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FINITE PRESHEAVES AND $\mathscr{A}$-FINITE GENERATION OF UNSTABLE ALGEBRAS MOD NILPOTENTS

by Geoffrey POWELL (*)

Abstract. — Inspired by the work of Henn, Lannes and Schwartz on unstable algebras over the Steenrod algebra modulo nilpotents, a characterization of unstable algebras that are $\mathscr{A}$-finitely generated up to nilpotents is given in terms of the associated presheaf. We do this by introducing the notion of a finite presheaf. In particular, this gives the natural characterization of the (co)analytic presheaves that are important in the theory of Henn, Lannes and Schwartz. An important source of examples is provided by unstable algebras of finite transcendence degree.

For unstable Hopf algebras, it is shown that the associated presheaf is finite if and only if its growth function is polynomial. This leads to a description of unstable Hopf algebras modulo nilpotents in the spirit of Henn, Lannes and Schwartz.

Résumé. — En s'inspirant du travail de Henn, Lannes et Schwartz sur la catégorie des algèbres instables sur l'algèbre de Steenrod localisée loin des nilpotents, une caractérisation des algèbres instables qui sont $\mathscr{A}$-finiment engendrées à nilpotents près est donnée en termes du préfaisceau associé en utilisant la notion d'un préfaisceau fini, qui est introduite dans cet article. Ceci permet une caractérisation naturelle des préfaisceaux (co)analytiques qui sont fondamentaux dans la théorie de Henn, Lannes et Schwartz. Une classe importante de préfaisceaux finis est fournie par les algèbres instables de degré de transcendance fini.

Pour les algèbres de Hopf instables, il est démontré que le préfaisceau associé est fini si et seulement si sa fonction de croissance est polynomiale. En particulier, ceci mène à une description de la catégorie des algèbres de Hopf, instables localisées loin des nilpotents.

1. Introduction

The work of Henn, Lannes and Schwartz [5] explains the relationship between unstable algebras over the mod p Steenrod algebra $\mathscr{A}$ and presheaves

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of profinite sets on the category $\mathscr{V}_f$ of finite-dimensional $\mathbb{F}_p$-vector spaces, where $\mathbb{F}_p$ is the prime field. This is based upon Lannes' T-functor; namely, for K an unstable algebra, there is a presheaf $\mathfrak{g}K$ given by

$$V \mapsto \mathrm{Hom}_{\mathscr{K}}(K, H^*(BV; \mathbb{F}_p)),$$

V an $\mathbb{F}_p$-vector space, where $\mathscr{K}$ is the category of unstable algebras. Motivating examples of unstable algebras are given by singular cohomology $H^*(Y; \mathbb{F}_p)$, for Y a topological space. Another presheaf is provided by homotopy classes of maps out of BV, $V \mapsto [BV, Y]$. These presheaves are related by Lannes' theory, notably in relation to the Sullivan conjecture.

One of the key results of [5, Part II] gives a characterization for an unstable algebra K to be Noetherian up to nilpotents in terms of the associated presheaf $\mathfrak{g}K$ (this is recalled here in Section 2.6). Likewise, the notion of transcendence degree is treated in terms of $\mathfrak{g}K$. The relevance of these to the current theory is explained below.

In the context of unstable algebras, the more general notion of being $\mathscr{A}$-finitely generated (see Section 2.3) is known to be of importance. For example, for an H-space, Castellana, Crespo and Scherer [2] showed that this condition imposes strong conditions on its structure as an H-space. One of the main objectives of this paper is to relax the condition of being $\mathscr{A}$-finitely generated in the spirit of Henn, Lannes and Schwartz, by considering unstable algebras that are $\mathscr{A}$-finitely generated up to nilpotents, and to give a characterization in terms of the associated presheaf.

This is carried out in the first part of the paper, by introducing the notion of a finite presheaf. This is given in terms of presheaves of $\mathbb{F}_p$-vector spaces. The latter form an abelian category and such a presheaf is said to be finite if it has a finite composition series. A set-valued presheaf X is defined to be finite if there exists an embedding $X \hookrightarrow F_X$ with F_X a finite presheaf of $\mathbb{F}_p$-vector spaces.

This definition may seem somewhat *ad hoc* at first view, in particular due to its dependence on using presheaves of $\mathbb{F}_p$-vector spaces. As shown in the second part of the paper, this condition on F_X can be relaxed: it suffices that F_X be a presheaf of finite p-groups that is a polynomial functor in the sense of Baues and Pirashvili [1] (generalizing the Eilenberg–MacLane notion of a polynomial functor to an abelian category [3], as recalled in Appendix A); this is a consequence of Theorem 8.7.

The characterization of $\mathscr{A}$-finitely generated up to nilpotents is as follows:

THEOREM (Corollary 4.4). — *For K an unstable algebra, the following are equivalent:*

(1) *K is $\mathscr{A}$-finitely generated up to nilpotents;*
(2) *$\mathfrak{g}K$ is a finite presheaf.*

This allows the theory of [5] to be revisited and made slightly more precise (see Theorem 4.6).

The class of finite presheaves is, as of yet, imperfectly understood; a reassuring general fact is the following, for unstable algebras of finite transcendence degree:

THEOREM (Corollary 5.9). — *Let K be an unstable algebra of finite transcendence degree. The following are equivalent:*

(1) *K is $\mathscr{A}$-finitely generated up to nilpotents;*
(2) *$\mathfrak{g}K$ takes values in finite sets.*

In general it is not easy to determine whether a given presheaf X taking values in finite sets is a finite presheaf. A necessary condition is that the associated growth function defined on $\mathbb{N}$ by $\gamma_X(t) := \log_p |X(\mathbb{F}_p^t)|$ should have polynomial growth. As illustrated by Example 6.9, this is not sufficient.

Such growth functions occur in the work of Lannes and Schwartz [10] and of Grodal [4], notably in relation to the study of finite Postnikov systems, for which (under suitable hypotheses), the growth functions are shown to have polynomial growth. Indeed, revisiting these results in the light of subsequent developments of the theory provided one of the motivations for introducing the notion of a finite presheaf.

Even in the case of two-stage Postnikov systems, there remain basic open questions such as: for which two-stage Postnikov systems Y is $\mathfrak{g}H^*(Y;\mathbb{F}_p)$ a finite presheaf?

When Y is an H-space, then the answer is yes, tying in with the work of [2]. Indeed, in general in this theory, the situation is much better when one restricts to unstable Hopf algebras (for these, see Section 7). This is the subject of the second part of the paper.

For instance, one has the following, which should be of independent interest:

THEOREM (Theorem 8.7 and Theorem 8.12). — *For H a connected unstable Hopf algebra over $\mathbb{F}_p$, the following are equivalent:*

(1) *the underlying unstable algebra of H is $\mathscr{A}$-finitely generated up to nilpotents;*
(2) *$\mathfrak{g}H$ is a finite presheaf;*

(3) $\mathfrak{g}H$ *takes values in finite p-groups and is polynomial;*
(4) $\gamma_{\mathfrak{g}H}$ *has polynomial growth.*

As a consequence, a Henn–Lannes–Schwartz style characterization of connected unstable Hopf algebras up to nilpotents is given (see Theorem 9.7).

Part 1. Finite presheaves

2. Dramatis Personæ

This section introduces the presheaf categories that are central to the paper and recalls the notion of finite and polynomial abelian presheaves. The relationship with unstable algebras is explained, in particular recalling some of the salient points of Henn–Lannes–Schwartz theory [5].

2.1. Presheaves on $\mathbb{F}$-vector spaces

Fix a prime p and let $\mathscr{V}_f \subset \mathscr{V}$ denote the full subcategory of finite-dimensional spaces in the category $\mathscr{V}$ of $\mathbb{F}$-vector spaces, where $\mathbb{F}$ denotes the prime field of characteristic p.

Notation 2.1. — Denote by

(1) $\widehat{\mathscr{V}_f}$ the category of presheaves of sets on $\mathscr{V}_f$ (i.e. contravariant functors from $\mathscr{V}_f$ to sets);
(2) $\widehat{\mathscr{V}_f}^{\mathrm{profin}}$ the category of presheaves of profinite sets on $\mathscr{V}_f$, equipped with the forgetful functor $\widehat{\mathscr{V}_f}^{\mathrm{profin}} \to \widehat{\mathscr{V}_f}$;
(3) $(\widehat{\mathscr{V}_f})_c \subset \widehat{\mathscr{V}_f}$ (respectively $(\widehat{\mathscr{V}_f}^{\mathrm{profin}})_c \subset \widehat{\mathscr{V}_f}^{\mathrm{profin}}$) the full subcategory of connected objects, namely X such that $|X(0)| = 1$;
(4) $\mathscr{F}$ the category of functors from $\mathscr{V}_f^{\mathrm{op}}$ to $\mathscr{V}$, with abelian structure inherited from $\mathscr{V}$.

Remark 2.2. — In the literature (eg. [5, 6]), $\mathscr{F}$ usually denotes *covariant* functors. However, since vector space duality $V \mapsto V^\sharp$ restricts to an equivalence of categories $\mathscr{V}_f^{\mathrm{op}} \cong \mathscr{V}_f$, the choice of variance is of little import. In particular, to simplify notation, for F a covariant functor, the associated contravariant functor $V \mapsto F(V^\sharp)$ will again be denoted by F.

The following is standard:

LEMMA 2.3. — *For $F \in \mathrm{ob}\mathscr{F}$, there is a canonical direct sum decomposition $F \cong F(0) \oplus \overline{F}$, where $F(0)$ is considered as a constant functor and $\overline{F}$ is constant-free (i.e. $\overline{F}(0) = 0$).*

The following results are clear.

LEMMA 2.4. — *For $\widehat{\mathscr{V}_f}$,*

(1) *the coproduct $\amalg$ and product $\prod$ are inherited from the category of sets;*
(2) *the product restricts to the product of $(\widehat{\mathscr{V}_f})_c$;*
(3) *the coproduct of $(\widehat{\mathscr{V}_f})_c$ is given by the wedge product $\vee$.*

There is a forgetful functor $\mathscr{F} \to \widehat{\mathscr{V}_f}$ that retains only the underlying set of a vector space. For $F \in \mathrm{ob}\mathscr{F}$, the underlying presheaf lies in $(\widehat{\mathscr{V}_f})_c$ if and only if F is constant-free.

LEMMA 2.5. — *The forgetful functor $\mathscr{F} \to \widehat{\mathscr{V}_f}$ admits as left adjoint, $X \mapsto \mathbb{F}[X]$, so that there is an adjunction*

$$\mathbb{F}[-] : \widehat{\mathscr{V}_f} \rightleftarrows \mathscr{F}.$$

The adjunction unit is the natural inclusion $X \hookrightarrow \mathbb{F}[X]$.

Hence, the forgetful functor preserves products: in particular, the underlying presheaf of $F \oplus G$ (for $F, G \in \mathrm{ob}\mathscr{F}$) is $F \times G$.

Notation 2.6. — For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ and $x \in X(0)$, let $X_x \subset X$ denote the connected presheaf with $X_x(V)$ the fibre over x of the surjection $X(V) \to X(0)$ induced by $0 \hookrightarrow V$.

LEMMA 2.7. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$,*

(1) *there is a natural isomorphism $X \cong \amalg_{x \in X(0)} X_x$;*
(2) *if $X \in \mathrm{ob}(\widehat{\mathscr{V}_f})_c$ and $Y \in \mathrm{ob}\widehat{\mathscr{V}_f}$, there is a natural bijection*

$$\mathrm{Hom}_{\widehat{\mathscr{V}_f}}(X, Y) \cong \amalg_{y \in Y(0)} \mathrm{Hom}_{(\widehat{\mathscr{V}_f})_c}(X, Y_y).$$

Proof. — The first statement is clear. For the second, for a map of presheaves $f : X \to Y$, the image $f(x) \in Y(0)$ of the basepoint $x \in X(0)$ determines the connected component of the image of f. □

2.2. Polynomial functors

The following general definition applies for example to the category $\mathscr{F}$:

Definition 2.8. — *An object F of an abelian category is finite if it has a finite composition series.*

The Eilenberg–MacLane definition of a polynomial functor [3] (see Appendix A, where a general definition is given that covers the abelian case) also applies to $\mathscr{F}$, and one has:

Proposition 2.9 ([6]). — *A functor $F \in \mathrm{ob}\mathscr{F}$ is finite if and only if it is polynomial and takes values in $\mathscr{V}_f$.*

Example 2.10. — The nth symmetric power functor S^n and the nth exterior power functor Λ^n are both polynomial functors in $\mathscr{F}$ of degree n.

