Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Jeremy Daniel

On somes characteristic classes of flat bundles in complex geometry

Tome 69, nº 2 (2019), p. 729-751.
<http://aif.centre-mersenne.org/item/AIF_2019__69_2_729_0>

Ann. Inst. Fourier, Grenoble
69, 2 (2019) 729-751

ON SOMES CHARACTERISTIC CLASSES OF FLAT BUNDLES IN COMPLEX GEOMETRY

by Jeremy DANIEL

Abstract. — On a compact Kähler manifold X, any semisimple flat bundle carries a harmonic metric. It can be used to define some characteristic classes of the flat bundle, in the cohomology of X. We show that these cohomology classes come from an infinite-dimensional space, constructed with loop groups, an analogue of the period domains used in Hodge theory.

Résumé. — Sur une variété kählérienne compacte X, tout fibré plat semi-simple admet une métrique harmonique. On peut grâce à elle définir certaines classes caractéristiques du fibré plat, dans la cohomologie de X. Nous montrons que ces classes de cohomologie proviennent d'un espace de dimension infinie construit à partir de groupes de lacets, cet espace étant un analogue des domaines de périodes de la théorie de Hodge.

1. Introduction

It is an open problem to determine which finitely presented groups Γ can be realized as the fundamental group of a compact Kähler manifold. Some obstructions come from classical Hodge theory, the simplest of which asserting that $b_1(\Gamma)$ has to be even, because of the Hodge decomposition on $H^1(X)$. Other results are obtained *via* non-abelian Hodge theory, see e.g [19, Section 4] or [1, Chapter 7] for a general survey.

It would in particular be valuable to obtain restrictions on the cohomology groups $H^k(\Gamma)$ for $k \geqslant 2$. In this direction, the following conjecture has focused a lot of attention:

Conjecture 1.1 (Carlson-Toledo). — *If Γ is infinite and is the fundamental group of a compact Kähler manifold, then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that $H^2(\Gamma', \mathbb{R}) \neq 0$.*

Keywords: harmonic bundles, non-abelian Hodge theory, flat bundles, loop groups, period domains.
2010 *Mathematics Subject Classification:* 57R20, 22E67, 58A14.

Let Γ be such a group and assume that Γ admits a nonrigid irreducible representation in $\mathrm{GL}_n(\mathbb{C})$, that is, a representation which can be deformed to a non-conjugate one. Then, a theorem of Reznikov ([18, Proposition 9.1]) asserts that $H^2(\Gamma,\mathbb{R}) \neq 0$. This was proved in another way in [8] and we will explain briefly the proof since it is one motivation of the present paper.

Let $\Gamma = \pi_1(X)$ and ρ be as above and denote by E_ρ the flat bundle on X associated with ρ. By the theorem of Corlette and Donaldson, there exists a harmonic metric h on E_ρ; we denote by $\phi \in \mathcal{A}^{1,0}(X, \mathrm{End}(E_\rho))$ the Higgs field and by $\phi^* \in \mathcal{A}^{0,1}(X, \mathrm{End}(E_\rho))$ its adjoint with respect to h, see Section 2.1 for these notions. The differential form

$$\beta_{2,\rho} := \frac{1}{2\pi\,\mathrm{i}} \operatorname{Tr}(\phi \wedge \phi^*) \tag{1.1}$$

is closed and defines a cohomology class in $H^2(X,\mathbb{R})$. Using the machinery of loop groups and infinite-dimensional period domains developed in [8], we have proved that this class is *integral* on the universal cover $\pi : \tilde{X} \to X$:

$$\pi^*\beta_{2,\rho} \in H^2(\tilde{X},\mathbb{Z}). \tag{1.2}$$

The cohomologies of Γ, X and $\tilde{X}$ are related by a spectral sequence that gives the following exact sequence in low-degree cohomology:

$$0 \to H^2(\Gamma,\mathbb{R}) \to H^2(X,\mathbb{R}) \to H^2(\tilde{X},\mathbb{R})^\Gamma.$$

If ρ and ρ' are in the same connected component in the moduli space of representations, then $\pi^*[\beta_{2,\rho} - \beta_{2,\rho'}]$ has to be zero in $H^2(\tilde{X},\mathbb{R})$ because of equation (1.2). Hence, $[\beta_{2,\rho} - \beta_{2,\rho'}]$ lives in $H^2(\Gamma,\mathbb{R})$. On the other hand, one shows that the function $\rho \mapsto [\beta_{2,\rho}] \in H^2(X,\mathbb{R})$ cannot be constant unless ρ is rigid; this concludes the proof.

Our first motivation for this paper is to generalize equation (1.2) in the following way. The differential forms

$$\beta_{2k,\rho} := \frac{1}{(2\pi\,\mathrm{i})^k} \operatorname{Tr}\left((\phi \wedge \phi^*)^k\right) \tag{1.3}$$

are closed and define cohomology classes in $H^{2k}(X,\mathbb{R})$. We will show that the pullback of these classes on the universal cover $\tilde{X}$ are integral, after a suitable normalization. We can now explain the results and proofs of this paper.

Very roughly, the infinite-dimensional period domain $\mathcal{D}$ is a complex Hilbert manifold, which is homogeneous under the action of a real loop group, that we denote $\Lambda_\sigma \mathrm{GL}(n,\mathbb{C})$. The representation $\rho : \Gamma \to \mathrm{GL}(n,\mathbb{C})$ can be lifted to a representation $\hat{\rho} : \Gamma \to \Lambda_\sigma \mathrm{GL}(n,\mathbb{C})$. The main theorem of [8] states that the datum of E_ρ endowed with the harmonic metric h is equivalent to the datum of a holomorphic map $F : \tilde{X} \to \mathcal{D}$, which is equivariant under $\hat{\rho}$ and satisfies another condition of *horizontality*. The map F is called the *period map*.

In Section 3, we give explicit de Rham representatives for the cohomology of a connected component $\mathcal{D}_0$ of $\mathcal{D}$:

Theorem 1.2. — *The cohomology algebra $H^\bullet(\mathcal{D}_0,\mathbb{Q})$ is a polynomial algebra generated by elements $(z_{2k})_{k=1,\dots,n-1}$ in $H^\bullet(\mathcal{D}_0,\mathbb{Q})$. De Rham representatives, that are $(\Lambda_\sigma \mathrm{GL}(n,\mathbb{C}))_0$-invariant, are given by*

$$(1.4)\quad (z_{2k})_\gamma(\delta_1\gamma,\dots,\delta_{2k}\gamma) = \frac{1}{(2\pi\,\mathrm{i})^{k+1}}\cdot\int_{S^1}\sum_\sigma \varepsilon(\sigma)\,\mathrm{Tr}\left(\xi_{\sigma(1)}(\theta)\dots\xi_{\sigma(2k-1)}(\theta)\xi'_{\sigma(2k)}(\theta)\right)\mathrm{d}\theta.$$

Here, $\xi_i(\theta)=\gamma^{-1}(\theta)\delta_i\gamma(\theta)$.

We slightly abuse notations by considering elements of $\mathcal{D}$ as loops; this will be valid under some identifications explained in Section 3.

We remark that a similar result is well known for the based loop group $\Omega\,\mathrm{SU}(n)$, see e.g. [16, Proposition 4.11.3]. To a large extent, Theorem 1.2 should be considered as a twisted version of this result.

The next step consists in relating the forms $\beta_{2k,\rho}$ defined in equation (1.3) with the forms z_{2k}. We recall that $F:\tilde{X}\to\mathcal{D}$ is the period map and we denote by $\pi:\tilde{X}\to X$ the universal cover of X.

Theorem 1.3.

$$F^*(z_{2k}) = 2(-1)^{k+1}\pi^*\beta_{2k,\rho}.$$

The proof of this theorem is a direct computation that uses the tools of [8]. In particular this gives the desired integrality statement. We emphasize that such an integrality statement has no reason to be true on X itself. Indeed, for $k=1$, one shows that the class $\beta_{2,\rho}$ has to vary when the representation ρ can be deformed. A similar statement should be true for $k\geqslant 2$, but it does not look so easy to construct explicit examples.

The question of which representations ρ give integral cohomology classes $[\beta_{2k,\rho}]$, beginning with the case $k=1$ on a curve, seems interesting. In this direction, we show the following:

Theorem 1.4. — *If the flat bundle E_ρ underlies a variation of complex Hodge structures, then the classes $[\beta_{2k,\rho}]$ live in $H^\bullet(X,\mathbb{Z})$.*

This follows from a classical computation of the curvature of the Hodge bundles.

Organization of the paper. In Section 2, we recall some facts about characteristic classes of flat bundles and we explain the consequences of the existence of a harmonic metric. Some results about Kamber–Tondeur classes are included but are not needed for our results. Section 3 is devoted to the period domain $\mathcal{D}$; we recall its relation with harmonic metrics and then study its cohomology. In the final section 4, we prove Theorems 1.3 and 1.4.

