal{D}})=\mathfrak{U}^{ma}_{\Delta_+(\{\sigma(i),\sigma(j)\})}(k')/Z'_{\mathcal{D}'}$$

for all $i,j\in I$ because

$$\mathfrak{U}^{ma+}_A=\mathfrak{U}^{ma}_{\Delta_+(J)}(k)\ltimes\mathfrak{U}^{ma}_{\Delta_+\setminus\Delta_+(J)}(k)\quad\text{for all } J\subseteq I,$$

and similarly for $\mathfrak{U}^{ma+}_{A'}$. As $\mathfrak{U}^{ma}_{\Delta_+(\{i,j\})}(k)=\mathfrak{U}^{ma+}_{A|_{\{i,j\}}}(k)$, it thus remains to prove that $k\cong k'$.

Since each panel of the building X_+ of $\mathfrak{G}^{pma}_{\mathcal{D}}(k)/Z'_{\mathcal{D}}$ (respectively, X'_+ of $\mathfrak{G}^{pma}_{\mathcal{D}'}(k')/Z'_{\mathcal{D}'}$) is of cardinality $|k|+1$ (respectively, $|k'|+1$)(see for instance [1, Chapter 7]), and since $X_+=X'_+$ (as simplicial complexes) by the above discussion, we deduce that $|k| = |k'| =: q$, and hence that $k \cong \mathbb{F}_q \cong k'$. This concludes the proof of the lemma. $\square$

Remark 6.6. — In the notation of Lemma 6.5, if α lifts to an isomorphism $\alpha\colon \mathfrak{G}^{pma}_{\mathcal{D}}(k) \to \mathfrak{G}^{pma}_{\mathcal{D}'}(k')$ and if $\mathfrak{G}^{pma}_{\mathcal{D}}(k)$ is of rank 2 (that is, $|I| = 2$), then Lemma 6.4 (1) and (2) implies that

$$\gamma\alpha(\mathfrak{U}^{ma+}_{A}(k)) = \mathfrak{U}^{ma+}_{A'}(k')$$

for some inner automorphism γ of $\mathfrak{G}^{pma}_{\mathcal{D}'}(k')$.

Lemma 6.7. — *Let $A = (a_{ij})_{i,j\in I}$ and $B = (b_{ij})_{i,j\in I}$ be GCM indexed by I and let k be a finite field with $p = \operatorname{char} k > M_A, M_B$. Assume that the groups $\mathfrak{U}^{ma+}_{A}(k)$ and $\mathfrak{U}^{ma+}_{B}(k)$ are isomorphic. Then the following hold:*

(1) $\sum_{\mathrm{ht}(\alpha)=n} \dim \mathfrak{g}(A)_\alpha = \sum_{\mathrm{ht}(\alpha)=n} \dim \mathfrak{g}(B)_\alpha$ *for all $n \geqslant 1$.*
(2) *If $I = \{i, j\}$, then $B = \left(\begin{smallmatrix} 2 & a_{ij} \\ a_{ji} & 2 \end{smallmatrix}\right)$ or $B = \left(\begin{smallmatrix} 2 & a_{ji} \\ a_{ij} & 2 \end{smallmatrix}\right)$.*

Proof. — Let $\alpha\colon \mathfrak{U}^{ma+}_{A}(k) \to \mathfrak{U}^{ma+}_{B}(k)$ be an isomorphism. Then α maps $\mathfrak{U}^{ma}_{A,n}(k)$ to $\mathfrak{U}^{ma}_{B,n}(k)$ for each $n \geqslant 1$ by Proposition 6.2, and hence induces isomorphisms of the quotients

$$\mathfrak{U}^{ma}_{A,n}(k)/\mathfrak{U}^{ma}_{A,n+1}(k) \cong \mathfrak{U}^{ma}_{B,n}(k)/\mathfrak{U}^{ma}_{B,n+1}(k) \quad \text{for all } n \geqslant 1.$$

In turn, this yields isomorphisms of the additive groups

$$\bigoplus_{\mathrm{ht}(\alpha)=n} \mathfrak{n}^+_k(A)_\alpha \cong \bigoplus_{\mathrm{ht}(\alpha)=n} \mathfrak{n}^+_k(B)_\alpha$$

by [28, Lemme 3.3 (e)]. Hence (1) follows from the fact that if $d_n(A) = \sum_{\mathrm{ht}(\alpha)=n} \dim \mathfrak{g}(A)_\alpha$, then $|k|^{d_n(A)}$ is the cardinality of $\bigoplus_{\mathrm{ht}(\alpha)=n} \mathfrak{n}^+_k(A)_\alpha$.