Remark 2.11. — The polynomial degree of a functor $F \in \mathrm{ob}\mathscr{F}$ that takes finite-dimensional values can be characterized in terms of its growth function (see Proposition 6.3).

The inclusion of the full subcategory of functors of $\mathscr{F}$ of Eilenberg–MacLane polynomial degree at most $n \in \mathbb{N}$ admits a left adjoint q_n (which is considered as a functor $q_n : \mathscr{F} \to \mathscr{F}$), so that the adjunction unit $F \twoheadrightarrow q_n F$ is the universal map to a polynomial functor of degree at most n. (See [8, Section 4.2] and the references therein.)

2.3. Unstable algebras

One motivation for considering profinite presheaves comes from the study of the category $\mathscr{K}$ of unstable algebras over the mod p Steenrod algebra $\mathscr{A}$ [5].

As usual, the category of unstable modules over $\mathscr{A}$ is denoted $\mathscr{U}$ and the Steenrod–Epstein (or Massey–Peterson) enveloping algebra functor by $U : \mathscr{U} \to \mathscr{K}$. The reader is referred to [15] for the basic theory of unstable modules and algebras, together with the localization of $\mathscr{U}$ away from $\mathscr{N}il$, the subcategory of nilpotent unstable modules, and the associated localization functor $\mathscr{U} \to \mathscr{U}/\mathscr{N}il$. The theory and applications of this nil-localization have been developed in [5, Part I], [6] and in subsequent work by many authors.

In [5, Part II], the analogous (non-abelian) theory for unstable algebras was developed, leading to the construction of the localized category $\mathscr{K}/\mathscr{N}il$. Recall the terminology introduced by Quillen:

Definition 2.12 ([5, Part II]). — *A morphism $\varphi : K \to L$ in $\mathscr{K}$ is*

(1) *an F-monomorphism if, $\forall\, x \in \ker\varphi$, $\exists\, n \in \mathbb{N}$ such that $x^n = 0$;*

(2) *an F-epimorphism if, $\forall\, y \in L$, $\exists\, m \in \mathbb{N}$ such that $y^{p^m} \in$ image φ;*
(3) *an F-isomorphism if φ satisfies both the above.*

The localization $\mathscr{K} \to \mathscr{K}/\mathscr{N}il$ is universal amongst functors $\mathscr{K} \to \mathscr{C}$ which send F-isomorphisms to isomorphisms of $\mathscr{C}$.

DEFINITION 2.13. — *An unstable algebra $K \in \mathrm{ob}\mathscr{K}$ is $\mathscr{A}$-finitely generated if there exists a finitely generated unstable module M and a morphism of unstable modules $M \to K$ such that the induced map of unstable algebras $UM \to K$ is surjective.*

Notation 2.14. — Let $\mathscr{K}^+$ denote the full subcategory of connected unstable algebras (K such that $K^0 = \mathbb{F}$).

Remark 2.15. — A connected unstable algebra $K \in \mathrm{ob}\mathscr{K}^+$ is canonically augmented, hence the augmentation ideal $\overline{K}$ and the module of indecomposables, $QK := \overline{K}/\overline{K}^2$, are defined.

In this case, K is $\mathscr{A}$-finitely generated if and only if QK is a finitely generated $\mathscr{A}$-module.

Remark 2.16. — The condition that the cohomology of a space is $\mathscr{A}$-finitely generated imposes strong conditions, notably when working with H-spaces; see [2, Theorem 7.3] for example.

2.4. From unstable algebras to presheaves

The following is clear:

PROPOSITION 2.17. — *Every unstable algebra K is the colimit of its $\mathscr{A}$-finitely generated sub unstable algebras.*

DEFINITION 2.18. — *Let*
(1) $\mathfrak{g} : \mathscr{K}^{\mathrm{op}} \to \widehat{\mathscr{V}_f}^{\mathrm{profin}}$ *be the functor defined by*

$$\mathfrak{g}K(V) := \mathrm{Hom}_{\mathscr{K}}(K, H^*(BV)),$$

where $H^*(BV)$ denotes the mod p group cohomology of V;
(2) $\mathfrak{g} : (\mathscr{K}^+)^{\mathrm{op}} \to (\widehat{\mathscr{V}_f}^{\mathrm{profin}})_c$ *denote the restriction to connected objects.*

Here the profinite structure arises from Proposition 2.17.

The following is a fundamental fact, following from the analogous result for $\mathscr{U}$ (as in [5, Part I] and [6]), together with Lannes' linearization principle.

PROPOSITION 2.19 ([5]). — *The functor $\mathfrak{g}$ factors naturally*

$$\mathfrak{g} : (\mathscr{K}/\mathscr{N}il)^{\mathrm{op}} \to \widehat{\mathscr{V}_f}^{\mathrm{profin}}.$$

Notation 2.20. — For Y a topological space, let $\mathfrak{g}_{\mathrm{top}}Y \in \mathrm{ob}\widehat{\mathscr{V}_f}$ denote the presheaf $\mathfrak{g}_{\mathrm{top}}Y(V) := [BV, Y]$.

The presheaves $\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{top}}$ are intimately related (compare [10]):

THEOREM 2.21 ([9]). — *For Y a topological space, mod p cohomology induces a morphism of presheaves $\mathfrak{g}_{\mathrm{top}}Y \to \mathfrak{g}H^*Y$ that is an isomorphism if Y is connected, nilpotent, $\pi_1 Y$ is finite and $H^*(Y)$ is of finite type.*

2.5. From presheaves to unstable algebras

For simplicity of presentation, suppose that $p = 2$ in this section. The odd primary case is treated by passage to objects concentrated in even degrees (cf. [5], for example).

DEFINITION 2.22. — *Let $\kappa : (\widehat{\mathscr{V}_f}^{\mathrm{profin}})^{\mathrm{op}} \to \mathscr{K}$, the associated unstable algebra functor, be defined by*

$$\kappa : X \mapsto \mathrm{Hom}_{\widehat{\mathscr{V}_f}^{\mathrm{profin}}}(X, S^*)$$

where the commutative $\mathbb{F}$-algebra structure of κX is induced by the commutative algebra structure of S^* and Steenrod operations act via $\mathrm{Hom}_{\mathscr{F}}(S^*, S^*)$ (cf. [6]).

Example 2.23. — For $n \in \mathbb{N}$, $\kappa S^n \cong UF(n)$, the free unstable algebra on a generator of degree n (see [7]), thus $\kappa S^n \cong H^*(K(\mathbb{F}, n); \mathbb{F})$.

To stress the relationship between presheaves and unstable algebras, recall the following part of [5, Theorem II.1.5]:

PROPOSITION 2.24. — *For $K \in \mathrm{ob}\mathscr{K}$, there is a natural transformation $K \to \kappa\mathfrak{g}K$ that is an F-isomorphism.*

The following is recorded for later use:

PROPOSITION 2.25. — *Let $X \in \widehat{\mathscr{V}_f}$ take values in finite sets. Then the unstable algebra κX has finite type.*

Proof. — In degree n, $(\kappa X)^n = \mathrm{Hom}_{\widehat{\mathscr{V}_f}}(X, S^n) \cong \mathrm{Hom}_{\mathscr{F}}(q_n\mathbb{F}[X], S^n)$. Now, by construction, $q_n\mathbb{F}[X]$ is a polynomial functor which takes finite-dimensional values, hence is finite, by Proposition 2.9; likewise, S^n is finite. It follows that $\mathrm{Hom}_{\mathscr{F}}(q_n\mathbb{F}[X], S^n)$ is a finite-dimensional vector space, whence the result. □

2.6. Finite transcendence degree and Noetherian unstable algebras

Recall (see [5, Section II.2]) that the transcendence degree of an unstable algebra K is the transcendence degree of its underlying graded algebra (namely the supremum of the cardinalities of finite subsets of algebraically independent homogeneous elements of K). Transcendence degree is invariant under F-isomorphism.

DEFINITION 2.26. — *For $d \in \mathbb{N}$, let $\mathscr{K}_d$ be the full subcategory of unstable algebras of transcendence degree at most d and $\mathscr{K}_d/\mathscr{N}il$ the corresponding full subcategory of $\mathscr{K}/\mathscr{N}il$.*

Notation 2.27. — For $d \in \mathbb{N}$, let $\mathscr{PS}\text{–End}(\mathbb{F}_p^d)$ denote the category of profinite right $\mathrm{End}(\mathbb{F}^d)$-sets.

THEOREM 2.28 ([5, Theorems II.2.7, II.2.8]). — *For $d \in \mathbb{N}$, the functor $\mathfrak{g}$ induces an equivalence of categories $(\mathscr{K}_d/\mathscr{N}il)^{\mathrm{op}} \cong \mathscr{PS}\text{–End}(\mathbb{F}^d)$ with inverse κ_d induced by κ.*

Henn, Lannes and Schwartz [5, Definition II.5.8] also introduce the notion of a Noetherian $\mathrm{End}(\mathbb{F}^d)$-set (necessarily finite) which allows them to give a characterization of unstable algebras that are Noetherian up to nilpotents:

THEOREM 2.29 ([5, Theorem II.7.1]). — *Let $d \in \mathbb{N}$.*

(1) *For K a Noetherian unstable algebra, $\mathfrak{g}_d K$ is a Noetherian $\mathrm{End}(\mathbb{F}^d)$-set.*
(2) *If S is a Noetherian $\mathrm{End}(\mathbb{F}^d)$-set, then $\kappa_d S$ is a Noetherian unstable algebra of transcendence degree at most d.*

In particular, this leads to the following:

DEFINITION 2.30. — *An unstable algebra K is Noetherian up to nilpotents if it has finite transcendence degree and, for any $d \in \mathbb{N}$, $\mathfrak{g}_d K$ is a Noetherian $\mathrm{End}(\mathbb{F}^d)$-set.*

3. Finite presheaves and coanalyticity

This section introduces the notion of a finite presheaf. Its relevance for unstable algebras is explained in Section 4.

3.1. Definitions and first properties

DEFINITION 3.1. — *An object X of $\widehat{\mathscr{V}_f}$ is finite if there exists a finite functor $F_X \in \mathrm{ob}\mathscr{F}$ and a monomorphism*

$$X \hookrightarrow F_X$$

in $\widehat{\mathscr{V}_f}$. The full subcategory of finite presheaves is denoted $\widehat{\mathscr{V}_f}^{\mathrm{fin}}$.

LEMMA 3.2. — *A presheaf X of $\widehat{\mathscr{V}_f}$ is finite if and only if $|X(0)| < \infty$ and, for each $x \in X(0)$, there exists a constant-free finite functor $F_x \in \mathscr{F}$ and a monomorphism $X_x \hookrightarrow F_x$.*

Proof. — First suppose that X is a finite presheaf, so that there exists $X \hookrightarrow F_X$ where $F_X \in \mathrm{ob}\mathscr{F}$ is a finite functor. Then $X(0)$ is a finite set and, for any $x \in X$, the map

$$X_x \hookrightarrow X \hookrightarrow F_X \twoheadrightarrow \overline{F_X}$$

is a monomorphism (by Lemma 2.7).

Conversely, suppose that $X(0)$ is a finite set and that there exist injective maps $f_x : X_x \hookrightarrow F_x$, $\forall\, x \in X(0)$, with $F_x \in \mathrm{ob}\mathscr{F}$ finite and constant-free. Then the map

$$X \hookrightarrow F_X := \mathbb{F}[X(0)] \oplus \bigoplus_{x \in X(0)} F_x$$

defined on X_x by the constant map $X_x \to \mathbb{F}[X(0)]$ to $[x]$, the map f_x to F_x and the zero map to the components F_y, $y \neq x$, is an injection into a finite functor of $\mathscr{F}$, exhibiting X as a finite functor. □

Remark 3.3. — The definition of a finite presheaf extends verbatim to $\widehat{\mathscr{V}_f}^{\mathrm{profin}}$. This leads to no increased generality, since a finite presheaf necessarily takes values in finite sets.