Notations. By a vector bundle, we mean a complex vector bundle of finite rank. If $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is a representation of the fundamental group of a complex manifold X, we write E_ρ for the induced flat bundle and D for the flat connection. Gothic letters are used to denote the Lie algebras of the corresponding Lie groups. Depending on the context, the letter $\mathfrak{p}$ denotes the space of Hermitian matrices in $\mathfrak{gl}(n, \mathbb{C})$ (resp. in $\mathfrak{sl}(n, \mathbb{C})$).

Acknowledgments. I thank Bruno Klingler, Takuro Mochizuki and Carlos Simpson for some useful discussions about the topic of this paper. I also thank the referee for their careful reading.

2. Preliminaries

We are interested in flat bundles over complex manifolds which carry a harmonic metric. Our aim is to emphasize that the vanishing of the Kamber–Tondeur classes, which are odd degree characteristic classes of a flat bundle, and the existence of the even degree characteristic classes β_{2k} are both related to the harmonic metric. This suggests that the characteristic classes β_{2k} might be defined in a setting of pure differential geometry, when the Kamber–Tondeur classes vanish.

2.1. Harmonic metrics

Definitions. Let X be a complex manifold and let $\rho : \pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ be a linear representation. If h is a Hermitian metric on the flat bundle E_ρ, there is a unique decomposition of the flat connection

$$D = \nabla + \omega \tag{2.1}$$

such that:

- ∇ is a metric connection for h: $\mathrm{d}h(u, v) = h(\nabla u, v) + h(u, \nabla v)$;

- ω is a Hermitian 1-form with values in $\mathrm{End}(E_\rho)$: $h(\omega u, v) = h(u, \omega v)$;

for every local sections u and v of E_ρ.

We decompose ∇ and ω in $(1,0)$ and $(0,1)$-types:

$$\nabla = \partial + \bar{\partial}$$
$$\omega = \phi + \phi^*.$$

We emphasize that ϕ and ϕ^* are adjoint, meaning that $\phi(Y)$ and $\phi^*(\overline{Y})$ are h-adjoint, for every Y in $T^{1,0}X$.

Definition 2.1. — *The metric h is* harmonic *if the differential operator $\bar{\partial} + \phi$ has vanishing square. In this case, we call ϕ the* Higgs field.

By considerations of types, this is equivalent to the three following conditions:

- $\bar{\partial}^2 = 0$; by the Koszul–Malgrange integrability theorem, $\bar{\partial}$ is then the Dolbeault operator of a holomorphic structure on E_ρ;
- $\bar{\partial}\phi = 0$; saying that ϕ is holomorphic for this structure;
- $\phi \wedge \phi = 0$.

Lemma 2.2. — *If h is a harmonic metric, then $\nabla\phi = \nabla\phi^* = 0$.*

Proof. — Since D is a flat connection,

$$D^2 = (\nabla^2 + \omega \wedge \omega) + \nabla\omega = 0.$$

Both terms in the middle have to vanish since the first one is anti-Hermitian with respect to h, whereas the second is Hermitian. Decomposing $\nabla\omega$ in types gives

$$\partial\phi + (\bar{\partial}\phi + \partial\phi^*) + \bar{\partial}\phi^* = 0$$

and the three terms have to vanish separately. Since $\bar{\partial}\phi = 0$ is one of the conditions for a harmonic metric, the four terms vanish. □

Existence result. If X is a compact Kähler manifold, the existence of a harmonic metric is not very restrictive as shown by the following theorem.

Theorem (Corlette–Donaldson). — *If X is a compact Kähler manifold, the flat bundle E_ρ admits a harmonic metric if and only if the representation ρ is semisimple.*

The harmonic metric is unique up to multiplication by a positive constant on each irreducible flat subbundle of E_ρ.

We emphasize that ϕ and ϕ^* do not depend on the choice of a harmonic metric. This existence result gives one of the two directions needed for the fundamental theorem in non-abelian Hodge theory, that gives a correspondence between flat and Higgs bundles, see e.g. [19] or [13].

2.2. Characteristic classes of flat bundles

Chern classes. Let X be a differentiable manifold and E be a vector bundle over X. The Chern classes of E are the simplest characteristic classes that one can define; they live in the even degree cohomology of X. We briefly recall their topological and geometrical constructions for analogy with the characteristic classes of flat bundles defined below; a basic reference for this material is [15].

The vector bundle E is associated with a $\mathrm{GL}(n,\mathbb{C})$-principal bundle $P \to X$. We denote by $\mathrm{EGL}(n,\mathbb{C}) \to \mathrm{BGL}(n,\mathbb{C})$ the universal $\mathrm{GL}(n,\mathbb{C})$-principal bundle. Then, there exists a map $f : X \to \mathrm{BGL}(n,\mathbb{C})$, unique up to homotopy, such that P is isomorphic to the pullback bundle $f^* \mathrm{EGL}(n,\mathbb{C})$. The cohomology algebra of $\mathrm{BGL}(n,\mathbb{C})$ is given by

$$H^\bullet(\mathrm{BGL}(n,\mathbb{C}),\mathbb{Z}) = \mathbb{Z}[c_1,\dots,c_n],$$

where c_k is of degree $2k$. The Chern classes of E are given by

$$c_k(E) := f^*(c_k) \in H^{2k}(X,\mathbb{Z}).$$

We can also use Chern–Weil theory to construct the Chern classes. Let P_k be the invariant polynomials on the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ such that, for any x in $\mathfrak{gl}(n,\mathbb{C})$,

$$\det\left(I - t\frac{x}{2\pi\,\mathrm{i}}\right) = \sum_{k=0}^{n} P_k(x)t^k.$$

Let ∇ be an arbitrary connection on E. Its curvature F_∇ is a 2-form with values in $\mathfrak{gl}(n,\mathbb{C})$. We define $c_k(E,\nabla) = P_k(F_\nabla)$. This is a closed $2k$-form on X and it is a de Rham representative of $c_k(E)$ in $H^{2k}(X,\mathbb{R})$. In particular, the cohomology class of $c_k(E,\nabla)$ is independent of the connection ∇.

Kamber–Tondeur classes. From now on, we consider the case where $E = E_\rho$ is the flat bundle associated to a representation $\rho : \pi_1(X) \to \mathrm{GL}(n,\mathbb{C})$. The Chern classes of E vanish since one can compute them with the flat connection D, whose curvature F_D vanishes. We can nevertheless define other characteristic classes, both in a topological and geometrical ways. These Kamber–Tondeur classes, also known as Borel classes, first appeared in [12]. They can be considered as the imaginary parts of the Cheeger–Simons differential characters of [7]. The following discussion is inspired from [2, p. 304–306] .

Since E_ρ is a flat bundle, it is associated with a $\mathrm{GL}(n,\mathbb{C})_\delta$ principal bundle $P_\rho \to X$, where $\mathrm{GL}(n,\mathbb{C})_\delta$ is the group $\mathrm{GL}(n,\mathbb{C})$ endowed with the discrete topology. There is a map $f : X \to \mathrm{BGL}(n,\mathbb{C})_\delta$, unique up to homotopy, such that $P_\rho \to X$ is isomorphic to the pullback of the universal bundle $\mathrm{EGL}(n,\mathbb{C})_\delta \to \mathrm{BGL}(n,\mathbb{C})_\delta$. By definition, the cohomology $H^\bullet(\mathrm{BGL}(n,\mathbb{C})_\delta)$ is the cohomology of the discrete group $H^\bullet(\mathrm{GL}(n,\mathbb{C})_\delta)$. One can also consider the *continuous cohomology* $H_c^\bullet(\mathrm{GL}(n,\mathbb{C}))$ computed by using only continous cochains, and there is a forgetful map

$$H_c^\bullet(\mathrm{GL}(n,\mathbb{C})) \to H^\bullet(\mathrm{GL}(n,\mathbb{C})_\delta);$$

we refer to [22] and the references therein. The Kamber–Tondeur classes are in the image of the composite map

$$H_c^\bullet(\mathrm{GL}(n,\mathbb{C})) \to H^\bullet(\mathrm{GL}(n,\mathbb{C})_\delta) = H^\bullet(\mathrm{BGL}(n,\mathbb{C})_\delta) \to H^\bullet(X). \tag{2.2}$$

By the Van Est isomorphism (see e.g. [22, p. 519]), the continuous cohomology $H_c^\bullet(\mathrm{GL}(n,\mathbb{C}))$ is equal to the cohomology of the manifold $\mathrm{U}(n)$. In Subsection 3.2, we will recall that $H^\bullet(\mathrm{U}(n))$ is an exterior algebra on generators $\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}$, where α_{2k+1} is of degree $2k+1$. The Kamber–Tondeur classes $\alpha_{2k+1}(E_\rho)$ can be defined as the image of these generators α_{2k+1} under the map (2.2), up to some normalization constants.

