Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Baptiste Devyver

Heat kernel and Riesz transform of Schrödinger operators

Tome 69, n° 2 (2019), p. 457-513.

<http://aif.centre-mersenne.org/item/AIF_2019__69_2_457_0>

Ann. Inst. Fourier, Grenoble
69, 2 (2019) 457-513

HEAT KERNEL AND RIESZ TRANSFORM OF SCHRÖDINGER OPERATORS

by Baptiste DEVYVER (*)

Abstract. — The goal of this article is two-fold: in the first part, we give a purely analytic proof of the Gaussian estimates for the heat kernel of Schrödinger operators $\Delta+\mathcal{V}$ whose potential $\mathcal{V}$ is "small at infinity" in a (weak) integral sense. Our results improve known results that have been proved by probabilistic techniques, and shed light on the hypotheses that are required on the potential $\mathcal{V}$. In a second part, we prove sharp boundedness results for the Riesz transform with potential $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$. A very simple characterization of p-non-parabolicity in terms of lower bounds for the volume growth, which is of independent interest, is also presented.

Résumé. — Le but de cet article est double : dans une première partie, nous donnons une preuve analytique des estimées Gaussiennes pour un opérateur de Schrödinger $\Delta+\mathcal{V}$ dont le potential $\mathcal{V}$ est « petit à l'infini » en un sens (faible) intégral. Nos résultats améliorent des résultats connus, qui avaient été prouvés précédemment par des techniques probabilistes, et éclairent les hypothèses qui doivent être faites sur le potentiel $\mathcal{V}$. Dans une seconde partie, nous prouvons des résultats optimaux concernant l'action de la transformée de Riesz avec potentiel $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ sur les espaces L^p. Une charactérisation particulièrement simple de la non-p-parabolicité en terme de borne inférieure de la croissance du volume, qui a un intérêt en tant que tel, est aussi obtenue.

1. Introduction

Notation. — For two real functions g and h, we will write $g \lesssim h$ if there is a positive constant C such that

$$Cg \leqslant h.$$

Keywords: Heat kernel, Schrödinger operators, Riesz transform, p-non-parabolicity.
2010 *Mathematics Subject Classification:* 35Kxx, 31Exx, 58Jxx.
(*) It is a pleasure to thank Y. Pinchover for interesting discussions, and G. Carron for comments on the manuscript. The research in this work has been carried out while the author was supported by post-doctoral fellowships both at the Technion (Haifa, Israel) and at the University of British Columbia (Vancouver, Canada). He thanks both institution for having provided him with excellent research environment.

We also write $g \sim h$ if there exists a positive constant C such that

$$C^{-1}g \leqslant h \leqslant Cg.$$

Convention. — The Laplacian Δ is taken with the sign convention that makes it non-negative. For example, on $\mathbb{R}^n$ endowed with the Euclidean metric, $\Delta = -\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$.

In this article, we shall be concerned with two related topics: heat kernel estimates, and Riesz transform of Schrödinger operators, both in the Riemannian manifold setting. On the one hand, heat kernel estimates for Schödinger operators have been investigated for quite a long time (see e.g. [24, 25, 29, 51, 52, 53, 54, 55, 56]), both by probabilistic and analytic methods. The main question is to describe the behavior of the heat semigroup $e^{-t(\Delta+\mathcal{V})}$, under assumptions on the potential $\mathcal{V}$. A more restrictive but still important question is to understand which assumptions on the potential ensure Gaussian estimates for the heat kernel of the Schrödinger perator; this corresponds to a heat diffusion that resembles the one of the Laplacian in the Euclidean space. On the other hand, the work of T. Coulhon and X. T. Duong [19] has made it clear that heat kernel estimates have important consequences for the Riesz transform. Loosely speaking, Gaussian upper-estimates estimates for the heat kernel of the (scalar) Laplacian imply boundedness of the Riesz transform $\mathrm{d}\Delta^{-1/2}$ on L^p for $p \in (1,2)$, while Gaussian estimates for the heat kernel of the Hodge Laplacian $\vec{\Delta} = dd^* + d^*d$, acting on 1-forms imply boundedness of the Riesz transform on L^p for $p \in (2,\infty)$ (see [19, 20]). Also, as a consequence of the Bochner formula and domination theory for semi-groups, if $\mathcal{V}(x)$ is the negative part of the smallest eigenvalue of the (symmetric) Ricci curvature operator at $x \in M$, then Gaussian estimates for the Schrödinger operator $\Delta + \mathcal{V}$ imply that the heat kernel of $\vec{\Delta}$ has Gaussian estimates, and thus the Riesz transform is bounded on L^p for $p \in (2,\infty)$ (see [24, 26]). From this last example, we see that understanding the behavior of the heat kernel of a Schrödinger operator is important for the study of the Riesz transform $\mathrm{d}\Delta^{-1/2}$.

The first part of this article deals with heat kernel estimates of Schrödinger operators and is of interest in its own right. We consider a complete, non-compact, (weighted) Riemannian manifold M, endowed with a measure μ, absolutely continuous with respect to the Riemannian measure. The (weighted) Laplacian is defined as $\Delta = \Delta_\mu = -\mathrm{div}(\nabla\cdot)$, where $-\mathrm{div}$ is the formal adjoint of ∇ for the measure μ: more precisely, for every function u

and vector field X, both smooth and compactly supported,

$$-\int_M \operatorname{div}(X)u\,\mathrm{d}\mu = \int_M (X, \nabla u)\,\mathrm{d}\mu.$$

Then, if $\mathcal{V}: M \to \mathbb{R}$ is a potential on M, one can consider the associated Schrödinger-type operator $\Delta + \mathcal{V}$. As we will be concerned with *global* properties (particularly when we will apply our results to the study of the Riesz transform) we will always assume the potential to be *smooth*, in order not to be bothered with local regularity issues and add extra technical difficulties to our results; as will be apparent from the proofs, most of our results actually hold under weaker regularity assumptions on $\mathcal{V}$. The Friedrichs extension provides us with a self-adjoint extension of $\Delta+\mathcal{V}$, which will still be denoted by $\Delta+\mathcal{V}$ for convenience. By the spectral theorem, one can thus look at the associated heat operator $e^{-t(\Delta+\mathcal{V})}$, which turns out to be an integral operator:

$$e^{-t(\Delta+\mathcal{V})}f(x) = \int_M p_t^{\mathcal{V}}(x,y)f(y)\,\mathrm{d}y,\ x \in M,\ f \in C_0^\infty(M),$$

and by elliptic regularity, the heat kernel $p_t^{\mathcal{V}}(x,y)$ is regular. An interesting question to ask is whether $p_t^{\mathcal{V}}$ satisfies the so-called Gaussian estimates, meaning that the heat propagates in a similar fashion as in the Euclidean space for the usual Laplacian. More specifically, one can ask whether Gaussian upper-estimates are satisfied for $p_t^{\mathcal{V}}$:

$$(UE_{\mathcal{V}})\quad |p_t^{\mathcal{V}}(x,y)| \lesssim \frac{1}{V(x,\sqrt{t})}\exp\left(-c\frac{d^2(x,y)}{t}\right),\quad \forall\,(x,y)\in M^2,\ \forall\, t>0,$$

or whether a two-sided Gaussian estimates (or, Li-Yau estimate) holds:

$$(LY_{\mathcal{V}})\quad \frac{1}{V(x,\sqrt{t})}\exp\left(-c_1\frac{d^2(x,y)}{t}\right) \lesssim |p_t^{\mathcal{V}}(x,y)| \lesssim \frac{1}{V(x,\sqrt{t})}\exp\left(-c_2\frac{d^2(x,y)}{t}\right),$$

for every $(x,y)\in M^2$, and $t>0$. Here, $V(x,r)$ denotes the μ-volume of the geodesic ball of center x and radius r. In the special case of the Laplacian itself, that is when $\mathcal{V}=0$, these Gaussian estimates have been completely characterized in terms of functional inequalities by A. Grigoryan and L. Saloff-Coste. In order to recall their results, we first need to introduce some more notations and definitions. The measure μ is called *doubling* if

$$(D)\qquad V(x,2r)\lesssim V(x,r),\ \text{ for } \mu-\text{a.e. } x\in M,\ \forall\, r>0.$$

As a consequence of (D), there exist two exponents $0<\nu'\leqslant\nu$ such that

$$(D_{\nu,\nu'}) \qquad \left(\frac{r}{s}\right)^{\nu'} \lesssim \frac{V(x,r)}{V(x,s)} \lesssim \left(\frac{r}{s}\right)^{\nu},$$

for all $r\geqslant s>0$ and $x\in M$. Let us remark that by the Bishop–Gromov volume comparison theorem, if μ is the Riemannian measure on M and the Ricci curvature on M is non-negative, then one can take $\nu=N$, the topological dimension of M, in $(D_{\nu,\nu'})$. We also introduce the non-collapsing of the volume of balls of radius 1, which may or may not hold on M:

$$(NC) \qquad 1\lesssim V(x,1), \qquad \forall\, x\in M.$$

Thanks to the work of J. Cheeger, M. Gromov and M. Taylor [13], if the Riemann curvature is bounded on M and μ is the Riemannian measure, then (NC) is equivalent to a lower bound of the injectivity radius of M. In other words, if the Riemann curvature is bounded on M, then (NC) is equivalent to M having bounded geometry. Under milder assumptions on M (for example, if μ is the Riemannian measure, Ricci curvature bounded from below is enough), (NC) is equivalent to a local Sobolev inequality, as we shall explain later. This is a much weaker requirement on M, however we shall work in full generality and not assume (NC) in general.

Let us consider the heat semi-group $e^{-t\Delta}$, and its kernel $p_t(x,y)$. Let us introduce on- and off-diagonal estimates for $p_t(x,y)$:

$$(DUE) \qquad p_t(x,x)\lesssim \frac{1}{V(x,\sqrt{t})}, \qquad \forall\, x\in M,\ \forall\, t>0.$$

and

$$(UE) \quad p_t(x,y)\lesssim \frac{1}{V(x,\sqrt{t})}\exp\left(-c\frac{d^2(x,y)}{t}\right), \qquad \forall\,(x,y)\in M^2,\ \forall\, t>0.$$

It is a well-known fact (see e.g. [50]), using the Gaffney–Davies estimates or equivalently, the finite speed propagation for the wave equation, that under (D), (DUE) and (UE) are equivalent. We also consider the two-sided Gaussian estimates (or Li-Yau estimates) for p_t: for all $x,y\in M$ and $t>0$,

$$(LY) \qquad \frac{1}{V(x,\sqrt{t})}e^{-c_1\frac{\mathrm{d}^2(x,y)}{t}} \lesssim p_t(x,y)\lesssim \frac{1}{V(x,\sqrt{t})}e^{-c_2\frac{\mathrm{d}^2(x,y)}{t}}.$$

By the work of A. Grigoryan and L. Saloff-Coste (see e.g. [48, Theorem 5.4.12]), (LY) is equivalent to the conjunction of (D) together with the scaled L^2 Poincaré inequalities for the measure μ:

$$(P) \qquad \int_B |u-u_B|^2\,\mathrm{d}\mu \lesssim r^2\int_M |\nabla u|^2\,\mathrm{d}\mu,$$

for every ball $B = B(x,r)$ in M, and every $u \in C^\infty(B)$; here, $u_B = \frac{1}{\mu(B)}\int_B u\,\mathrm{d}\mu$. Also, it is known from [30] that (UE) together with (D) is equivalent to a family of relative Faber–Krahn inequalities: there exists $\alpha > 0$ such that

$$(RFK) \qquad \frac{1}{r^2}\left(\frac{V(x,r)}{\mu(\Omega)}\right)^\alpha \lesssim \lambda_1(\Omega),$$

for any ball $B(x,r)$ and any relatively compact, open, subset $\Omega \Subset B(x,r)$; here, $\lambda_1(\Omega)$ is the first eigenvalue of $\Delta = \Delta_\mu$ on Ω with Dirichlet boundary conditions.

Another notion that will be important in order to further discuss heat kernels of Schrödinger operators, is the notion of parabolicity of M. There are various equivalent definitions of parabolicity on a manifold; we refer the reader to the survey [31] for proofs and references, and limit ourselves to stating some well-known facts. The manifold M is called *parabolic* if

$$\int_0^\infty p_t(x,y)\,\mathrm{d}t = \infty,$$

for some (or equivalently, all) $x \neq y$ in M. If M is not parabolic, it is called *non-parabolic*. Non-parabolicity of M is equivalent to the existence of a positive, minimal Green function $G(x,y)$ for the Laplacian, which is then defined by the following formula:

$$G(x,y) = \int_0^\infty p_t(x,y)\,\mathrm{d}t, \qquad x \neq y.$$

The positive, minimal Green function can also be constructed by an exhaustion procedure, which we describe now. In the whole paper, we fix $\{\Omega_k\}_{k=0}^\infty$ an *exhaustion* of M, i.e. a sequence of smooth, relatively compact domains of M such that $\Omega_0 \neq \emptyset$, $\bar{\Omega}_k \subset \Omega_{k+1}$ and

$$(1.1) \qquad M = \bigcup_{k=0}^\infty \Omega_k.$$

If we denote by G^{Ω_n} the Green function of Δ on Ω_n with Dirichlet boundary conditions at the boundary, then it is well-known that the sequence $(G^{\Omega_n})_{n\mathbb{N}}$ is increasing, and converges pointwise to G outside of the diagonal $\{(x,x), ; x \in M\}$. Yet another characterization of parabolic manifolds is in terms of a Liouville property: M is parabolic if and only if every non-negative superharmonic function is constant. Lastly, let us mention that M is parabolic (resp. non-parabolic) if and only if the Brownian motion on M is recurrent (resp. transient). As an example, $\mathbb{R}$ and $\mathbb{R}^2$ are parabolic, while $\mathbb{R}^n$, $n \geqslant 3$ is non-parabolic.

A closely related notion to parabolicity exists for a *non-negative* Schrödinger operator. Actually, we shall present this theory in a slightly more general setting, but again, we will only state some facts, and refer the reader to the survey [45] more details. Instead of just the (weighted) Laplacian, we shall consider an operator P of the form

$$P = -\mathrm{div}(A\nabla\cdot) + c, \tag{1.2}$$

where A is a locally elliptic, smooth section of the vector bundle of symmetric endomorphisms of the tangent bundle TM, and c is a smooth, real function on M. The quadratic form q associated to P is defined by

$$q(u) = \int_M \Big((A\nabla u, \nabla u) + cu^2\Big)\,\mathrm{d}\mu, \quad u \in C_0^\infty(M).$$

If we assume that $P+\mathcal{V}$ is non-negative (that is, its quadratic form is non-negative), then we can consider the property of having positive, minimal Green functions for $P+\mathcal{V}$. If such Green functions exist, then the operator $P+\mathcal{V}$ is called *subcritical*. If not, it is called *critical*. In the case of $P=\Delta$, $\mathcal{V}=0$, we recover the definition of parabolicity/non-parabolicity; that is, Δ is subcritical (resp. critical) if and only if M is non-parabolic (resp. parabolic). For the sake of brevity, we will call a potential $\mathcal{V}$ critical (resp. subcritical) if $\Delta+\mathcal{V}$ is a critical (resp. subcritical) operator. As for the case of Δ, for general $P+\mathcal{V}$ there are various equivalent definition of subcriticality/criticality. In terms of heat kernels, the criticality of $P+\mathcal{V}$ is equivalent to

$$\int_0^\infty p_t^{P+\mathcal{V}}(x,y)\,\mathrm{d}t = \infty,$$

for some (or equivalently, all) $x \neq y$ in M, and in the subcritical case the positive, minimal Green function can be obtained by the formula:

$$G_{P+\mathcal{V}}(x,y) = \int_0^\infty p_t^{P+\mathcal{V}}(x,y)\,\mathrm{d}t, \qquad x \neq y.$$

As before, $G_{P+\mathcal{V}}$ can also be obtained as the pointwise limit of the increasing sequence of functions $G^{\Omega_n}_{P+\mathcal{V}}$, where by definition $G^{\Omega_n}_{P+\mathcal{V}}$ is the Green function of $P+\mathcal{V}$ on Ω_n with Dirichlet boundary conditions. The characterization of criticality in terms of Liouville property writes as follows: $P+\mathcal{V}$ is critical if and only if the cone of positive functions u satisfying $(P+\mathcal{V})u \geqslant 0$ in the weak sense is one-dimensional, generated by an element φ; in this case, φ is called the ground state of $P+\mathcal{V}$, and it necessarily satisfies $(P+\mathcal{V})\varphi = 0$. In connection with this Liouville property, note that the celebrated Allegretto–Piepenbrink theorem asserts that $P+\mathcal{V}$ is non-negative if and only if there exists a positive function φ such that

$(P+\mathcal{V})\varphi=0$. A notion that is closely related (but, in general, different) to subcriticality is the notion of *strong subcriticality*, which was introduced by E. B. Davies and B. Simon [25]: the potential $\mathcal{V}$ is called *strongly subcritical* with respect to P, if there exists $\varepsilon\in[0,1)$ such that

$$\varepsilon^{-1}\int_M \mathcal{V}_-u^2 \leqslant q(u)+\mathcal{V}_+u^2,\ \ \forall\, u\in C_0^\infty(M) \tag{1.3}$$

(recall that q denotes the quadratic form of P). In the case $P=\Delta$, if (1.3) holds we will just say that $\mathcal{V}$ is strongly subcritical. This definition gives a quantitative information concerning the posivity of the quadratic form of $P+\mathcal{V}$. For example, it implies that the inequality $P+\mathcal{V}\geqslant(1-\varepsilon)P$ holds at the level of quadratic forms, that is the quadratic form of $P+\mathcal{V}$ is bounded from below by the quadratic form of $(1-\varepsilon)P$. It can be shown that in general, strong subcriticality of $\mathcal{V}$ with respect to P implies that the operator $P+\mathcal{V}$ is subcritical, but the converse is not true in general. However, the converse holds under more restrictive assumption on P and on $\mathcal{V}$. More details will be given later on.