LEMMA 3.4. — *Let $X, Y \in \mathrm{ob}\widehat{\mathscr{V}_f}$ be finite presheaves. Then*

1. *$X \amalg Y$ is finite;*
2. *$X \times Y$ is finite;*
3. *if $U \subset X$ is a subobject, then U is finite;*
4. *if X, Y are connected, with respective basepoints x, y, then $X \vee Y$ is finite.*

Proof. — The case of $X \amalg Y$ follows easily from Lemma 3.2. For $X \times Y$, the inclusions $X \hookrightarrow F_X$ and $Y \hookrightarrow F_Y$, with $F_X, F_Y \in \mathrm{ob}\mathscr{F}$ finite, induce $X \times Y \hookrightarrow F_X \oplus F_Y$ by cartesian product.

The preservation of finiteness under passage to subobjects is clear. Considering $X \vee Y$ as the subobject of $X \times Y$ given by $X \times y \cup x \times Y$ gives the final statement. □

3.2. The degree of a finite presheaf

PROPOSITION 3.5. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}^{\mathrm{profin}}$, the following conditions are equivalent:*

(1) *X is finite;*
(2) *X takes values in finite sets and there exists $n \in \mathbb{N}$ such that the composite $X \hookrightarrow \mathbb{F}[X] \twoheadrightarrow q_n\mathbb{F}[X]$ is a monomorphism.*

The degree of a finite X is the least such n.

Proof. — The condition that X takes finite values is necessary; under this hypothesis, for any n, $q_n\mathbb{F}[X]$ is a finite functor (by Proposition 2.9), hence the existence of a monomorphism $X \hookrightarrow q_n\mathbb{F}[X]$ implies that X is finite.

Conversely, suppose that X is finite, so that there is a finite functor F_X and an inclusion $X \hookrightarrow F_X$. Let n be the polynomial degree of F_X, then the induced linear map $\mathbb{F}[X] \to F_X$ (provided by Lemma 2.5) factors across $q_n\mathbb{F}[X]$, from which the result follows. □

The relevance of the functor $q_n \circ \mathbb{F}[-]$ is explained by the following straightforward proposition, using:

Notation 3.6. — For $n \in \mathbb{N}$, denote by

(1) $\mathscr{F}_n^{\mathrm{fin}} \subset \mathscr{F}$ the full subcategory with objects finite functors of polynomial degree at most n;
(2) $(\widehat{\mathscr{V}_f}^{\mathrm{fin}})_n \subset \widehat{\mathscr{V}_f}$ the full subcategory of finite presheaves of degree at most n.

PROPOSITION 3.7. — *For $n \in \mathbb{N}$, the forgetful functor $\mathscr{F} \to \widehat{\mathscr{V}_f}$ restricts to $\mathscr{F}_n^{\mathrm{fin}} \to (\widehat{\mathscr{V}_f}^{\mathrm{fin}})_n$ and admits left adjoint $q_n \circ \mathbb{F}[-] : (\widehat{\mathscr{V}_f}^{\mathrm{fin}})_n \to \mathscr{F}_n^{\mathrm{fin}}$.*

Remark 3.8. — The restriction to $(\widehat{\mathscr{V}_f}^{\mathrm{fin}})_n$ serves to restrict to presheaves taking values in finite sets. (The functor $q_n \circ \mathbb{F}[-] : \widehat{\mathscr{V}_f} \to \mathscr{F}$ does not take values in finite functors.)

COROLLARY 3.9. — *For $X, Y \in \mathrm{ob}\widehat{\mathscr{V}_f}$ such that X takes values in finite sets and Y is finite, $\mathrm{Hom}_{\widehat{\mathscr{V}_f}}(X, Y)$ is a finite set.*

Proof. — By hypothesis, there exists $n \in \mathbb{N}$ such that the natural morphism $Y \hookrightarrow q_n\mathbb{F}[Y]$ is injective, hence there is a monomorphism:

$$\mathrm{Hom}_{\widehat{\mathscr{V}_f}}(X, Y) \hookrightarrow \mathrm{Hom}_{\widehat{\mathscr{V}_f}}(X, q_n\mathbb{F}[Y]) \cong \mathrm{Hom}_{\mathscr{F}}(q_n\mathbb{F}[X], q_n\mathbb{F}[Y]),$$

where the second isomorphism is given by Proposition 3.7.

Now $q_n\mathbb{F}[X]$ and $q_n\mathbb{F}[Y]$ are both finite functors of $\mathscr{F}$ (by Proposition 2.9) hence $\mathrm{Hom}_{\mathscr{F}}(q_n\mathbb{F}[X], q_n\mathbb{F}[Y])$ is a finite-dimensional $\mathbb{F}$-vector space, thus a finite set. □

Definition 3.10. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ that takes values in finite sets and $n \in \mathbb{N}$, let $X_n \in \mathrm{ob}\widehat{\mathscr{V}_f}$ denote the image of $X \to q_n\mathbb{F}[X]$, equipped with the canonical surjection $X \twoheadrightarrow X_n$.*

Proposition 3.11. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ that takes values in finite sets, the natural surjection $X \twoheadrightarrow X_n$ is the universal map to a finite presheaf of degree n.*

These maps form a tower

$$\begin{array}{ccccccccc} & & & & X & & & & \\ & & & \swarrow & \downarrow & \searrow & & & \\ \cdots & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots . \end{array}$$

Proof. — The key point is to check that X_n is finite of degree at most n. This is clear from the commutative diagram

$$\begin{array}{ccc} X_n & \hookrightarrow & q_n\mathbb{F}[X] \\ \downarrow & \nearrow & \uparrow \\ \mathbb{F}[X_n] & \twoheadrightarrow & q_n\mathbb{F}[X_n], \end{array}$$

where the top monomorphism is given by the construction of X_n, the dashed arrow by $\mathbb{F}$-linear extension and the dotted arrow by the polynomial degree adjunction.

The universality of $X \twoheadrightarrow X_n$ follows from Proposition 3.7. □

3.3. Coanalyticity of presheaves of sets

Even under the hypothesis that X takes values in finite sets, the induced map $X \to \lim_{\leftarrow} X_n$ given by Proposition 3.11 need not be an isomorphism.

Example 3.12. — Consider the functor $V \mapsto I_{\mathbb{F}}(V) := \mathbb{F}^V$ in $\mathscr{F}$, and take its constant-free summand $\overline{I_{\mathbb{F}}}$. Forgetting the linear structure gives a connected presheaf taking values in finite sets.

The linearization $\mathbb{F}[\overline{I_{\mathbb{F}}}]$ admits *no* non-trivial map to a finite functor. It is straightforward to reduce to proving this for $q_n\mathbb{F}[-] \circ \overline{I_{\mathbb{F}}}$, for all $n \in \mathbb{N}$. Using the identification of the subquotients of the filtration associated to $\mathbb{F}[-] \twoheadrightarrow q_n\mathbb{F}[-]$, one shows that it is sufficient to show that functors of the form $\overline{I_{\mathbb{F}}}^{\otimes i}$, for $0 < i \in \mathbb{N}$, admit no finite quotients. Now Kuhn's embedding theorem [6] implies that it suffices to show that $\mathrm{Hom}_{\mathscr{F}}(\overline{I_{\mathbb{F}}}^{\otimes i}, S^t) = 0$ for all $t \in \mathbb{N}$ and $i > 0$. This follows from the case $i = 1$ by using the exponential property of the symmetric power functors; the case $i = 1$ is well-known, since the structure of $\overline{I_{\mathbb{F}}}$ is known.

PROPOSITION 3.13. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ taking values in finite sets, the map*

$$X \to \lim_{\leftarrow n} X_n$$

is a bijection if and only if, for each $V \in \mathrm{ob}\mathscr{V}_f$, $\exists\, n_V$ such that

$$X(V) \overset{\cong}{\twoheadrightarrow} X_{n_V}(V).$$

In general one must consider the category $\widehat{\mathscr{V}_f}^{\mathrm{profin}}$ of presheaves on $\mathscr{V}_f$ with values in profinite sets.

The following should be compared with the definition in [5, Part II] of an *analytic* functor from $\mathscr{V}_f$ to $\mathscr{P}\mathscr{S}^{\mathrm{op}}$, the opposite of the category of profinite sets. (Henn, Lannes and Schwartz work with the opposite of the category of presheaves, whence their terminology *analytic* rather than *coanalytic* here.)

DEFINITION 3.14. — *An object X of $\widehat{\mathscr{V}_f}^{\mathrm{profin}}$ is coanalytic if*

$$X \cong \lim_{\leftarrow i \in \mathscr{I}} X(i),$$

where the indexing category $\mathscr{I}$ is cofiltered and small and $X(i)$ are finite presheaves.

Let $\widehat{\mathscr{V}_f}^{\omega} \subset \widehat{\mathscr{V}_f}^{\mathrm{profin}}$ denote the full subcategory of coanalytic functors.

DEFINITION 3.15. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ (respectively $X \in \mathrm{ob}\widehat{\mathscr{V}_f}^{\mathrm{profin}}$), let $X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}$ denote the full subcategory of the undercategory $X/\widehat{\mathscr{V}_f}$ (resp. $X/\widehat{\mathscr{V}_f}^{\mathrm{profin}}$) with objects $X \to Y$ with Y finite.*

LEMMA 3.16. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ or $X \in \mathrm{ob}\widehat{\mathscr{V}_f}^{\mathrm{profin}}$,*

(1) *$X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}$ has finite morphism sets;*

(2) *given $X \to Y_i$, $i \in \{1,2\}$ of $X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}$, there is a diagram of morphisms of $X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}$:*

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow & \searrow & \\ Y_1 & \longleftarrow & Y_1 \times Y_2 & \longrightarrow & Y_2, \end{array}$$

in which the horizontal arrows are the projections;

(3) *any object $X \to Y$ of $X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}$ is the range of a morphism in $X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}$:*

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y' & \hookrightarrow & Y \end{array}$$

where $Y' \subset Y$ is a sub-presheaf and $X \twoheadrightarrow Y'$ is surjective.

(4) *for morphisms $f, g : (X \to Y_1) \rightrightarrows (X \to Y_2)$ of $X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}$, the morphism $(X \twoheadrightarrow Y_1') \to (X \to Y_1)$ given by the factorization of $X \to Y_1$ equalizes f, g.*

In particular, the category $X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}$ is cofiltered.

Proof. — The first statement is an immediate consequence of Corollary 3.9 and the second follows from the categorical definition of the product. The factorization of a morphism of presheaves is clear and applies in the final statement, using the categorical property of a surjection. □

DEFINITION 3.17. — *Let $X^\omega \in \mathrm{ob}\widehat{\mathscr{V}_f}^{\omega}$ denote the presheaf of profinite sets given by*

$$X^\omega := \varprojlim_{X \to Y \in X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}} Y,$$

equipped with the natural (continuous) coanalytic completion *map $X \to X^\omega$.*

The following is clear from the definitions:

PROPOSITION 3.18. — *A presheaf $X \in \mathrm{ob}\widehat{\mathscr{V}_f}^{\mathrm{profin}}$ is coanalytic if and only if the natural map $X \to X^\omega$ is an isomorphism.*

Example 3.19. — For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$, the category $X/\widehat{\mathscr{V}_f}^{\mathrm{fin}}$ can be the discrete category with one object. Consider the presheaf $\overline{I_{\mathbb{F}}}$ of Example 3.12, where

it was shown that there are no non-trivial maps from $\overline{I_{\mathbb{F}}}$ to a finite presheaf. It follows that $(\overline{I_{\mathbb{F}}})^{\omega} = *$.

4. The relationship with unstable algebras

The relationship between finite presheaves and $\mathscr{A}$-finite generation up to nilpotents is made explicit in this section, leading to a conceptual restatement of one of the main results of [5, Part II].