An explicit construction of the Kamber–Tondeur classes can be given by using an auxiliary metric h. Here, we follow closely [9]. Let h be a Hermitian metric on E_ρ and let ω be the 1-form with values in $\mathrm{End}(E_\rho)$ defined by equation (2.1). Then, up to some normalization constants, the Kamber–Tondeur classes $\alpha_{2k+1}(E_\rho)$ are the cohomology classes of the closed forms

$$\alpha_{2k+1}(E_\rho, h) := \mathrm{Tr}(\omega^{2k+1}). \tag{2.3}$$

Characteristic classes and harmonic metrics. We now assume that X is a complex manifold and that the flat bundle E_ρ admits a harmonic metric. The following result first appeared in [17] as a crucial step for proving the Bloch conjecture.

Proposition 2.3. — *If a flat bundle E_ρ over X admits a harmonic metric, then the Kamber–Tondeur classes $\alpha_{2k+1}(E_\rho)$ vanish for $k \geqslant 1$.*

Proof. — We use the harmonic metric h in equation (2.3) to compute de Rham representatives of the Kamber–Tondeur classes. We recall that $\omega = \phi + \phi^*$ and that $\phi \wedge \phi = \phi^* \wedge \phi^* = 0$. Hence, in the wedge product

ω^{2k+1}, only the terms $(\phi\wedge\phi^*)\wedge\ldots(\phi\wedge\phi^*)\wedge\phi$ and $(\phi^*\wedge\phi)\wedge\cdots\wedge(\phi^*\wedge\phi)\wedge\phi^*$ do not vanish. However, their trace vanishes since

$$\begin{aligned}\operatorname{Tr}\left((\phi\wedge\phi^*)\wedge\ldots(\phi\wedge\phi^*)\wedge\phi\right) &= \operatorname{Tr}\left(\phi\wedge(\phi\wedge\phi^*)\wedge\ldots(\phi\wedge\phi^*)\right)\\ &= \operatorname{Tr}\left((\phi\wedge\phi)\wedge\phi^*\wedge\ldots\right) = 0.\end{aligned}$$ ☐

On the other hand, we can use the harmonic metric to define other cohomology classes. This paper deals with these characteristic classes.

Proposition 2.4. — *Let E_ρ be a flat bundle over X that admits a harmonic metric. Then, the differential forms*

$$\beta_{2k,\rho} := \frac{1}{(2\pi\,\mathrm{i})^k}\operatorname{Tr}\left((\phi\wedge\phi^*)^k\right) \tag{2.4}$$

are closed and define real cohomology classes.

Proof. — If ω is a form with values in $\operatorname{End}(E_\rho)$ and ∇ is an arbitrary connection on E_ρ, then

$$\mathrm{d}\operatorname{Tr}(\omega) = \operatorname{Tr}(\nabla\omega).$$

We can use for ∇ the metric connection given in equation (2.1). By Lemma 2.2, we obtain that $d\beta_{2k,\rho}$ is closed.

That these classes are real follows easily from the properties of the trace and the fact that ϕ and ϕ^* are adjoint. ☐

We can compute the behaviour of these forms under direct sum and tensor product of flat bundles:

Proposition 2.5. — *Let E_ρ and $E_{\rho'}$ be flat bundles over X, with harmonic metrics h_{E_ρ} and $h_{E_{\rho'}}$. Then the flat bundles $E_\rho\oplus E_{\rho'}$ (resp. $E_\rho\otimes E_{\rho'}$) carry the harmonic metrics $h_{E_\rho}\oplus h_{E_{\rho'}}$ (resp. $h_{E_\rho}\otimes h_{E_{\rho'}}$).*

Moreover, for these metrics, the following equations hold:

$$\beta_{2k,\rho\oplus\rho'} = \beta_{2k,\rho} + \beta_{2k,\rho'} \tag{2.5}$$

$$\beta_{2k,\rho\otimes\rho'} = \operatorname{rk}(E_{\rho'})\,\beta_{2k,\rho} + \operatorname{rk}(E_\rho)\,\beta_{2k,\rho'}. \tag{2.6}$$

Proof. — The statement about the harmonic metrics is an easy computation, see e.g. [19, p. 18] for the tensor product. The Higgs field of $E_\rho\oplus E_{\rho'}$ is $(\phi_\rho,\phi_{\rho'})$ and its adjoint is $(\phi^*_\rho,\phi^*_{\rho'})$. Equation (2.5) is then straightforward.

The Higgs field ϕ of $E_\rho\oplus E_{\rho'}$ is $\phi_\rho\otimes 1 + 1\oplus\phi_{\rho'}$ and its adjoint ϕ^* is $\phi^*_\rho\otimes 1 + 1\oplus\phi^*_{\rho'}$. Hence, the wedge product $\phi\wedge\phi^*$ is given by

$$\phi\phi^* = \phi_\rho\phi^*_\rho\otimes 1 + 1\otimes\phi_{\rho'}\phi_{\rho'^*} + \phi_\rho\otimes\phi^*_{\rho'} - \phi^*_\rho\otimes\phi'_\rho.$$

By taking the k-th power, we get something of the form

$$(\phi\phi^*)^k = (\phi_\rho\phi^*_\rho)^k\otimes 1 + 1\otimes(\phi_{\rho'}\phi^*_{\rho'})^k + \cdots$$

and one has to show that the sum of the terms in the dots has no trace. This is a tedious computation and we prefer to postpone the proof of equation (2.6) to the end of Section 4.1, where we will use the results of this paper. □

3. Infinite-dimensional period domain

We recall some of the results of [8] that give the relation between flat bundles with harmonic metric and the period domain $\mathcal{D}$. We then focus on the study of its cohomology and give explicit de Rham generators.

3.1. Results of [8]

Period domain. Let $\Lambda\,\mathrm{GL}(n,\mathbb{C})$ be the loop group of $\mathrm{GL}(n,\mathbb{C})$, that is the set of maps from S^1 to $\mathrm{GL}(n,\mathbb{C})$ that satisfy some regularity conditions. The group structure is given by pointwise multiplication.

Remark 3.1. — The precise regularity that we ask is in general irrelevant. We assume that a real number $s > 3/2$ is fixed and, unless otherwise stated, elements in $\Lambda\,\mathrm{GL}(n,\mathbb{C})$ are loops with H^s Sobolev regularity, in particular, these loops are of $\mathcal{C}^1$-class. With this convention, $\Lambda\,\mathrm{GL}(n,\mathbb{C})$ is a complex Hilbert–Lie group.

We denote by $\sigma : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C}), A \mapsto (A^*)^{-1}$ the Cartan involution of $\mathrm{GL}(n,\mathbb{C})$.

Definition 3.2. — *The* twisted loop group *is*

$$\Lambda_\sigma\,\mathrm{GL}(n,\mathbb{C}) := \{\gamma \in \Lambda\,\mathrm{GL}(n,\mathbb{C}) \mid \sigma(\gamma(\lambda)) = \gamma(-\lambda), \lambda \in S^1\}.$$

The period domain *is*

$$\mathcal{D} := \Lambda_\sigma\,\mathrm{GL}(n,\mathbb{C})/\,\mathrm{U}(n),$$

where $\mathrm{U}(n)$ *is identified to the closed subgroup of constant loops with values in* $\mathrm{U}(n)$.

The period domain is a real Hilbert manifold. It has in fact an invariant complex structure J. Indeed, at the base-point, vectors in the tangent space can be decomposed in Fourier series as

$$X(\lambda) = \sum_{n<0} A_n\lambda^n + \sum_{n>0} A_n\lambda^n,$$

with A_n in $\mathfrak{gl}(n,\mathbb{C})$ satisfying $A_{-n} = (-1)^{n+1}A_n^*$. The complex structure J acts on X as

$$(J.X)(\lambda) = i\sum_{n<0} A_n\lambda^n - i\sum_{n>0} A_n\lambda^n.$$

It follows easily from the polar decomposition

$$\mathrm{GL}(n,\mathbb{C}) = \exp(\mathfrak{p}).\,\mathrm{U}(n)$$

that the period domain is diffeomorphic to the space $\Omega_\sigma\,\mathrm{GL}(n,\mathbb{C})$ defined as

$$\Omega_\sigma\,\mathrm{GL}(n,\mathbb{C}) := \{\gamma \in \Lambda_\sigma\,\mathrm{GL}(n,\mathbb{C}) \mid \gamma(1) \in \exp(\mathfrak{p})\}.$$

Period map. We shall explain how these infinite-dimensional spaces are related to flat bundles with a harmonic metric, over a complex manifold X. Given such a bundle E_ρ, we define connections D_λ, for λ in S^1, by

$$D_\lambda := \nabla + \lambda^{-1}\phi + \lambda\phi^*.$$

One checks that D_λ is flat, we denote its monodromy by

$$\rho_\lambda : \pi_1(X) \to \mathrm{GL}(n,\mathbb{C}).$$

This circle of representations can be considered as a single representation with values in $\Lambda\,\mathrm{GL}(n,\mathbb{C})$:

$$\hat\rho : \pi_1(X) \to \Lambda\,\mathrm{GL}(n,\mathbb{C}), \gamma \mapsto (\lambda \mapsto \rho_\lambda(\gamma)).$$

In fact, $\hat\rho$ takes its values in $\Lambda_\sigma\,\mathrm{GL}(n,\mathbb{C})$. We call $\hat\rho$ the *total monodromy* of the harmonic bundle.