After these definitions, let us come back to heat kernel estimates of Schrödinger operators $\Delta+\mathcal{V}$. Recall that the potential $\mathcal{V}$ is assumed to be smooth for simplicity. In [24], it is shown that if $\mathcal{V}$ is negative and small in an integral sense, then the heat kernel of $\Delta+\mathcal{V}$ has Gaussian estimates. Later, building on ideas of Grigoryan [32], Takeda [53] proved by probabilistic methods that the smallness of $\mathcal{V}$ *at infinity* (in an integral sense) is enough to have the two-sided Gaussian estimates for the heat kernel of $\Delta+\mathcal{V}$. The strongest result available in the literature follows from the combination of works of Chen [15] and Takeda [52, 53]; to explain their results, let us assume that M is a non-parabolic manifold, so that it has a positive minimal Green kernel $G(x,y)$. As follows from Chen and Takeda's work, a two-sided Gaussian estimate for the heat kernel of $\Delta+\mathcal{V}$ holds provided M is a non-parabolic manifold satisfying the scaled L^2-Poincaré inequalities, and the potential $\mathcal{V}$ is regular enough and satisfies the following conditions (warning: we use a terminology and a sign convention for the Laplacian that differ from Takeda's, and assume smoothness of the potential):

(1) $\mathcal{V}$ is *strongly subcritical*, i.e. there exists $\varepsilon\in[0,1)$ such that

$$\varepsilon^{-1}\int_M \mathcal{V}_-u^2 \leqslant \int_M |\nabla u|^2+\mathcal{V}_+u^2,\ \ \forall\, u\in C_0^\infty(M).$$

(2) $\mathcal{V}_-$ is in the *Kato class at infinity* $K^\infty(M)$, i.e.

$$\lim_{R\to\infty}\sup_{x\in M}\int_{M\setminus B(x_0,R)} G(x,y)\mathcal{V}_-(y)\,\mathrm{d}y=0.$$

(3) $\mathcal{V}_+$ is *H-bounded*, i.e.

$$\sup_{x\in M}\int_M G(x,y)\mathcal{V}_+(y)\,\mathrm{d}y<\infty.$$

Here, we have denoted $\mathcal{V}_+ := \max(\mathcal{V},0)$ and $\mathcal{V}_- = \max(-\mathcal{V},0)$. The proof follows an idea introduced by Grigoryan [32] (who first proved the theorem in the case of positive potentials): first, one proves that there exists a positive function h, solution of $(\Delta+\mathcal{V})h=0$, such that, for some constant $C>0$,

$$C^{-1}\leqslant h\leqslant C. \tag{1.4}$$

Then, one uses the function h to perform the *h-transform* $h^{-1}(\Delta+\mathcal{V})h = \Delta_{h^2}$, a weighted Laplacian for which one knows how to estimate the heat kernel. The main point is to prove the estimate (1.4), and this is where the assumptions on $\mathcal{V}$ come into play. However, we feel that Chen and Takeda's arguments, being of probabilistic nature and not easily accessible to analysts, have not been well understood in the analytic community; in particular, the fact that if one is interested only in upper-estimates (and not two-sided estimates) for the heat kernel of $\Delta+\mathcal{V}$, then the Poincaré inequalities are too strong a requirement, has been overlooked (see e.g. [2, 3, 14]). Also, it is not clear from Takeda's argument whether the strong subcriticality assumption on $\mathcal{V}$ is really necessary, or if it can be replaced by the weaker assumption of subcriticality. Subcriticality and strong subcriticality both aim to quantify to which extent the operator $\Delta+\mathcal{V}$ is non-negative, or in other words the negative part of the potential $\mathcal{V}$ is not too big (compared to $\Delta+\mathcal{V}_+$). Recall that strong subcriticality is indeed stronger than subcriticality in general. The distinction between $\mathcal{V}$ strongly subcritical and $\Delta+\mathcal{V}$ subcritical is an important but quite subtle point; in fact, the equivalence of these two conditions for various classes of potentials (which unfortunately do not cover potentials satisfying the hypotheses in Takeda's theorem) has been investigated by several authors (e.g. [15, 52]). Our first result in this article is an elementary analytic proof of an improvement of Takeda's result; one of the features of the proof is that, contrary to Chen and Takeda's arguments, it will make it quite clear why the Kato class at infinity comes into play. Before we present our result, let us introduce a definition: we will say that $\mathcal{V}_-$ satisfies the condition $(K^\infty,1)$ if

$$\lim_{R\to\infty}\sup_{x\in M}\int_{M\setminus B(x_0,R)} G(x,y)\mathcal{V}_-(y)\,\mathrm{d}y<1.$$

This condition is weaker than requiring $\mathcal{V}_-$ to be in the Kato class at infinity. We will show that condition $(K^\infty,1)$ is the natural condition to require on the negative part of the potential, by proving the following result (see Theorem 4.1):

Theorem 1.1. — *Let M be a complete, non-parabolic (weighted) manifold with doubling measure (D), such that the heat kernel corresponding to the Laplacian has Gaussian upper-estimates (UE). Let $\mathcal{V}$ be a smooth, real potential, such that*

(1) *The operator $\Delta + \mathcal{V}$ is subcritical, i.e. $\mathcal{V}$ is subcritical.*
(2) *$\mathcal{V}_+$ is H-bounded.*
(3) *$\mathcal{V}_-$ satisfies the condition $(K^\infty, 1)$.*

Then, the Gaussian upper-estimate for the heat kernel of $\Delta + \mathcal{V}$ holds. If moreover the scaled L^2 Poincaré inequalities (P) are satisfied, then the two-sided Gaussian estimates hold for the heat kernel of $\Delta + \mathcal{V}$.

Thus, the Kato class at infinity can be replaced by the weaker condition $(K^\infty, 1)$. Note also that the subcriticality assumption on $\mathcal{V}$ is weaker than the strong subcriticality assumed by Takeda; in fact, as we have explained before, the relationship between subcriticality and strong subcriticality is often not well understood in the literature. We emphasize that our proof is entirely analytic and quite elementary, and we hope that it will help clarifying the picture in the analytic community. As Takeda, we follow Grigoryan's idea and show the existence of a positive function h, solution of $(\Delta + \mathcal{V})h = 0$, such that (1.4) holds.

As a consequence of Theorem 1.1, we will prove a boundedness result for the Riesz transform on a manifold whose Ricci curvature is "small at infinity" in an integral sense (Corollary 4.3). Let us mention that our approach owes much in spirit to the theory of perturbation of Schrödinger operators developed by Y. Pinchover and M. Murata (see in particular, [39, 40, 41, 42, 44]).

In the second part of the article, we will present some results concerning the Riesz transform with potential $\mathrm{d}(\Delta + \mathcal{V})^{-1/2}$, also in the Riemannian manifold setting. When the potential $\mathcal{V}$ is non-negative and lies in a reverse Hölder class, the operator $\mathrm{d}(\Delta + \mathcal{V})^{1/2}$ has been studied by Shen [49], Auscher and Ben Ali [4], and Badr and Ben Ali [6]. For potentials that can take negative values, Guillarmou and Hassell [35] have proved a sharp boundedness result for $\mathrm{d}(\Delta + \mathcal{V})^{-1/2}$ on an asymptotically conic manifold, if the potential $\mathcal{V}$ decays at rate $O(|x|^{-3})$ as x goes to infinity. More precisely, under these assumptions Guillarmou and Hassell prove that (in absence of zero-modes and zero-resonances and in dimension larger than 3), the operator $\mathrm{d}(\Delta + \mathcal{V})^{-1/2}$ is bounded on L^p *if and only if* $p \in (1, n)$, where n

is the dimension of the manifold. Their proof uses the difficult techniques of the $b-$calculus of Melrose and his coauthors, and relies crucially on the very precise description of the geometry of the manifold at infinity, and on the precise decay rate at infinity of the potential $\mathcal{V}$. Thus, there is no hope to extend their proof to a more general setting. It is however natural to try to extend Guillarmou–Hassell's result to more general situations. In this direction, Assaad [2], and Assaad and Ouhabaz [3] have obtained partial results in a much more general setting. Recall that we denote by $V(x,r)$ the volume of the geodesic ball of center x and radius $r>0$. A consequence of Assaad–Ouhabaz' results is the following:

Theorem 1.2 ([3]). — *Let M be a complete, non-compact manifold with doubling measure (D), and such that the heat kernel corresponding to the Laplacian has Gaussian upper-estimates (UE). Assume that the Riesz tranform $\mathrm{d}\Delta^{-1/2}$ on M is bounded on L^p for all $p\in(2,\delta)$. Let $\mathcal{V}$ be a strongly subcritical potential, such that the heat kernel of $\Delta+\mathcal{V}$ has Gaussian upper-estimates, satisfying the following smallness condition: for some r_1, $r_2>2$,*

$$\int_0^1 \left\|\frac{|V|^{1/2}}{V(\,\cdot\,,\sqrt{s})^{1/r_1}}\right\|_{r_1}\frac{\mathrm{d}s}{\sqrt{s}}+\int_1^\infty \left\|\frac{|V|^{1/2}}{V(\,\cdot\,,\sqrt{s})^{1/r_2}}\right\|_{r_2}\frac{\mathrm{d}s}{\sqrt{s}}<\infty. \tag{1.5}$$

Let $r=\min(r_1,r_2)$. Then, $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p, for all $p\in(2,\min(\delta,r))$.

Assaad and Ouhabaz remark that when the volume is Euclidean, i.e. $V(x,s)\sim s^n$, $s>0$, then (1.5) holds for $r=n$, if the potential satisfies the more familiar condition $\mathcal{V}\in L^{\frac{n}{2}-\varepsilon}$, $\varepsilon>0$. As we shall see, when the volume is not Euclidean, the condition (1.5) is not completely transparent, because there are actually geometric restrictions on the exponent r_2, and hence on r, that can be taken in terms of the $p-$parabolicity of M. More specifically, whatever the potential $\mathcal{V}\neq 0$, r_2 in (1.5) is necessarily less or equal to the so-called parabolic dimension of M. Moreover, contrary to the above-mentioned result by Guillarmou and Hassell, Assaad and Ouhabaz do not prove whether or not the obtained range of boundedness $(2,r)$ for $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is the largest possible even when $\delta=\infty$. One of the main goals of this article is to complement their work: under additional conditions on the potential and the manifold, we show that their result provides the largest interval of boundedness possible. We will also shed light on condition (1.5), by relating it to the geometry of M, and will provide natural, explicit conditions on $\mathcal{V}$ and on M for its validity.

Roughly speaking, our results prove in great generality that the following phenomenon occurs: when one adds a non-zero, smooth, fast decaying potential to the Riesz transform, the range of boundedness shrinks to an extent that is determined by the volume growth of geodesic balls of large radius in M. In particular, the range of boundedness depends only on the *global* geometry of M.

Let us now present our results concerning the Riesz transform with potential with greater details. Like Assaad and Ouhabaz, we will work on manifolds with doubling measure, and such that the heat kernel corresponding to the Laplacian has Gaussian upper-estimates. We first recall some additional notions from potential theory. Let $p \in (1,\infty)$, and introduce the (weighted) p-Laplacian $\Delta_p(u) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$. The notion of p-parabolicity of M has recently proved important to study the boundedness of the Riesz transform, cf. [11, 27]. As its linear ($p=2$) counterpart, it has several equivalent definitions, some of which we now recall (see [21] for more details and references). We say that M is *p-parabolic* if and only if every positive supersolution of Δ_p is constant (Liouville property). Here, by supersolution of Δ_p, we mean a function $u \in W^{1,p}_{loc}(M)$, such that for every non-negative $\psi \in C_0^\infty(M)$,

$$\int_M |\nabla u|^{p-2}(\nabla u, \nabla \psi)\,\mathrm{d}\mu \geqslant 0.$$

An equivalent definition of p-parabolicity is that the p-capacity of every relatively compact, open subset of M is zero. Recall that the *p-capacity* of U is defined as

$$\mathrm{Cap}_p(U) = \inf_u \int_M |\nabla u|^p\,\mathrm{d}\mu,$$

where the infimum is taken over all smooth (or equivalently, Lipschitz) functions u with compact support in M, such that $u \geqslant 1$ on U. If M is not p-parabolic, it is called *p-non-parabolic*. Another characterization of p-non-parabolicity, related to the above capacity characterization, is the existence of a positive function ρ such that the following L^p Hardy-type inequality holds (see [47]):

$$\int_M \rho |u|^p\,\mathrm{d}\mu \leqslant \int_M |\nabla u|^p\,\mathrm{d}\mu, \qquad \forall\, u \in C_0^\infty(M). \tag{1.6}$$

It is well-known that volume growth estimates are related to p-parabolicity: it is shown in [21, Corollary 3.2] that for $p \in (1,\infty)$, a necessary condition for M to be p-non-parabolic is that for some (all) $x \in M$,

$$\int_1^\infty \left(\frac{t}{V(x,t)}\right)^{1/p-1} \mathrm{d}t < \infty. \tag{V_p}$$

It is also known that in general, (V_p) is not sufficient. However, (V_p) is known to be sufficient if in addition M satisfies (D) together with scaled L^p Poincaré inequalities ([36]), or if M has *uniform* volume growth and satisfies L^p pseudo-Poincaré inequalities ([21, Proposition 3.4]). In particular, if the Ricci curvature is non-negative on M, then (V_p) is equivalent to the p-non-parabolicity of M. It is also true that if $p=2$ and M satisfies (D) and (UE), then without further assumptions (V_2) is equivalent to the non-parabolicity of M (see [31, Theorem 11.1]). We now introduce the *parabolic dimension* of M, for which it will be useful to first define a family of local Poincaré inequalities: for every $1 \leqslant p < \infty$, and every $x \in M$, $r > 0$, there exists a constant C_r such that for all $x \in M$ and all $u \in C_0^\infty(B(x,r))$,

$$(P_{loc}) \qquad \int_{B(x,r)} |u - u_{B(x,r)}|^p \,\mathrm{d}\mu \leqslant C_r \int_{B(x,r)} |\nabla u|^p \,\mathrm{d}\mu,$$

where $u_{B(x,r)} = \frac{1}{V(x,r)} \int_{B(x,r)} u(y)\,\mathrm{d}\mu(y)$. As a consequence of [9], the inequalities (P_{loc}) are satisfied for all $1 \leqslant p < \infty$ if μ is the Riemannian measure, and the Ricci curvature is bounded from below on M. Let

$$\mathfrak{I} = \{p \in (1,\infty) \,:\, M \text{ is } p-\text{parabolic}\}.$$

By an observation in [21, p. 1152–1153], if M satisfies (P_{loc}) for all $1 \leqslant p < \infty$, then r-parabolicity implies s-parabolicity for every $r \leqslant s$ (we were not aware of this fact in [27]). In particular, $\mathfrak{I}$ is an interval (which is also closed in all cases known by the author). Following [16], let us define

$$\kappa(M) = \inf \mathfrak{I},$$

the *parabolic dimension* of M (notice that the term “hyperbolic dimension” has been used instead in [27]). Recall the exponents ν and ν' from $(D_{\nu,\nu'})$, then by the fact that (V_p) is necessary for the p-non-parabolicity of M, we see that

$$\kappa \leqslant \nu.$$

We will see later, as a consequence of Theorem 5.1 that under (D), (UE) and (P_{loc}), the inequality $\kappa \geqslant \nu'$ also holds. Let us highlight these two facts as a lemma:

Lemma 1.3. — *Let M satisfying (D), (UE) and (P_{loc}) for all $1 \leqslant p < \infty$. Recall the exponents ν and ν' from $(D_{\nu,\nu'})$, and let κ be the parabolic dimension of M. Then*

$$\nu' \leqslant \kappa \leqslant \nu.$$

One of our results is the following very simple characterization of κ (see Theorem 5.1):

Theorem 1.4. — *Assume that M be a complete, non-compact manifold having doubling measure (D), satisfying the local Poincaré inequalities (P_{loc}) for all $1 \leqslant p < \infty$, and such that the heat kernel corresponding to the Laplacian has Gaussian estimates (UE). Then, the parabolic dimension κ is equal to the supremum of p's such that the following volume lower estimates hold: there is constant C_p such that, for some (all) x_0 in M and all $r \geqslant 1$,*

$$V(x_0, r) \geqslant C_p r^p.$$

We will also characterize the reverse doubling exponent ν' in terms of lower bounds of the p-capacities of geodesic balls (see Theorem 5.6):

Theorem 1.5. — *Assume that M has doubling measure (D) and the heat kernel corresponding to the Laplacian has Gaussian estimates (UE). Let $p_0 \in (2, \infty]$. The following are equivalent:*

(1) *For every $p \in (1, p_0)$, and every $x \in M$, $r > 0$, the p-capacity $\mathrm{Cap}_p(B(x, r))$ of the geodesic ball $B(x, r)$ has the lower estimate:*

$$\mathrm{Cap}_p(B(x, r)) \geqslant C_p \frac{V(x, r)}{r^p}.$$

(2) *For all $p \in (1, p_0)$, and every $x \in M$, $t \geqslant r > 0$, the following reverse doubling volume estimate holds:*

$$\frac{V(x, t)}{V(x, r)} \geqslant C_p \left(\frac{t}{r}\right)^p.$$

These results have independent interest from the rest of the paper, and greatly improve results of Coulhon–Holopainen–Saloff-Coste [21], and Holopainen [36]. In light of Theorem 1.5, we make the following definition:

Definition 1.6. — *Let M be a Riemannian manifold with doubling measure (D) and whose heat kernel corresponding to the Laplacian has Gaussian estimates (UE). Recall the parabolic dimension κ of M. We will say that M is* parabolic regular *if, for every $p < \kappa$,*

$$\mathrm{Cap}_p(B(x, r)) \geqslant C_p \frac{V(x, r)}{r^p}, \qquad \forall\, x \in M, \forall\, r > 0.$$

For example, the Euclidean space is parabolic regular.

Coming back to the Riesz tranform with potential, we will show the following result (see Theorem 7.8):

Theorem 1.7. — *Let M be a complete, non-parabolic manifold having doubling measure (D), satisfying the local Poincaré inequalities (P_{loc}) for all $1 \leqslant p < \infty$, and such that the heat kernel corresponding to the Laplacian* *has Gaussian upper-estimates* (*UE*)*. Assume that M is parabolic regular in the sense of Definition 1.6, that κ, the parabolic dimension of M, is strictly greater than* 2 *and that M is κ-parabolic. Assume that there is* $\delta > 0$ *such that* $\mathrm{d}\Delta^{-1/2}$*, the Riesz transform on M, is bounded on L^p for all* $p \in (1,(1+\delta)\nu)$*. Let $\mathcal{V}$ be a smooth, subcritical potential such that*

$$\mathcal{V} \in L^{\frac{\nu'}{2}-\varepsilon}\left(M, \frac{\mathrm{d}x}{V(x,1)}\right) \cap L^{\frac{\nu}{2}+\varepsilon}\left(M, \frac{\mathrm{d}x}{V(x,1)}\right),$$

for some $\varepsilon > 0$*. Then, the Riesz transform with potential* $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ *is bounded on L^p*, if and only if $p \in (1,\kappa)$.

The main novelty of Theorem 1.7 is the *negative* result, i.e. that $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is *not* bounded on L^p for $p \geqslant \kappa$. There are very few such optimal negative results for Riesz tranforms in the literature. In fact, apart from the previously mentionned result by Guillarmou and Hassell, we are only aware of one such result: using the L^p cohomology and the L^p Hodge projector, and building on earlier results from [12], Carron shows in [11] the following result: if M is a non-parabolic manifold having at least two ends (meaning, roughly speaking, that there are two ways for a continuous path on M to "go to infinity", much like a continuous path of real numbers can either tend to $+\infty$ or to $-\infty$), then the Riesz transform without potential $\mathrm{d}\Delta^{-1/2}$ is unbounded on L^p or $L^{\frac{p}{p-1}}$ provided $p > 2$ and M is p-parabolic. Notice that the recent works [27], [11] demonstrate the relevance of the p-parabolicity in Riesz transform problems. Moreover, concerning the *boundedness* part of Theorem 1.7, we recover and even generalize Guillarmou–Hassell result with a quite elementary proof, that is different from the one of Assaad and Ouhabaz (see the proof of Corollary 7.4).

Let us mention that while this article was being written, Chen, Magniez and Ouhabaz [14] have independently proved an unboundedness result for the Riesz transform with potential, that is significantly weaker than our Theorem 1.7: they essentially prove that *if there exists* a positive, bounded solution h of $(\Delta+\mathcal{V})h = 0$, then the Riesz transform with potential $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ cannot be bounded on L^p, for $p > \nu$. Let us make two remarks concerning their result: first, they do not provide conditions on $\mathcal{V}$ and on M for the existence of such a function h, and limit themselves to quote results of Simon [51] for the case of $\mathbb{R}^n$ with $\mathcal{V} \in L^{\frac{n}{2}\pm\varepsilon}$, and Takeda [52] for positive H-bounded potentials. This allows them to prove that, on $\mathbb{R}^n$, if $\mathcal{V} \in L^{\frac{n}{2}\pm\varepsilon}$ such that $\mathcal{V}_-$ is strongly subcritical, then the Riesz transform with potential $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ cannot be bounded on L^p, for $p > n$. Note that even in this very particular case, contrary our Theorem 1.7, it is not possible to conclude from their result that $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is not bounded on L^n. Secondly, there are examples where the "gap" between our result and their is much more important: in fact, there are examples of manifolds satisfying (D) and (UE), and for which $\kappa<\nu$. In Remark 7.3, we present an example of such a manifold with $\kappa=3$ but $\nu=4$, and that has non-negative Ricci curvature, hence Riesz transform without potential bounded on L^p for all $p\in(1,\infty)$. This example satisfies all the hypotheses of Theorem 1.7, hence, for a non-negative, smooth, compactly supported potential $\mathcal{V}$, we obtain unboundedness of $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ on L^p for $p\geqslant 3$, whereas from the results of [14] one concludes only unboundedness on L^p for $p>4$. In particular, let us stress that the result of Chen, Magniez and Ouhabaz is far from being optimal. This comes from the fact that their argument (which is completely different from ours) is *local*, since they obtain a contradiction to a local Morrey-type inequality of the form:

$$|h(x)-h(x')|\leqslant C\|\mathrm{d}h\|_p,$$

if $p>\nu$ and the distance between x and x' is less than 1. The idea of using a Morrey inequality to show an unboundedness result for the Riesz transform actually goes back to an article of Coulhon and Duong [19], in which such an inequality is employed to show that $\mathrm{d}\Delta^{-1/2}$ is unbounded on L^p for all $p>n$, on the connected sum of several Euclidean spaces. But it is well-known in this case that this approach cannot yield the unboundedness of $\mathrm{d}\Delta^{-1/2}$ on L^n; actually unboundedness on L^n was shown by a completely different argument in [12]. On the contrary, our argument using the p-parabolicity, is *global* in nature, and allows us to get a sharp result. It is also completely new: indeed, it does not use either Morrey-type inequalities, or any of Carron's argument in [11] (in particular, the L^p cohomology), or the b-calculus as in [12] and [35]. Instead, the idea consists in proving the *boundedness* on L^p of the auxiliary operator $\mathrm{d}(h^{-1}(\Delta+\mathcal{V})^{-1/2})$, where h is a positive solution of $(\Delta+\mathcal{V})h=0$ satisfying (1.4). The boundedness of $\mathrm{d}(h^{-1}(\Delta+\mathcal{V})^{-1/2})$, in turn, is proved using properties of h that follows from the (integral) smallness assumption on $\mathcal{V}$ at infinity, and a perturbation result of Coulhon and Dungey [18].