Assume now that $I = \{i, j\}$. For $X \in \{A, B\}$, let $\mathfrak{i}^+(X)$ be the ideal of the free Lie algebra $\tilde{\mathfrak{n}}^+(X)$ generated by the Serre relations $x^+_{ij}(X) = \operatorname{ad}(e_i)^{1+|X_{ij}|}e_j$ and $x^+_{ji}(X) = \operatorname{ad}(e_j)^{1+|X_{ji}|}e_i$. For each $n \geqslant 1$, let also $\tilde{\mathfrak{n}}^+_n(X)$ denote the subspace of elements of $\tilde{\mathfrak{n}}^+(X)$ of total degree n, that is, the linear span of all brackets of the form $[e_{i_1}, \ldots, e_{i_n}]$ ($i_s \in I$). In particular, since $\mathfrak{i}^+(X)$ is graded,

$$\tilde{\mathfrak{n}}^+_n(X)/\mathfrak{i}^+_n(X) = \mathfrak{n}^+_n(X) \quad \text{for all } n \geqslant 1$$

as vector spaces, where $\mathfrak{i}^+_n(X) := \mathfrak{i}^+(X) \cap \tilde{\mathfrak{n}}^+_n(X)$ and $\mathfrak{n}^+_n(X) := \bigoplus_{\mathrm{ht}(\alpha)=n} \mathfrak{n}^+(X)_\alpha$. The above discussion now implies that

$$\begin{aligned}\dim \mathfrak{i}^+_n(A) = \dim \tilde{\mathfrak{n}}^+_n(A) - \dim \mathfrak{n}^+_n(A) &= \dim \tilde{\mathfrak{n}}^+_n(B) - \dim \mathfrak{n}^+_n(B) \\ &= \dim \mathfrak{i}^+_n(B) \quad \text{for all } n \geqslant 1.\end{aligned}$$

If $|a_{ij}| = |a_{ji}| = m$, then $\dim \mathfrak{i}_n^+(A) = 0$ for all $n \leqslant m+1$, while $\dim \mathfrak{i}_{m+2}^+(A) = 2$. The corresponding assertion for B then implies that $|b_{ij}| = |b_{ji}| = m$, proving (2) in this case.

Assume now that $a_{ij} \neq a_{ji}$, say $m = |a_{ij}| < |a_{ji}| = m'$. Then $\dim \mathfrak{i}_n^+(A) = 0$ for all $n \leqslant m+1$, while $\dim \mathfrak{i}_{m+2}^+(A) = 1$. Again, the corresponding assertion for B implies that $m = |b_{ij}| < |b_{ji}|$ or that $m = |b_{ji}| < |b_{ij}|$. Say $m = |b_{ij}| < |b_{ji}| = m''$. For $X \in \{A, B\}$, let $\mathfrak{i}_{ij}^+(X)$ denote the ideal of $\tilde{\mathfrak{n}}^+(X)$ generated by $x_{ij}^+(X) = \mathrm{ad}(e_i)^{1+m} e_j$. Assume for a contradiction that $m' \neq m''$, say $m' < m''$ (the case $m' > m''$ being similar). Then

$$\begin{aligned}\dim \mathfrak{i}_{m'+2}^+(A) &= \dim(\mathfrak{i}_{ij}^+(A) \cap \tilde{\mathfrak{n}}_{m'+2}^+(A)) + 1\\ &= \dim(\mathfrak{i}_{ij}^+(B) \cap \tilde{\mathfrak{n}}_{m'+2}^+(B)) + 1\\ &= \dim \mathfrak{i}_{m'+2}^+(B) + 1,\end{aligned}$$

yielding the desired contradiction. This concludes the proof of (2). □

THEOREM 6.8. — *Let k, k' be finite fields, and $A = (a_{ij})_{i,j\in I}$ and $B = (b_{ij})_{i,j\in J}$ be GCM. Assume that $p = \mathrm{char}\, k > M_A, M_B$ and that all rank 2 subgroups of $\mathfrak{G}_A^{pma}(k)$ and $\mathfrak{G}_B^{pma}(k')$ are GK-simple.*