By using the flat connection D, we can develop the harmonic metric h and get a ρ-equivariant map $f : \tilde X \to \mathrm{GL}(n,\mathbb{C})/\,\mathrm{U}(n)$, the target space being the space of Hermitian metrics. There is a projection map

$$p : \mathcal{D} = \Lambda_\sigma\,\mathrm{GL}(n,\mathbb{C})/\,\mathrm{U}(n) \to \mathrm{GL}(n,\mathbb{C})/\,\mathrm{U}(n),$$

which is induced from the evaluation map at $1 \in S^1$.

The main theorem of [8] is the following:

Theorem 3.3. — *If E_ρ is a flat bundle with a harmonic metric, one can define* a period map $F : \tilde X \to \mathcal{D}$. *This map is holomorphic and equivariant under the total monodromy of the harmonic bundle. Moreover, F lifts the developing map of the metric f, with respect to the projection map p.*

Of crucial importance for us will be the identification of the differential of the period map. The following proposition is a reformulation of the discussion in paragraph 3.2.4 of [8].

Proposition 3.4. — *Let x be in $\tilde{X}$ and let γ_x be in $\Lambda_\sigma \operatorname{GL}(n,\mathbb{C})$ such that $F(x) = \gamma_x \cdot o$, where o is the base-point in $\mathcal{D}$. We write $\gamma_x^{-1} : T_{F(x)}\mathcal{D} \to T_o\mathcal{D}$ for the isomorphism induced by γ_x^{-1}.*

Using Fourier series in $T_o\mathcal{D}$, the map $\gamma_x^{-1} \circ d_xF$ is of the form

$$\gamma_x^{-1} \circ d_xF = \tilde{\phi}_x\lambda^{-1} + \tilde{\phi}_x^*\lambda,$$

where $\tilde{\phi}_x$ is 1-form on $T_x\tilde{X}$ with values in $\operatorname{End}(\mathbb{C}^n)$ and $\tilde{\phi}_x^*$ is its adjoint with respect to the standard metric on $\mathbb{C}^n$.

Moreover, there is an isometry λ_x from E_x endowed with the harmonic metric to $\mathbb{C}^n$ endowed with the standard metric, such that $\tilde{\phi}_x : T_xX \to \operatorname{End}(\mathbb{C}^n)$ and $\phi_x : T_xX \to \operatorname{End}(E_x)$ are conjugated under λ_x.

Remark 3.5. — We see that the differential of F (translated to the base-point of $\mathcal{D}$) only involves the powers -1 and 1 of λ. This is the property of *horizontality* of the period map.

3.2. Cohomological aspects

Homotopy type. The Lie group $\operatorname{GL}(n,\mathbb{C})$ retracts on its maximal compact subgroup $\operatorname{U}(n)$ by the map $r(\exp(X)k) = k$, where X is in $\mathfrak{p}$ and k is in $\operatorname{U}(n)$. We can use this retraction pointwise on a loop γ in $\Lambda_\sigma \operatorname{GL}(n,\mathbb{C})$ to show that the period domain $\mathcal{D}$ has the homotopy type of $\Lambda_p \operatorname{U}(n)/\operatorname{U}(n) \cong \Omega_p \operatorname{U}(n)$, where the index p, for *periodic*, stands for the loops that satisfy $\gamma(\lambda) = \gamma(-\lambda)$. By reparametrization, the space of periodic loops is homeomorphic to the space of loops, so that:

Lemma 3.6. — *The period domain $\mathcal{D}$ has the homotopy type of $\Omega \operatorname{U}(n)$.*

Since $\pi_1(\operatorname{U}(n)) = \mathbb{Z}$, $\pi_0(\Omega \operatorname{U}(n)) = \mathbb{Z}$: this is well-known for continuous loops and can be proved by approximation arguments for other regularities. We recall from [8] that $\mathcal{D}$ can be seen as an open subset in $\Omega \operatorname{U}(n)$. Under this embedding, $\pi_0(\mathcal{D}) = 2\mathbb{Z}$, since a loop in $\Lambda_\sigma \operatorname{GL}(n,\mathbb{C})$ has an even winding number.

We will only be interested in the connected component of the basepoint in $\mathcal{D}$, that we denote $\mathcal{D}_0$.

Lemma 3.7. — *The space $\mathcal{D}_0$ has the homotopy type of the quotient space $\Lambda_\sigma \operatorname{SL}(n,\mathbb{C})/\operatorname{SU}(n)$ (equivalently of $\Omega \operatorname{SU}(n)$).*

Proof. — Let $j : \operatorname{GL}(n,\mathbb{C}) \to \mathbb{C}^* \times \operatorname{SL}(n,\mathbb{C})$ be the diffeomorphism given by

$$j(A) = (\det A, A \times D_A),$$

where D_A is the diagonal matrix

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & (\det A)^{-1} \end{pmatrix}.$$

A loop γ in $\Lambda\,\mathrm{GL}(n,\mathbb{C})$ is in $\Lambda_\sigma\,\mathrm{GL}(n,\mathbb{C})$ if and only if the loop $j\circ\gamma$ is in $\Lambda_\sigma\mathbb{C}^*\times\Lambda_\sigma\,\mathrm{SL}(n,\mathbb{C})$. The connected component of $\Lambda_\sigma\mathbb{C}^*$ has the homotopy type of the space $\Lambda_0 S^1$ of loops in S^1 with winding number zero. Since this space is contractible, we get that the connected component of $\Lambda_\sigma\,\mathrm{GL}(n,\mathbb{C})$ has the homotopy type of $\Lambda_\sigma\,\mathrm{SL}(n,\mathbb{C})$. We conclude the proof by taking quotients; the homotopy equivalence $\Lambda_\sigma\,\mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)\cong\Omega\,\mathrm{SU}(n)$ follows from Lemma 3.6. □

Cohomology of $\mathrm{SL}(n,\mathbb{C})$**.** Let $\omega=g^{-1}dg$ be the Maurer–Cartan form of $\mathrm{SL}(n,\mathbb{C})$. For $k=1,\dots,n-1$, the forms

$$x_{2k+1}=\frac{1}{(2\pi\,\mathrm{i})^{k+1}}\,\mathrm{Tr}(\omega^{2k+1}) \tag{3.1}$$

are bi-invariant and define *real* cohomology classes in $H^\bullet(\mathrm{SL}(n,\mathbb{C}),\mathbb{R})$. These cohomology classes are in fact rational (see for instance [6, (3.10)]). For future reference, we note that

$$\frac{1}{6}x_3 \text{ is in } H^3(\mathrm{SL}(n,\mathbb{C}),\mathbb{Z})\cong\mathbb{Z} \tag{3.2}$$

and is in fact an integral generator.

It is well-known that

$$H^\bullet(\mathrm{SL}(n,\mathbb{C}),\mathbb{Q})\cong\Lambda[x_3,\dots,x_{2n-1}],$$

where the right hand side is a free exterior algebra over the cohomology classes x_{2k+1}, see e.g. [4, Theorem 8.2], and the references therein.

Path fibration for the period domain. We will need explicit de Rham generators of the cohomology of $\mathcal{D}_0$. By Lemma 3.7, it has the homotopy type of $\Omega\,\mathrm{SU}(n)$, whose cohomology is well-known, see [16, Section 4.11]. We could use this result on $\Omega\,\mathrm{SU}(n)$ to solve our problem. We prefer to compute it directly, by defining an analogue of the path fibration that is used for the loop space $\Omega\,\mathrm{SU}(n)$.

We define the *twisted path space* $P^c_\sigma\,\mathrm{SL}(n,\mathbb{C})$ as

$$P^c_\sigma\mathrm{SL}(n,\mathbb{C}):=\{\gamma:[0,\pi]\to\mathrm{SL}(n,\mathbb{C})\ \text{ s.t. }\gamma\text{ is continuous and }\gamma(0)\in\exp(\mathfrak{p})\}$$

and we denote by $\Omega_\sigma^c \operatorname{SL}(n,\mathbb{C})$ the continuous version of $\Omega_\sigma \operatorname{SL}(n,\mathbb{C})$: that is the space of continuous loops $\gamma : S^1 \to \operatorname{SL}(n,\mathbb{C})$, such that $\sigma(\gamma(\lambda)) = \gamma(-\lambda)$ for any λ in S^1 and $\gamma(0)$ is in $\exp(\mathfrak{p})$.