As a by-product of our approach, we can also prove an alternative L^p inequality, in the case where $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is *not* bounded on L^p. More precisely, we get the following result (see Theorem 7.1):

THEOREM 1.8. — *Let M be a complete, non-parabolic manifold, having doubling measure (D), local Poincaré inequalities (P_{loc}) for all $1\leqslant p<\infty$, and such that the heat kernel corresponding to the Laplacian has Gaussian* *upper-estimates (UE). Assume that the Riesz transform* $\mathrm{d}\Delta^{-1/2}$ *on* M *is bounded on* L^p*, for all* $p \in (1,\infty)$*. Let* $\mathcal{V}$ *be a smooth, subcritical potential such that*

$$\mathcal{V} \in L^{\frac{\nu'}{2}-\varepsilon}\left(M, \frac{\mathrm{d}x}{V(x,1)}\right) \cap L^\infty(M),$$

for some $\varepsilon > 0$*. Then, there is a positive function* h *satisfying* (1.4)*, such that* $(\Delta + \mathcal{V})h = 0$ *and such that, for every* $p \in (2,\infty)$*, the following inequality takes place:*

$$\|\mathrm{d}u - u[\mathrm{d}(\log h)]\|_p \leqslant C\|(\Delta+\mathcal{V})^{1/2}u\|_p, \ \ \forall\, u \in C_0^\infty(M). \tag{1.7}$$

To the author's knowledge, such an alternative L^p inequality is completely new, even in the very special case of $\mathbb{R}^n$ with a potential $\mathcal{V}$ decaying at infinity as $r^{-\alpha}$, $\alpha > 2$. It is interesting in that it quantifies precisely to which extent $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ fails to be bounded on L^p, for $p \geqslant \kappa$.

To conclude this introduction, we emphasize again that a key point, used in many of our proofs, is that roughly speaking, if $\mathcal{V}$ is "small at infinity" in an integral sense, then there exists a function h, bounded above and below by positive constants, such that $(\Delta+\mathcal{V})h = 0$. That the existence of such a function h has consequences for the Riesz transform $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is reminiscent of [35], and also to some extend of [12] : indeed, in [12], the unboundedness of the Riesz transform on L^p, $p \geqslant n$ ($p > 2$ if $n = 2$) on the connected sum $\mathbb{R}^n \sharp \mathbb{R}^n$ of two Euclidean spaces relies on the existence of a non-zero harmonic function with gradient in L^2.

The plan of this article is as follows: in Section 2, we introduce the setting. In Section 3, we prove a general perturbation result for positive solutions of a Schrödinger operator. In Section 4, we use the results of Section 3 to prove Theorem 1.1. We discuss some consequences for the Riesz transform $\mathrm{d}\Delta^{-1/2}$. Section 5 is devoted to a characterizations of p-non-parabolicity, based on volume growth, and which is of independent interest (Theorems 1.4 and 1.5). In Section 6, we introduce a natural scale of weighted L^p spaces, to define an appropriate notion of "smallness at infinity" for a potential, in the case where the underlying Riemannian manifold does not satisfy a global Sobolev inequality. Finally, in Section 7, we use the results of Sections 1–6 to prove Theorems 1.7 and 1.8.

2. Preliminaries

2.1. Sobolev inequalities

In order to compare our results with existing results in the literature, and present simpler cases of our theorems, we will sometimes assume that

M satisfies a global Sobolev inequality of dimension ν:

$$(S^\nu) \qquad \|u\|_{\frac{2\nu}{\nu-2}} \lesssim \|\nabla u\|_2.$$

The Sobolev inequality (S^ν) implies the following mapping properties for the operators $\Delta^{-\frac{\alpha}{2}}$, as well as the following Gagliardo–Nirenberg type inequalities (see [23]):

$$(2.1) \qquad \Delta^{-\frac{\alpha}{2}} \,:\, L^p \to L^q,$$

for every $1 < p < \frac{\nu}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\nu}$; and

$$(2.2) \qquad \|u\|_\infty \leqslant C(\nu, r, s)\|\Delta u\|_{r/2}^{\theta}\|u\|_{s/2}^{1-\theta}, \qquad \forall\, u \in C_0^\infty(M),$$

for all $s \geqslant r > \nu$, and $\theta = \frac{\nu/s}{1-(\nu/r)+(\nu/s)}$.

It is well-known that the Sobolev inequality (S^ν) is related to the volume growth. In fact (see [48, Theorem 3.1.5]), if (S^ν) holds, then

$$(2.3) \qquad r^\nu \lesssim V(x, r), \qquad \forall\, x \in M, \forall\, r > 0,$$

which implies that the exponent ν in (S^ν) must be greater or equal to the topological dimension of M. This rules out some interesting manifolds that satisfy (D) and (UE): for example, a complete, non-compact manifold with non-negative Ricci curvature (hence, satisfying (D) and (UE)), satisfies the Sobolev inequality (S^ν) if and only if it has *maximal* volume growth, i.e. $V(x, r) \sim r^\nu$, for all $x \in M$, $r > 0$ (ν being the topological dimension of M). An obvious consequence of (2.3) is that the Sobolev inequality (S^ν) implies the non-collapsing of balls (NC). Under mild geometric assumptions, the non-collapsing of balls (NC) is in fact equivalent to a local Sobolev inequality, as we explain now. Assume that M satisfies the doubling condition (D) for balls of radius less than 1, and the heat kernel estimate (UE) for times less than 1. For example, this is the case if μ is the Riemannian measure, and the Ricci curvature on M is bounded from below. Then (NC) implies the following ultracontractivity estimate for small times:

$$\|e^{-t\Delta}\|_{1,\infty} \lesssim t^{-\frac{\nu}{2}}, \qquad \forall\, t \in (0, 1),$$

where ν is the exponent in $(D_{\nu,\nu'})$. By the work of Varopoulos (see [23, Theorem II.4.2]), this ultracontractivity estimate is equivalent to a local Sobolev inequality:

$$(S^\nu_{loc}) \qquad \|u\|_{\frac{2\nu}{\nu-2}} \lesssim \|\nabla u\|_2 + \|u\|_2, \qquad \forall\, u \in C_0^\infty(M).$$

So, (NC) implies (S^ν_{loc}), where ν is the exponent in $(D_{\nu,\nu'})$, under the assumption that M satisfies the doubling condition (D) for balls of radius less than 1, and the heat kernel estimate (UE) for times less than 1. Conversely, assume that M satisfies (S^{ν}_{loc}). Then, by [48, Theorem 3.2.13],

$$r^{\nu} \lesssim V(x,r), \qquad \forall\, x \in M, \forall\, r \leqslant 1,$$

and in particular (NC) holds. Hence, under (D) (for balls of small radius), (UE) (for small times), the non-collapsing of balls (NC) is equivalent to the local Sobolev inequality (S^{ν}_{loc}). Note that these hypotheses on M are in particular satisfied if μ is the Riemannian measure, and the Ricci curvature of M is bounded from below.

2.2. Criticality and perturbation theory for Schrödinger operators

Let us recall that we have fixed $\{\Omega_k\}_{k=0}^{\infty}$ an *exhaustion* of M, i.e. a sequence of smooth, relatively compact domains of M such that $\Omega_0 \neq \emptyset$, $\bar{\Omega}_k \subset \Omega_{k+1}$ and

$$M = \bigcup_{k=0}^{\infty} \Omega_k.$$

We also denote

$$\Omega_k^* := M \setminus \Omega_k,$$

and we associate to this exhaustion a sequence of smooth cut-off functions $\{\chi_k\}_{k=0}^{\infty}$ such that $\chi_k \equiv 1$ on Ω_k, $\chi_k \equiv 0$ on Ω_{k+1}^*, and $0 \leqslant \chi_k \leqslant 1$ on M.

We consider P a general Schrödinger-type elliptic operator on M in divergence form (1.2), and q its quadratic form. As we have recalled in the introduction, the positivity and criticality theory of these operators is well-established. Let us denote by $\mathcal{C}_P(M)$ the cone of positive (smooth) functions u satisfying $Pu = 0$. Recall that P is defined to be *non-negative* if its associated quadratic form q is non-negative, and by the celebrated Allegretto–Piepenbrink theorem, this is equivalent to $\mathcal{C}_P(M) \neq \emptyset$. If $\mathcal{V}$ is a real potential, we have already defined in the introduction the concepts of criticality/subcriticality of $P + \mathcal{V}$, as well as the concept of strong subcriticality of $\mathcal{V}$ with respect to P. As has been mentioned, in general strong subcriticality implies subcriticality, but the converse is not true. However, under additional assumptions, the two concepts are equivalent; in this respect, we think the following lemma is enlightening:

LEMMA 2.1. — *Assume that M satisfies the Sobolev inequality (S^{ν}), and $\mathcal{V}_-$ belongs to $L^{\frac{\nu}{2}}$. Then, $\mathcal{V}$ is subcritical if and only if it is strongly subcritical (with respect to Δ).*

Proof. — It is shown in [26, Definition 6] that if (S^ν) holds and $\mathcal{V}_- \in L^{\frac{\nu}{2}}$, then $\mathcal{V}_-$ is strongly subcritical if and only if $\mathrm{Ker}_{H_0^1}(\Delta+\mathcal{V}) = \{0\}$, where H_0^1 is the completion of $C_0^\infty(M)$ for the norm $\sqrt{\mathcal{Q}}(u) = \left(\int_M |\nabla u|^2 + \mathcal{V}_+ u^2\right)^{1/2}$. But, if there exists $\varphi \in \mathrm{Ker}_{H_0^1}(\Delta+\mathcal{V})\setminus\{0\}$, then for every sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_0^\infty(M)$, converging to φ in H_0^1,

$$\lim_{n\to\infty} \mathcal{Q}(\varphi_n) - \int_M \mathcal{V}_- \varphi_n^2 = 0. \tag{2.4}$$

Also, the Sobolev inequality implies that $H_0^1 \hookrightarrow L^{\frac{2\nu}{\nu-2}}$, hence

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{\frac{2\nu}{\nu-2}} = 0. \tag{2.5}$$

Then, (2.4) and (2.5) imply that $(\varphi_n)_{n\in\mathbb{N}}$ is a *null-sequence* (see [46, Definition 1.1]), hence $\Delta + \mathcal{V}$ is critical. Therefore, $\mathcal{V}$ is subcritical if and only if $\mathcal{V}_-$ is strongly subcritical. □

The perturbation theory by a potential for Schrödinger-type operators has been the topic of active research over the past 30 years. Various classes of perturbations have been introduced, in order to prove results such as the stability of the Martin boundary, or the (semi-)equivalence of the Green functions. Recall that semi-equivalence of the Green functions means that the Green functions of the operator and its perturbation are equivalent, provided the pole is fixed (but the constants of equivalence may depend on the pole). Actually, both an analytic and a probabilistic approach to the perturbation theory have been developped in parallel. For example, the equivalence of the Green functions of P and of $P+\mathcal{V}$ when both operators are subcritical and $\mathcal{V}$ is a *small perturbation* ($S^\infty(M)$ class in the probabilistic terminology), first proved by analytic means by Y. Pinchover in [41], has been later rediscovered with a probabilistic proof by Z. Q. Chen [15] and M. Takeda [52], building on earlier work by Z. Zhao [57]. Let us also mention the work of A. Ancona [1], who proves the equivalence of the Green functions of two elliptic operators when the difference of their coefficients is small in a quantitative sense. We warn the reader that the terminology of the perturbation classes in the probabilistic community is often different from the ones in the analytic community.

We now introduce some known classes of perturbation, as well as two new ones that are close to some classes introduced by Murata in [39], and that are tailored to the purposes of this article. Since we are not interested in local regularity issues, we recall that the potential $\mathcal{V}$ will be assumed to be smooth, which simplifies the definitions given below (indeed, in the case where $\mathcal{V}$ is a measure one needs to consider local Kato classes as well).

Let P be subcritical of the form (1.2). We say that $\mathcal{V}$ is a *small perturbation* of P if

$$\lim_{k\to\infty}\sup_{x,y\in\Omega_k^*}\int_{\Omega_k^*}\frac{G_P(x,z)|\mathcal{V}(z)|G_P(z,y)}{G_P(x,y)}\mathrm{d}z=0. \tag{2.6}$$

Small perturbations have first been introduced in [41]. We say that $\mathcal{V}$ is a *G-bounded* perturbation of P if

$$\sup_{x,y\in M}\int_M\frac{G_P(x,z)|\mathcal{V}(z)|G_P(z,y)}{G_P(x,y)}\mathrm{d}z<\infty \tag{2.7}$$

The notion of G-bounded perturbation has been introduced in [39]. Of course, if $\mathcal{V}$ is a small perturbation then it is $G-$bounded. Let $h\in\mathcal{C}_P(M)$, i.e. h is a positive solution of $Pu=0$. We say that $\mathcal{V}$ is in the *Kato class at infinity* with respect to (P,h), denoted $K^\infty(M,P,h)$, if

$$\lim_{k\to\infty}\sup_{x\in\Omega_k^*}\int_{\Omega_k^*}\frac{G_P(x,y)|\mathcal{V}(y)|h(y)}{h(x)}\mathrm{d}y=0. \tag{2.8}$$

In the case $P=\Delta$ and $h\equiv\mathbb{1}$, we will simply speak of the Kato class at infinity of M, denoted $K^\infty(M)$. For example, if $P=\Delta$, $M=\mathbb{R}^n$ for $n\geqslant 3$, and for $|x|\geqslant A$,

$$|\mathcal{V}(x)|\leqslant\frac{\varphi(|x|)}{|x|^2},$$

where φ is a non-increasing, continuous function which satisfies

$$\int_A^\infty\frac{\varphi(s)}{s}\,\mathrm{d}s<\infty,$$

then $\mathcal{V}\in K^\infty(\mathbb{R}^n)$ (see [43, Lemma 2.3]).

More generally, we introduce the following new definition: for $\varepsilon>0$, we will say that $\mathcal{V}$ satisfies the condition $(K^\infty,M,P,h,\varepsilon)$ if

$$\lim_{k\to\infty}\sup_{x\in\Omega_k^*}\int_{\Omega_k^*}\frac{G_P(x,y)|\mathcal{V}(y)|h(y)}{h(x)}\mathrm{d}y<\varepsilon. \tag{2.9}$$

When $P=\Delta$ and $h\equiv\mathbb{1}$, we will simply speak of the condition (K^∞,ε). By the Maximum Principle, in (2.6), (2.8) and (2.9), one can replace the supremum over Ω_k^* by the supremum over M (see [39, Lemma 2.1]). Finally, we introduce a notion closely related to the $H-$boundedness introduced by M. Murata in [39]: for a positive solution h of $Pu=0$, we say that $\mathcal{V}$ is $(H,h)-$*bounded* if

$$\sup_{x\in M}\int_{x\in M}\frac{G_P(x,y)|\mathcal{V}(y)|h(y)}{h(x)}\mathrm{d}y<\infty. \tag{2.10}$$

If $\mathcal{V}$ is $(H,h)-$bounded, then we define

$$\|\mathcal{V}\|_{H,h} := \sup_{x\in M}\int_{x\in M}\frac{G_P(x,y)|\mathcal{V}(y)|h(y)}{h(x)}\mathrm{d}y < \infty.$$

Obviously, if $\mathcal{V}$ satisfies condition $K^\infty(M,P,h,\varepsilon)$ for some $\varepsilon>0$, then $\mathcal{V}$ is $(H,h)-$bounded. In particular, if $\mathcal{V}$ is in the Kato class at infinity $K^\infty(M,P,h)$, then $\mathcal{V}$ is $(H,h)-$bounded. Furthermore, it is a direct application of Martin's theory that if $\mathcal{V}$ is a small (resp. $G-$bounded) perturbation of P, then for every $h\in\mathcal{C}_P(M)$, $\mathcal{V}$ is in $K^\infty(M,P,h)$ (resp., $\mathcal{V}$ is $(H,h)-$bounded) (see [39, 40, 41]).

We warn again the reader that equivalent classes may be found in the literature, under different names. To conclude this discussion, let us give a particular but important example of potentials in $K^\infty(M)$:

Example 2.2. — Assume that M satisfies the Sobolev inequality (S^ν), and let $\mathcal{V}\in L^{\frac{n}{2}\pm\varepsilon}$, for some $\varepsilon>0$. Then $\mathcal{V}\in K^\infty(M)$.

Remark 2.3. — This example will be generalized later (in Proposition 6.7) to manifolds which only satisfy (D) and (UE), but not the Sobolev inequality (S^ν).