4.1. $\mathscr{A}$-finite generation up to nilpotents

The following is the natural extension of Definition 2.13 to working modulo nilpotents.

DEFINITION 4.1. — *An unstable algebra $K \in \mathrm{ob}\mathscr{K}$ is $\mathscr{A}$-finitely generated up to nilpotents if there exists an unstable algebra L that is $\mathscr{A}$-finitely generated and an F-epimorphism $L \overset{F-\mathrm{epi}}{\rightsquigarrow} K$.*

Remark 4.2. — Unlike the case of $\mathscr{A}$-finite generation, for $K \in \mathrm{ob}\mathscr{K}^{+}$, the definition of $\mathscr{A}$-finitely generated up to nilpotents cannot be given in terms of QK.

For example, consider the unstable algebra $K := U(\bigoplus_{n\geqslant 2} F(n))$ at the prime $p = 2$, so that $QK \cong \bigoplus_{m\geqslant 1} \Sigma F(m)$; K is not $\mathscr{A}$-finitely generated up to nilpotents. Now consider the unstable algebra $\mathbb{F} \oplus \bigoplus_{m\geqslant 1} \Sigma F(m)$ with trivial algebra structure; this has the same module of indecomposables, but is F-isomorphic to $\mathbb{F}$, hence is $\mathscr{A}$-finitely generated up to nilpotents.

The following is an immediate consequence of Kuhn's embedding theorem [6] that a finite functor $F \in \mathscr{F}$ embeds in a finite direct sum of symmetric power functors.

PROPOSITION 4.3. — *A presheaf $X \in \mathrm{ob}\widehat{\mathscr{V}_f}^{\mathrm{profin}}$ is finite if and only if there is a finite direct sum $\bigoplus_{i\in\mathscr{I}} S^{n_i}$ of symmetric power functors and a monomorphism $X \hookrightarrow \bigoplus_{i\in\mathscr{I}} S^{n_i}$ in $\widehat{\mathscr{V}_f}^{\mathrm{profin}}$.*

COROLLARY 4.4. — *Let $K \in \mathrm{ob}\mathscr{K}$ be an unstable algebra. The following conditions are equivalent:*

(1) *$\mathfrak{g}K \in \mathrm{ob}\widehat{\mathscr{V}_f}^{\mathrm{profin}}$ is finite;*
(2) *K is $\mathscr{A}$-finitely generated up to nilpotents.*

Proof. — An immediate consequence of Proposition 4.3, Proposition 2.24 and Example 2.23, using the fact that κ sends inclusions to F-epimorphisms [5]. □

Remark 4.5. — As observed by a referee, the key ingredient here is the fact that an unstable module M is $\mathscr{A}$-finitely generated modulo nilpotents if and only if the associated functor $V \mapsto (T_V M)^0$ (where T_V is Lannes' T-functor) is finite (this is related to Kuhn's embedding theorem). Using this, Corollary 4.4 has a short direct proof, by applying the functor $\mathfrak{g}$ to a morphism of unstable algebras of the form $UM \to K$.

4.2. Reinterpreting $\mathscr{K}/\mathscr{N}il$

The above leads to the following refinement of [5, Theorem II.1.5]:

THEOREM 4.6. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}^{\mathrm{profin}}$, the natural map $X \to X^\omega$ induces an isomorphism of unstable algebras $\kappa X^\omega \to \kappa X$.*

In particular,

(1) *X is coanalytic if and only if $X \cong \mathfrak{g}\kappa X$;*

(2) *$\mathfrak{g}$ induces an equivalence of categories $(\mathscr{K}/\mathscr{N}il)^{\mathrm{op}} \xrightarrow{\cong} \widehat{\mathscr{V}_f}^\omega$.*

5. Finitely generated presheaves are finite

5.1. The rank filtration

For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ and $n \in \mathbb{N}$, the sections $X(\mathbb{F}^n)$ have a natural right action of $\mathrm{End}(\mathbb{F}^n)$, which restricts to a right $\mathrm{Aut}(\mathbb{F}^n)$-action.

The following result is standard (and corresponds to the skeletal filtration of [5, Part II]):

PROPOSITION 5.1. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$, there is a natural rank filtration*

$$X_{\leqslant 0} \subset X_{\leqslant 1} \subset \ldots \subset X_{\leqslant n} \subset X_{\leqslant n+1} \subset \ldots \subset X$$

such that $X = \bigcup X_{\leqslant n}$, where $X_{\leqslant n}$ is the image of the evaluation map:

$$X(\mathbb{F}^n) \times_{\mathrm{End}(\mathbb{F}^n)} \mathrm{Hom}(-, \mathbb{F}^n) \to X.$$

DEFINITION 5.2. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ and $n \in \mathbb{N}$, let $X_{\mathrm{reg}}(n)$ be the set of* regular elements *of $X(\mathbb{F}^n)$, namely the right $\mathrm{Aut}(\mathbb{F}^n)$-set given by*

$$X_{\mathrm{reg}}(n) := X(\mathbb{F}^n) \backslash X_{\leqslant n-1}(\mathbb{F}^n).$$

The following is related to the Key Lemma, [5, Lemma II.2.1] and its associated results.

LEMMA 5.3. — *For $X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ and $1 \leqslant n \in \mathbb{N}$,*

(1) *the quotient presheaf $X_{\leqslant n}/X_{\leqslant n-1}$ is naturally isomorphic to*

$$V \mapsto * \amalg \left(X_{\mathrm{reg}}(n) \times_{\mathrm{Aut}(\mathbb{F}^n)} \mathrm{Surj}(V, \mathbb{F}^n) \right)$$

(where $\mathrm{Surj}(V, \mathbb{F}^n) \subset \mathrm{Hom}(V, \mathbb{F}^n)$ is the set of surjective morphisms) considered as a quotient presheaf of $X(\mathbb{F}^n) \times_{\mathrm{End}(\mathbb{F}^n)} \mathrm{Hom}(-, \mathbb{F}^n) \to X$.

(2) *there is a natural isomorphism of $\mathrm{Aut}(V)$-sets:*

$$X_{\leqslant n}(V) \backslash X_{\leqslant n-1}(V) \cong X_{\mathrm{reg}}(n) \times_{\mathrm{Aut}(\mathbb{F}^n)} \mathrm{Surj}(V, \mathbb{F}^n).$$

Proof. — By definition, $X_{\leqslant n}$ is the image

$$X(\mathbb{F}^n) \times_{\mathrm{End}(\mathbb{F}^n)} \mathrm{Hom}(-, \mathbb{F}^n) \twoheadrightarrow X_{\leqslant n} \hookrightarrow X$$

of the map induced by evaluation. From the definition of $X_{\mathrm{reg}}(n)$, this induces a surjection

$$* \amalg \Big(X_{\mathrm{reg}}(n) \times_{\mathrm{Aut}(\mathbb{F}^n)} \mathrm{Surj}(-, \mathbb{F}^n) \Big) \twoheadrightarrow X_{\leqslant n}/X_{\leqslant n-1}. \tag{5.1}$$

Now $\mathrm{Surj}(V, \mathbb{F}^n)$ is a free left $\mathrm{Aut}(\mathbb{F}^n)$-set with cosets $\overline{\mathrm{Surj}(V, \mathbb{F}^n)}$ in bijection with the set of codimension n subspaces of V. Hence, as sets,

$$X_{\mathrm{reg}}(n) \times_{\mathrm{Aut}(\mathbb{F}^n)} \mathrm{Surj}(V, \mathbb{F}^n) \cong X_{\mathrm{reg}}(n) \times \overline{\mathrm{Surj}(V, \mathbb{F}^n)}.$$

For the first point, it remains to show that the map (5.1) is injective. By construction, this is true for sections with $\dim V \leqslant n$ and, for $V = \mathbb{F}^n$, both sides of (5.1) identify with the set $* \amalg X_{\mathrm{reg}}(n)$.

For the general case, given two sections $x \neq y \in X_{\mathrm{reg}}(n) \times_{\mathrm{Aut}(\mathbb{F}^n)} \mathrm{Surj}(V, \mathbb{F}^n)$, using the fact that any surjection admits a section, there exists a morphism $\varphi : \mathbb{F}^n \hookrightarrow V$ such that one of the following hold:

(1) $x\varphi \in X_{\mathrm{reg}}(n)$ and $y\varphi = *$, (in the case that x, y correspond to different cosets in $\overline{\mathrm{Surj}(V, \mathbb{F}^n)}$);
(2) $x\varphi \neq y\varphi \in X_{\mathrm{reg}}(n)$ (in the remaining case).

This implies the required injectivity.

The second statement is an immediate consequence. □

Remark 5.4. — A morphism of presheaves $f : X \to Y$ does not in general restrict to a morphism $X_{\mathrm{reg}}(n) \to Y_{\mathrm{reg}}(n)$.

PROPOSITION 5.5. — *For a morphism of presheaves $f : X \to Y$ in $\widehat{\mathscr{V}_f}$, the following conditions are equivalent:*

(1) *f is a monomorphism $X \hookrightarrow Y$;*

(2) *for each $n \in \mathbb{N}$, f restricts to a monomorphism $X_{\mathrm{reg}}(n) \hookrightarrow Y_{\mathrm{reg}}(n)$.*

Proof. — If f is a monomorphism, it is sufficient to show that, for each $n \in \mathbb{N}$, f sends $X_{\mathrm{reg}}(n)$ to $Y_{\mathrm{reg}}(n)$. Suppose that $x \in X_{\mathrm{reg}}(n)$; if $f(x)$ is not regular, then there exists a non-invertible $\alpha \in \mathrm{End}(\mathbb{F}^n)$ such that $f(x) = f(x)\alpha$. By injectivity of f, it follows that $x = x\alpha$, a contradiction.

The converse is established by induction on the rank filtration. For $V \in \mathrm{ob}\mathscr{V}_f$, as in the proof of Lemma 5.3,

$$X_{\leqslant n}(V) \backslash X_{\leqslant n-1}(V) \cong X_{\mathrm{reg}}(n) \times \overline{\mathrm{Surj}(V, \mathbb{F}^n)}.$$

Hence, the inclusion $X_{\mathrm{reg}}(n) \hookrightarrow Y_{\mathrm{reg}}(n)$ of right $\mathrm{Aut}(\mathbb{F}^n)$-sets induces a monomorphism

$$X_{\leqslant n}(V) \backslash X_{\leqslant n-1}(V) \hookrightarrow Y_{\leqslant n}(V) \backslash Y_{\leqslant n-1}(V)$$

and thus, by induction upon n, $X_{\leqslant n}(V) \hookrightarrow Y_{\leqslant n}(V)$. The result follows by passage to the colimit as $n \to \infty$. □

Corollary 5.6. — *For a morphism of presheaves $f : X \to Y$ in $\widehat{\mathscr{V}_f}$ such that $X = X_{\leqslant n}$, f is a monomorphism if and only if*

$$f : X(\mathbb{F}^n) \hookrightarrow Y(\mathbb{F}^n)$$

is a monomorphism of sets.

Proof. — By hypothesis, $X_{\mathrm{reg}}(k) = \emptyset$ for $k > n$, hence the result follows from Proposition 5.5. □

5.2. Finite generation implies finite

Notation 5.7. — For Z a finite right $\mathrm{End}(\mathbb{F}^n)$-set, denote by

(1) $X_Z \in \mathrm{ob}\widehat{\mathscr{V}_f}$ the induced presheaf $V \mapsto Z \times_{\mathrm{End}(\mathbb{F}^n)} \mathrm{Hom}(V, \mathbb{F}^n)$
(2) $G_Z \in \mathrm{ob}\mathscr{F}$ the induced functor $V \mapsto \mathbb{F}[Z] \otimes_{\mathrm{End}(\mathbb{F}^n)} \mathbb{F}[\mathrm{Hom}(V, \mathbb{F}^n)]$

equipped with the morphism $X_Z \to G_Z$ of $\widehat{\mathscr{V}_f}$ induced by the canonical inclusion of right $\mathrm{End}(\mathbb{F}^n)$-sets $Z \hookrightarrow \mathbb{F}[Z]$.