Let $\chi : P_\sigma^c \operatorname{SL}(n,\mathbb{C}) \to \operatorname{SL}(n,\mathbb{C}), \gamma \mapsto \gamma(0)\cdot\gamma(\pi)$.

Lemma 3.8. — *The map χ is a topological fibration; its fiber over the neutral element is homeomorphic to $\Omega_\sigma^c \operatorname{SL}(n,\mathbb{C})$.*

Proof. — We emphasize that the only regularity condition on loops is the continuity. The fact that χ is a topological fibration works as in the usual setting of untwisted loop groups. The fiber over the neutral elements is made of loops γ such that $\gamma(\pi) = \gamma(0)^{-1} = \sigma(\gamma(0))$. Such a loop defines a loop $\tilde\gamma$ in $\Omega_\sigma^c \operatorname{SL}(n,\mathbb{C})$ by $\tilde\gamma(\mathrm{e}^{\mathrm{i}\,\theta}) = \gamma(\theta)$ if θ is in $[0,\pi]$ and $\tilde\gamma(\mathrm{e}^{\mathrm{i}\,\theta}) = \sigma(\gamma(\theta-\pi))$ if θ is in $[\pi, 2\pi]$. The condition on γ implies that $\tilde\gamma$ is continuous; the map $\gamma \mapsto \tilde\gamma$ is clearly a homeomorphism from the fiber over the neutral element to $\Omega_\sigma^c \operatorname{SL}(n,\mathbb{C})$. □

The topological fibration

$$\begin{tikzcd} \Omega_\sigma^c \operatorname{SL}(n,\mathbb{C}) \arrow[r] & P_\sigma^c \operatorname{SL}(n,\mathbb{C}) \arrow[d, "\chi"] \\ & \operatorname{SL}(n,\mathbb{C}) \end{tikzcd}$$

gives a spectral sequence

$$E_2^{p,q} = H^p(\operatorname{SL}(n,\mathbb{C})) \otimes H^q(\Omega_\sigma^c \operatorname{SL}(n,\mathbb{C})) \Rightarrow H^{p+q}\left(P_\sigma^c(\operatorname{SL}(n,\mathbb{C}))\right).$$

Since $P_\sigma^c(\operatorname{SL}(n,\mathbb{C}))$ is contractible, the spectral sequence abuts to the cohomology of a point. We get the following proposition, as in the untwisted case.

Lemma 3.9. — *The cohomology of $\Omega_\sigma^c \operatorname{SL}(n,\mathbb{C})$ is given by*

$$H^\bullet(\Omega_\sigma^c \operatorname{SL}(n,\mathbb{C}),\mathbb{Q}) \cong \mathbb{Q}[y_2, y_4, \ldots, y_{2n-2}],$$

where y_{2k} is a rational class of degree $2k$ that transgresses x_{2k+1}.

Proof. — The proof of this lemma is an application of the notion of *transgression* that we briefly explain; a formal treatment can be found in [3, Paragraph 5].

Consider a representative of the class x_{2k+1}; we still denote it by x_{2k+1}. Its pullback $\chi^* x_{2k+1}$ is a cocycle, hence a coboundary since $P_\sigma^c \operatorname{SL}(n,\mathbb{C})$ has no cohomology. We write $\chi^* x_{2k+1} = \partial\gamma_{2k}$. Let y_{2k} be the class in $H^{2k}(\Omega_\sigma^c \operatorname{SL}(n,\mathbb{C}))$ of the restriction $\iota^*\gamma_{2k}$, where $\iota : \Omega_\sigma^c \operatorname{SL}(n,\mathbb{C}) \to P_\sigma^c \operatorname{SL}(n,\mathbb{C})$ is the inclusion. We say that y_{2k} *transgresses* (the class) x_{2k+1}.

It is then an exercise using the spectral sequence to show that the classes y_{2k} are generators of the cohomology algebra of $\Omega^c_\sigma(\mathrm{SL}(n,\mathbb{C}))$. □

Lemma 3.10. — *The inclusion* $\Omega_\sigma \mathrm{SL}(n,\mathbb{C}) \to \Omega^c_\sigma \mathrm{SL}(n,\mathbb{C})$ *is a homotopy equivalence.*

Proof. — This is proved in great generality in [21, Theorem 4.6], for untwisted free loops spaces. This paper also deals with based loop spaces in Section 4.3

The interested reader can mimick the proofs for the twisted version. □

Proposition 3.11. — *The cohomology of* $\mathcal{D}_0$ *is given by*

$$H^\bullet(\mathcal{D}_0,\mathbb{Q}) \cong \mathbb{Q}[y_2, y_4, \ldots, y_{2n-2}],$$

where y_{2k} *is a rational class of degree* $2k$ *that* transgresses x_{2k+1}.

Proof. — This immediately follows from the previous lemmas and the homotopy equivalence $\mathcal{D}_0 \cong \Omega_\sigma \mathrm{SL}(n,\mathbb{C})$. □

One has to be careful with the use of the word *transgression* in the proposition since the period domain $\mathcal{D}_0$ appears only up to homotopy equivalence in the twisted path fibration.

3.3. Proof of Theorem 1.2

Our aim is to give explicit de Rham representatives for the classes y_{2k} that transgress the generators x_{2k+1} of the cohomology of $\mathrm{SL}(n,\mathbb{C})$. We proceed in three steps.

Transgressed classes in the twisted path fibration. Let

$$e : [0,\pi] \times P^c_\sigma \mathrm{SL}(n,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$$

be the evaluation map. We consider singular cocycles in $\mathrm{SL}(n,\mathbb{C})$ that represent the cohomology classes $x_3, \ldots, x_{2n-1}$ and we denote them in the same way. On the space $[0,\pi] \times P^c_\sigma \mathrm{SL}(n,\mathbb{C})$, we have the cocycles $e^* x_{2k+1}$ and we define the cochains $\tilde{y}_{2k}$ by

$$\tilde{y}_{2k} := (e^* x_{2k+1}) \backslash [0,\pi] \in C^{2k}(P^c_\sigma \mathrm{SL}(n,\mathbb{C}),\mathbb{Q}). \qquad (3.3)$$

In this formula, the backslash stands for the *slant product* (see e.g. [14, Chapter 13]) which is the analogue of the operation of integration along fibers in differential geometry. We denote by y_{2k} the restriction of $\tilde{y}_{2k}$ on $\Omega^c_\sigma \mathrm{SL}(n,\mathbb{C})$, that is identified to the fiber over the neutral element in the twisted path fibration.

Lemma 3.12. — *The cochains y_{2k} are cocycles and their cohomology classes transgress the classes $-x_{2k+1}$.*

Proof. — Using the discussion on transgression in the proof of Lemma 3.9, we have to compute $\partial\tilde{y}_{2k}$.

$$\begin{aligned}\partial\tilde{y}_{2k} &= \partial\left((e^* x_{2k+1})\backslash[0,\pi]\right)\\ &= -(e^* x_{2k+1})\backslash\partial[0,\pi] \quad \text{since } x_{2k+1} \text{ is closed}\\ &= -(p_\pi^*(x_{2k+1}) - p_0^*(x_{2k+1})),\end{aligned}$$

where p_0 and p_π are the evaluation maps $P_\sigma^c\,\mathrm{SL}(n,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$ at 0 and π.

On the other hand, $\chi^*(x_{2k+1}) = p_\pi^*(x_{2k+1}) + p_0^*(x_{2k+1})$. The map p_0 factorizes through $\exp(\mathfrak{p})$; we write $p_0 = \iota\circ\pi_0$, where $\pi_0 : P_\sigma\,\mathrm{SL}(n,\mathbb{C}) \to \exp(\mathfrak{p})$ and $\iota : \exp(\mathfrak{p}) \hookrightarrow \mathrm{SL}(n,\mathbb{C})$. Since $\exp(\mathfrak{p})$ has no cohomology, there exists a cochain γ_{2k} in $\exp(\mathfrak{p})$ such that $p_0^*(x_{2k+1}) = \pi_0^*(\partial\gamma_{2k})$.

We get that $\partial(\tilde{y}_{2k} - 2\pi_0^*\gamma_{2k}) = \chi^*(-x_{2k+1})$. The restriction of $\tilde{y}_{2k} - 2\pi_0^*\gamma_{2k}$ to the fiber $\Omega_\sigma^c\,\mathrm{SL}(n,\mathbb{C})$ is (cohomologous to) y_{2k}. This proves the lemma. □

Proposition 3.13. — *On $\Omega_\sigma^c\,\mathrm{SL}(n,\mathbb{C})$, the singular cocycle y_{2k} is cohomologous to $-\frac{1}{2}((e_{S^1})^* x_{2k+1})\backslash S^1$, where*

$$e_{S^1} : S^1\times\Omega_\sigma^c\,\mathrm{SL}(n,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$$

is the evaluation map.