Proof. — Let

$$u(x) = \int_M G(x,y)|\mathcal{V}(y)|\mathrm{d}y,$$

that is, $u=\Delta^{-1}|\mathcal{V}|$. Then by the fact that $\mathcal{V}\in L^{\frac{n}{2}-\varepsilon}$ and (2.1), there is $s>n$ defined by $\frac{2}{s}=\frac{1}{\frac{n}{2}-\varepsilon}-\frac{2}{n}$, such that $u\in L^{s/2}$. Also, $\Delta u=\mathcal{V}\in L^{r/2}$ with $\frac{r}{2}=\frac{n}{2}+\varepsilon$, therefore by (2.2), we deduce that

$$\|u\|_\infty \leqslant C(n,\varepsilon)\|\mathcal{V}\|^{\theta}_{\frac{n}{2}+\varepsilon}\|u\|^{1-\theta}_{\frac{s}{2}} \leqslant C(n,\varepsilon)\|\mathcal{V}\|^{\theta}_{\frac{n}{2}+\varepsilon}\|\mathcal{V}\|^{1-\theta}_{\frac{n}{2}-\varepsilon}. \tag{2.11}$$

Let $\mathcal{V}_k=\mathcal{V}\chi_k$, then

$$\lim_{k\to\infty}\|\mathcal{V}_k\|_{\frac{n}{2}\pm\varepsilon}=0.$$

Applying (2.11) with $\mathcal{V}_k$ instead of $\mathcal{V}$, and letting $k\to\infty$, we deduce that

$$\lim_{k\to\infty}\sup_{x\in M}\int_{\Omega_k^*}G(x,y)|\mathcal{V}(y)|\mathrm{d}y=0,$$

i.e. $\mathcal{V}$ belongs to $K^\infty(M)$. □

2.3. h-transform

We recall a standard procedure to eliminate the zero-order term of an operator P of the form (1.2). Let $h\in\mathcal{C}_P(M)$, and define a map

$$T_h : v \to hv\,. \tag{2.12}$$

Notice that T_h is an isometry between $L^2(M, h^2 \mathrm{d}\nu)$ and $L^2(M, \mathrm{d}\nu)$. The operator $P_h := T_h^{-1} \circ P \circ T_h$, that is,

$$P_h u = \frac{P(hu)}{h} \tag{2.13}$$

is called the *h-transform* (or *Doob transform*) of P. Notice that

$$P_h \mathbb{1} = 0.$$

Also, it is not hard to see that P_h is explicitly given by

$$P_h u = -\frac{1}{h^2} \mathrm{div}(h^2 A(x) \nabla u), \tag{2.14}$$

and P_h is self-adjoint on $L^2(M, h^2 \mathrm{d}\nu)$. Moreover, P and P_h are unitary equivalent. It is also easy to prove that P_h is subcritical if and only if P is subcritical, and in this case, the corresponding Green function satisfies

$$G_{P_h}(x, y) = \frac{G_P(x, y)}{h(x)h(y)}.$$

On the other hand, in the critical case $\mathbb{1}$ is the ground state of the equation $P_h u = 0$ in M. Finally, notice that the use of the $h-$transform allows to rewrite (2.9), i.e. the condition $(K^\infty, P, h, \varepsilon)$, as

$$\lim_{k \to \infty} \|P_h^{-1} |\mathcal{V}|\|_{L^\infty(\Omega_k^*) \to L^\infty(\Omega_k^*)} < \varepsilon.$$

3. Perturbation result for positive solutions of a Schrödinger operator

In [39], Murata introduced the class of *semi-small perturbations*, and proved that if $\mathcal{V}$ is a semi-small perturbation of a subcritical operator P, then the minimal Martin boundaries of P and $P + \mathcal{V}$ are homeomorphic. By the Martin representation theorem, every positive solution h of $Pu = 0$ in M can be written as

$$h(x) = \int_{\partial_m(M,P)} K(x, \xi) \, \mathrm{d}\nu(\xi),$$

for some probability measure $\mathrm{d}\nu$ on the minimal Martin boundary $\partial_m(M, P)$. Here, $K(x, \xi)$ denotes the Martin kernel. This implies that if $\mathcal{V}$ is a semi-small perturbation of P, then there is a bijection that preserves order from the cones of positive solutions of P, $\mathcal{C}_P(M)$, into $\mathcal{C}_{P+\mathcal{V}}(M)$. In this section, we will be concerned with the following related problem:

Problem 3.1. — *Let P in the form* (1.2) *be subcritical. If h is a positive solution of $Pu=0$ in M, under which conditions on $\mathcal{V}$ does there exist $g\sim h$ such that $(P+\mathcal{V})g=0$?*

In the case $P=\Delta$ and $h\equiv \mathbb{1}$, an answer to Problem 3.1 can be extracted from Takeda's article [53]. Takeda's arguments are of probabilist nature; the main idea goes back to the pioneering work of B. Simon [51], who first proved by a probabilistic argument that on $\mathbb{R}^n$, $n\geqslant 3$, if $\mathcal{V}\in L^{\frac{n}{2}-\varepsilon}\cap L^{\frac{n}{2}+\varepsilon}$ then the existence of $g\sim \mathbb{1}$ solution of $(\Delta+\mathcal{V})u=0$, is equivalent to $\Delta+\mathcal{V}$ being subcritical. Problem 3.1 was also studied by Pinchover, who solved it for small perturbations (see [40, Lemma 2.4], [41, Lemma 1.1]). In this section, we will present an answer to Problem 3.1, more precisely we will give an analytic proof of the following result:

Theorem 3.2. — *Let P be subcritical. Let h be a positive solution of $Pu=0$ in M, and let $\mathcal{V}$ be a potential such that $\mathcal{V}_-$ satisfies $(K^\infty,P,h,1)$ and $\mathcal{V}_+$ is $(H,h)-$bounded. Assume that $P+\mathcal{V}$ is subcritical. Then there exists $g\sim h$, positive solution of $(P+\mathcal{V})u=0$. Furthermore, g satisfies*

$$g(x)=h(x)-\int_M G_P(x,y)\mathcal{V}(y)g(y)\,\mathrm{d}y.$$

Remark 3.3. — Theorem 3.2 applies in particular for potentials $\mathcal{V}$ in the Kato class at infinity $K^\infty(M,P,h)$.

Remark 3.4. — In fact, as the proof will show, the following lower estimate of g holds (compare with Equation (3.2) in [53]):

$$e^{-\|\mathcal{V}_+\|_{H,h}}\leqslant \frac{g}{h}.$$

In [53], in the case $P=\Delta$, $h\equiv\mathbb{1}$ and under the extra assumption that $\mathcal{V}_-$ is strongly subcritical with respect to $\Delta+\mathcal{V}$, an upper-bound, with a probabilistic flavor, for g is given:

$$\frac{g}{h}\leqslant \sup_{x\in M}\mathbf{E}_x\exp\left(-\int_0^\infty \mathcal{V}(B_s)\,\mathrm{d}s\right),$$

where B_s is the Brownian motion on M, and $\mathbf{E}_x$ is the conditional expectation, starting from x.

In the case $P=\Delta$ and $h\equiv 1$, and $\mathcal{V}_-$ strongly subcritical with respect to $\Delta+\mathcal{V}_+$, Theorem 3.2 follows from Theorem 1, Equation (3.2), as well as Lemma 2, in [53]. It was not noticed in [53] that the strong subcriticality of $\mathcal{V}_-$ can be replaced by the weaker assumption that $\Delta+\mathcal{V}$ is subcritical. Also, our assumption that $\mathcal{V}_-$ satisfies $(K^\infty,P,h,1)$ is weaker than the assumption that $\mathcal{V}_-$ belongs to $K^\infty(M,P,h)$. In the case where $\mathcal{V}$ is a small perturbation of P (which is a much stronger condition), Theorem 3.2 follows from [40, Lemma 2.4] and [41, Lemma 2.4].

Let us start with the following lemma, which is essentially well-known (see [44, Lemma 3.3]), but whose proof is provided since it will be instrumental in the proof of Theorem 3.2:

Lemma 3.5. — *Assume that $\mathcal{V}$ is $(H,h)-$bounded and that*

$$\|\mathcal{V}\|_{H,h} < \frac{1}{2}.$$

Then there exists $g \sim h$ solution of $(P+\mathcal{V})u = 0$. Furthermore, if $\mathcal{V}$ is non-positive, then the existence of g is guaranteed as soon as

$$\|\mathcal{V}\|_{H,h} < 1.$$

For the convenience of the reader, we give the proof of Lemma 3.5:

Proof of Lemma 3.5. — let

$$\varepsilon := \sup_{x\in M} \int_M \frac{G_P(x,y)|\mathcal{V}(y)|h(y)}{h(x)} \mathrm{d}y < 1.$$

We want to define g by the formula

$$g = (I + P^{-1}\mathcal{V})^{-1} h.$$

In fact, let us define g by the Neumann series

$$g = \sum_{k=0}^{\infty} (-1)^k (P^{-1}\mathcal{V})^k h.$$

If the series converges, then it is easy to see that g is solution of $(P+\mathcal{V})u = 0$. Define $h_k := (P^{-1}\mathcal{V})^k h$. Then,

$$h_k(x) = \int_M G_P(x,y)\mathcal{V}(y)h_{k-1}(y)\,\mathrm{d}y,$$

and by an easy induction,

$$|h_k| \leqslant \varepsilon^k h.$$

Hence

$$\left(1 - \sum_{k=1}^{\infty} \varepsilon^k\right) h \leqslant g \leqslant \left(\sum_{k=0}^{\infty} \varepsilon^k\right) h,$$

that is

$$\frac{1-2\varepsilon}{1-\varepsilon} h \leqslant g \leqslant \frac{1}{1-\varepsilon} h,$$

hence the result in the general case. In the case where $\mathcal{V}$ is non-positive, then $P^{-1}\mathcal{V} \leqslant 0$, which implies that $h \leqslant g$ from the definition of g. Thus, by the previous computation,

$$h \leqslant g \leqslant \frac{1}{1-\varepsilon} h,$$

hence $g \sim h$ as soon as $\varepsilon < 1$. □

Proof of Theorem 3.2. — We split the proof into two parts.

Step 1: case $\mathcal{V} \geqslant 0$. — Without loss of generality, one can assume that $\mathcal{V} \neq 0$, that is $\|\mathcal{V}\|_{H,h} > 0$. For $t \geqslant 0$, define

$$g_t(x) = h(x) - t \int_M G_{P+t\mathcal{V}}(x,y)\mathcal{V}(y)h(y)\,\mathrm{d}y.$$

We will employ the following lemma:

Lemma 3.6. — *For every $t \geqslant 0$, g_t is a positive solution of $(P+t\mathcal{V})u = 0$. Furthermore, let $0 \leqslant t_0 < t_1 < t_2 < \infty$, and define $\alpha \in (0,1)$ by*

$$t_1 = (1-\alpha)t_0 + \alpha t_2.$$

Then

$$g_{t_1} \leqslant g_{t_0}^{1-\alpha} g_{t_2}^{\alpha}.$$

Assuming for the moment the result of Lemma 3.6, let us finish the proof of Step 1. We apply Lemma 3.6 with $t_0 = 0$, $t_1 = \varepsilon$ and $t_2 = t > \varepsilon$. It yields

$$g_t^{\varepsilon/t} \geqslant g_\varepsilon h^{-1+\varepsilon/t}.$$

Since $\mathcal{V} \geqslant 0$, one has $G_{P+t\mathcal{V}} \leqslant G_P$, therefore, for all $x \in M$,

$$\begin{aligned}\int_M G_{P+t\mathcal{V}}(x,y)\mathcal{V}(y)h(y)\,\mathrm{d}y &\leqslant \int_M G_P(x,y)\mathcal{V}(y)h(y)\,\mathrm{d}y \\ &\leqslant \|\mathcal{V}\|_{H,h}\, h(x).\end{aligned}$$

Consequently, if $C_\varepsilon = 1 - \varepsilon\|\mathcal{V}\|_{H,h}$, one has for $\varepsilon < \|\mathcal{V}\|_{H,h}^{-1}$,

$$C_\varepsilon h \leqslant g_\varepsilon \leqslant h.$$

Thus

$$g_t \geqslant C_\varepsilon^{t/\varepsilon} h = e^{-C'_\varepsilon t} h,$$

where $C'_\varepsilon = -\varepsilon^{-1}\log(1-\varepsilon\|\mathcal{V}\|_{H,h})$. Letting $\varepsilon \to 0$, one gets

$$g_t \geqslant e^{-t\|\mathcal{V}\|_{H,h}} h.$$

Applying this for $t = 1$ and defining $g = g_1$, we find that g is a positive solution of $(P+\mathcal{V})u = 0$ such that

$$e^{-\|\mathcal{V}\|_{H,h}} \leqslant \frac{g}{h} \leqslant 1.$$

This concludes the proof in the case where $\mathcal{V} \geqslant 0$.

Step 2: general case. — Since $\mathcal{V}_-$ satisfies condition $(K^\infty, P, h, 1)$, we can fix $k \in \mathbb{N}$ such that

$$\sup_{x\in M} \int_{\Omega_k^*} \frac{G_P(x,y)\mathcal{V}_-(y)h(y)}{h(x)}\,\mathrm{d}y < 1.$$

Define

$$\mathcal{V}_{-,0} = \mathcal{V}_-\chi_k,\ \mathcal{V}_{-,\infty} = \mathcal{V}_- - \mathcal{V}_{-,0}.$$

Notice that $\mathcal{V}_{-,0}$ has compact support, and that $\|\mathcal{V}_{-,\infty}\|_{H,h} < 1$. By Step 1, there exists $g_1 \sim h$, solution of $(P + t\mathcal{V}_+)u = 0$, and g_1 is given by

$$g_1 = h - (P + \mathcal{V}_+)^{-1}\mathcal{V}_+ h.$$

Since $G_{P+\mathcal{V}_+} \leqslant G_P$,

$$\sup_{x\in M} \int_M \frac{G_{P+\mathcal{V}_+}(x,y)\mathcal{V}_{-,\infty}(y)h(y)}{h(x)}\,\mathrm{d}y < 1.$$

By Lemma 3.5, there exists $g_2 \sim g_1$ solution of $(P + \mathcal{V}_+ - \mathcal{V}_{-,\infty})u = 0$, and g_2 is given by

$$g_2 = (I - (P + \mathcal{V}_+)^{-1}\mathcal{V}_{-,\infty})^{-1} g_1.$$

Define

$$g = g_2 + (P + \mathcal{V})^{-1}\mathcal{V}_{-,0}\, g_2.$$

Obviously, g satisfies $(P + \mathcal{V})g = 0$, and $g_2 \leqslant g$. We are going to show that $g \leqslant Cg_2$, and for this purpose it is clearly enough to show that there is a constant C such that for every $x \in M$,

$$\int_M G_{P+\mathcal{V}}(x,y)\mathcal{V}_{-,0}(y)g_2(y)\,\mathrm{d}y \leqslant Cg_2(x). \tag{3.1}$$

Denote $L = P + \mathcal{V}_+ - \mathcal{V}_{-,\infty}$. By [40], since $\mathcal{V}_{-,0}$ has compact support, there is a constant C such that

$$C^{-1}G_{P+\mathcal{V}} \leqslant G_L \leqslant CG_{P+\mathcal{V}}.$$

Consequently,

$$\int_M G_{P+\mathcal{V}}(x,y)\mathcal{V}_{-,0}(y)g_2(y)\,\mathrm{d}y \leqslant C \int_M G_L(x,y)\mathcal{V}_{-,0}(y)g_2(y)\,\mathrm{d}y.$$

Denote $f(y) = \mathcal{V}_{-,0}(y)g_2(y)$, then f is non-negative and has support included in Ω_k. Notice that $u(x) = \int_M G_L(x,y)f(y)\,\mathrm{d}y$ is a positive solution of $Lv = f$ on M, and

$$u(x) = \lim_{n\to\infty} u_n,$$

with

$$u_n = \int_M G_L^{\Omega_n}(x,y)f(y)\,\mathrm{d}y.$$

Notice that for $n \geqslant k$, u_n is solution of $Lu = 0$ in $\Omega_n \setminus \Omega_k$, and vanishes identically on the boundary of Ω_n. Therefore, by the Maximum Principle, there exists a constant

$$C_n = \frac{\sup_{\partial\Omega_k} g_2}{\inf_{\partial\Omega_k} u_n}$$

such that

$$u_n(x) \leqslant C_n g_2(x), \qquad \forall\, x \in \Omega_n \setminus \Omega_k.$$

Notice that $(u_n)_{n=0}^{\infty}$ is increasing, so that $C_n \leqslant C_k$. Letting $n \to \infty$, one deduces that

$$u(x) \leqslant C_k g_2(x), \qquad \forall\, x \in \Omega_k^*.$$

Since u and g_2 are positive and continuous on Ω_k, one can increase C_k to ensure that

$$u(x) \leqslant C g_2(x), \qquad \forall\, x \in M.$$

Thus, (3.1) holds. This implies that

$$g \sim g_2,$$

and since $g_1 \sim h$ and $g_2 \sim g_1$, one concludes that

$$g \sim h.$$

In order to finish the proof of Theorem 3.2, one has to prove that g satisfies the equation

$$g(x) = h(x) - \int_M G_P(x,y)\mathcal{V}(y)g(y)\,\mathrm{d}y. \tag{3.2}$$

For $n \geqslant N$, denote by h_n the restriction of h to Ω_n, and define $g_{1,n}$, $g_{2,n}$ and g_n by

$$g_{1,n}(x) = h_n(x) - \int_{\Omega_n} G^{\Omega_n}_{P+\mathcal{V}_+}(x,y)\mathcal{V}_+(y)h_n(y)\,\mathrm{d}y,$$
$$g_{2,n} = (I - T_n)^{-1} g_{1,n},$$

where T_n is the operator

$$T_n u(x) = \int_{\Omega_n} G^{\Omega_n}_{P+\mathcal{V}_+}(x,y)\mathcal{V}_{-,\infty}(y)u(y)\,\mathrm{d}y,$$

and

$$g_n(x) = g_{2,n}(x) + \int_{\Omega_n} G^{\Omega_n}_{P+\mathcal{V}}(x,y)\mathcal{V}_{-,0}(y)g_{2,n}(y)\,\mathrm{d}y.$$

It follows from the hypotheses that there is a constant C such that for every $n \geqslant N$ and $i = 1,2$,

$$C^{-1}h \leqslant g_{i,n} \leqslant Ch,$$

and

$$C^{-1}h \leqslant g_n \leqslant Ch.$$

Since $\mathcal{V}$ is (H,h)-bounded, the Dominated Convergence Theorem implies that pointwise,

$$\lim_{n\to\infty} g_{i,n} = g_i,\ i = 1, 2,$$

and

$$\lim_{n\to\infty} g_n = g.$$

Define

$$w_n(x) = h_n(x) - \int_{\Omega_n} G_P^{\Omega_n}(x,y)\mathcal{V}(y)g_n(y)\,\mathrm{d}y,$$

and notice that by the fact that $\mathcal{V}$ is (H,h)-bounded and the Dominated Convergence Theorem, as $n \to \infty$, w_n converges pointwise to

$$w(x) = h(x) - \int_M G_P(x,y)\mathcal{V}(y)g(y)\,\mathrm{d}y.$$

One wishes to show that $w = g$. Notice that for every $n \geqslant N$, w_n and g_n are solutions of the following Dirichlet problem:

$$\begin{cases} Pu = -\mathcal{V}g_n \text{ in } \Omega_n \\ u|_{\partial\Omega_n} = h|_{\partial\Omega_n} \end{cases}$$

The Maximum Principle implies that

$$w_n = g_n.$$

Passing to the limit as $n \to \infty$ gives

$$w = g,$$

that is, (3.2). □

Proof of Lemma 3.6. — We first prove the inequality

$$g_{t_1} \leqslant g_{t_0}^{1-\alpha} g_{t_2}^{\alpha}. \tag{3.3}$$

Let $\{\Omega_n\}_{n=0}^{\infty}$ be an exhaustion of M, and for $t \geqslant 0$, define

$$g_{t,n}(x) = h|_{\Omega_n}(x) - t\int_M G_{P+t\mathcal{V}}^{\Omega_n}(x,y)\mathcal{V}(y)h(y)\,\mathrm{d}y.$$

Since $\{G_{P+t\mathcal{V}}^{\Omega_n}\}_{n\in\mathbb{N}}$ is non-increasing and converges pointwise to $G_{P+t\mathcal{V}}$ as $n \to \infty$, the Monotone Convergence Theorem implies that $\{g_{t,n}\}_{n=0}^{\infty}$

converges pointwise to g_t as $n \to \infty$. Let us remark that $g_{t,n}$ is solution of the following Dirichlet problem in Ω_n:

$$\begin{cases}(P+t\mathcal{V})u = 0 \text{ in } \Omega_n, \\ \qquad u|_{\partial\Omega_n} = h|_{\partial\Omega_n}.\end{cases}$$

Let us consider

$$u_n = g_{n,t_0}^{1-\alpha} g_{n,t_2}^{\alpha}.$$

By an easy computation (see the proof of [42, Theorem 3.1] or [28, Lemma 5.1]), u_n is a supersolution of $(P + t_1\mathcal{V})$ in Ω_n, i.e. $(P + t_1\mathcal{V})u_n \geqslant 0$, and u_n is equal to h on the boundary of Ω_n. Therefore, by the Maximum Principle,

$$u_n \geqslant g_{n,t_1},$$

that is

$$g_{n,t_1} \leqslant g_{n,t_0}^{1-\alpha} g_{n,t_2}^{\alpha}.$$

Letting $n \to \infty$, one finds (3.3). Let us show that (3.3) implies that g_t is positive for all $t > 0$. Take $0 < \varepsilon < \|\mathcal{V}\|_{H,h}^{-1}$, then

$$C_\varepsilon h \leqslant g_\varepsilon \leqslant h,$$

where $C_\varepsilon = 1 - \varepsilon\|\mathcal{V}\|_{H,h} > 0$. By (3.3) with $t_0 = 0$, $t_1 = \varepsilon$ and $t_2 = t$, one has

$$g_\varepsilon \leqslant h^{1-\varepsilon/t} g_t^{\varepsilon/t},$$

therefore

$$C_\varepsilon^{t/\varepsilon} h \leqslant g_t,$$

which implies that g_t is positive. This finishes the proof. □

To conclude this section, we give a simple analytic proof of a result of Takeda [52] and Chen [15]; it provides conditions under which the strong subcriticality of $\mathcal{V}$ with respect to P is equivalent to the subcriticality of $P + \mathcal{V}$:

Theorem 3.7. — *Let P be subcritical of the form* (1.2). *Assume that $\mathcal{V}$ is a potential such that $\mathcal{V}_-$ is a small perturbation of P and $\mathcal{V}_+$ is $G-$bounded. Then, $P+\mathcal{V}$ is subcritical if and only if $\mathcal{V}$ is strongly subcritical with respect to P.*

Proof. — The proof relies on the following:

Lemma 3.8. — *Let $\mathcal{V}_-$ be a small perturbation of P and $\mathcal{V}_+$ is $G-$bounded with respect to P, and assume that $L = P + \mathcal{V}$ is subcritical. Then the Green function G_P of P is equivalent to the Green function G_L of L. That is, there is a positive constant C such that, for every $x \neq y$,*

$$C^{-1}G_L(x,y) \leqslant G(x,y) \leqslant CG_L(x,y).$$

Lemma 3.8 follows from [41, Lemma 2.4] and [42, Corollary 3.6]. Let h be a positive solution of $Pu = 0$. Since $\mathcal{V}_-$ is a small perturbation of P, in particular it is (H, h)-bounded:

$$\sup_{x\in M}\int_M G_P(x,y)\mathcal{V}_-(y)h(y)\,\mathrm{d}y < \infty.$$

By Lemma 3.8, it follows that

$$\sup_{x\in M}\int_M G_L(x,y)\mathcal{V}_-(y)h)(y)\,\mathrm{d}y < \infty.$$

According to Lemma 3.5, for $\varepsilon < \|\mathcal{V}\|_{H,\mathbb{1}}^{-1}$, there is $g \sim h$ (in particular, positive) solution of $(L - \varepsilon\mathcal{V}_-)u = 0$. By the Allegretto–Piepenbrink Theorem, it follows that $L - \varepsilon\mathcal{V}_-$ is non-negative, i.e. the inequality

$$\varepsilon\int_M \mathcal{V}_-u^2 \leqslant q(u) + \mathcal{V}_+u^2 - \mathcal{V}_-u^2, \qquad \forall\, u \in C_0^\infty(M),$$

is satisfied, where q is the quadratic form of P. Equivalently,

$$\int_M \mathcal{V}_-u^2 \leqslant (1+\varepsilon)^{-1}\left\{q(u) + \mathcal{V}_+u^2\right\}, \qquad \forall\, u \in C_0^\infty(M).$$

This shows that $\mathcal{V}_-$ is strongly subcritical. □

4. Heat kernel estimates

In this section, we give consequences of Theorem 3.2 for the heat kernel of $\Delta + \mathcal{V}$. Recall that we denote by $p_t^{\mathcal{V}}$ the heat kernel of $\Delta + \mathcal{V}$. Our main result in this section is the following:

Theorem 4.1. — *Let (M, μ) be a non-parabolic weighted manifold with heat kernel satisfying (D) and (UE). Let $\mathcal{V}$ be a smooth subcritical potential, such that $\mathcal{V}_-$ satisfies the condition $(K^\infty, 1)$ and $\mathcal{V}_+$ is $(H, \mathbb{1})-$bounded. Then the Gaussian upper-estimate $(UE_{\mathcal{V}})$ for the heat kernel of $\Delta + \mathcal{V}$ holds. If moreover the scaled Poincaré inequalities (P) are satisfied, then the full Li-Yau estimates $(LY_{\mathcal{V}})$ hold for the heat kernel of $\Delta + V$.*

Remark 4.2. — In fact, using domination theory, it will be apparent from the proof that $(UE_{\mathcal{V}})$ holds under the following weaker assumption on $\mathcal{V}_+$: it is enough to assume that $\mathcal{V}_+ \geqslant W$ for some $(H, \mathbb{1})-$bounded potential W, such that $W - \mathcal{V}_-$ is subcritical. In the forthcoming paper [17], $(UE_{\mathcal{V}})$ is proved with no assumption on $\mathcal{V}_+$.

In the case where (P) is satisfied, Theorem 4.1 has been proved in [53, Theorem 2] under the stronger assumptions that $\mathcal{V}$ is strongly subcritical instead of subcritical, and that $\mathcal{V}_- \in K^\infty(M)$ instead of satisfying condition $(K^\infty, 1)$. As we have already mentionned for Theorem 3.2, the result of Theorem 4.1 holds under much weaker local regularity hypotheses on $\mathcal{V}$, but we will not pursue this here. Let us also mention that under (P), Theorem 4.1 has first been proved in [32, Theorem 10.5] under the assumption that $\mathcal{V} \geqslant 0$. Also, the idea that Theorem 4.1 follows from Theorem 3.2 originally comes from [32]; the proof uses a very useful device traditionally called the *Doob transform* or *h-transform* with respect to a positive solution h of $(\Delta + \mathcal{V})h = 0$, and that allows one to pass from the Schrödinger operator $\Delta + \mathcal{V}$ to the weighted Laplacian $\Delta_{h^2\mu}$. Recall that the h-transform has been discussed in Section 2.3. In our proof of Theorem 4.1, we will use the same idea, and indeed, besides weakening the hypotheses on the potential $\mathcal{V}$, the only new element in our proof of Theorem 4.1 is that one can treat upper-bounds of the heat kernel *only*, using the caracterization of the Gaussian upper-estimates for a (weighted) Laplacian in terms of relative Faber–Krahn inequalities.

Proof of Theorem 4.1. — Let $P = \Delta + \mathcal{V}$ (recall that Δ is a notation for the weighted Laplacian Δ_μ). By Theorem 3.2, there exists $h \sim 1$ solution of $Pu = 0$. Let us consider the h-transform P_h:

$$P_h = h^{-1}(\Delta + \mathcal{V})h,$$

which is self-adjoint on $L^2(\Omega, h^2\, \mathrm{d}\mu)$. By (2.14), the operator P_h is nothing but the weighted Laplacian $\Delta_{h^2\mu}$. Furthermore, the heat kernel $p_t^h(x, y)$ of P_h on $L^2(\Omega, h^2\, \mathrm{d}\mu)$ is given by

$$p_t^h(x, y) = \frac{p_t^{\mathcal{V}}(x, y)}{h(x)h(y)}.$$

Since $h \sim 1$, it is thus enough to prove the Gaussian upper-estimates (resp. the two-sided Gaussian estimates) for $p_t^h(x, y)$. Let us start by proving the upper-bound. Since (D) and (UE) hold, (M, μ) satisfies the relative Faber–Krahn inequalities (RFK). Since $h \sim 1$, it follows that $(M, h^2\mu)$ also satisfies the relative Faber–Krahn inequality (RFK). Therefore, by [30], we conclude that the heat kernel of $P_h = \Delta_{h^2\mu}$ has Gaussian upper-estimates. The case where (M, μ) satisfies the scaled L^2 Poincaré inequalities (P) follows the same idea: according to the work of Grigoryan and Saloff-Coste (see e.g. [48, Theorem 5.4.12]), the two-sided Gaussian estimates for $p_t^h(x, y)$ are equivalent to doubling together with the scaled L^2 Poincaré inequalities, both for the measure $h^2\mu$. But since $h \sim 1$, these are a direct consequence of the corresponding inequalities (P) and (D) for the measure μ. This concludes the proof of Theorem 4.1. $\square$

Upper-estimates for the heat kernel of Schrödinger operators have consequences for the boundedness of the Riesz transform, as we explain now. Ten years ago, it was discovered by Coulhon and Duong [20] (see also Sikora [50]) that Gaussian estimates for the heat kernel of the Hodge–De Rham Laplacian $\vec{\Delta} = dd^\star + d^\star d$ acting on $1-$forms have consequences for the boundedness on L^p, $p \in (2,\infty)$ of the Riesz transform $\mathrm{d}\Delta^{-1/2}$. More precisely, they show that if M satisfies (D) and (UE), and if the heat kernel of $\vec{\Delta}$ acting on 1-forms has Gaussian estimates, then the Riesz transform $\mathrm{d}\Delta^{-1/2}$ is bounded on L^p, for all $p \in (1,\infty)$. The Bochner formula asserts that $\vec{\Delta}$ acting on $1-$forms can be written as

$$\vec{\Delta} = -\nabla^*\nabla + \mathrm{Ric},$$

where for every $x \in M$, Ric_x is an symmetric endomorphism of the fiber $T^*_x M$, canonically associated with the Ricci curvature. Define a potential $\mathcal{V}$ by the requirement that $\mathcal{V}(x)$ is the lowest eigenvalue of Ric_x. As a consequence of the domination theory, for every $t > 0$ and $x, y \in M$,

$$\|e^{-t\vec{\Delta}}(x,y)\| \leqslant |e^{-t(\Delta+\mathcal{V})}(x,y)|.$$

Therefore, if the heat kernel of $\Delta + \mathcal{V}$ has Gaussian estimates, so does the heat kernel of $\vec{\Delta}$. This has been used in [24] to prove that if M satisfies (D) and (UE), if $\mathcal{V}_-$ is strongly positive with respect to Δ and, if for $\delta > 0$ small enough,

$$\sup_{x\in M} \int_0^\infty \int_M \frac{1}{V(x,\sqrt{t})} e^{-\frac{\mathrm{d}^2(x,y)}{t}} \mathcal{V}_-(y)\,\mathrm{d}y\mathrm{d}t < \delta, \tag{4.1}$$

then $e^{-t\vec{\Delta}}$ has Gaussian estimates and the Riesz transform is bounded on L^p for every $p \in (1,\infty)$. Notice that (4.1) is closely related (see [10, Theorem 2.9]) to the validity of

$$\sup_{x\in M} \int_M G(x,y)\mathcal{V}_-(y)\,\mathrm{d}y < \varepsilon, \tag{4.2}$$

for ε small enough (indeed, if M satisfies the Poincaré inequalities (P), then (4.1) for small δ is equivalent to (4.2) for small ε). Under (P), Takeda [53, Theorem 2] proves a far better result: in order that $e^{-t\vec{\Delta}}$ has Gaussian estimates, it is enough that $\mathcal{V}$ is strongly positive, $\mathcal{V}_-$ is in the Kato class at infinity $K^\infty(M)$, and $\mathcal{V}_+$ is $(H, \mathbb{1})$-bounded. That is, the smallness of $\mathcal{V}$ in an integral sense is required only at infinity, and not globally as in (4.1) or (4.2). As a consequence of Theorem 4.1, we can get rid of the extra hypothesis (P):

Corollary 4.3. — *Let M be a non-parabolic Riemannian manifold, endowed with its Riemannian measure, satisfying (D) and (UE). Let $\mathcal{V}(x)$ be the lowest eigenvalue of Ric_x; assume that $\mathcal{V}$ is subcritical, that $\mathcal{V}_-$ satisfies the condition $(K^\infty, 1)$, and that $\mathcal{V}_+$ is $(H, \mathbb{1})$-bounded. Then the Riesz transform on M is bounded on L^p for every $p \in (1, \infty)$.*

Remark 4.4. — The hypothesis that is needed for $\mathcal{V}_+$ comes from the fact that in order to apply Theorem 4.1, one needs $\mathcal{V}_+$ to be $(H, \mathbb{1})$-bounded. However, as one should expect, it is not necessary; this is proved in the forthcoming paper [17].

Corollary 4.3 greatly improves on [24, Corollary 3.1]. In light of [27, Theorem 4], it is natural to make the following conjecture:

Conjecture 4.5. — *Let M be a non-parabolic Riemannian manifold, endowed with its Riemannian measure satisfying (D) and (UE) such that $\vec{\Delta}$ is strongly positive, and $|\mathrm{Ric}_-|$ satisfies $(K^\infty, 1)$. Then the heat kernel of the Hodge Laplacian has Gaussian estimates, and the Riesz transform is bounded on L^p for every $p \in (1, \infty)$.*

Remark 4.6. — As this article was being written, Conjecture 4.5 has been proved in the forthcoming work [17].

5. A criterion for p-non-parabolicity

As we have already mentioned in the Preliminaries, the p-non-parabolicity of a manifold is tightly related to its volume growth. The aim of this section is to prove the following characterization of p-non-parabolicity in terms of volume growth:

Theorem 5.1. — *Let M be a manifold satisfying (UE) and (D), and let $p_0 \in (2, \infty]$. The following are equivalent:*

(1) *For all $p \in (1, p_0)$, M is p-non-parabolic.*
(2) *For all $p \in (1, p_0)$, and for some (all) point $x_0 \in M$, there is a constant $C = C(x_0, p)$ such that for all $t \geqslant 1$,*

$$V(x_0, t) \geqslant C t^p.$$

Remark 5.2. — Notice that in contrast to [36] or [21, Proposition 3.4], no *global* Poincaré-type inequality is required. Furthermore, the volume growth condition is particularly simple. The drawback is that the above equivalence may not hold at the boundary of the interval, i.e. for p_0 itself. However, in all the examples known by the author, the set of p's such that M is p-non-parabolic is an *open* set, so it does not seem to be too serious a restriction.

Corollary 5.3. — *Let M be a manifold satisfying (UE) and (D), and (P_{loc}), and assume that κ, the parabolic dimension of M, satisfies $\kappa > 2$. Then κ is the supremum of p's having the following property: for some (all) x_0 in M, there is a constant $C = C(p, x_0)$ such that, for all $t \geqslant 1$,*

$$V(x_0, t) \geqslant C(p, x_0) t^p.$$

Before proving Theorem 5.1, we need some preliminary results. For $p \in (1, \infty)$, let us introduce the following volume condition: for some (all) $x \in M$,

$$(\tilde{V}_p) \qquad \int_1^\infty \frac{\mathrm{d}t}{V(x,t)^{1/p}} < \infty.$$

Lemma 5.4. — *Assume that $(\tilde{V}_p)$ holds. Then, $\Delta^{-1/2} : L^p \to L^p_{loc}$ is bounded.*

Proof. — Let us start by some preliminary observations. Write

$$\Delta^{-1/2} = \int_0^1 e^{-t\Delta} \frac{\mathrm{d}t}{\sqrt{t}} + \int_1^\infty e^{-t\Delta} \frac{\mathrm{d}t}{\sqrt{t}} = R + S.$$

Using the fact that $e^{-t\Delta}$ is a contraction semi-group on L^p, $p \in [1, \infty]$, we see that R is bounded on L^p, $p \in [1, \infty]$. Hence, $\Delta^{-1/2} : L^p \to L^p_{loc}$ if and only if $S : L^p \to L^p_{loc}$. We are going to see that S is in fact bounded from L^p to L^∞_{loc}. Introduce the notation: for $x \in M$, and a continuous function $f : M \to \mathbb{R}$, we denote

$$|f|_{L^\infty(x)} := |f(x)|.$$

Thus, $\|S\|_{L^p \to L^\infty(x)} \leqslant C$ means that for every function f in L^p,

$$|Sf(x)| \leqslant C \|f\|_p.$$

We estimate

$$\|S\|_{L^p, L^\infty(x)} \leqslant \int_1^\infty \|e^{-t\Delta}\|_{L^p, L^\infty(x)} \frac{\mathrm{d}t}{\sqrt{t}}.$$

From the Gaussian estimate satisfied by $e^{-t\Delta}$,

$$\|e^{-t\Delta}\|_{L^p, L^\infty(x)} \leqslant \frac{C}{V(x, \sqrt{t})^{1/p}}$$

(this follows by interpolation from the obvious cases $p = 1, \infty$). Therefore,

$$\|S\|_{L^p, L^\infty(x)} \leqslant C \int_1^\infty \frac{\mathrm{d}t}{\sqrt{t} V(x, \sqrt{t})^{1/p}} = C \int_1^\infty \frac{\mathrm{d}t}{V(x,t)^{1/p}}.$$

By hypothesis, the integral $\int_1^\infty \frac{\mathrm{d}t}{V(x,t)^{1/p}}$ converges for some (all) $x \in M$. It is easy to see that the convergence is uniform with respect to x, if x belongs to a fixed compact set. Consequently, $S : L^p \to L^\infty_{loc}$, and therefore

$$\Delta^{-1/2} : L^p \to L^p_{loc}.$$

□

There is also a link between p-non-parabolicity and the fact that $\Delta^{-1/2}$ is bounded from L^p to L^p_{loc}. Indeed, let us recall the following result from [27, Proposition 2.2]:

Proposition 5.5. — *Let $p \in (1, \infty)$ such that M is p-non-parabolic. Assume that the Riesz transform $\mathrm{d}\Delta^{-1/2}$ is bounded on L^p. Then*

$$\Delta^{-1/2} : L^p \to L^p_{loc}$$

is a bounded operator. Conversely, if the Riesz transform is bounded on $L^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, and if

$$\Delta^{-1/2} : L^p \to L^p_{loc}$$

is a bounded operator, then M is p-non-parabolic.

Proof of Theorem 5.1. — Consider the following four assertions:

(i) For all $p \in (1, p_0)$, M is p-non-parabolic.
(ii) For all $p \in (1, p_0)$, (V_p) is satisfied.
(iii) For all $p \in (1, p_0)$, $(\tilde{V}_p)$ is satisfied.
(iv) For all $p \in (1, p_0)$, $\Delta^{-1/2} : L^p \to L^p_{loc}$.

We are going to show the chain of implications

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)},$$

and that (2) is equivalent to (iii), which will prove the theorem. First, by [21, Corollary 3.2], (i) implies (ii).

Let us assume (ii), and let $p \in (1, p_0)$. By Hölder's inequality,

$$\int_1^\infty \frac{\mathrm{d}t}{V(x,t)^{1/p}} \leqslant \left(\int_1^\infty \left(\frac{t}{V(x,t)} \right)^{\frac{q}{p(q-1)}} \right)^{\frac{q-1}{q}} \left(\int_1^\infty t^{-\frac{q}{p}} \mathrm{d}t \right)^{\frac{1}{q}}.$$

If $p < q < p_0$, then the second integral converges. For the first one, define r by

$$\frac{q}{p(q-1)} = \frac{1}{r-1}$$

If q is close enough to p, then $p < r < p_0$. By (ii), (V_r) is satisfied, that is

$$\int_1^\infty \left(\frac{t}{V(x,t)}\right)^{\frac{1}{r-1}} \mathrm{d}u < \infty.$$

Therefore, the integral $\int_1^\infty \frac{\mathrm{d}t}{V(x,t)^{1/p}}$ converges. This shows (ii) $\Rightarrow$ (iii).

The implication (iii) $\Rightarrow$ (iv) is the result of Lemma 5.4.

Assume (iv). Since M satisfies (UE) and (D), by [19] the Riesz transform $\mathrm{d}\Delta^{-1/2}$ is bounded on L^s, $s \in (1,2]$. Hence, by Proposition 5.5, M is p-non-parabolic for all $p \in (2,p_0)$. This shows (iv) $\Rightarrow$ (i).

It remains to show that (2) is equivalent to (iii). Obviously, (2) implies (iii). The converse is elementary, and has been proved in [14]. For the sake of completeness, we reproduce the argument here. Let us assume that $(\tilde{V}_p)$ holds. Let us denote $f(t) = V(x_0,t)^{-1/p}$, then f is non-negative, non-increasing and integrable over $(1,\infty)$. It follows that for every $t > 1$,

$$(t-1)f(t) \leqslant \int_1^t f(u)\,\mathrm{d}u \leqslant \int_1^\infty f(u)\,\mathrm{d}u = C < \infty.$$

Therefore,

$$f(t) \leqslant C(t-1)^{-1},$$

which implies that for every $t \geqslant 2$,

$$V(x_0,t) \geqslant C_p t^p.$$

This shows that (iii) implies (2), and concludes the proof of the theorem. $\square$

In fact, using the same ideas as in the proof of Theorem 5.1, one can also treat *relative* volume estimates:

THEOREM 5.6. — *Let M satisfying (D) and (UE). Let $p_0 \in (2,\infty]$, $r_0 > 0$ and $x_0 \in M$. Consider the following two inequalities, where C_p is a positive constant:*

(1) *for all $p \in (1,p_0)$,*

$$\mathrm{Cap}_p(B(x,r)) \geqslant C_p \frac{V(x,r)}{r^p}.$$

(2) *for all $p \in (1,p_0)$,*

$$\frac{V(x,t)}{V(x,r)} \geqslant C_p \left(\frac{t}{r}\right)^p, \qquad \forall\, t \geqslant r.$$

Then (1) *and* (2)*, for all $r \geqslant r_0$ and $x = x_0$, are equivalent.*

Remark 5.7. — One can also show the equivalence of (1) and (2) in Theorem 5.6 under one of the following alternative conditions on x and r:

- For all $x \in M$, and $r = r_0$.
- For all $x \in M$ and all $r \geqslant r_0$.