Theorem 5.8. — *For Z a finite right $\mathrm{End}(\mathbb{F}^n)$-set, there exists $t \in \mathbb{N}$ such that the composite*

$$X_Z \to G_Z \twoheadrightarrow q_t G_Z$$

is a monomorphism. In particular, X_Z is finite of degree at most t.

Proof. — By Corollary 5.6, it suffices to exhibit $t \in \mathbb{N}$ such that the natural surjection $G_Z \twoheadrightarrow q_t G_Z$ is a bijection when evaluated on $\mathbb{F}^n$. The functor G_Z is a quotient of a finite direct sum of copies of the functor $\mathbb{F}[\mathrm{Hom}(-, \mathbb{F}^n)]$. The latter takes finite-dimensional values and is dual to a locally finite functor, namely is the inverse limit of its finite quotients. The same therefore holds for G_Z, which implies the existence of such a t.

The final statement follows, since $q_t G_Z$ is a finite functor of polynomial degree at most t. □

Recall the theory of unstable algebras of finite transcendence degree from Section 2.6, in particular Theorem 2.28.

COROLLARY 5.9. — *Let K be an unstable algebra of finite transcendence degree. Then K is $\mathscr{A}$-finitely generated up to nilpotents if and only if $\mathfrak{g}K$ takes values in finite sets.*

Proof. — If K is $\mathscr{A}$-finitely generated up to nilpotents, then $\mathfrak{g}K$ is finite, by Corollary 4.4, hence takes values in finite sets.

Conversely, if K has finite transcendence degree d and $\mathfrak{g}K$ takes values in finite sets, then $\mathfrak{g}K \cong X_{\mathfrak{g}K(\mathbb{F}^d)}$, where $\mathfrak{g}K(\mathbb{F}^d)$ is a finite right $\mathrm{End}(\mathbb{F}^d)$-set. The presheaf $X_{\mathfrak{g}K(\mathbb{F}^d)}$ is finite by Theorem 5.8, hence K is $\mathscr{A}$-finitely generated up to nilpotents, by Corollary 4.4. □

Remark 5.10. — Corollary 5.9 applies when $\mathfrak{g}K(\mathbb{F}^d)$ is a Noetherian right $\mathrm{End}(\mathbb{F}^d)$-set (see Theorem 2.29). However, the Corollary shows that, for $\mathscr{A}$-finite generation up to nilpotents (as opposed to Noetherian up to nilpotents), only the much weaker condition of finite values is required.

5.3. Examples

The prime p is taken to be 2 in this section.

Example 5.11. — For $n \in \mathbb{N}$, let $\mathrm{Gr}_{\leqslant n} \in \mathrm{ob}\widehat{\mathscr{V}_f}$ denote the presheaf defined by $\mathrm{Gr}_{\leqslant n}(V) := \mathrm{GL}_n \setminus \mathrm{Hom}(V, \mathbb{F}^n)$. A linear embedding $\mathbb{F}^{n-1} \hookrightarrow \mathbb{F}^n$ induces an inclusion (independent of the choice) $\mathrm{Gr}_{\leqslant n-1} \hookrightarrow \mathrm{Gr}_{\leqslant n}$. Let Gr_n denote the presheaf defined by the pushout diagram in $\widehat{\mathscr{V}_f}$

$$\begin{array}{ccc} \mathrm{Gr}_{\leqslant n-1} & \hookrightarrow & \mathrm{Gr}_{\leqslant n} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Gr}_n\,. \end{array}$$

Hence Gr_n identifies as the presheaf $* \amalg \overline{\mathrm{Surj}(-, \mathbb{F}^n)}$, using the notation of the proof of Lemma 5.3.

Each of these presheaves is connected and takes values in finite sets. Moreover, $\mathrm{Gr}_{\leqslant n}$ is induced by the right $\mathrm{End}(\mathbb{F}^n)$-set $\mathrm{GL}_n \setminus \mathrm{Hom}(\mathbb{F}^n, \mathbb{F}^n)$ (which is Noetherian), whereas Gr_n is induced (for $n > 0$) by the pointed $\mathrm{End}(\mathbb{F}^n)$-set $*_0 \amalg *_n$, where $*_n$ is regular (at the prime 2, this can be identified with $\Lambda^n(\mathbb{F}^n)$). The latter is *not* Noetherian for $n > 1$.

The unstable algebra $\kappa(\mathrm{Gr}_{\leqslant n})$ is the Dickson algebra $D(n) := H^*(B\mathbb{F}^n; \mathbb{F})^{\mathrm{GL}_n}$ whereas $\kappa(\mathrm{Gr}_n)$ is $\mathbb{F} \oplus \omega_n D(n)$, where ω_n denotes the top Dickson invariant. For $n = 1$ these coincide, whereas for $n > 1$ they differ: indeed $D(n)$ is a Noetherian algebra whereas, for $n > 1$, $\mathbb{F} \oplus \omega_n D(n)$ is easily seen not to be Noetherian or even $\mathscr{A}$-finitely generated. However, Theorem 5.8 implies that $\mathbb{F} \oplus \omega_n D(n)$ is $\mathscr{A}$-finitely generated up to nilpotents.

Example 5.12. — Consider the finite presheaf Gr_2 over $\mathbb{F}$. It is a basic calculation that $q_2\mathbb{F}[\mathrm{Gr}_2] \cong \Lambda^2 \oplus \mathbb{F}$, so that there is a canonical inclusion

$$\mathrm{Gr}_2 \hookrightarrow \Lambda^2 \oplus \mathbb{F}$$

in $\widehat{\mathscr{V}_f}$. Since Gr_2 is connected, it follows that there is an inclusion $\mathrm{Gr}_2 \hookrightarrow \Lambda^2$ (as a presheaf of sets, $\mathbb{F} \oplus \Lambda^2$ identifies as $\Lambda^2 \amalg \Lambda^2$). This gives one approach to showing that $\mathrm{End}_{\widehat{\mathscr{V}_f}}(\mathrm{Gr}_2)$ is the monoid underlying $\mathbb{F}$.

The inclusion $\mathrm{Gr}_2 \hookrightarrow \Lambda^2$ corresponds to the canonical F-epimorphism in $\mathscr{K}$:

$$U(\Lambda^2(F(1))) \to \mathbb{F} \oplus \omega_2 D(2).$$

Example 5.13. — The following example complements Example 3.19. Consider the presheaves $\mathrm{Gr}_{\leqslant n}$ of Example 5.11 and form the colimit

$$\mathrm{Gr}_{\leqslant \infty} := \bigcup_n \mathrm{Gr}_{\leqslant n}$$

in $\widehat{\mathscr{V}_f}$. This exhibits $\mathrm{Gr}_{\leqslant \infty}$ as a colimit of connected presheaves induced from Noetherian $\mathrm{End}(\mathbb{F}^n)$-sets, whereas $\mathrm{Gr}_{\leqslant \infty}$ is not finitely generated, although it does take values in finite sets. The fact that the augmentation ideal $\overline{D(n)} \subset D(n)$ is zero in degrees $< 2^{n-1}$ implies that $(\mathrm{Gr}_{\leqslant \infty})^\omega = *$.

6. The growth function γ

The growth functions introduced in this section provide a useful first approximation to the property of finiteness for presheaves.

6.1. The function γ_X

DEFINITION 6.1. — *For $\emptyset \neq X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ that takes values in finite sets, let $\gamma_X : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be the growth function defined by*

$$t \mapsto \gamma_X(t) := \log_p |X(\mathbb{F}^t)|.$$

Remark 6.2. — If $F \in \mathrm{ob}\mathscr{F}$ takes finite-dimensional values, then $\gamma_F(t) = \dim F(\mathbb{F}^t)$, so that γ_F coincides with the growth function considered in [6].

The following characterization of polynomial degree in terms of the growth function is useful:

PROPOSITION 6.3 ([6]). — *Let $F \in \mathrm{ob}\mathscr{F}$ be a functor that takes finite-dimensional values. The following are equivalent:*

(1) *F is polynomial of degree d;*
(2) *γ_F is a polynomial function of degree d.*

PROPOSITION 6.4. — *Suppose that $\emptyset \neq X \in \mathrm{ob}\widehat{\mathscr{V}_f}$ is finite of degree d. Then*

(1) *$|X(V)| < \infty$ for every $V \in \mathrm{ob}\mathscr{V}_f$;*
(2) *$\gamma_X(t) = O(t^d)$.*

Proof. — By hypothesis, $\emptyset \neq X \hookrightarrow F_X$, where $F_X \in \mathrm{ob}\mathscr{F}$ is a finite functor of degree d. Hence $\gamma_X \leqslant \gamma_{F_X}$; since F_X is of degree d, γ_{F_X} is a polynomial function of degree d, by Proposition 6.3. □

6.2. Applications

Such growth functions play an important role in the work of Grodal [4] and Lannes and Schwartz [10].

Notation 6.5 ([4]). — For functions $f, g : \mathbb{N} \to \mathbb{R}$, write $f \lesssim g$ if, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $f(t) \leqslant (1+\varepsilon)g(t)$ for all $t \geqslant N$.

THEOREM 6.6 ([4, Theorem 3.3]). — *Let E be a connected, nilpotent finite Postnikov system with finite $\pi_1 E$ and finitely-generated homotopy groups. Then there exist $0 < c, C \in \mathbb{N}$ such that, as functions of t:*

$$ct^d \lesssim \log_p |\mathfrak{g}_{\mathrm{top}} E(\mathbb{F}^t)| \lesssim Ct^d,$$

where for $k := \sup\{i | \pi_i(E)_p \neq 0\}$, $d = k$ if $\pi_k E$ has p-torsion, otherwise $d = k - 1$.

Remark 6.7. — This should be compared with the argument used in the proof of [10, Theorem 0.1], using the fact (see Theorem 2.21) that, under suitable hypotheses upon the topological space E, cohomology induces an isomorphism of presheaves $\mathfrak{g}_{\mathrm{top}}E \xrightarrow{\cong} \mathfrak{g}H^*E$. In this situation, Grodal's theorem implies that $\gamma_{\mathfrak{g}H^*E}(t) = O(t^d)$.

However, the arguments of [4, 10] only provide bounds on the growth function; in particular, they do not show that $\mathfrak{g}H^*E$ is a finite presheaf (see Example 6.9 below).

We do, however, have the following:

Corollary 6.8. — *Let E satisfy the hypotheses of Theorem 6.6 and suppose that $\mathfrak{g}_{\mathrm{top}}E$ is a finite presheaf. Then $\mathfrak{g}_{\mathrm{top}}E$ has degree at least d (for d as in the Theorem).*

Example 6.9. — The property that γ_X is a polynomial function does not imply that X is a finite presheaf.

Consider any finite, constant-free functor $0 \neq F \in \mathscr{F}$ of polynomial degree $d \geqslant 2$, so that $\gamma_F(t) = O(t^d)$. To be concrete, we take $F = \Lambda^2$ over the field $\mathbb{F}$.

The rank filtration of Proposition 5.1 provides the decomposition of the underlying set

$$F(V) = \coprod_n F_{\leqslant n}(V) \backslash F_{\leqslant n-1}(V)$$

and

$$F_{\leqslant n}(V) \backslash F_{\leqslant n-1}(V) \cong F_{\mathrm{reg}}(n) \times_{\mathrm{Aut}(\mathbb{F}^n)} \mathrm{Surj}(V, \mathbb{F}^n)$$

by Lemma 5.3 and, for $n \geqslant 1$, the quotient $F_n(V) := F_{\leqslant n}(V)/F_{\leqslant n-1}(V)$ in presheaves identifies with

$$V \mapsto * \amalg \Big(F_{\mathrm{reg}}(n) \times_{\mathrm{Aut}(\mathbb{F}^n)} \mathrm{Surj}(V, \mathbb{F}^n) \Big).$$

Consider the connected presheaf $\tilde{F} := \bigvee_{n \geqslant 1} F_n$. For $V \in \mathrm{ob}\mathscr{V}_f$, by construction, $\tilde{F}(V)$ has the same underlying finite set as $F(V)$, but a very different $\mathrm{End}(V)$-structure.