If $\alpha\colon \mathfrak{G}_A^{pma}(k)/Z'_A \to \mathfrak{G}_B^{pma}(k')/Z'_B$ is an isomorphism of topological groups, then $k \cong k'$, and there exist an inner automorphism γ of $\mathfrak{G}_B^{pma}(k')/Z'_B$ and a bijection $\sigma\colon I \to J$ such that

1. *$\gamma\alpha(\mathfrak{U}_{A|_{\{i,j\}}}^{ma+}(k)) = \mathfrak{U}_{B|_{\{\sigma(i),\sigma(j)\}}}^{ma+}(k')$ for all distinct $i, j \in I$.*
2. *$B|_{\{\sigma(i),\sigma(j)\}} \in \left\{\left(\begin{smallmatrix} 2 & a_{ij} \\ a_{ji} & 2\end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & a_{ji} \\ a_{ij} & 2\end{smallmatrix}\right)\right\}$ for all distinct $i, j \in I$.*

Proof. — Since all rank 2 subgroups of $\mathfrak{G}_A^{pma}(k)$ and $\mathfrak{G}_B^{pma}(k')$ are GK-simple by assumption, (1) follows from Lemma 6.5 and (2) follows from Lemma 6.7. □

Remark 6.9. — In the notation of Theorem 6.8, if α lifts to an isomorphism $\alpha\colon \mathfrak{G}_A^{pma}(k) \to \mathfrak{G}_B^{pma}(k')$ and if $\mathfrak{G}_A^{pma}(k)$ is of rank 2, then the conclusion of Theorem 6.8 holds without any GK-simplicity assumption using Remark 6.6 and Lemma 6.7.

We conclude this section with two further observations on the isomorphism problem, using the results from the previous sections.

LEMMA 6.10. — *Let $A = (a_{ij})_{i,j\in I}$ and $B = (b_{ij})_{i,j\in I}$ be GCM, and let $k = \mathbb{F}_q$ with $\mathrm{char}\, k = p$. If $M_A < p$ and $M_B \geqslant q$, then $\mathfrak{U}_A^{ma+}(k)$ and $\mathfrak{U}_B^{ma+}(k)$ are not isomorphic.*

Proof. — By Proposition 6.2, the quotient of $\mathfrak{U}_A^{ma+}(k)$ by its commutator subgroup has cardinality $q^{|I|}$. On the other hand, it follows from Proposition 4.8 that the quotient of $\mathfrak{U}_A^{ma+}(k)$ by its commutator subgroup has cardinality strictly larger than $q^{|I|}$. This proves the claim. □

Proposition 6.11. — *Let $A = (a_{ij})_{i,j\in I}$ and $B = (b_{ij})_{i,j\in I}$ be GCM with $B \leqslant A$, and let k be a finite field with* $\operatorname{char} k > M_A$. *If $\mathfrak{U}_A^{ma+}(k)$ and $\mathfrak{U}_B^{ma+}(k)$ are isomorphic, then $B = A$.*

Proof. — Since $\mathfrak{U}_A^{ma+}(k)$ is a finitely generated residually finite prop-p group by [2, §2.2], it is Hopfian, in the sense that every surjective homomorphism from $\mathfrak{U}_A^{ma+}(k)$ to itself is an isomorphism (see Lemma 6.12 below). Assume now for a contradiction that $B \neq A$. Then by Theorem 3.6, there is a surjective group homomorphism $\widehat{\pi}_{AB}\colon \mathfrak{U}_A^{ma+}(k) \to \mathfrak{U}_B^{ma+}(k)$ with nontrivial kernel. Hence $\mathfrak{U}_A^{ma+}(k)$ and $\mathfrak{U}_B^{ma+}(k)$ cannot be isomorphic, for this would contradict the fact that $\mathfrak{U}_A^{ma+}(k)$ is Hopfian. □

The following lemma and its proof are a straightforward adaptation of [14, Theorem 4.10].