The proof is the same.

Remark 5.8. — The result of Theorem 5.6 shows that in [11, Theorem A], the second part does not improve on the first one.

DEFINITION 5.9. — *Let $p \in (1, \infty)$. A manifold M such that, for some positive constant C_p, for all $x \in M$ and all $r \geqslant 1$,*

$$\mathrm{Cap}_p(B(x,r)) \geqslant C_p \frac{V(x,r)}{r^p},$$

will be called p-regular. *Recall the parabolic dimension κ. We will call M* parabolic regular *if, for every $p < \kappa$, M is p-regular.*

Proof. — Let us introduce a third inequality:

(3)
$$\int_r^\infty \left(\frac{V(x,r)}{V(x,t)}\right)^{1/p} \mathrm{d}t \leqslant C_p\, r.$$

We show the equivalence of (1), (2) and (3) under the condition (b), the two other cases being similar. We first show that (1) and (3) are equivalent. According to [21], the p-capacity of $\bar{B}(x_0, r)$ can be estimated by

$$\mathrm{Cap}_p(\bar{B}(x_0,r)) \leqslant \left(\int_r^\infty \left(\frac{t}{V(x_0,t)}\right)^{\frac{1}{p-1}} \mathrm{d}t\right)^{1-p}.$$

Assume that (1) holds true for all $1 < p < p_0$ and all $r \geqslant 1$, at the point $x = x_0$. One obtains that for all $r \geqslant 1$,

$$\int_r^\infty \left(\frac{tV(x_0,r)}{r^p V(x_0,t)}\right)^{\frac{1}{p-1}} \mathrm{d}t \leqslant C_p^{1-p}.$$

The argument based on Hölder's inequality and used in the proof of Theorem 5.1 (see the implication (2) $\Rightarrow$ (3)) shows that for every $p \in (1, p_0)$, (3) is satisfied for all $r \geqslant r_0$, at the point x_0.

Conversely, assume that (3) is satisfied for all $r \geqslant r_0$, at the point x_0. As in the proof of Proposition 5.4, let us introduce the operators:

$$R_r = \int_0^{r^2} e^{-t\Delta} \frac{\mathrm{d}t}{\sqrt{t}},$$

and

$$S_r = \int_{r^2}^\infty e^{-t\Delta} \frac{\mathrm{d}t}{\sqrt{t}},$$

so that

$$\Delta^{-1/2} = R_r + S_r.$$

Using the uniform boundedness of $e^{-t\Delta}$ on L^p, we see that

$$\|R_r\|_{p,p} \leqslant Cr.$$

We want to estimate $\|S_r\|_{L^p\to L^p(B(x_0,r))}$. The computations done in the proof of Proposition 5.4 show that

$$\|S_r\|_{L^p\to L^\infty(x)} \leqslant C\int_r^\infty \frac{\mathrm{d}t}{V(x,t)^{1/p}}.$$

Using (D), we see that for every $x\in B(x_0,r)$,

$$\begin{aligned}\|S_r\|_{L^p\to L^\infty(x)} &\leqslant \frac{C}{V(x_0,r)^{1/p}}\int_r^\infty \left(\frac{V(x_0,r)}{V(x,t)}\right)^{1/p}\mathrm{d}t\\ &\leqslant \frac{C}{V(x_0,r)^{1/p}}\int_r^\infty \left(\frac{V(x_0,r)}{V(x_0,t)}\right)^{1/p}\mathrm{d}t.\end{aligned}$$

Thus,

$$\begin{aligned}\|S_r\|_{L^p\to L^p(B(x_0,r))} &\leqslant V(x_0,r)^{1/p}\|S_r\|_{L^p\to L^\infty(B(x_0,r))}\\ &\leqslant \int_r^\infty \left(\frac{V(x_0,r)}{V(x_0,t)}\right)^{1/p}\mathrm{d}t,\end{aligned}$$

and by (3) one obtains

$$\|S_r\|_{L^p\to L^p(B(x_0,r))} \leqslant Cr.$$

Consequently,

$$\|\Delta^{-1/2}\|_{L^p\to L^p(B(x_0,r))} \leqslant Cr.$$

We now recall an argument from [27, Proposition 2.2] (we refer to this paper for more details): from the above inequality, one can conclude that

$$\|u\|_{L^p(B(x_0,r))} \leqslant Cr\|\Delta^{1/2}u\|_p, \qquad \forall\, u\in C_0^\infty(M).$$

Since M satisfies (UE) and (D), by [19] the Riesz transform $\mathrm{d}\Delta^{-1/2}$ is bounded on L^s, $s\in(1,2]$. In particular, it is bounded on $L^{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$. But it is well-known (see [20, Proposition 2.2]) that the boundedness of the Riesz transform on $L^{p'}$, implies the inequality

$$\|\Delta^{1/2}u\|_p \leqslant C\|\nabla u\|_p, \qquad \forall\, u\in C_0^\infty(M).$$

Thus,

$$\|u\|_{L^p(B(x_0,r))} \leqslant Cr\|\nabla u\|_p, \qquad \forall\, u\in C_0^\infty(M).$$

Using this inequality for a sequence $(u_n)_{n\in\mathbb{N}}$ of C_0^∞ functions, such that $u_n \geqslant 1$ on $B(x_0,r)$ and

$$\lim_{n\to\infty}\int_M |\nabla u_n|^p = \mathrm{Cap}_p(\bar{B}(x_0,r)),$$

one finds that

$$V(x_0,r)^{1/p} \leqslant Cr\left(\mathrm{Cap}_p(B(x_0,r))\right)^{1/p},$$

that is (1). This concludes the proof of the equivalence between (1) and (3). We now prove the equivalence between (2) and (3). Of course, (2) implies (3). The converse uses the same idea as in the proof of Theorem 5.1: for $r \geqslant r_0$, denote $f_r(t) = \left(\frac{V(x_0,r)}{V(x_0,t)}\right)^{1/p}$, then f_r is non-negative, non-increasing, so that by (2),

$$(t-r)f_r(t) \leqslant \int_r^t f_r(s)\,\mathrm{d}s \leqslant \int_r^\infty f_r(s)\,\mathrm{d}s \leqslant Cr.$$

Therefore,

$$f_r(t) \leqslant Cr(t-r)^{-1}, \qquad \forall\, t \geqslant r,$$

that is,

$$\left(\frac{V(x_0,r)}{V(x_0,t)}\right) \geqslant C_p\left(\frac{t-r}{r}\right)^p, \qquad \forall\, t\geqslant r,$$

which by (D) implies (2). □

6. Weighted spaces

In general, a manifold M for which (D) and (UE) hold, do not need to satisfy the Sobolev inequality (S^ν), nor the non-collapsing of balls (NC). In particular, in general the operators $\Delta^{-\alpha}$ do not behave well on the L^p spaces. In order to overcome these difficulties, weighted spaces have to be considered. Weighted estimates on the L^p spaces for the heat kernel have recently been considered in [8] (see also [3]), and their equivalence with weighted resolvent estimates on L^p spaces has been demonstrated. Here, we will push this idea one step further, and introduce a natural class of weighted L^p spaces. For $p\in[1,\infty]$, define

$$L^p_V(M) := L^p\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right).$$

Notice that $L^\infty_V(M) = L^\infty(M)$. Note also that by Bishop–Gromov, if the Ricci curvature of M is bounded from below, and M satisfies the non-collapsing (NC), then $V(x,1)\sim 1$ and $L^p_V(M)$ identifies to $L^p(M)$. Let us recall the following result from [3] or [8]:

Proposition 6.1. — *Let $1 \leqslant p \leqslant q \leqslant \infty$. Let δ and γ be real numbers so that $\delta + \gamma = \frac{1}{p} - \frac{1}{q}$. Let P be a non-negative, self-adjoint operator on $L^2(M, \mu)$, such that the Gaussian estimates hold for its heat kernel. Then*

$$\sup_{t>0} \|V(\,\cdot\,, \sqrt{t})^{\gamma} e^{-tP} V(\,\cdot\,, \sqrt{t})^{\delta}\|_{p,q} < \infty.$$

We will need a similar estimate for the gradient of the heat kernel:

Proposition 6.2. — *Assume (D) and (UE), and that for some $q \in (1, \infty)$,*

$$\sup_{t>0} \sqrt{t} \|\nabla e^{-t\Delta}\|_{q,q} < \infty.$$

Then for every $1 \leqslant p < q$, and δ, γ real numbers so that $\delta + \gamma = \frac{1}{p} - \frac{1}{q}$,

$$\sup_{t>0} \sqrt{t} \|V(\,\cdot\,, \sqrt{t})^{\gamma} \nabla e^{-t\Delta} V(\,\cdot\,, \sqrt{t})^{\delta}\|_{p,q} < \infty.$$

Remark 6.3. — By analyticity on L^p of the heat semi-group of Δ, if the Riesz transform is bounded on L^q, then the gradient estimate

$$\sup_{t>0} \sqrt{t} \|\nabla e^{-t\Delta}\|_{q,q} < \infty$$

holds. Conversely, it is shown in [5] that if the above gradient estimate holds, and M satisfies (D) and (P), then the Riesz transform is bounded on L^p, for all $p \in (1, q)$.

Proof. — Denote $V_{\sqrt{t}}(x) := V(x, \sqrt{t})$. Writing $\nabla e^{-t\Delta} = \nabla e^{-\frac{t}{2}\Delta} e^{-\frac{t}{2}\Delta}$ and using Proposition 6.1 and (D), it is easy to see that the result holds true if $\gamma = 0$. In order to prove the result for all γ, we will use some ideas from [8]. By complex interpolation for the family of operators

$$T_z = V(\,\cdot\,, \sqrt{t})^{\gamma_1 z + (1-z)\gamma_2} \nabla e^{-t\Delta} V(\,\cdot\,, \sqrt{t})^{\delta_1 z + (1-z)\delta_2},$$

and using $\sup_{t>0} \sqrt{t} \|\nabla e^{-t\Delta}\|_{q,q} < \infty$, it is enough to prove the result for $p = 2$. Therefore, let us take δ, γ real numbers so that $\delta + \gamma = \frac{1}{2} - \frac{1}{q}$. Define $\Phi = F_a$ to be the Fourier transform of $t \mapsto (1 - t^2)_+^a$. It can be checked (see the proof of Proposition 4.1.6. in [8]) that the following transmutation formula holds:

$$e^{-t\Delta} = \int_0^\infty F_a(\sqrt{st\Delta}) s^{a+\frac{1}{2}} e^{-s/4} \,\mathrm{d}s.$$

Therefore, using $(D_{\nu,\nu'})$,

$$\begin{aligned} &\sqrt{t} \left\| V_{\sqrt{t}}^{\gamma} \nabla e^{-t\Delta} V_{\sqrt{t}}^{\delta} \right\|_{2,q} \\ &\leqslant \int_0^\infty \sqrt{st} \left\| V_{\sqrt{st}}^{\gamma} \nabla F_a(\sqrt{st\Delta}) V_{\sqrt{st}}^{\delta} \right\|_{2,q} \left(\sqrt{s} + \frac{1}{\sqrt{s}} \right)^{\nu(|\gamma|+|\delta|)} s^a e^{-s/4} \,\mathrm{d}s. \end{aligned}$$

Hence, if a is big enough,

$$\sup_{t>0}\sqrt{t}\left\|V_{\sqrt{t}}^{\gamma}\nabla e^{-t\Delta}V_{\sqrt{t}}^{\delta}\right\|_{2,q}\leqslant\sup_{t>0}\sqrt{t}\left\|V_{\sqrt{t}}^{\gamma}\nabla F_a(\sqrt{t\Delta})V_{\sqrt{t}}^{\delta}\right\|_{2,q}.$$

Setting $T_r=\nabla F_a(r\sqrt{\Delta})$, it follows from the finite propagation speed property for the wave equation that if f_1 has support in B_1, f_2 has support with B_2 and $\mathrm{d}(B_1,B_2)>r$, then $\langle T_rf_1,f_2\rangle=0$. By [8, Proposition 4.1.1.],

$$\|V_r^{\gamma}T_rV_r^{\delta}\|_{2,q}\leqslant C\left\|T_rV_r^{\frac{1}{2}-\frac{1}{q}}\right\|_{2,q}.$$

By a straightforward adaptation of [8, Lemma 4.1.4.], if

$$\sup_{\lambda}|(1+\lambda^2)^{N+1}\Phi(\lambda)|<\infty,$$

then

$$\left\|\nabla\Phi(\sqrt{t\Delta})V_{\sqrt{t}}^{\frac{1}{2}-\frac{1}{q}}\right\|_{2,q}\leqslant C\left\|\nabla(I+t\Delta)^{-N}V_{\sqrt{t}}^{\frac{1}{2}-\frac{1}{q}}\right\|_{2,q}.$$

One easily checks that the condition $\sup_\lambda|(1+\lambda^2)^{N+1}\Phi(\lambda)|<\infty$ is satisfied for $\Phi=F_a$ if $2N+1\leqslant a$. Therefore, if a is large enough and $a\geqslant 2N+1$,

$$\sup_{t>0}\sqrt{t}\left\|V_{\sqrt{t}}^{\gamma}\nabla e^{-t\Delta}V_{\sqrt{t}}^{\delta}\right\|_{2,q}\leqslant C\sup_{t>0}\sqrt{t}\left\|\nabla(I+t\Delta)^{-N}V_{\sqrt{t}}^{\frac{1}{2}-\frac{1}{q}}\right\|_{2,q}.$$

We now use

$$(I+t\Delta)^{-N}=\frac{1}{\Gamma(N)}\int_0^\infty e^{-s}s^{N-1}e^{-st\Delta}\,\mathrm{d}s,$$

so that, using $(D_{\nu,\nu'})$,

$$\begin{aligned}&\sup_{t>0}\sqrt{t}\left\|\nabla(I+t\Delta)^{-N}V_{\sqrt{t}}^{\frac{1}{2}-\frac{1}{q}}\right\|_{2,q}\\&\quad\leqslant C\int_0^\infty e^{-s}s^{N-\frac{3}{2}}\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)^{\nu(|\delta|+|\gamma|)}\sqrt{st}\left\|\nabla e^{-st\Delta}V_{\sqrt{st}}^{\frac{1}{2}-\frac{1}{q}}\right\|_{2,q}\mathrm{d}s.\end{aligned}$$

Hence, if N is big enough,

$$\sup_{t>0}\sqrt{t}\left\|\nabla(I+t\Delta)^{-N}V_{\sqrt{t}}^{\frac{1}{2}-\frac{1}{q}}\right\|_{2,q}\leqslant C\sup_{t>0}\sqrt{t}\left\|\nabla e^{-t\Delta}V_{\sqrt{t}}^{\frac{1}{2}-\frac{1}{q}}\right\|_{2,q}.$$

Consequently, if one chooses N and a big enough satisfying $a\geqslant 2N+1$, one obtains

$$\sup_{t>0}\sqrt{t}\left\|V_{\sqrt{t}}^{\gamma}\nabla e^{-t\Delta}V_{\sqrt{t}}^{\delta}\right\|_{2,q}\leqslant\sup_{t>0}\sqrt{t}\left\|\nabla e^{-t\Delta}V_{\sqrt{t}}^{\frac{1}{2}-\frac{1}{q}}\right\|_{2,q}.$$

This last quantity is finite by the remark we made at beginning of the proof, and this concludes the proof of Proposition 6.2. □

We now rephrase Proposition 6.1 and Proposition 6.2 in term of action of the heat kernel on the weighted spaces L^p_V. Let $P=\Delta+\mathcal{V}$ be a Schrödinger operator. Using $(D_{\nu,\nu'})$, we see that if the heat kernel of P has Gaussian estimates, then

$$\begin{aligned}
&\left\|V(\,\cdot\,,\sqrt{t})^{-1/q}e^{-tP}V(\,\cdot\,,\sqrt{t})^{1/p}\right\|_{p,q}\\
&\quad=\left\|\left(\frac{V(\,\cdot\,,\sqrt{t})}{V(\,\cdot\,,1)}\right)^{-1/q}V(\,\cdot\,,1)^{-1/q}e^{-tP}V(\,\cdot\,,1)^{1/p}\left(\frac{V(\,\cdot\,,\sqrt{t})}{V(\,\cdot\,,1)}\right)^{1/p}\right\|_{p,q}\\
&\quad\geqslant C\left(\varphi_{p,q}(t)\right)^{-1}\|e^{-tP}\|_{L^p_V,L^q_V},
\end{aligned}$$

where

$$\varphi_{p,q}(t)=\begin{cases}t^{-\frac{\nu'}{2p}+\frac{\nu}{2q}}, & t\geqslant 1\\ t^{-\frac{\nu}{2p}+\frac{\nu'}{2q}}, & t\leqslant 1\end{cases}$$

Thus, one can rephrase Proposition 6.1 in the following way:

Corollary 6.4. — *Assume (D) and let $P=\Delta+\mathcal{V}$ be a Schrödinger operator whose heat kernel have Gaussian estimates. Let $1\leqslant p\leqslant q\leqslant\infty$, then for every $t>0$,*

$$\|e^{-tP}\|_{L^p_V,L^q_V}\lesssim\varphi_{p,q}(t).$$

Concerning Proposition 6.2, one has the following:

Corollary 6.5. — *Assume (D) and (UE), and that for some $q\in(1,\infty)$,*

$$\sup_{t>0}\sqrt{t}\|\nabla e^{-t\Delta}\|_{q,q}<\infty.$$

Then for every $1\leqslant p\leqslant q$,

$$\sqrt{t}\|\nabla e^{-t\Delta}\|_{L^p_V,L^q_V}\lesssim\varphi_{p,q}(t).$$

In Section 7, we will need the following slight generalization of a perturbation result by T. Coulhon and N. Dungey [18, Theorem 2.1]:

Proposition 6.6. — *Let M satisfying (D) and (UE). Let $H=-\mathrm{div}A\nabla$, A symmetric, and $a=A-Id$, and assume that for some $q\in[1,\infty)$, $a\in L^q_V\cap L^\infty$. Let $p_0>2$ such that $\nabla\Delta^{-1/2}$ and $\nabla(I+H)^{-1/2}$ are bounded on L^p for every $p\in(2,p_0)$, and such that $\nabla H^{-1/2}$ is bounded on L^p, for $p\in(p_0',2)$. Then $\nabla H^{-1/2}$ is bounded on L^p for every $p\in(2,p_0)$.*

T. Coulhon and N. Dungey show this result under the stronger assumptions that $a\in L^q\cap L^\infty$ and that for every $t\geqslant 1$, $\|e^{-t\Delta}\|_{1,\infty}\leqslant Ct^{-D/2}$. Such an ultracontractivity estimate for $e^{-t\Delta}$ is known to be equivalent to a Nash inequality at infinity ([22]), however in this paper, we want to work in greater generality and avoid this type of hypothesis. Therefore, we have to work with the weighted spaces L^p_V.