To show that the presheaf $\tilde{F}$ is not finite, it suffices to show that, for $G \in \mathrm{ob}\mathscr{F}$ a finite, constant-free functor, there is no non-trivial map from F_n to G for $n \gg 0$.

Now $\mathrm{Hom}_{\widehat{\mathscr{V}_f}}(F_n, G) \cong \mathrm{Hom}_{\mathscr{F}}(\mathbb{F}[F_n], G)$ and $\mathbb{F}[F_n]$ splits as $\mathbb{F} \oplus \overline{\mathbb{F}[F_n]}$ where $\overline{\mathbb{F}[F_n]}(V) = 0$ if $\dim V < n$, by construction of F_n. Since G is constant-free, $\mathrm{Hom}_{\mathscr{F}}(\mathbb{F}[F_n], G) \cong \mathrm{Hom}_{\mathscr{F}}(\overline{\mathbb{F}[F_n]}, G)$ and, since G is finite, it follows by connectivity arguments that $\mathrm{Hom}_{\mathscr{F}}(\overline{\mathbb{F}[F_n]}, G) = 0$ for $n \gg 0$.

Part 2. Unstable Hopf algebras and presheaves of p-groups

7. Hopf algebras in $\mathscr{K}$

7.1. Preliminaries

As usual, $\mathscr{K}$ denotes the category of unstable algebras over the mod p Steenrod algebra $\mathscr{A}$.

DEFINITION 7.1. — *Let $\mathscr{H}_{\mathscr{K}}$ be the category of cogroup objects in $\mathscr{K}$.*

(1) *An object of $\mathscr{H}_{\mathscr{K}}$ is a commutative $\mathbb{F}$-Hopf algebra H, such that the underlying algebra is an unstable algebra over the Steenrod algebra, and the structure morphisms $\Delta : H \to H \otimes H$ (the diagonal or coproduct) and $\chi : H \to H$ (the conjugation or antipode) are morphisms of modules over the Steenrod algebra.*

(2) *A morphism $H_1 \to H_2$ of $\mathscr{H}_{\mathscr{K}}$ is a morphism of $\mathbb{F}$-Hopf algebras that is $\mathscr{A}$-linear.*

Let $\mathscr{H}^+_{\mathscr{K}} \subset \mathscr{H}_{\mathscr{K}}$ denote the full subcategory of connected objects (H such that $H^0 = \mathbb{F}$).

Remark 7.2. — It is not assumed that the coproduct Δ is cocommutative.

Here we focus upon connected unstable Hopf algebras. This is not a serious restriction, since the general case can be treated by using the following:

LEMMA 7.3. — *Let $H \in \mathrm{ob}\mathscr{H}_{\mathscr{K}}$ be a Hopf algebra concentrated in degree zero with $\dim H^0 < \infty$. Then $H \cong \mathbb{F}^{\mathrm{Spec}\, H^0}$, where $\mathrm{Spec}\, H^0$ is a finite group.*

The following is a key fact:

PROPOSITION 7.4. — *Let $H \in \mathrm{ob}\mathscr{H}^+_{\mathscr{K}}$ and $K \subset H$ be a sub unstable algebra such that K is $\mathscr{A}$-finitely generated. Then there exists $H_K \subset H$ in $\mathscr{H}^+_{\mathscr{K}}$ such that*

(1) *H_K is $\mathscr{A}$-finitely generated as an unstable algebra;*

(2) *$K \subset H_K$ as unstable algebras.*

Proof. — By hypothesis, there exists a finite graded vector subspace $V_K \subset \overline{K}$ such that the induced morphism of unstable algebras $U(F(V_K)) \to K$ is surjective, where $F(V_K)$ is the free unstable module on V_K.

The vector space V_K is contained within a finite-dimensional sub $\mathbb{F}$-coalgebra $C_K \subset H$. (This is a standard fact; in this graded connected

setting, the proof is straightforward.) Let H_K denote the sub unstable algebra of H generated by C_K; by construction, this contains K. Moreover, since the coproduct is $\mathscr{A}$-linear, it is straightforward to check that H_K is stable under the coproduct. In the connected setting, stability under the conjugation is automatic [11], hence H_K is an object of $\mathscr{H}^+_{\mathscr{K}}$, as required. □

Recall that cokernels exist in $\mathscr{H}^+_{\mathscr{K}}$ (see [12, 13] for example). In particular, if $K \hookrightarrow H$ is a monomorphism of $\mathscr{H}^+_{\mathscr{K}}$, then the cokernel $H \twoheadrightarrow H/\!\!/K$ has underlying unstable algebra given by the pushout in $\mathscr{K}$:

$$\begin{array}{ccc} K & \longrightarrow & H \\ \downarrow & & \downarrow \\ \mathbb{F} & \longrightarrow & H \otimes_K \mathbb{F}. \end{array}$$

LEMMA 7.5. — *Let $H \in \mathrm{ob}\mathscr{H}^+_{\mathscr{K}}$ such that $\mathfrak{g}H$ takes values in finite sets. Then the Hopf algebra structure of H induces a natural group structure on $\mathfrak{g}H$, so that $\mathfrak{g}H$ takes values in finite groups.*

Let $H' \subset H$ be a sub Hopf algebra in $\mathscr{H}^+_{\mathscr{K}}$ and consider the associated sequence $H' \hookrightarrow H \twoheadrightarrow H/\!\!/H'$. This induces a short exact sequence of finite groups:

$$\mathfrak{g}(H/\!\!/H') \overset{\lhd}{\hookrightarrow} \mathfrak{g}H \twoheadrightarrow \mathfrak{g}H'$$

Proof. — The first statement is a formal consequence of the natural isomorphism $\mathfrak{g}(K \otimes L) \cong \mathfrak{g}K \times \mathfrak{g}L$, for $K, L \in \mathscr{K}$ and the fact that H is a cogroup object in $\mathscr{K}$.

The underlying sequence of pointed sets

$$\mathfrak{g}(H/\!\!/H') \hookrightarrow \mathfrak{g}H \twoheadrightarrow \mathfrak{g}H'$$

is exact as pointed sets, since the underlying unstable algebra of $H/\!\!/H'$ is $H \otimes_{H'} \mathbb{F}$. The morphisms are respectively injective and surjective as morphisms of pointed sets, by [5, Corollary II.1.4]. In particular, both $\mathfrak{g}H'$ and $\mathfrak{g}(H/\!\!/H')$ take values in finite sets.

The naturality of the group structure implies that these are group morphisms; the result follows. □

7.2. The primitive filtration

Recall that, if H is a connected Hopf algebra with augmentation ideal $\overline{H}$, then the module of primitives PH is the kernel of the reduced diagonal

$$\overline{H} \overset{\overline{\Delta}}{\to} \overline{H} \otimes \overline{H}.$$

LEMMA 7.6. — *For $H \in \mathrm{ob}\mathscr{H}^+_{\mathscr{K}}$, the module of primitives $PH \subset H$ is a non-trivial sub unstable module and the canonical morphism*

$$UPH \to H,$$

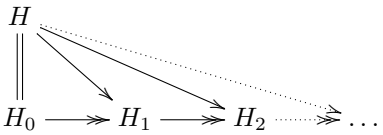
where $U : \mathscr{U} \to \mathscr{K}$ is the enveloping algebra functor, is a morphism of $\mathscr{H}^+_{\mathscr{K}}$ when UPH is equipped with the primitively-generated Hopf algebra structure.

Moreover, this morphism is injective; UPH is the largest primitively-generated sub Hopf algebra of H.

Proof. — This is standard. The injectivity statement follows from the natural identification $PH \cong PUPH$ together with the fact that a non-zero element of minimal degree in the kernel is necessarily primitive. □

As usual, one has the primitive filtration of an object of $\mathscr{H}^+_{\mathscr{K}}$ (cf. [12], for example).

DEFINITION 7.7. — *For $H \in \mathrm{ob}\mathscr{H}^+_{\mathscr{K}}$, recursively define the natural sequence of quotients in $\mathscr{H}^+_{\mathscr{K}}$:*



where, for each $n \in \mathbb{N}$, H_{n+1} is the cokernel in $\mathscr{H}^+_{\mathscr{K}}$ of $UPH_n \hookrightarrow H_n$.

LEMMA 7.8. — *For $H \in \mathrm{ob}\mathscr{H}^+_{\mathscr{K}}$,*

1. *$\lim_{\to} H_n = \mathbb{F}$;*
2. *if H is $\mathscr{A}$-finitely generated (as an unstable algebra), $H_n = \mathbb{F}$ for $n \gg 0$.*

Proof. — Straightforward. □

The nilpotent filtration behaves well when working modulo nilpotents, due to the following Lemma:

LEMMA 7.9. — *Let $H \to H'$ be a morphism of $\mathscr{H}^+_{\mathscr{K}}$ such that the underlying morphism of unstable algebras is an F-isomorphism (equivalently induces an isomorphism in $\mathscr{U}/\mathscr{N}il$). Then the induced map $PH \to PH'$ is an isomorphism in $\mathscr{U}/\mathscr{N}il$.*

Proof. — By definition, $PH \to PH'$ fits into a commutative diagram in $\mathscr{U}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & PH & \longrightarrow & \overline{H} & \xrightarrow{\overline{\Delta}} & \overline{H} \otimes \overline{H} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & PH' & \longrightarrow & \overline{H'} & \xrightarrow[\overline{\Delta}]{} & \overline{H'} \otimes \overline{H'} \end{array}$$

in which the rows are exact. After passage to $\mathscr{U}/\mathscr{N}il$, the two right hand vertical arrows are isomorphisms, by the hypothesis, hence so is the left hand one. □

PROPOSITION 7.10. — *Let $H \to H'$ be a morphism of $\mathscr{H}^+_{\mathscr{K}}$ such that the underlying morphism of unstable algebras is an F-isomorphism and $\mathfrak{g}H$ takes values in finite groups. Then the induced morphisms $H_n \to H'_n$ of the primitive filtration are F-isomorphisms.*

Proof. — By induction, it suffices to prove the case $n = 1$. The morphism $H \to H'$ induces a commutative diagram of sequences in $\mathscr{H}^+_{\mathscr{K}}$:

$$\begin{array}{ccccc} UPH & \hookrightarrow & H & \twoheadrightarrow & H_1 \\ \downarrow & & \downarrow & & \downarrow \\ UPH' & \hookrightarrow & H' & \twoheadrightarrow & H'_1. \end{array}$$

By hypothesis, $H \to H'$ is an F-isomorphism; Lemma 7.9 implies that $PH \to PH'$ is an isomorphism in $\mathscr{U}/\mathscr{N}il$ and this implies that $UPH \to UPH'$ is an F-isomorphism (by [7]).

Applying Lemma 7.5 gives a morphism between short exact sequences of functors to finite groups. It follows that $\mathfrak{g}H'_1 \to \mathfrak{g}H_1$ is an isomorphism, hence that $H_1 \to H'_1$ is an F-isomorphism. □

PROPOSITION 7.11. — *Let $H \in \mathrm{ob}\mathscr{H}^+_{\mathscr{K}}$ have underlying unstable algebra that is $\mathscr{A}$-finitely generated up to nilpotents. Then $\mathfrak{g}H$ takes values naturally in finite p-groups.*

Proof. — By hypothesis, there exists a sub unstable algebra $K \subset H$ such that K is $\mathscr{A}$-finitely generated and the inclusion is an F-epimorphism. Proposition 7.4 provides $K \subset H_K \subset H$ with $H_K \in \mathrm{ob}\mathscr{H}^+_{\mathscr{K}}$ that is also $\mathscr{A}$-finitely generated. Clearly $H_K \subset H$ is also an F-epimorphism, hence we may suppose without loss of generality that H is $\mathscr{A}$-finitely generated as an unstable algebra, so that the primitive filtration is finite. (Alternatively, Proposition 7.10 can be used.)