Lemma 6.12. — *Let G be a finitely generated residually finite pro-p group. Then G is Hopfian, i.e. every surjective homomorphism $G \to G$ is an isomorphism.*

Proof. — Let $\theta\colon G \to G$ be a surjective homomorphism, and let K be the kernel of θ. Let $n \in \mathbb{N}^*$. By [9, Proposition 1.6 and Theorem 1.17], there are only finitely many subgroups of G of index n, say $M_1, \dots, M_r$. Then the subgroups $L_i := \theta^{-1}(M_i)$ $(i = 1, \dots, r)$ are pairwise distinct and of index n in G. Thus $\{M_1, \dots, M_r\} = \{L_1, \dots, L_r\}$. In particular,

$$K \subseteq \bigcap_{i=1}^{r} L_i = \bigcap_{i=1}^{r} M_i,$$

and since n was arbitrary, we deduce that K is contained in the intersection of all finite-index subgroups of G. Since G is residually finite, this implies that $K = \{1\}$, as desired. □

Remark 6.13. — Lemma 6.12 also holds when G is a finitely generated residually finite profinite group. Indeed, the main result of [23] (which relies on the classification of finite simple groups) asserts that finite-index subgroups of a finitely generated profinite group G are automatically open, and hence G has only finitely many subgroups of index n for any given $n \in \mathbb{N}^*$ by [9, Proposition 1.6]. The proof of Lemma 6.12 thus also holds in that case.

7. Zassenhaus–Jennings–Lazard series

This section is devoted to the proof of Proposition D. The general reference for this section is [9, Chapter 11].

Given a group G, as well as some positive natural number n, we write G^n for the subgroup of G generated by the elements of the form g^n, $g \in G$. We also let $\gamma_n(G)$ denote the lower central series of G:

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{n+1}(G) = [G, \gamma_n(G)] \quad \text{for all } n \geqslant 1.$$

[Here, we consider lower central series in the category of abstract groups; as noticed in Remark 6.1, when G is a finitely generated pro-p group, this coincides with the lower central series defined at the beginning of Section 6.]

Let $k = \mathbb{F}_q$ be a finite field of characteristic p, let A be a GCM, and set $G := \mathfrak{U}_A^{ma+}(k)$. Then G is a prop-p group. Set $\Gamma_n = \gamma_n(G)$, and let $D_n = D_n(G)$ be the series of characteristic subgroups of G defined by $D_1 := G$ and for $n > 1$,

$$D_n := D_{n^*}^p \cdot \prod_{i+j=n} \big[D_i, D_j\big],$$

where $n^* := \lceil n/p \rceil$ is the least integer r such that $pr \geqslant n$. The series $(D_n)_{n\geqslant 1}$ is called the *Zassenhaus–Jennings–Lazard series* of G. The subgroups D_n are also called the *dimension subgroups* of G.

For each $n \geqslant 1$, the quotient $L_n := D_n/D_{n+1}$ is an elementary abelian p-group. We view it as a vector space over $\mathbb{F}_p$ and write the group operation additively. Then

$$L := \bigoplus_{n=1}^{\infty} L_n$$

is a graded Lie algebra over $\mathbb{F}_p$ for the Lie bracket

$$(\overline{x}, \overline{y}) := [x, y] D_{i+j+1} \in L_{i+j},$$

where $\overline{x} = xD_{i+1} \in L_i$ and $\overline{y} = yD_{j+1} \in L_j$ (see [9, p. 280]). It is called the *Zassenhaus–Jennings–Lazard Lie algebra* of G. Note that the p-operation

$$[p]\colon\ L_i \to L_{pi} : \overline{x} = xD_{i+1} \mapsto \overline{x}^{[p]} := x^p D_{pi+1}$$

extends to a p-operation on L, turning L into a restricted Lie algebra ([9, Theorem 12.8]).

Lemma 7.1. — *$\mathfrak{U}_n^{ma}(k)^p \subseteq \mathfrak{U}_{np}^{ma}(k)$ for all $n \geqslant 1$.*

Proof. — We realise as usual $\mathfrak{U}_A^{ma+}(k)$ inside $\widehat{\mathcal{U}}_k^+$. For each $m \geqslant 1$, we set

$$\widehat{\mathcal{U}}_{\geqslant m}^+ := \prod_{\mathrm{ht}(\alpha)\geqslant m} (\mathcal{U}_\alpha^+ \otimes_{\mathbb{Z}} k) \subseteq \widehat{\mathcal{U}}_k^+.$$