Proof. — The only part of the proof of [18, Theorem 2.1] where the ultracontractivity estimate $\|e^{-t\Delta}\|_{1,\infty} \leqslant Ct^{-D/2}$ and the hypothesis $a \in L^q$ are used, is to show that for $p \in (2, p_0)$, there is $\varepsilon > 0$ such that

$$\|a\nabla(I+t\Delta)^{-1}\|_{p,p} \leqslant Ct^{-\frac{1}{2}-\varepsilon}, \qquad \forall\, t>0.$$

Let us show how to prove a similar estimate in our context, which allows us to make Coulhon and Dungey's proof work. Write

$$(I+t\Delta)^{-1} = \int_0^\infty e^{-s}e^{-st\Delta}\,\mathrm{d}s,$$

so that

$$\|a\nabla(I+t\Delta)^{-1}\|_{p,p} \leqslant \int_0^\infty e^{-s}\|a\nabla e^{-st\Delta}\|_{p,p}\,\mathrm{d}s.$$

Using Hölder's inequality,

$$\|a\nabla e^{-st\Delta}\|_{p,p} \leqslant \|aV(\,\cdot\,,1)^{-1/q}\|_q \|V(\,\cdot\,,1)^{1/q}\nabla e^{-st\Delta}\|_{p,r},$$

where r is defined by $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Notice that

$$\|aV(\,\cdot\,,1)^{-1/q}\|_q = \|a\|_{L^q_V} < \infty.$$

Since $a \in L^q_V \cap L^\infty$, by interpolation we can assume that q is large enough so that $2 < r < p_0$. Then, the Riesz transform $\nabla\Delta^{-1/2}$ is bounded on L^r, and by a classical argument involving analyticity of the heat semi-group $e^{-t\Delta}$ on L^p,

$$\sup_{t>0} \sqrt{t}\|\nabla e^{-t\Delta}\|_{r,r} < \infty.$$

By $(D_{\nu,\nu'})$ and Proposition 6.2,

$$\begin{aligned}
\|V(\,\cdot\,,1)^{1/q}\nabla e^{-st\Delta}\|_{p,r} &= \left\| \left(\frac{V(\,\cdot\,,\sqrt{st})}{V(\,\cdot\,,1)}\right)^{-1/q} V(\,\cdot\,,\sqrt{st})^{1/q}\nabla e^{-st\Delta}\right\|_{p,r} \\
&\leqslant C(st)^{-\frac{\nu'}{2q}-\frac{1}{2}} + C(st)^{-\frac{\nu}{2q}-\frac{1}{2}} \\
&\leqslant C(st)^{-\frac{1}{2}-\varepsilon_1} + C(st)^{-\frac{1}{2}-\varepsilon_2},
\end{aligned}$$

where $\varepsilon_1 = \frac{\nu'}{2q}$, $\varepsilon_2 = \frac{\nu}{2q}$. Therefore, if q is big enough so that $\varepsilon_1 < \frac{1}{2}$, $\varepsilon_2 < \frac{1}{2}$,

$$\|a\nabla(I+t\Delta)^{-1}\|_{p,p} \leqslant C(t^{-\frac{1}{2}-\varepsilon}), \qquad \forall\, t>0,$$

with $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. With this at hand, the proof of Proposition 6.6 follows the lines of the proof of [18, Theorem 2.1]. □

Let us conclude this section by presenting a sufficient condition for a potential to belong to the Kato class at infinity, which generalizes Example 2.2 to manifolds that do not satisfy (S^ν):

Proposition 6.7. — *Let M satisfying (D) and (UE), and let $\mathcal{V} \in L_V^{\frac{\nu'}{2}-\varepsilon} \cap L_V^{\frac{\nu}{2}+\varepsilon}$, for some $\varepsilon > 0$. Then $\mathcal{V} \in K^\infty(M)$, and there is $q \in [1,\infty)$ such that $\Delta^{-1}|\mathcal{V}| \in L_V^q$.*

Proof. — Let us show that for some constant C independent of $\mathcal{V}$,

$$\|\Delta^{-1}|\mathcal{V}|\|_\infty \leqslant C\left(\|\mathcal{V}\|_{L_V^{\frac{\nu'}{2}-\varepsilon}} + \|\mathcal{V}\|_{L_V^{\frac{\nu}{2}+\varepsilon}}\right). \tag{6.1}$$

Write

$$\Delta^{-1} = \int_0^\infty e^{-t\Delta}\,\mathrm{d}t,$$

so that

$$\|\Delta^{-1}|\mathcal{V}|\|_\infty \leqslant \int_0^\infty \|e^{-t\Delta}|\mathcal{V}|\|_\infty\,\mathrm{d}t.$$

Using Corollary 6.4 with $P = \Delta$, we get

$$\begin{aligned}\|\Delta^{-1}|\mathcal{V}|\|_\infty &\leqslant \left(\int_0^1 t^{-\frac{\nu}{\nu+2\varepsilon}}\,\mathrm{d}t\right)\|\mathcal{V}\|_{L_V^{\frac{\nu}{2}+\varepsilon}} + \left(\int_1^\infty t^{-\frac{\nu'}{\nu'-2\varepsilon}}\,\mathrm{d}t\right)\|\mathcal{V}\|_{L_V^{\frac{\nu'}{2}-\varepsilon}} \\ &\leqslant C\left(\|\mathcal{V}\|_{L_V^{\frac{\nu'}{2}-\varepsilon}} + \|\mathcal{V}\|_{L_V^{\frac{\nu}{2}+\varepsilon}}\right),\end{aligned}$$

which shows (6.1). For $k \geqslant 0$, let $\mathcal{V}_k = \mathcal{V}\chi_k$. Then

$$\lim_{k\to\infty}\left(\|\mathcal{V}_k\|_{L_V^{\frac{\nu'}{2}-\varepsilon}} + \|\mathcal{V}_k\|_{L_V^{\frac{\nu}{2}+\varepsilon}}\right) = 0,$$

which implies, by (6.1) applied to $\mathcal{V}_k$,

$$\lim_{k\to\infty}\sup_{x\in M}\int_{\Omega_k^*} G(x,y)|\mathcal{V}(y)|\,\mathrm{d}y = 0,$$

i.e. $\mathcal{V} \in K^\infty(M)$. Let us now check that $\Delta^{-1}\mathcal{V} \in L_V^q$ for some $q < \infty$. We use again Corollary 6.4, to get

$$\begin{aligned}\|\Delta^{-1}|\mathcal{V}|\|_{L_V^q} \leqslant \left(\int_0^1 t^{-\frac{\nu}{\nu+2\varepsilon}+\frac{\nu}{2q}}\,\mathrm{d}t\right)\|\mathcal{V}\|_{L_V^{\frac{\nu}{2}+\varepsilon}}& \\ + \left(\int_1^\infty t^{-\frac{\nu'}{\nu'-2\varepsilon}+\frac{\nu'}{2q}}\,\mathrm{d}t\right)\|\mathcal{V}\|_{L_V^{\frac{\nu'}{2}-\varepsilon}}&.\end{aligned}$$

Consequently, if q is big enough so that

$$-\frac{\nu'}{\nu'-2\varepsilon} + \frac{\nu'}{2q} < -1,$$

then $\|\Delta^{-1}|\mathcal{V}|\|_{L^q_V} < \infty$. □

7. Riesz transform with potential

From now on, μ is assumed to be the Riemannian measure on M.

In this section, we will obtain boundedness and unboundedness results for the Riesz transform with potential $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$. Let us start by introducing a *local class* of potentials: we say that V satisfies condition (L_p) if there is a constant C such that for every $x \in M$,

$$(L_p) \qquad \|\mathcal{V}\|_{L^p(B(x,1))} \lesssim V(x,1)^{1/p'},$$

where $\frac{1}{p}+\frac{1}{p'}=1$. If M has Ricci curvature bounded from below and (NC) holds, then the condition (L_p) for $p=\infty$, is equivalent to $\mathcal{V} \in L^\infty$. Let us now state our main technical result, from which will be derived various consequences:

Theorem 7.1. — *Let M be a 2-non-parabolic manifold, satisfying (D) and (UE). Assume that for some $p_0 > 2$, the Riesz transform $\mathrm{d}\Delta^{-1/2}$ on M is bounded on L^p, for all $p \in (1,p_0)$. Recall the exponents ν,ν' from $(D_{\nu,\nu'})$. Let $\mathcal{V}$ be a subcritical potential, and consider the following two assumptions:*

(1) *$p_0 \leqslant \nu$, $\nu' > 2$, the Ricci curvature on M is bounded from below, and $\mathcal{V} \in L^{\frac{\nu'}{2}-\varepsilon}\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right) \cap L^{\frac{\nu}{2}+\varepsilon}\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right)$ satisfies (L_p) for some $p > N$, $N > 2$ being the topological dimension of M.*
(2) *$p_0 > \nu$, and $\mathcal{V} \in L^{q_1}\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right) \cap L^{q_2}\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right)$, where*

$$q_1 = \frac{\nu'}{2} - \varepsilon,\ q_2 = \max\left(\frac{p_0}{2}, \frac{\nu p_0}{p_0+\nu'} + \varepsilon\right).$$

Assume that either (1) or (2) is satisfied. Let $h \sim 1$ be the solution of $(\Delta+\mathcal{V})u=0$, provided by Theorem 3.2. Then the operator

$$\mathrm{d}(\Delta+\mathcal{V})^{-1/2} - (\mathrm{d}\log h)(\Delta+\mathcal{V})^{-1/2}$$

is bounded on L^p, for every $p \in [2,p_0)$. Equivalently, for every $p \in [2,p_0)$, the following inequality holds:

$$(7.1) \qquad \|\mathrm{d}u - u(\mathrm{d}\log h)\|_p \leqslant C\|(\Delta+\mathcal{V})^{1/2}u\|_p, \qquad \forall\, u \in C_0^\infty(M).$$

One of the main features of Theorem 7.1 is to provide an alternative inequality (7.1) in the case the Riesz transform $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is unbounded on L^p. In turns, as we shall see, the validity of the inequality (7.1) gives a necessary condition (the $p-$non-parabolicity of M) for the boundedness of $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ on L^p, which will turn out to be sufficient under further assumptions on $\mathcal{V}$.

Corollary 7.2. — *Under the assumptions of Theorem 7.1 and if $\mathcal{V}\not\equiv 0$, a necessary condition for $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ to bounded on L^p, for some $p\in(2,p_0)$ is that M is p-non-parabolic.*

Remark 7.3. — In [14, Theorem 6.1], a related result to Corollary 7.2 is proved: more precisely, the authors prove that if $\mathcal{V}\geqslant 0$ and there exists a positive, bounded function h such that $(\Delta+\mathcal{V})h=0$, then a necessary condition for $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ to be bounded on L^p is that $p\leqslant\nu$. Notice that unlike our result, the existence of $h>0$ bounded, solution of $(\Delta+\mathcal{V})h=0$ is *assumed*, and moreover the necessary condition is $p>\nu$, instead of M being p-non-parabolic. Thus, even in the case of a smooth, compactly supported non-negative potential $\mathcal{V}$ in $\mathbb{R}^n$, it is not possible from their result to conclude that $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ cannot be bounded on L^n. Notice also that in some cases, our Corollary 7.2 yields a much sharper condition than $p>\nu$: indeed, there are examples for which κ, the parabolic dimension, is strictly less than ν. For instance, consider the Taub-NUT metric g on $\mathbb{R}^4$ (see [37]): it is a complete metric with zero Ricci curvature such that

$$V(x,r)\sim r^3,\qquad \forall\, x\in\mathbb{R}^4,\,\forall\, r\geqslant 1.$$

In fact, at infinity the metric is asymptotic to the product metric on $\mathbb{R}^3\times S^1$. Thus, $(\mathbb{R}^4,g)$ has parabolic dimension κ equals to 3, and by Bakry's celebrated result [7], the Riesz transform on $(\mathbb{R}^4,g)$ is bounded on L^p for all $p\in(1,\infty)$. Thus, our Corollary 7.2 applies, and gives the necessary condition $p<3$, for the boundedness of $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$. On the other hand, looking at balls of small radius in $(D_{\nu,\nu'})$, one sees that $4\leqslant\nu$, and in fact $4=\nu$ by Bishop–Gromov. Thus, the necessary condition provided by [14, Theorem 6.1] is only $p\leqslant 4$.

Notice also that while the proof of [14] is *local*, and the idea consists in using the function h to contradict a local L^p Morrey inequality. This argument goes back to [18], where it was used to prove that the Riesz transform is unbounded on L^p for $p>n$, on the connected sum of Euclidean spaces. On the contrary, our argument is different, and is of *global* nature, which allows us to get the optimal result.

As a consequence of Theorem 7.1, one can give a quick proof of the following result, which can also be obtained (except the borderline case $p=n$) as a combination of [2] and [14], with different arguments:

Corollary 7.4. — *Let M be a manifold satisfying the Sobolev inequality (S^ν) for some $\nu > 2$, and having Euclidean volume growth:*

$$V(x,r) \sim r^\nu,$$

for all $x \in M$ and $r > 0$. Let $\mathcal{V} \in L^{\frac{\nu}{2}+\varepsilon} \cap L^{\frac{\nu}{2}-\varepsilon}$, $\varepsilon > 0$ be subcritical. Assume that the Riesz transform $\mathrm{d}\Delta^{-1/2}$ is bounded on L^p, $p \in (1,(1+\delta)\nu+\varepsilon)$, $\delta > 0$. Then $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p if and only if $p \in (1,\nu)$.

Remark 7.5. — From Corollary 7.4, one recovers the first half of a result by C. Guillarmou and A. Hassell [34, Theorem 1.5], which states that if M is an asymptotically conic manifold of dimension $\nu \geqslant 3$, and $\mathcal{V} = O(r^{-3})$, then $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p if and only if $p \in (1,\nu)$. In particular, we obtain an elementary (i.e. without using the $b-$calculus) proof of Guillarmou and Hassell's result, which is also of perturbative nature.

In [3, Theorem 3.9], a boundedness result for the Riesz transform is proved. Actually, looking closely at the proof and using our Proposition 6.7, together with Theorems 4.1 and 5.1, one can improve it and show:

Theorem 7.6. — *Let M satisfying (D), (UE) and (P_{loc}). Let κ the parabolic dimension of M, and assume that $\kappa > 2$. Assume also that the Riesz transform on M is bounded on L^p, for all $p \in (1,p_0)$. Let $\mathcal{V} \in L^{\frac{\nu'}{2}-\varepsilon}\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right) \cap L^{\frac{\nu}{2}+\varepsilon}\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right)$ be subcritical such that for all $p < \kappa$, close enough to κ,*

$$\int_1^\infty \left\| \frac{|\mathcal{V}|^{1/2}}{V(\,\cdot\,,t)^{1/p}} \right\|_p \mathrm{d}t < \infty.$$

Then, $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p, for every $p \in (1,\min(\kappa,p_0))$.

Remark 7.7.

(1) Actually, as follows from Theorem 5.1, $p \leqslant \kappa$ is *necessary* for having

$$\int_1^\infty \left\| \frac{|\mathcal{V}|^{1/2}}{V(\,\cdot\,,t)^{1/p}} \right\|_p \mathrm{d}t < \infty.$$

(2) Assume that M satisfies (P_{loc}). If $\mathcal{V} \in L^\infty$ is subcritical with compact support, by Theorem 5.1 one has, for all $p \in (1,\kappa)$,

$$\int_1^\infty \left\| \frac{|\mathcal{V}|^{1/2}}{V(\,\cdot\,,t)^{1/p}} \right\|_p \mathrm{d}t < \infty.$$

Therefore, $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p for all $p \in (1,\min(\kappa,p_0))$. This shows that Corollary 7.2 is *optimal*.

One can in fact complement Theorem 7.6, and extend Corollary 7.4 to more general manifolds; in the next theorem, which is one of the main results of this article, we prove a sharp boundedness result for the Riesz transform with potential:

Theorem 7.8. — *Let M satisfying (D), (UE) and (P_{loc}). Recall the exponents ν,ν' from $(D_{\nu,\nu'})$. Let κ be the parabolic dimension of M, assume that $\kappa>2$ and that M is κ-parabolic. Assume also that M is p-regular, for all $p<\kappa$, close enough to κ. Assume that the Riesz transform on M is bounded on L^{p_0} for some $p_0>\nu$. Let $\mathcal{V}\in L^{\frac{\nu'}{2}-\varepsilon}\left(M,\frac{\mathrm{d}\mu(x)}{V(x,1)}\right)\cap L^{\frac{\nu}{2}+\varepsilon}\left(M,\frac{\mathrm{d}\mu(x)}{V(x,1)}\right)$ be subcritical. Then $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p if and only if $p\in(1,\kappa)$.*

Remark 7.9. — The fact that $p<\kappa$ is sufficient for the boundedness on L^p of $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ follows from Theorem 7.6, together with our Theorems 4.1 and Theorems 5.1. Therefore, in a sense it is a consequence of the results of [3]. The converse, however, is the main point of our result, and is entirely new.

The rest of this subsection is devoted to the proof of Theorem 7.1, Corollaries 7.2 and 7.4, and Theorem 7.8. One of the main technical ingredients in the proof of Theorem 7.1 is a perturbation result by Coulhon and Dungey ([18, Theorem 4.1]).

Proof. — Let $P=\Delta+\mathcal{V}$, and D be the first-order differential operator:

$$Du=\mathrm{d}(h^{-1}u),$$

where $h\sim 1$ is the positive solution of $Pu=0$ given by Theorem 3.2. Consider the operator T_h which is multiplication by h, and the h-transform $P_h=T_h^{-1}PT_h$. By (2.14), P_h is the weighted Laplacian Δ_{h^2}, self-adjoint on $L^2(M,h^2\mu)\simeq L^2(M,\mathrm{d}\mu)$. It is clear by spectral theory that

$$P_h^{-1/2}=T_h^{-1}P^{-1/2}T_h.$$

Notice also that

$$T_h^{-1}DT_h=h^{-1}d.$$

Consequently,

$$T_h^{-1}DP^{-1/2}T_h=h^{-1}\mathrm{d}\Delta_{h^2}^{-1/2}.$$

Notice that T_h is an isometry from $L^p(M,\mathrm{d}\mu)$ to $L^p(M,h^p\mathrm{d}\mu)$. Given that $h\sim 1$, there is a natural identification $L^p(M,h^p\mathrm{d}x)\simeq L^p(M,\mathrm{d}x)$, and so $DP^{-1/2}$ is bounded on $L^p(M,\mathrm{d}x)$ if and only if $\mathrm{d}\Delta_{h^2}^{-1/2}$ is bounded on $L^p(M,\mathrm{d}x)$. We claim that

Lemma 7.10. — *The operator* $\mathrm{d}\Delta_{h^2}^{-1/2}$ *is bounded on* L^p*, for every* $p \in [2, p_0)$.