Dévissage using the primitive filtration together with Lemma 7.5 allows reduction to the case where H is primitively generated. If $H \cong UPH$, then $\mathfrak{g}H$ is given by the underlying set-valued functor of $V \mapsto \mathrm{Hom}_{\mathscr{U}}(PH, H^*(BV))$. It is straightforward to check that the group structure of $\mathfrak{g}H$ corresponds to the elementary abelian p-group structure of $\mathrm{Hom}_{\mathscr{U}}(PH, H^*(BV))$; in particular, $\mathfrak{g}H$ takes values in finite p-groups. □

8. Functors to finite p-groups

Motivated by Proposition 7.11, this section studies presheaves of finite p-groups.

8.1. p-finiteness

DEFINITION 8.1. — *Let $\mathscr{G}_p^f$ denote the category of functors from $\mathscr{V}_f^{\mathrm{op}}$ to the category of finite p-groups (i.e. presheaves of finite p-groups).*

Remark 8.2. — The category $\mathscr{F}$ is a full subcategory of $\mathscr{G}_p^f$. Forgetting the group structure gives a faithful functor $\mathscr{G}_p^f \to \widehat{\mathscr{V}_f}$.

Recall that the Frattini subgroup $\Phi(G) \subset G$ of a finite group G is the intersection of all maximal proper subgroups of G. If G is a finite p-group then

$$\Phi G = [G, G]G^p$$

and ΦG is the minimal normal subgroup of G such that the quotient $G/\Phi G$ is p-elementary abelian.

The above description makes it clear that a morphism of finite p-groups, $G_1 \to G_2$ restricts to $\Phi G_1 \to \Phi G_2$ and thus induces a morphism of $\mathbb{F}$-vector spaces

$$G/\Phi G_1 \to G/\Phi G_2.$$

DEFINITION 8.3. — *For G a finite p-group, let $\Phi_n G$ denote the p-derived series (or Frattini series) of G, defined recursively by $\Phi_0 G = G$ and $\Phi_{n+1} G = \Phi(\Phi_n G)$.*

For a finite p-group, this series is finite (i.e. $\Phi_N G = \{e\}$ for $N \gg 0$).

LEMMA 8.4. — *Let $f : G_1 \to G_2$ be a morphism of finite p-groups, then for $n \in \mathbb{N}$, f restricts to a morphism*

$$\Phi_n f : \Phi_n G_1 \to \Phi_n G_2,$$

thus induces a natural morphism of graded $\mathbb{F}$-vector spaces:

$$\bigoplus_{n\geqslant 0}\Phi_n f : \bigoplus_{n\geqslant 0}\Phi_n G_1/\Phi_{n+1}G_1 \to \bigoplus_{n\geqslant 0}\Phi_n G_2/\Phi_{n+1}G_2.$$

Proof. — A straightforward induction upon n. □

PROPOSITION 8.5. — *Let $\mathfrak{G} \in \mathrm{ob}\mathscr{G}_p^f$ be a presheaf of finite p-groups. The p-derived series induces a natural series*

$$\ldots \subset \Phi_{n+1}\mathfrak{G} \subset \Phi_n\mathfrak{G} \subset \ldots \subset \mathfrak{G}$$

such that, evaluated on $V \in \mathrm{ob}\mathscr{V}_f$, $(\Phi_n\mathfrak{G})(V) = \Phi_n(\mathfrak{G}(V))$.

The associated graded

$$\bigoplus_{n\geqslant 0}\Phi_n\mathfrak{G}/\Phi_{n+1}\mathfrak{G}$$

is an $\mathbb{N}$-graded functor with values in $\mathscr{V}_f$. For any $V \in \mathrm{ob}\mathscr{V}_f$,

$$\Phi_i\mathfrak{G}/\Phi_{i+1}\mathfrak{G}(V) = 0, \quad \text{for } i \gg 0.$$

Proof. — An immediate consequence of the naturality of the p-derived series, established in Lemma 8.4. □

DEFINITION 8.6. — *A functor $\mathfrak{G} \in \mathrm{ob}\mathscr{G}_p^f$ to the category of finite p-groups is p-finite if $\bigoplus_{n\geqslant 0}\Phi_n\mathfrak{G}/\Phi_{n+1}\mathfrak{G}$ is a finite functor, considered as an object of the category $\mathscr{F}$ (forgetting the grading).*

The notion of a polynomial functor[(1)] to the category of groups (*à la* Baues–Pirashvili [1]) is recalled in Section A.

THEOREM 8.7. — *For $\mathfrak{G} \in \mathrm{ob}\mathscr{G}_p^f$, the following conditions are equivalent:*

(1) *$\mathfrak{G}$ is p-finite;*
(2) *$\mathfrak{G}$ is polynomial;*
(3) *Both the following conditions are satisfied:*
   (a) *$\Phi_N\mathfrak{G} = 0$ for $N \gg 0$ (uniformly);*
   (b) *each $\Phi_i\mathfrak{G}/\Phi_{i+1}\mathfrak{G}$ is a finite functor of $\mathscr{F}$;*
(4) *$\mathfrak{G}$ has a composition series;*
(5) *the growth function $\gamma_{\mathfrak{G}}$ satisfies $\gamma_{\mathfrak{G}}(t) = O(t^d)$ for some $d \in \mathbb{N}$.*

If $\mathfrak{G}$ takes values in finite abelian p-groups, this is equivalent to

- *$\mathfrak{G}$ is finite as a functor to the abelian category of finite abelian p-groups.*

(1) The author is grateful to Christine Vespa for pointing out that this notion of polynomial provides an equivalent condition.

Proof. — The result is proved by reducing to the case where $\mathscr{G}$ takes values in $\mathbb{F}$-vector spaces, where the result is standard, for instance by applying Proposition 2.9 and Proposition 6.3.

The equivalence of p-finiteness and polynomiality is proved by using the thickness of the polynomial property established in Corollary A.9. □

The relevance of p-finiteness is shown by the following result:

COROLLARY 8.8. — *Let $\mathfrak{G} \in \mathrm{ob}\mathscr{G}_p^f$ be a functor to finite p-groups such that the underlying presheaf $\mathfrak{G} \in \mathrm{ob}\widehat{\mathscr{V}_f}$ is finite. Then $\mathfrak{G}$ is p-finite.*

Proof. — The hypothesis that the underlying presheaf is finite implies that $\gamma_{\mathfrak{G}} = O(t^d)$ for some $d \in \mathbb{N}$, by Proposition 6.4, hence the result follows from Theorem 8.7. □

Example 8.9. — Corollary 8.8 applies to the case $\mathfrak{G} = \mathfrak{g}H$, where $H \in \mathrm{ob}\mathscr{H}_{\mathscr{K}}^+$ is $\mathscr{A}$-finitely generated up to nilpotents.

8.2. Coanalyticity of p-finite functors to p-groups

For any $\mathfrak{G} \in \mathrm{ob}\mathscr{G}_p^f$, composition with the group ring functor $\mathbb{F}[-]$ (and forgetting the ring structure) gives a functor $\mathbb{F}[\mathfrak{G}] \in \mathscr{F}$ that takes finite-dimensional values. Example 3.12 shows that such a functor need not be *coanalytic* (the inverse limit of its finite quotients).

This issue is resolved when one imposes p-finiteness:

PROPOSITION 8.10. — *Let $\mathfrak{G} \in \mathrm{ob}\mathscr{G}_p^f$ be p-finite. Then $\mathbb{F}[\mathfrak{G}] \in \mathrm{ob}\mathscr{F}$ is coanalytic.*

Proof. — Let $I\mathfrak{G} \subset \mathbb{F}[\mathfrak{G}]$ denote the augmentation ideal (kernel in $\mathscr{F}$ of the augmentation $\mathbb{F}[\mathfrak{G}] \to \mathbb{F}$). The powers of the augmentation ideal induces a decreasing filtration

$$\ldots \subset I^{k+1}\mathfrak{G} \subset I^{k+1}\mathfrak{G} \subset \ldots \subset I\mathfrak{G} \subset \mathbb{F}[\mathfrak{G}]$$

and hence an inverse system of quotients $\mathbb{F}[\mathfrak{G}] \twoheadrightarrow \mathbb{F}[\mathfrak{G}]/I^k\mathfrak{G}$.

To establish the result, it suffices to show that

- each quotient $\mathbb{F}[\mathfrak{G}]/I^k\mathfrak{G}$ is finite;
- $\mathbb{F}[\mathfrak{G}] \cong \lim_{\leftarrow} \mathbb{F}[\mathfrak{G}]/I^k\mathfrak{G}$.

This is proved using standard results on this filtration (see [14], for example).

For the first statement, since $I^k\mathfrak{G}/I^{k+1}\mathfrak{G}$ is a quotient of $(I\mathfrak{G}/I^2\mathfrak{G})^{\otimes k}$, it suffices to show that $I\mathfrak{G}/I^2\mathfrak{G}$ is a finite functor. But the latter is equivalent

to the functor $\mathfrak{G}_{ab} \otimes \mathbb{F} \cong \mathfrak{G}/\Phi\mathfrak{G}$. The p-finiteness hypothesis implies that this is a finite functor of $\mathscr{F}$: it takes finite-dimensional values and has polynomial growth (by Theorem 8.7), hence is finite.

For the second statement, it suffices to show that, for any $V \in \mathrm{ob}\mathscr{V}_f$, there exists $k_V \in \mathbb{N}$ such that $I^{k_V}\mathfrak{G}(V) = 0$ (here k_V depends upon $\mathfrak{G}$); this follows from Lemma 8.11 below. □

LEMMA 8.11. — *Let G be a finite p-group, and IG be the augmentation ideal of the mod p group ring $\mathbb{F}[G]$. Then IG is nilpotent, i.e. there exists $N \in \mathbb{N}$ such that $I^N G = 0$.*

Proof. — This is a standard result. It is proved by induction upon the order of the finite p-group G, the inductive step relying on the fact that the centre of a finite p-group is non-trivial. □

8.3. p-finiteness versus finiteness

The following theorem implies that the two notions of finiteness for $\mathfrak{G} \in \mathrm{ob}\mathscr{G}_p^f$ coincide.

THEOREM 8.12. — *Let $\mathfrak{G} \in \mathrm{ob}\mathscr{G}_p^f$ be a functor to finite p-groups such that $\mathfrak{G}(0) = \{e\}$. Then $\mathfrak{G}$ is p-finite if and only if the underlying presheaf $\mathfrak{G}$ in $\widehat{\mathscr{V}_f}$ is finite.*

Proof. — The implication $\Leftarrow$ is given by Corollary 8.8.

For $\Rightarrow$, since $\mathbb{F}[\mathfrak{G}] \in \mathscr{F}$ is coanalytic (by Proposition 8.10), it is possible to carry out the argument by passing to unstable (Hopf) algebras. Namely, consider the unstable algebra $H := \kappa\mathfrak{G}$; by Proposition 2.25, this has finite type. It follows that the group structure of $\mathfrak{G}$ gives H the structure of a Hopf algebra in $\mathscr{K}$ and, by construction, $\mathfrak{g}H \cong \mathfrak{G}$ as p-group valued functors.

The proof is by induction upon the length of the p-derived series of $\mathfrak{G}$, using the correspondence between short exact sequences of Hopf algebras and short exact sequences of group-valued functors given by Lemma 7.5. If $\Phi\mathfrak{G} = \{e\}$, then $\mathfrak{G}$ actually belongs to $\mathscr{F}$ and p-finite is equivalent to finite.

For the inductive step, consider the projection $\mathfrak{G} \twoheadrightarrow \mathfrak{G}/\Phi\mathfrak{G}$. Applying κ gives an inclusion of Hopf algebras in $\mathscr{H}_{\mathscr{K}}^+$

$$H' := \kappa(\mathfrak{G}/\Phi\mathfrak{G}) \hookrightarrow H$$

and hence a short exact sequence in $\mathscr{H}^+_{\mathscr{K}}$:

$$H' \hookrightarrow H \twoheadrightarrow H/\!\!/H'.$$

By Lemma 7.5, one has that $\mathfrak{g}(H/\!\!/H') \cong \Phi\mathfrak{G}$. Hence, by induction, H' and $H/\!\!/H'$ are both $\mathscr{A}$-finitely generated up to nilpotents.