Proof. — Loops γ in $\Omega_\sigma^c\,\mathrm{SL}(n,\mathbb{C})$ satisfy $\sigma(\gamma(\lambda)) = \gamma(-\lambda)$ and the path c in $P_\sigma^c\,\mathrm{SL}(n,\mathbb{C})$ that corresponds to γ is given by $\gamma_{|[0,\pi]}$. Since the circle S^1 decomposes as two half-circles, we have to show that both half-circles give the same contribution. On the other hand, $\sigma^* x_{2k+1} = x_{2k+1}$ in cohomology since the inclusion $\mathrm{SU}(n) \to \mathrm{SL}(n,\mathbb{C})$ is a homotopy equivalence. It easily follows that integrating over S^1 is the same as integrating twice over the half-circle parametrized by $[0,\pi]$. □

Transgressed classes in de Rham cohomology. We can eventually work on $\Omega_\sigma\,\mathrm{SL}(n,\mathbb{C})$. We recall that we assume that loops are in a H^s Sobolev space, with $s > \frac{3}{2}$, so that loops are of $\mathcal{C}^1$-class. In particular, the evaluation map e_{S^1} appearing in Proposition 3.13 is of $\mathcal{C}^1$-class and we can give de Rham representatives of the cohomology classes y_{2k}.

Since we are only interested in having generators of the cohomology of $\Omega_\sigma\,\mathrm{SL}(n,\mathbb{C})$, we will not care about the sign of y_{2k}, so that y_{2k} transgresses either x_{2k} or $-x_{2k}$.

Proposition 3.14. — *De Rham representatives for y_{2k} are given by the differential forms*

$$(y_{2k})_\gamma = \left(k+\frac{1}{2}\right)\frac{1}{(2\pi\,\mathrm{i})^{k+1}}\int_{S^1}\mathrm{Tr}\left(\omega(\gamma'(\theta))\omega_\theta^{2k}\right)\mathrm{d}\theta. \tag{3.4}$$

Here, ω_θ is the 1-form on $\Omega_\sigma\,\mathrm{SL}(n,\mathbb{C})$ given by $(\omega_\theta)_\gamma(\delta\gamma) = \gamma^{-1}(\theta)(\delta\gamma)(\theta)$.

Proof. — Let γ be a loop in $\Omega^c_\sigma\,\mathrm{SL}(n,\mathbb{C})$. By Proposition 3.13, de Rham representatives for y_{2k} are given by

$$(y_{2k})_\gamma(\delta_1\gamma,\dots,\delta_{2k}\gamma) = \frac{1}{2}\int_{S^1}((e_{S^1})^*x_{2k+1})_{\gamma(\theta)}(\delta\theta,\delta_1\gamma,\dots,\delta_{2k}\gamma)\mathrm{d}\theta.$$

One computes easily that

$$((e_{S^1})^*x_{2k+1})_{\gamma(\theta)}(\delta\theta,\delta_1\gamma,\dots,\delta_{2k}\gamma) = x_{2k+1}(\gamma'(\theta),\delta_1\gamma(\theta),\dots,\delta_{2k}\gamma(\theta)).$$

Also, observe that by the commutativity property of the trace,

$$\begin{aligned}\mathrm{Tr}&\left(\omega^{2k+1}(\gamma'(\theta),\delta_1\gamma(\theta),\dots,\delta_{2k}\gamma(\theta))\right)\\ &= (2k+1)\,\mathrm{Tr}\left(\omega(\gamma'(\theta))\omega^{2k}(\delta_1\gamma(\theta),\dots,\delta_{2k}\gamma(\theta))\right).\end{aligned}$$

Using the definition of the forms x_{2k+1} (equation (3.1)), we conclude this computation. □

Invariant forms. Our final step is to give other de Rham representatives for the transgressed classes. As forms on $\mathcal{D}_0$, they will be invariant under the action of $\Lambda_\sigma\,\mathrm{GL}(n,\mathbb{C})_0$.

Lemma 3.15. — *The forms $(k+\frac{1}{2})^{-1}y_{2k}$ are cohomologous to the forms z_{2k} given by*

$$\begin{aligned}&(z_{2k})_\gamma(\delta_1\gamma,\dots,\delta_{2k}\gamma)\\ &\quad= \frac{1}{(2\pi\,\mathrm{i})^{k+1}}\cdot\int_{S^1}\sum_\sigma\varepsilon(\sigma)\,\mathrm{Tr}\left(\xi_{\sigma(1)}(\theta)\dots\xi_{\sigma(2k-1)}(\theta)\xi'_{\sigma(2k)}(\theta)\right)\mathrm{d}\theta,\end{aligned} \tag{3.5}$$

where $(\xi_i)_\gamma(\theta) = \gamma^{-1}(\theta)\delta_i\gamma(\theta)$.

Proof. — We define a $(2k-1)$-form μ_{2k-1} by

$$\mu_{2k-1} = \frac{1}{(2\pi\,\mathrm{i})^{k+1}}\int_{S^1}\mathrm{Tr}\left(\omega(\gamma'(\theta))\omega_\theta^{2k-1}\right)\mathrm{d}\theta.$$

We compute its differential:

$$\begin{aligned}
\mathrm{d}\mu_{2k-1} &= \frac{1}{(2\pi\,\mathrm{i})^{k+1}}\int_{S^1}\operatorname{Tr}\left(\mathrm{d}(\gamma^{-1}(\theta)\gamma'(\theta)\omega_\theta^{2k-1})\right)\mathrm{d}\theta\\
&= \frac{1}{(2\pi\,\mathrm{i})^{k+1}}\int_{S^1}\operatorname{Tr}\Big(\left(-\omega_\theta\omega(\gamma'(\theta))+\gamma^{-1}(\theta)\mathrm{d}\gamma'(\theta)\right)\wedge\omega_\theta^{2k-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad -\omega(\gamma'(\theta))\omega_\theta^{2k}\Big)\,\mathrm{d}\theta\\
&= \frac{1}{(2\pi\,\mathrm{i})^{k+1}}\int_{S^1}\operatorname{Tr}\left(\gamma^{-1}(\theta)\mathrm{d}\gamma'(\theta)\wedge\omega_\theta^{2k-1}\right)\mathrm{d}\theta\\
&= \frac{1}{(2\pi\,\mathrm{i})^{k+1}}\int_{S^1}\operatorname{Tr}\left(\left((\gamma^{-1}(\theta)\mathrm{d}\gamma(\theta))'+\omega(\gamma'(\theta))\omega_\theta\right)\wedge\omega_\theta^{2k-1}\right)\mathrm{d}\theta,
\end{aligned}$$

so that $\mathrm{d}\mu_{2k-1} = (k+\frac{1}{2})^{-1}y_{2k} - z_{2k}$. □

Now, we can prove Theorem 1.2 of the introduction.

Proof of Theorem 1.2. — First, we insist that elements in $\mathcal{D}_0$ are not loops themselves and the forms z_{2k} given in the Theorem should for the moment be considered as forms on $\Omega_\sigma\,\mathrm{SL}(n,\mathbb{C})$. Now, we remark that these forms z_{2k} can be defined on $\Lambda_\sigma\,\mathrm{SL}(n,\mathbb{C})$ with the same formula and they are clearly invariant under left multiplication by $\Lambda_\sigma\,\mathrm{SL}(n,\mathbb{C})$. They are also invariant under *right* multiplication by the action of $\mathrm{SU}(n)$: this follows immediately from the commutativity property of the trace and the fact that, since a loop k in $\mathrm{SU}(n)$ is constant, multiplication by k commutes with derivation.

This shows that the forms z_{2k} are well-defined in $\Lambda_\sigma\,\mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)$ by taking *any* γ that lifts it to $\Lambda_\sigma\,\mathrm{SL}(n,\mathbb{C})$. This concludes the proof for the period domain corresponding to the group $\mathrm{SL}(n,\mathbb{C})$. The similar statement for the connected component $\mathcal{D}_0$ of the period domain for $\mathrm{GL}(n,\mathbb{C})$ follows from Lemma 3.7. □

4. Characteristic classes of harmonic bundles

Now that we have described explicitly the cohomology of the period domain, we can compute the pullback of the cohomogy classes by the period map. This will give a proof of Theorem 1.3 and an integrality result as an easy corollary. We then study the special case of variations of Hodge structures and show that, in this case, the integrality result holds on X, and not only on its universal cover $\tilde{X}$.