Let $g \in \mathfrak{U}_n^{ma}(k)$. Then $g = 1 + x$ for some $x \in \widehat{\mathcal{U}}_{\geqslant n}^+$, and hence

$$g^p = (1+x)^p = 1 + x^p \in 1 + \widehat{\mathcal{U}}_{\geqslant np}^+.$$

In particular, $g^p \in \mathfrak{U}_{np}^{ma}(k)$, as desired. □

Lemma 7.2. — *$\Gamma_n \leqslant D_n \leqslant \mathfrak{U}_n^{ma}(k)$ for all $n \geqslant 1$.*

Proof. — The first inclusion follows by induction on n, since $\Gamma_1 = G = D_1$ and since if $\Gamma_n \subseteq D_n$, then

$$\Gamma_{n+1} = [G, \Gamma_n] \subseteq [D_1, D_n] \subseteq D_{n+1}.$$

Since $\big[\mathfrak{U}_i^{ma}(k), \mathfrak{U}_j^{ma}(k)\big] \subseteq \mathfrak{U}_{i+j}^{ma}(k)$ for all $i, j \geqslant 1$ by [28, Lemme 3.3], the second inclusion follows from Lemma 7.1 and the fact that $(D_n)_{n\geqslant 1}$ is the fastest descending series with $D_1 = G$ such that $D_i^p \leqslant D_{pi}$ and $[D_i, D_j] \leqslant D_{i+j}$ for all $i, j \geqslant 1$. □

Corollary 7.3. — *Assume that $p > M_A$. Then $\Gamma_n = D_n = \mathfrak{U}_n^{ma}(k)$ for all $n \geqslant 1$.*

Proof. — The equality $\Gamma_n = \mathfrak{U}_n^{ma}(k)$ follows from Remark 6.1 and Proposition 6.2. The lemma then follows from Lemma 7.2. □

For each $n \geqslant 1$, set $(\mathfrak{n}_k^+)_n := \bigoplus_{\mathrm{ht}(\alpha)=n} (\mathfrak{n}_k^+)_\alpha$. Then the quotient $L_n(\mathfrak{U}_A^{ma+}(k)) := \mathfrak{U}_n^{ma}(k)/\mathfrak{U}_{n+1}^{ma}(k)$ is isomorphic to the additive group of $(\mathfrak{n}_k^+)_n$ by [28, Lemme 3.3(e)]. We view it as an $\mathbb{F}_p$-vector space and write the group operation additively. Set

$$L(\mathfrak{U}_A^{ma+}(k)) := \bigoplus_{n=1}^{\infty} L_n(\mathfrak{U}_A^{ma+}(k)),$$

which we endow with the graded Lie algebra structure given by the Lie bracket

$$(\overline{x}, \overline{y}) := [x, y]\mathfrak{U}_{i+j+1}^{ma}(k)$$

for $\overline{x} = x\mathfrak{U}_{i+1}^{ma}(k) \in L_i(\mathfrak{U}_A^{ma+}(k))$ and $\overline{y} = y\mathfrak{U}_{j+1}^{ma}(k) \in L_j(\mathfrak{U}_A^{ma+}(k))$.

Lemma 7.4. — *Let k be a finite field of characteristic p. The map $\mathfrak{n}_k^+ \to L(\mathfrak{U}_A^{ma+}(k))$ mapping a homogeneous element $x \in \mathfrak{n}_k^+$ with $\mathrm{ht}(\deg(x)) = n$ to $([\exp]x)\mathfrak{U}_{n+1}^{ma}(k)$ defines an isomorphism of Lie algebras over $\mathbb{F}_p$.*

Proof. — This readily follows from the fact that if $x, y \in \mathfrak{n}_k^+$ are homogeneous with $\mathrm{ht}(\deg(x)) = i$ and $\mathrm{ht}(\deg(y)) = j$, then

$$\big[[\exp]x, [\exp]y\big] \equiv [\exp][x, y] \mod \mathfrak{U}_{i+j+1}^{ma}(k). \qquad \square$$

Corollary 7.5. — *If $p > M_A$, then $L = L(\mathfrak{U}_A^{ma+}(k)) \cong \mathfrak{n}_k^+$ as Lie algebras over $\mathbb{F}_p$.*

Proof. — This follows from Corollary 7.3 and Lemma 7.4. □

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