Since $H/\!\!/H'$ is $\mathscr{A}$-finitely generated up to nilpotents, there exists a connected unstable algebra $K \subset H$ that is $\mathscr{A}$-finitely generated and such that the composite $K \to H/\!\!/H'$ is an F-epimorphism.

Proposition 7.4 provides a sub unstable Hopf algebra $H_K \subset H$ that is $\mathscr{A}$-finitely generated as an unstable algebra and such that $K \subset H_K$ (as unstable algebras).

Let $H'' \subset H$ denote the sub Hopf algebra image in $\mathscr{H}^+_{\mathscr{K}}$ of

$$H' \otimes H_K \to H.$$

By construction, H'' is $\mathscr{A}$-finitely generated and the cokernel $H/\!\!/H''$ is nilpotent.

Applying the functor $\mathfrak{g}$ to the short exact sequence of $\mathscr{H}^+_{\mathscr{K}}$

$$H'' \hookrightarrow H \to H/\!\!/H'',$$

by Lemma 7.5 implies that $\mathfrak{g}H'' \cong \mathfrak{g}H$; since $\mathfrak{g}(H/\!\!/H') = *$. (This step relies crucially on having a short exact sequence of groups, not just pointed sets.)

This implies that H is $\mathscr{A}$-finitely generated modulo nilpotents, by Corollary 4.4. □

COROLLARY 8.13. — *For $H \in \mathrm{ob}\mathscr{H}^+_{\mathscr{K}}$, the underlying unstable algebra of H is $\mathscr{A}$-finitely generated up to nilpotents if and only if γH is p-finite.*

If H satisfies the above conditions and $K \subset H$ is a sub Hopf algebra in unstable algebras, then K is $\mathscr{A}$-finitely generated up to nilpotents.

9. Unstable Hopf algebras modulo nilpotents

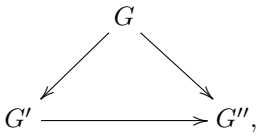
9.1. Coanalytic presheaves of p-profinite groups

The material of Section 8 leads to the appropriate notion of profinite p-group valued functors.

Notation 9.1.

(1) Let $\mathscr{G}_p^{\mathrm{fin}} \subset \mathscr{G}_p^f$ denote the full subcategory of p-finite objects.

(2) For $G \in \mathrm{ob}(\widehat{\mathscr{V}_f}^{\mathrm{profin}})_c$ that takes values in profinite p-groups, let $G/\mathscr{G}_p^{\mathrm{fin}}$ be the full subcategory of $G/(\widehat{\mathscr{V}_f}^{\mathrm{profin}})_c$ with objects morphisms $G \to G'$ with $G' \in \mathrm{ob}\mathscr{G}_p^{\mathrm{fin}}$ that are continuous group morphisms on sections and such that morphisms are commutative diagrams



where $G' \to G''$ is a morphism of $\mathscr{G}_p^{\mathrm{fin}}$.

(3) For G as above, set $G^\omega := \lim\limits_{\leftarrow\, G \to G' \in G/\mathscr{G}_p^{\mathrm{fin}}} G'$, equipped with the canonical map $G \to G^\omega$.

Remark 9.2. — Let $G(i)$ be a diagram of $\mathscr{G}_p^{\mathrm{fin}}$ indexed by a small, cofiltering category $\mathscr{I}$. Then $\lim\limits_{\leftarrow\, i \in \mathscr{I}} G(i)$ has underlying presheaf in $\widehat{\mathscr{V}_f}^{\omega}$ and takes values in p-profinite groups.

Definition 9.3.

(1) *For $G \in \mathrm{ob}(\widehat{\mathscr{V}_f}^{\mathrm{profin}})_c$ which takes values in profinite p-groups, say that G is p-coanalytic if the canonical map $G \to G^\omega$ is an isomorphism.*

(2) *Let $\mathscr{G}_p^{\omega+}$ for the category of connected, p-coanalytic sheaves and continuous group morphisms, equipped with the forgetful functor $\mathscr{G}_p^{\omega+} \to (\widehat{\mathscr{V}_f}^{\mathrm{profin}})_c$.*

Remark 9.4. — A p-coanalytic presheaf is, in particular, coanalytic. Hence the forgetful functor induces $\mathscr{G}_p^{\omega+} \to (\widehat{\mathscr{V}_f}^{\omega})_c$.

Lemma 9.5.

(1) *For $H \in \mathrm{ob}\mathscr{H}_{\mathscr{K}}^+$, $\mathfrak{g}H$ is p-coanalytic and $\mathfrak{g}$ induces a functor $\mathscr{H}_{\mathscr{K}}^+ \to \mathscr{G}_p^{\omega+}$.*

(2) *For $G \in \mathrm{ob}\mathscr{G}_p^{\omega+}$ $\kappa G \in \mathrm{ob}\mathscr{H}_{\mathscr{K}}^+$ and κ induces a functor $\mathscr{G}_p^{\omega+} \to \mathscr{H}_{\mathscr{K}}^+$.*

Proof. — For the first statement, use the fact that H is the colimit of its sub-Hopf algebras in $\mathscr{K}$ that are $\mathscr{A}$-finitely generated, by Proposition 7.4. The second is clear; the colimit of the associated diagram in $\mathscr{H}_{\mathscr{K}}^+$ lies in $\mathscr{H}_{\mathscr{K}}^+$. The functoriality is clear in both cases. □

9.2. Connected unstable Hopf algebras modulo nilpotents

The analogue of Theorem 4.6 for connected unstable Hopf algebras is now essentially tautological.

Notation 9.6. — Let $\mathscr{H}^+_{\mathscr{K}}/\mathscr{N}il$ denote the localization, defined as for $\mathscr{K}/\mathscr{N}il$ (cf. [5, Part II.1] and Section 2.3).

THEOREM 9.7. — *The functor* $\mathfrak{g}$ *induces an equivalence of categories*

$$(\mathscr{H}^+_{\mathscr{K}}/\mathscr{N}il)^{\mathrm{op}} \xrightarrow{\cong} \mathscr{G}_p^{\omega+}.$$

Proof. — From the construction and from Lemma 9.5, it is clear that $\mathfrak{g}$ induces a functor $\mathscr{H}^+_{\mathscr{K}}/\mathscr{N}il \to \mathscr{G}_p^{\omega+}$. The inverse functor is induced by κ. □

Example 9.8. — If $H \in \mathscr{H}^+_{\mathscr{K}}$ is primitively-generated, then $\mathfrak{g}H$ is a constant-free functor of $\mathscr{F}$ that is dual to a locally finite functor (considered as a pro-object).

Appendix A. Recollections on polynomial functors

A.1. Definitions

Let $(\mathscr{C}, 0)$ be a small pointed category, equipped with symmetric monoidal structure $(\mathscr{C}, \odot, 0)$ for which 0 is the unit. Following Baues and Pirashvili [1], consider the category of functors $\mathsf{Gp}^{\mathscr{C}}$ from $\mathscr{C}$ to the category of groups Gp, and the associated notion of polynomial functor. The trivial group is written e.

DEFINITION A.1. — *A functor* $F \in \mathrm{ob}\mathsf{Gp}^{\mathscr{C}}$ *is constant-free if* $F(0) = e$.

The cross-effects of [1] generalize the Eilenberg–MacLane notion of cross-effect from the abelian setting [3].

DEFINITION A.2. — *For* $1 \leqslant n \in \mathbb{N}$, *let* $\mathsf{cr}_n : \mathsf{Gp}^{\mathscr{C}} \to \mathsf{Gp}^{\mathscr{C}^{\times n}}$ *denote the functors defined recursively by*

(1) $\mathsf{cr}_1 F(C) := \ker\{F(C) \to F(0)\}$;
(2) $\mathsf{cr}_2 F(C, D) := \ker\{\mathsf{cr}_1 F(C \odot D) \to \mathsf{cr}_1 F(C) \times \mathsf{cr}_1 F(D)\}$ *and*
(3) $\mathsf{cr}_{n+1} F(C_1, C_2, \dots, C_{n+1}) := \mathsf{cr}_2\big(\mathsf{cr}_n F(-, C_3, \dots, C_{n+1})\big)(C_1, C_2)$, *for* $n \geqslant 2$.

(Here the morphisms are induced by the projections to 0, *considered as the terminal object of* $\mathscr{C}$.*)*

The following is standard, and allows usage of cr_1 to be avoided when considering constant-free functors.

LEMMA A.3. — *If $F \in \mathsf{Gp}^{\mathscr{C}}$ is constant-free, then*

(1) $\mathsf{cr}_1 F \cong F$;
(2) *the functor $C \mapsto \mathsf{cr}_2 F(C, D)$ is constant-free for any $D \in \mathrm{ob}\mathscr{C}$.*

LEMMA A.4. — *If $F \in \mathrm{ob}\mathsf{Gp}^{\mathscr{C}}$ is constant-free, then there is a natural isomorphism:*

$$\mathsf{cr}_n F(C_1, \dots, C_n) \cong \ker\big\{F(C_1 \odot \dots \odot C_n) \to \prod_{i=1}^{n} F(C_1 \odot \dots \odot \widehat{C_i} \odot \dots \odot C_n)\big\},$$

where $\widehat{C_i}$ indicates that the term is omitted.

Proof. — The result is proved by induction on n. For $n = 2$, by Lemma A.3, this follows from the Definition of cr_2. For $n > 2$, the inductive step proceeds as for the case $n = 3$, which is [1, Lemma 1.8]. □

DEFINITION A.5. — *For $n \in \mathbb{N}$, a functor $F \in \mathrm{ob}\mathsf{Gp}^{\mathscr{C}}$ is polynomial of degree $\leqslant n$ if $\mathsf{cr}_{n+1} F = e$ is the constant functor.*

The following is an immediate consequence of Lemma A.4:

LEMMA A.6. — *For $n \in \mathbb{N}$, a constant-free functor $F \in \mathrm{ob}\mathsf{Gp}^{\mathscr{C}}$ is polynomial of degree $\leqslant n$ if and only if the natural transformation*

$$F(C_1 \odot \dots \odot C_{n+1}) \to \prod_{i=1}^{n+1} F(C_1 \odot \dots \odot \widehat{C_i} \odot \dots \odot C_{n+1})$$

is injective.

A.2. Exactness of cross-effects

DEFINITION A.7. — *For $\mathscr{D}$ a small category, a sequence of functors of $\mathsf{Gp}^{\mathscr{D}}$, $F_1 \to F_2 \to F_3$ is short exact if, for all $D \in \mathrm{ob}\mathscr{D}$,*

$$F_1(D) \to F_2(D) \to F_3(D)$$

is a short exact sequence of groups.

The following generalizes the standard result for functors to abelian categories:

PROPOSITION A.8. — *Let $1 \leqslant n \in \mathbb{N}$. If $K \to G \to Q$ is a short exact sequence of $\mathsf{Gp}^{\mathscr{C}}$, then*

$$\mathsf{cr}_n K \to \mathsf{cr}_n G \to \mathsf{cr}_n Q$$

is a short exact sequence of $\mathsf{Gp}^{\mathscr{C}^{\times n}}$.

Proof. — The reduction to the case where the functors are constant-free is left to the reader. From the recursive definition of cross-effects, it is straightforward to reduce to the case $n = 2$. Now, for F constant-free, as in [1, Section 1], there is a natural short exact sequence of groups

$$\mathsf{cr}_2 F(C, D) \to F(C \odot D) \to F(C) \times F(D).$$

Hence the result follows by the nine (or 3×3) Lemma in the category Gp. □

This immediately provides the *thickness* of the polynomial property:

COROLLARY A.9. — *For $K \to G \to Q$ a short exact sequence of $\mathsf{Gp}^{\mathscr{C}}$ and $n \in \mathbb{N}$, G has polynomial degree $\leqslant n$ if and only if both K and Q have polynomial degree $\leqslant n$.*

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Geoffrey POWELL
Laboratoire angevin de recherche en mathématiques
(LAREMA), CNRS, Université d'Angers
2 bd Lavoisier
49045 Angers Cedex 01 (France)
geoffrey.powell@math.cnrs.fr