4.1. Identification of the cohomology classes

Let X be a complex manifold and $\rho : \pi_1(X) \to \mathrm{GL}(n,\mathbb{C})$ be a linear representation. We assume that the associated flat bundle E_ρ carries a harmonic metric h. By Section 3.1, there is a map $F : \tilde{X} \to \mathcal{D}$, which is holomorphic, equivariant under the total monodromy $\hat{\rho}$ and horizontal, see Proposition 3.4 and the remark following it. We can prove Theorem 1.3 of the introduction that we recall:

Theorem. — *The following equality of $2k$-forms holds:*

$$F^*(z_{2k}) = 2(-1)^{k+1}\pi^*\beta_{2k,\rho}.$$

Proof. — Let x be in $\tilde{X}$. By Proposition 3.4, the differential d_xF is given, as a function of θ, by

$$\gamma_x^{-1} \circ d_xF = \tilde{\phi}_x \,\mathrm{e}^{-\mathrm{i}\,\theta} + \tilde{\phi}_x^* \,\mathrm{e}^{\mathrm{i}\,\theta},$$

where $\gamma_x \in \Lambda_\sigma \mathrm{GL}(n,\mathbb{C})$ is such that $F(x) = \gamma(x).o$. Moreover, $\tilde{\phi}_x$ and $\tilde{\phi}_x^*$ are conjugated (by the same endomorphism) to ϕ_x and ϕ_x^*. Since the definition of the forms z_{2k} involves only a trace, we can do the computations as if $\gamma_x^{-1} \circ d_xF$ were equal to $\phi_x \,\mathrm{e}^{-\mathrm{i}\,\theta} + \phi_x^* \,\mathrm{e}^{\mathrm{i}\,\theta}$. We get that

$$\begin{aligned} &F^* z_{2k}(X_1,\ldots,X_{2k}) \\ &= -i\frac{1}{(2\pi\,\mathrm{i})^{k+1}} \cdot \int_{S^1} \mathrm{Tr}\Bigg(\sum_{\sigma\in S_n} \varepsilon(\sigma)(\phi(X_{\sigma(1)})\,\mathrm{e}^{-\mathrm{i}\,\theta} - \phi^*(X_{\sigma(1)})\,\mathrm{e}^{\mathrm{i}\,\theta}) \\ &\qquad \ldots (\phi(X_{\sigma(2k)})\,\mathrm{e}^{-\mathrm{i}\,\theta} - \phi^*(X_{\sigma(2k)})\,\mathrm{e}^{\mathrm{i}\,\theta})\Bigg)\,\mathrm{d}\theta. \end{aligned} \tag{4.1}$$

The factor $-i$ comes from the derivative. In order to compute this expression, we recall that $\phi \wedge \phi = 0$. This implies that a product involving two consecutive terms of the kind $\phi(X_{\sigma(l)})\,\mathrm{e}^{-\mathrm{i}\,\theta}\,\phi(X_{\sigma(l+1)})\,\mathrm{e}^{-\mathrm{i}\,\theta}$ will be cancelled with another product involved in another permutation; of course the same is true with ϕ^*. This gives:

$$\begin{aligned} &F^* z_{2k}(X_1,\ldots,X_{2k}) \\ &= (-1)^{k+1}\frac{1}{(2\pi\,\mathrm{i})^k} \cdot \mathrm{Tr}\Bigg(\sum_{\sigma\in S_n} \varepsilon(\sigma)(\phi(X_{\sigma(1)})\phi^*(X_{\sigma(2)})\ldots\phi^*(X_{\sigma(2k)}) \\ &\qquad - \phi^*(X_{\sigma(1)})\phi(X_{\sigma(2)})\ldots\phi(X_{\sigma(2k)})\Bigg). \end{aligned}$$

By the commutativity property of the trace, the two summands (including the minus sign) are the same, so that

$$F^* z_{2k}(X_1, \dots, X_{2k}) \\ = \frac{2(-1)^{k+1}}{(2\pi\,\mathrm{i})^k} \operatorname{Tr}\left(\sum_{\sigma \in S_n} \varepsilon(\sigma) \phi(X_{\sigma(1)}) \phi^*(X_{\sigma(2)}) \dots \phi^*(X_{\sigma(2k)}) \right).$$

By the definition of $\beta_{2k,\rho}$ (equation (2.4)), we get

$$F^* z_{2k} = 2(-1)^{k+1} \pi^* \beta_{2k,\rho}.$$

That concludes the proof of Theorem 1.3. □

Corollary 4.1. — *There exist universal rational constants q_{2k} such that, on the universal cover $\tilde{X}$:*

$$q_{2k} \pi^* [\beta_{2k,\rho}] \in H^{2k}(\tilde{X}, \mathbb{Z}).$$

Remark 4.2. — Following equation (3.2), one finds that $q_2 = \frac{1}{2}$. The reader who is interested in the other constants should consult page 237 of [5].

Using Theorem 1.3, we can finally prove Proposition 2.5, which deals about the behaviour of the characteristic classes under tensor products.

Proof of Proposition 2.5. — The idea of the proof is to transfer the computation of the characteristic classes of a tensor product of flat bundles to a computation in the cohomology of the linear group.

Let E be a harmonic bundle of rank n and let E' be a harmonic bundle of rank m, over some complex manifold X. For any positive integer i, we write $\mathcal{D}_i$ for the period domain

$$\mathcal{D}_i := \Lambda_\sigma \operatorname{GL}(i, \mathbb{C}) / U(i).$$

We have period maps $F_E : \tilde{X} \to \mathcal{D}_n$ and $F_{E'} : \tilde{X} \to \mathcal{D}_m$. Moreover, since the computation is local, we can assume that $X = \tilde{X}$.

We consider now the period domain $\mathcal{D}_{nm}$. The group $\operatorname{GL}(nm, \mathbb{C})$ can be thought as the group of automorphisms of the vector space $\mathbb{C}^n \otimes \mathbb{C}^m$ so that we get a map $\operatorname{GL}(n, \mathbb{C}) \times \operatorname{GL}(m, \mathbb{C}) \to \operatorname{GL}(nm, \mathbb{C})$. This induces a map of the corresponding twisted loop groups and finally a map of the period domains since $U(n) \times U(m)$ is sent to $U(nm)$. We write

$$j_{\mathcal{D}} : \mathcal{D}_n \times \mathcal{D}_m \to \mathcal{D}_{nm}$$

for this map.

The period map for the harmonic bundle $E \otimes E'$ is given by $j \circ (F_E, F_{E'})$. Let $z_{2k,nm}$ be the cohomology class in $H^{2k}(\mathcal{D}_{nm})$. By Theorem 1.3, if we compute $j_{\mathcal{D}}^* z_{2k,nm}$ in $H^{2k}(\mathcal{D}_n \times \mathcal{D}_m)$, we will obtain a relation between the characteristic classes on $E \otimes E'$ and those on E and E'.

The classes z_{2k} were defined by transgressing classes x_{2k+1} in the corresponding special linear groups. If we denote by

$$j_{SL} : \mathrm{SL}(n, \mathbb{C}) \times \mathrm{SL}(m, \mathbb{C}) \to \mathrm{SL}(nm, \mathbb{C})$$

the natural map between special linear groups, then it is clear that the maps j_{SL} and $j_{\mathcal{D}}$ appear in a commutative diagram that relates the twisted path fibrations of $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SL}(m, \mathbb{C})$ and $\mathrm{SL}(nm, \mathbb{C})$. This proves that the class $j_{\mathcal{D}}^* z_{2k,nm}$ transgresses the class $j_{\mathrm{SL}}^* x_{2k+1,nm}$ in $H^{2k+1}(\mathrm{SL}(n, \mathbb{C}) \times \mathrm{SL}(m, \mathbb{C}))$.

By the following lemma, $j_{\mathrm{SL}}^* x_{2k+1,nm} = m \cdot x_{2k+1,n} + n \cdot x_{2k+1,m}$. This proves that $j_{\mathcal{D}}^* z_{2k,nm}$ and $m \cdot z_{2k,n} + n \cdot z_{2k,m}$ transgress the same class, hence they are equal. This concludes the proof of Proposition 2.5. □

Lemma 4.3. — *The following equality holds:*

$$j_{\mathrm{SL}}^* x_{2k+1,nm} = m \cdot x_{2k+1,n} + n \cdot x_{2k+1,m}.$$

Proof. — We write ω_i the Maurer–Cartan form in $\mathrm{SL}(i, \mathbb{C})$. The pullback of ω_{nm} is given by

$$j^* \omega_{nm} = \omega_n \otimes 1 + 1 \otimes \omega_m.$$

We compute its $(2k+1)$-th power:

$$j^* \omega_{nm}^{2k+1} = \sum_{j=0}^{2k+1} \binom{2k+1}{j} \omega_n^j \otimes \omega_m^{2k+1-j}.$$

Taking the traces:

$$\mathrm{Tr}(j^* \omega_{nm}^{2k+1}) = \sum_{j=0}^{2k+1} \binom{2k+1}{j} \mathrm{Tr}(\omega_n^j) \wedge \mathrm{Tr}(\omega_m^{2k+1-j}).$$

In this sum, except for $j = 0$ or $j = 2k+1$, one of the two factors in the wedge product vanishes since the traces of the even powers of the Maurer–Cartan form vanish. This proves the lemma. □

4.2. Variations of Hodge structures

Definitions. We consider variations of (complex polarized) Hodge structures. It is a classical principle due to Simpson that flat bundles that carry a harmonic metric should be considered as a generalization of a variation of Hodge structures. This has been made more precise by the notions of *twistor structure* [20] and of *loop Hodge structure* [8].

Definition 4.4. — *Let E_ρ be a bundle, with flat connection D, that is endowed with a harmonic metric h. A* variation of Hodge structures *on (E_ρ, D, h) is a smooth vector bundle decomposition*

$$E_\rho = \bigoplus_{k\in\mathbb{Z}} E^k$$

such that:

- *The decomposition is orthogonal with respect to h;*
- *The metric connection ∇, see equation (2.1), preserves the decomposition;*
- *The Higgs field ϕ is a $(1,0)$-form with values in $\bigoplus_k \mathrm{Hom}(E_k, E_{k-1})$;*
- *Its adjoint ϕ^* is a $(0,1)$-form with values in $\bigoplus_k \mathrm{Hom}(E_k, E_{k+1})$.*

We observe that the last property is a consequence of the others. Let us write $\phi = \sum_k \phi_k$ and $\phi^* = \sum_k \phi_k^*$ for the adapted decompositions of the Higgs field and its adjoint.

Curvature computations. The bundle E_k has a holomorphic structure whose Dolbeault operator is $\bar\partial$, the $(0,1)$-part of ∇. The Chern connection of E_k with respect to this holomorphic structure and the harmonic metric is ∇.

Lemma 4.5. — *The Chern curvature of E^k is given by*

$$F(\nabla_{E^k}) = -(\phi_{k-1}^* \wedge \phi_k + \phi_{k+1} \wedge \phi_k^*).$$

Proof. — This is well-known, see e.g. [11]. The equality $D^2 = 0$ gives

$$F(\nabla) + \phi\wedge\phi^* + \phi^*\wedge\phi = 0.$$

We obtain the formula for $F(\nabla_{E^k})$ by restricting everything to E_k. □

Lemma 4.6. — *For any positive integer l,*

$$F(\nabla_{E^k})^l = (-1)^l \left((\phi_{k-1}^* \wedge \phi_k)^l + (\phi_{k+1}\wedge\phi_k^*)^l\right).$$

Proof. — By Lemma 4.5, it is sufficient to remark that

$$(\phi_{k-1}^*\wedge\phi_k)\wedge(\phi_{k+1}\wedge\phi_k^*) = (\phi_{k+1}\wedge\phi_k^*)\wedge(\phi_{k-1}^*\wedge\phi_k) = 0.$$

This follows from $\phi\wedge\phi = \phi^*\wedge\phi^* = 0$. □

Interpretation of the forms $\beta_{2k,\rho}$. We want to show that the forms $\beta_{2k,\rho}$ can be computed using the curvature of a bundle over X.

Definition 4.7. — *If $E_\rho = \bigoplus E_k$ is a variation of Hodge structures over X, we denote by $\mathcal{E}_\rho$ the bundle $\bigoplus E_k^{\oplus k}$.*

Remark 4.8. — A variant of the determinant of $\mathcal{E}_\rho$ is considered in [10], equation (7.13).

Proposition 4.9. — *The Chern character* $\mathrm{ch}(\mathcal{E}_\rho,\nabla)$ *of* $\mathcal{E}_\rho$ *is given by*

$$\mathrm{ch}_l(\mathcal{E}_\rho,\nabla) = -\frac{1}{l!}\beta_{2k}.$$

Proof. — By definition, $\mathrm{ch}_l(\mathcal{E}_\rho,\nabla) = \frac{1}{l!}\left(\frac{\mathrm{i}}{2\pi}\right)^l \mathrm{Tr}(F_\nabla(\mathcal{E}_\rho)^l)$. By Lemma 4.6, we get

$$\begin{aligned}\mathrm{ch}_l(\mathcal{E}_\rho,\nabla) &= \frac{(-1)^l}{l!}\left(\frac{\mathrm{i}}{2\pi}\right)^l \sum_k k\left(\mathrm{Tr}((\phi_{k-1}^*\wedge\phi_k)^l) + \mathrm{Tr}((\phi_{k+1}\wedge\phi_k^*)^l)\right)\\ &= \frac{(-1)^l}{l!}\left(\frac{\mathrm{i}}{2\pi}\right)^l \sum_k k\left(\mathrm{Tr}((\phi_{k-1}^*\wedge\phi_k)^l) - \mathrm{Tr}((\phi_k^*\wedge\phi_{k+1})^l)\right)\\ &= \frac{(-1)^l}{l!}\left(\frac{\mathrm{i}}{2\pi}\right)^l \sum_k \mathrm{Tr}((\phi_{k-1}^*\wedge\phi_k)^l)\\ &= -\frac{1}{l!}\beta_{2k}.\end{aligned}$$

□

We have proved Theorem 1.4 of the introduction:

Theorem. — *If the flat bundle* E_ρ *is endowed with a variation of Hodge structures, then the forms* $\beta_{2k,\rho}$ *have integral cohomology classes.*

BIBLIOGRAPHY

[1] J. Amorós, M. Burger, A. Corlette, D. Kotschick & D. Toledo, *Fundamental groups of compact Kähler manifolds*, Mathematical Surveys and Monographs, vol. 44, American Mathematical Society, 1996, xi+140 pages.

[2] J.-M. Bismut & J. Lott, "Flat vector bundles, direct images and higher real analytic torsion", *J. Am. Math. Soc.* **8** (1995), no. 2, p. 291-363.

[3] A. Borel, "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts", *Ann. Math.* **57** (1953), p. 115-207.

[4] ———, "Sur l'homologie et la cohomologies des groupes de Lie", *Am. J. Math.* **76** (1954), no. 2, p. 273-342.

[5] R. Bott & R. Seeley, "Some Remarks on the Paper of Callias", *Commun. Math. Phys.* **62** (1978), p. 235-245.

[6] J. Cheeger & J. Simons, "Characteristic Forms and Geometric Invariants", *Ann. Math.* **99** (1974), no. 1, p. 48-69.

[7] ———, "Differential characters and geometric invariants", in *Geometry and topology*, Lecture Notes in Mathematics, vol. 1167, Springer, 1985.

[8] J. Daniel, "Loop Hodge structures and harmonic bundles", *Algebr. Geom.* **4** (2017), no. 5, p. 603-643.

[9] S. Goette, “Torsion Invariants for Families”, in *From Probability to Geometry (II). Volume in honor of the 60th birthday of Jean-Michel Bismut*, Astérisque, vol. 328, Société Mathématique de France, 2009, p. 161-206.

[10] P. Griffiths, “Periods of integrals on algebraic manifolds, III (Some global differential-geometric properties of the period mapping)”, *Publ. Math., Inst. Hautes Étud. Sci.* **38** (1970), p. 125-180.

[11] P. Griffiths & W. Schmid, “Locally homogeneous complex manifolds”, *Acta Math.* **123** (1969), p. 253-302.

[12] F. Kamber & P. Tondeur, “Characteristic invariants of foliated bundles”, *Manuscr. Math.* **11** (1974), p. 51-89.

[13] J. Le Potier, “Fibrés de Higgs et systèmes locaux”, in *Séminaire Bourbaki, Vol. 1990/91*, Astérisque, vol. 201-203, Société Mathématique de France, 1990–1991, p. 221-268.

[14] W. S. Massey, *A Basic Course in Algebraic Topology*, Graduate Texts in Mathematics, vol. 127, Springer, 1991, xiv+428 pages.

[15] J. W. Milnor & J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, vol. 76, Princeton University Press, 1974, vii+331 pages.

[16] A. Pressley & G. Segal, *Loop Groups*, Oxford Mathematical Monographs, Oxford Science Publications, 1988, viii+318 pages.

[17] A. Reznikov, “All regulators of flat bundles are torsion”, *Ann. Math.* **141** (1995), no. 2, p. 373-386.

[18] ———, “The structures of Kähler groups, I: Second cohomology”, in *Motives, polylogarithms and Hodge theory. Part II: Hodge theory*, International Press Lecture Series, vol. 3, International Press, 2002, p. 717-730.

[19] C. Simpson, “Higgs bundles and local systems”, *Publ. Math., Inst. Hautes Étud. Sci.* **75** (1992), p. 5-95.

[20] ———, “Mixed twistor structures”, `https://arxiv.org/abs/alg-geom/9705006`, 1997.

[21] A. Stacey, “Constructing smooth manifolds of loop spaces”, *Proc. Lond. Math. Soc.* **99** (2009), no. 1, p. 195-216.

[22] J. D. Stasheff, “Continuous cohomology of groups and classifying spaces”, *Bull. Am. Math. Soc.* **84** (1978), no. 4, p. 513-530.

Manuscrit reçu le 6 septembre 2017,
révisé le 6 février 2018,
accepté le 13 mars 2018.

Jeremy DANIEL
Max Planck Institute
Bonn (Germany)
jdaniel.math@gmail.com