The proof of this claim relies on the above-mentioned perturbation result of Coulhon and Dungey [18, Theorem 4.1], and is postponed. Assuming the result of Lemma 7.10 for the moment, we obtain that $D(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p, $p \in (2, p_0)$. But

$$D(\Delta+\mathcal{V})^{-1/2} = \mathrm{d}(h^{-1})(\Delta+\mathcal{V})^{-1/2} + h^{-1}\mathrm{d}(\Delta+\mathcal{V})^{-1/2},$$

hence the operator

$$\mathrm{d}(\Delta+\mathcal{V})^{-1/2} - (\mathrm{d}\log h)(\Delta+\mathcal{V})^{-1/2}$$

is bounded on L^p, $p \in (2, p_0)$. It remains to prove the inequality (7.1). Let $v \in C_0^\infty(M)$. By the above, one has for every $u \in L^p$,

$$\|\mathrm{d}(\Delta+\mathcal{V})^{-1/2}u - (\mathrm{d}\log h)(\Delta+\mathcal{V})^{-1/2}u\|_p \leqslant C\|u\|_p. \tag{7.2}$$

We want to apply inequality (7.2) with the choice $u = (\Delta+\mathcal{V})^{1/2}v$. In order for this to be licit, one has to prove that $(\Delta+\mathcal{V})^{1/2}C_0^\infty \subset L^p$. Conjugating by h, one sees that this is equivalent to $\Delta_{h^2}^{1/2}C_0^\infty \subset L^p$. This has been proved in [27, Lemma 2.2]. Hence, plugging in (7.2) the function $u = (\Delta+\mathcal{V})^{1/2}v$, one finds (7.1). This concludes the proof of Theorem 7.1. □

Proof of Corollary 7.2. — Let us prove the first part. Assume that $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p for some $p \in (2, p_0)$. Let $D = d \circ h^{-1}$. We claim that for every $q \in [2, \infty)$,

$$\|(\Delta+\mathcal{V})^{1/2}u\|_q \leqslant C\|Du\|_q, \qquad \forall\, u \in C_0^\infty(M). \tag{7.3}$$

To show this, let us start with the inequality

$$\|\Delta_{h^2}^{1/2}v\|_q \leqslant C\|\mathrm{d}v\|_q, \qquad \forall\, v \in C_0^\infty(M). \tag{7.4}$$

Inequality (7.4) for every $q \in [2, \infty)$ follows from $h \sim 1$ and the fact that $\mathrm{d}\Delta_{h^2}^{-1/2}$ is bounded on L^q, $q \in (1, 2]$ (this will be proved in Lemma 7.12), together with a classical duality argument (see [20, Proposition 2.1]). Now, by (2.14),

$$h^{-1}(\Delta+\mathcal{V})^{1/2}h = \Delta_{h^2}^{1/2},$$

and we use inequality (7.4) with $u = hv$, to obtain that for every $u \in C_0^\infty(M)$,

$$\|h^{-1}(\Delta+\mathcal{V})^{1/2}u\|_q \leqslant C\|Du\|_q.$$

Since $h \sim 1$, we get (7.3). By Theorem 7.1, the operator $(\mathrm{d}h)(\Delta+\mathcal{V})^{-1/2}$ has to be bounded on L^p, therefore

$$\|(\mathrm{d}h)u\|_p \leqslant C\|(\Delta+\mathcal{V})^{1/2}u\|_p, \qquad \forall\, u \in C_0^\infty(M),$$

which, together with (7.3) for $q=p$, implies that

$$\|(\mathrm{d}h)u\|_p \leqslant C\|Du\|_p, \qquad \forall\, u \in C_0^\infty(M).$$

If we let $v=h^{-1}u$, we obtain that

$$\|(\mathrm{d}\log h)v\|_p \leqslant C\|dv\|_p, \qquad \forall\, u \in C_0^\infty(M).$$

Since $V \not\equiv 0$, h is not constant, and thus $\mathrm{d}\log h \not\equiv 0$. Thus, (1.6) holds with a non-zero $\rho = |\nabla \log h|^p$, and it follows that M is $p-$non-parabolic. □

Proof of Corollary 7.4. — if (S^ν) holds, then for $\mathcal{V}_- \in L^{\frac{\nu}{2}}$, subcriticality of $\mathcal{V}$ is equivalent to strong subcriticality of $\Delta+\mathcal{V}$ (see Lemma 2.1). Thus, there exists $\eta>0$ such that

$$\eta \int_M |\nabla u|^2 \leqslant \int_M |\nabla u|^2 + \mathcal{V}u^2, \qquad \forall\, u \in C_0^\infty(M).$$

Hence, $\Delta+\mathcal{V}$ satisfies the Sobolev inequality

$$\|u\|_{\frac{2\nu}{\nu-2}} \leqslant C\int_M |\nabla u|^2 + \mathcal{V}u^2, \qquad \forall\, u \in C_0^\infty(M).$$

By Example 2.2, a potential $\mathcal{V}$ belonging to $L^{\frac{\nu}{2}+\varepsilon} \cap L^{\frac{\nu}{2}-\varepsilon}$ is in the Kato class at infinity $K^\infty(M)$, therefore by Theorem 4.1, $e^{-t(\Delta+\mathcal{V})}$ has Gaussian estimates, and thus is bounded uniformly on L^1. According to [23, Theorem 2.7], the Sobolev inequality for $\Delta+\mathcal{V}$ and the fact that $e^{-t(\Delta+\mathcal{V})}$ is bounded uniformly on L^1 implies that for every $p\in(1,\nu)$, $(\Delta+\mathcal{V})^{-1/2}$ is bounded from L^p to L^q, $\frac{1}{q}=\frac{1}{p}-\frac{1}{\nu}$. By Theorem 7.1, $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p if and only if $(\mathrm{d}h)(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p. Thus, in order to conclude the proof, it is enough to show that $\mathrm{d}h \in L^\nu$. This will be proved in Lemma 7.11. □

Proof of Theorem 7.8. — By Corollary 7.2, $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is unbounded on L^p if M is p-parabolic. By Theorem 5.6, the hypothesis on capacities implies that for every $p<\kappa$, the following reverse volume estimate holds:

$$r^p \lesssim \frac{V(x,r)}{V(x,1)}, \qquad \forall\, x\in M, \forall\, r>0. \tag{7.5}$$

Hence, $\nu' > \kappa-\delta, \forall\, \delta>0$. Since $\mathcal{V} \in L^{\frac{\nu'}{2}-\varepsilon}\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right) \cap L^{\frac{\nu}{2}+\varepsilon}\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right)$, by interpolation and Lemma 1.3, one obtains that $\mathcal{V} \in L^{\frac{p}{2}}\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right)$, for every $p<\kappa$, close enough to κ. Therefore, one sees that for $p<\kappa$, close enough to κ,

$$\int_1^\infty \left\| \frac{|\mathcal{V}|^{1/2}}{V(\,\cdot\,,t)^{1/p}} \right\|_p \mathrm{d}t < \infty.$$

Also, by Proposition 6.7, $\mathcal{V} \in K^\infty(M)$. Thus, by Theorem 7.6, the Riesz transform $\mathrm{d}(\Delta+\mathcal{V})^{-1/2}$ is bounded on L^p for all $1 < p < \min(\kappa, p_0)$. This concludes the proof of Theorem 7.8. □

To prove Lemma 7.10, we will need the following result concerning the positive function h, solution of $(\Delta+\mathcal{V})u = 0$, whose existence is provided by Theorem 3.2.

Lemma 7.11. — *The function h satisfies the following properties:*

(1) *There exists $q \in [1,\infty)$ such that $h - 1 \in L^q\left(M, \frac{\mathrm{d}\mu(x)}{V(x,1)}\right)$.*
(2) *Under assumption* (1), $\mathrm{d}h \in L^\infty$.
(3) *Under assumption* (2), $\mathrm{d}h \in L^r_V$, *for $r < p_0$, close enough to p_0.*
(4) *Under the assumptions of Corollary 7.4,* $\mathrm{d}h \in L^\nu$.

Proof. — By Theorem 3.2, h satisfies the following equation:

$$1 = h + \Delta^{-1}\mathcal{V}h.$$

Therefore, $|1-h| \leqslant \Delta^{-1}|\mathcal{V}|h \lesssim \Delta^{-1}|\mathcal{V}|$. By Proposition 6.7, there exists $q \in [1,\infty)$ such that $\Delta^{-1}|\mathcal{V}| \in L^q_V$. Consequently, $1-h \in L^q_V$. Again according to Proposition 6.7, $\mathcal{V}$ is in $K^\infty(M)$, and by definition of $K^\infty(M)$, $\Delta^{-1}|\mathcal{V}| \in L^\infty$.

Let us now assume (1). By the gradient estimate of Cheng–Yau (see e.g. [38, Theorem 6.1]), as a consequence of the bound from below of the Ricci curvature, there is a constant C such that for every $x \in M$,

$$|\nabla \log G(x,y)| \leqslant C, \qquad \forall\, y \in M \setminus B(x,1), \tag{7.6}$$

and,

$$|\nabla \log G(x,y)| \leqslant C\mathrm{d}(x,y)^{-1}, \qquad \forall\, y \in B(x,1) \setminus \{x\}. \tag{7.7}$$

As a consequence of (D) and (UE), there holds:

$$G(x,y) \leqslant \int_{\mathrm{d}(x,y)}^\infty \frac{r\mathrm{d}r}{V(x,r)}$$

(see e.g. [33, Exercise 15.8]). Using $(D_{\nu,\nu'})$ and the fact that $\nu' > 2$, we see that there is a constant C such that for all $x \in M$,

$$\int_1^\infty \frac{r\mathrm{d}r}{V(x,r)} \leqslant \frac{C}{V(x,1)}.$$

Also, by Bishop–Gromov and our assumption on Ricci, for every $x \in M$ and $r \leqslant 1$,

$$\frac{V(x,r)}{V(x,1)} \geqslant Cr^N,\ V(x,r) \leqslant Cr^N, \tag{7.8}$$

where N is the dimension of M. Therefore, for all $(x,y)\in M^2$ such that $\mathrm{d}(x,y)\leqslant 1$,

$$G(x,y)\leqslant C\frac{\mathrm{d}(x,y)^{2-N}}{V(x,1)}+\frac{C}{V(x,1)}\leqslant C\frac{\mathrm{d}(x,y)^{2-N}}{V(x,1)}.$$

This implies, using (7.7) and (7.8), that for all $q<\frac{N}{N-1}$, there exists some constant C_q so that for every $x\in M$,

$$\|\nabla G(x,\cdot)\|_{L^q(B(x,1))}\leqslant C_q V(x_0,1)^{-\frac{1}{q}}. \tag{7.9}$$

Using the hypothesis that $\mathcal{V}$ satisfies (L_p) and (7.9) for $q=p'<\frac{N}{N-1}$, we get that there is a constant C such that for every $x\in M$,

$$\int_{B(x,1)}|\nabla G(x,y)|\,|\mathcal{V}(y)|\mathrm{d}y\leqslant C. \tag{7.10}$$

Also, by (7.6) and the fact that $\mathcal{V}$ is in $K^\infty(M)$,

$$\sup_{x\in M}\int_{M\setminus B(x,1)}|\nabla G(x,y)|\,|\mathcal{V}(y)|\mathrm{d}y<\infty. \tag{7.11}$$

Combining (7.10) and (7.11), one obtains that $|\mathrm{d}h|\leqslant|\mathrm{d}\Delta^{-1}\mathcal{V}h|\in L^\infty$.

Let us now assume (2). We write

$$\|\mathrm{d}h\|_{L^r_V}\leqslant\int_0^1\|\nabla e^{-t\Delta}\|_{L^{s_1}_V,L^r_V}\|\mathcal{V}\|_{L^{s_1}_V}\,\mathrm{d}t+\int_1^\infty\|\nabla e^{-t\Delta}\|_{L^{s_2}_V,L^r_V}\|\mathcal{V}\|_{L^{s_2}_V}\,\mathrm{d}t.$$

For $r<p_0$, using Corollary 6.5, one finds

$$\|\mathrm{d}h\|_{L^r_V}\leqslant\|\mathcal{V}\|_{L^{s_1}_V}\int_0^1\varphi_{s_1,r}(t)\,\frac{\mathrm{d}t}{\sqrt{t}}+\|\mathcal{V}\|_{L^{s_2}_V}\int_1^\infty\varphi_{s_2,r}(t)\,\frac{\mathrm{d}t}{\sqrt{t}}.$$

By definition of $\varphi_{p,q}$, the two integrals converge if and only if

$$-\frac{\nu}{2s_1}+\frac{\nu'}{2r}>-\frac{1}{2},\ -\frac{\nu'}{2s_2}+\frac{\nu}{2r}<-\frac{1}{2}.$$

Since r is arbitrarily close to p_0, it is enough to have these two inequalities satisfied for $r=p_0$, i.e.

$$-\frac{\nu}{s_1}+\frac{\nu'}{p_0}>-1,\ -\frac{\nu'}{s_2}+\frac{\nu}{p_0}<-1.$$

This is equivalent to

$$s_2<\frac{\nu' p_0}{p_0+\nu},\ s_1>\frac{\nu p_0}{p_0+\nu'}.$$

We now choose $s_2=\frac{\nu' p_0}{p_0+\nu}-\varepsilon$, and $s_1=\frac{\nu p_0}{p_0+\nu'}+\varepsilon$. By hypothesis, $\mathcal{V}\in L^{s_1}_V\cap L^{s_2}_V$, and the result is proved. $\square$

Proof of Lemma 7.10. — For $p=2$, by a direct consequence of the Green formula, $\mathrm{d}\Delta_{h^2}^{-1/2}$ is an isometry on $L^2(M,h^2\,\mathrm{d}\mu)\simeq L^2(M,\,\mathrm{d}\mu)$. For $p>2$, we want to apply Proposition 6.6, with $A=h^2$. Indeed, assuming that the hypotheses of Proposition 6.6 are satisfied for the choice $A=h^2$, we obtain that $d\mathcal{L}^{-1/2}=\mathrm{d}\Delta_{h^2}^{-1/2}$ is bounded on L^p for every $p\in(2,p_0)$, hence the result of Lemma 7.10. It thus remains to check that the hypotheses of Proposition 6.6 are fulfilled. We check this in the following sequence of lemmas.

Lemma 7.12. — *Under the assumptions of Theorem 7.1, the Riesz transform associated to the weighted Laplacian* $\mathrm{d}\Delta_{h^2}^{-1/2}$ *is bounded on* L^p *for every* $1<p\leqslant 2$.

Proof. — The boundedness on $L^2(M,\,\mathrm{d}\mu)\simeq L^2(M,\,h^2\mathrm{d}\mu)$ follows from the fact that by the Green formula, $\mathrm{d}\Delta_{h^2}^{-1/2}$ is an isometry on $L^2(\Omega,\,h^2\mathrm{d}\mu)$. Since $h\sim 1$, the relative Faber–Krahn inequality (RFK) is satisfied for the weighted Laplacian Δ_{h^2}, and consequently the heat kernel $e^{-t\Delta_{h^2}}$ has Gaussian upper-estimates. Also, the measure $h^2\mathrm{d}\mu$ is doubling since $\mathrm{d}\mu$ is. By [19, Theorem 1.1], the Riesz transform $\mathrm{d}\Delta_{h^2}^{-1/2}$ is bounded on L^p for every $1<p\leqslant 2$. □

Lemma 7.13. — *Under assumptions* (1) *or* (2)*, the local Riesz transform* $\mathrm{d}(\Delta_{h^2}+1)^{-1/2}$ *is bounded on* L^p*, for all* $p\in(2,p_0)$.

Proof. — Notice that $\mathrm{d}(\Delta_{h^2}+1)^{-1/2}$ is bounded on L^2. By interpolation, it is thus enough to prove the boundedness of $\mathrm{d}(\Delta_{h^2}+1)^{-1/2}$ on L^p for p close enough to p_0. Thus, for the rest of the proof, we assume that $p<p_0$ is close enough to p_0. Conjugating by h and using $h\sim 1$, we easily see that the boundedness of $\mathrm{d}(\Delta_{h^2}+1)^{-1/2}$ on L^p is equivalent to

$$D(\Delta+\mathcal{V}+1)^{-1/2}:L^p\to L^p,$$

where we recall that D is the differential operator of degree one defined by

$$D=d\circ h^{-1}.$$

Also,

$$D(\Delta+\mathcal{V}+1)^{-1/2}=\mathrm{d}(h^{-1})(\Delta+\mathcal{V}+1)^{-1/2}+h^{-1}\mathrm{d}(\Delta+\mathcal{V}+1)^{-1/2}. \tag{7.12}$$

It is thus enough to prove that $(\mathrm{d}h)(\Delta+\mathcal{V}+1)^{-1/2}$ and $\mathrm{d}(\Delta+\mathcal{V}+1)^{-1/2}$ are bounded on L^p. We start with $\mathrm{d}(\Delta+\mathcal{V}+1)^{-1/2}$. By a straightforward adaptation of the proof of [3, Theorem 3.9], if $\mathcal{V}$ satisfies

$$\int_0^1\left\|\frac{|\mathcal{V}|^{1/2}}{V(\,\cdot\,,\sqrt{t})^{\frac{1}{r_1}}}\right\|_{r_1}\frac{\mathrm{d}t}{\sqrt{t}}+\int_1^\infty\left\|\frac{|\mathcal{V}|^{1/2}}{V(\,\cdot\,,\sqrt{t})^{\frac{1}{r_2}}}\right\|_{r_2}e^{-t}\,\frac{\mathrm{d}t}{\sqrt{t}}<\infty, \tag{7.13}$$

then $(\Delta+1)^{1/2}(\Delta+\mathcal{V}+1)^{-1/2}$ is bounded on L^r, $r\in(1,\min(r_1,r_2))$. The additional e^{-t} in the second integral comes from the fact that one considers $\Delta+1$ and $\Delta+\mathcal{V}+1$ instead of Δ and $\Delta+\mathcal{V}$ as in [3]. Using $(D_{\nu,\nu'})$ and the hypothesis on $\mathcal{V}$, one sees that (7.13) is satisfied for $r_1=r_2=\nu+2\varepsilon$ in case (1), and $r_1=r_2=p_0$ in case (2). Also, by analyticity of $e^{-t\Delta}$ on L^p, $\Delta^{-1/2}(\Delta+1)^{1/2}$ is bounded on L^p. Thus, writing

$$\mathrm{d}(\Delta+\mathcal{V}+1)^{-1/2}=\left(\mathrm{d}\Delta^{-1/2}\right)\left(\Delta^{1/2}(\Delta+1)^{-1/2}\right)\left((\Delta+1)^{1/2}(\Delta+\mathcal{V}+1)^{-1/2}\right),$$

we see that $\mathrm{d}(\Delta+\mathcal{V}+1)^{-1/2}$ is bounded on L^p for all $p\in(1,p_0)$. Let us now treat the operator $(\mathrm{d}h)(\Delta+\mathcal{V}+1)^{-1/2}$, for which we distinguish two cases.

Case 1: assume that (1) *is satisfied.* — Let us first notice that $(\Delta+\mathcal{V}+1)^{-1/2}$ is bounded on L^p: indeed,

$$\|(\Delta+\mathcal{V}+1)^{-1/2}\|_{p,p}\leqslant C\int_0^\infty e^{-t}\|e^{-t(\Delta+\mathcal{V})}\|_{p,p}\mathrm{d}t, \tag{7.14}$$

furthermore by Theorem 4.1, $e^{-t(\Delta+\mathcal{V})}$ has Gaussian estimates and so it is uniformly bounded on L^p. So, the integral in (7.14) converges, therefore $(\Delta+\mathcal{V}+1)^{-1/2}$ is bounded on L^p. By Lemma 7.11, $dh\in L^\infty$. Therefore, $(\mathrm{d}h)(\Delta+\mathcal{V}+1)^{-1/2}$ is bounded on L^p.

Case 2: assume that (2) *is satisfied.* — Write

$$(\mathrm{d}h)(\Delta+\mathcal{V}+1)^{-1/2}=\int_0^\infty(\mathrm{d}h)e^{-t(\Delta+\mathcal{V})}e^{-t}\,\frac{\mathrm{d}t}{\sqrt{t}}.$$

Since $e^{-t(\Delta+\mathcal{V})}$ has Gaussian estimates, by Corollary 6.4 and Hölder's inequality, one has

$$\|(\mathrm{d}h)(\Delta+\mathcal{V}+1)^{-1/2}\|_{p,p}\leqslant\|\mathrm{d}h\|_{L^r_V}\int_0^\infty\varphi_{p,q}(t)e^{-t}\,\frac{\mathrm{d}t}{\sqrt{t}},$$

with $\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$. The integral $\int_0^\infty\varphi_{p,q}(t)e^{-t}\,\frac{\mathrm{d}t}{\sqrt{t}}$ converges if and only if

$$\frac{\nu'}{2q}-\frac{\nu}{2p}>-\frac{1}{2}. \tag{7.15}$$

Since $p_0>\nu$, (7.15) is satisfied if p is close enough to p_0, and q is big enough. By Lemma 7.11, $\mathrm{d}h\in L^r_V$, for $r<p_0$ close enough to p_0. Taking p close enough to p_0, and q big enough, one can arrange for $r<p_0$ as close as we want to p_0, as well as (7.15) satisfied. Therefore, the operator $(\mathrm{d}h)(\Delta+\mathcal{V}+1)^{-1/2}$ is bounded on L^p. □

BIBLIOGRAPHY

[1] A. Ancona, "First eigenvalues and comparison of Green's functions for elliptic operators on manifolds or domains", *J. Anal. Math.* **72** (1997), p. 45-92.

[2] J. Assaad, "Riesz transforms associated to Schrödinger operators with negative potentials", *Publ. Mat., Barc.* **55** (2011), no. 1, p. 123-150.

[3] J. Assaad & E. M. Ouhabaz, "Riesz transforms of Schrödinger operators on manifolds", *J. Geom. Anal.* **22** (2012), no. 4, p. 1108-1136.

[4] P. Auscher & B. Ben-Ali, "Maximal inequalities and Riesz transform estimates on L^p spaces for Schrödinger operators with nonnegative potentials", *Ann. Inst. Fourier* **57** (2007), no. 6, p. 1975-2013.

[5] P. Auscher, T. Coulhon, X. T. Duong & S. Hofmann, "Riesz transform on manifolds and heat kernel regularity", *Ann. Sci. Éc. Norm. Supér.* **37** (2004), no. 6, p. 911-957.

[6] N. Badr & B. Ben-Ali, "L^p boundedness of the Riesz transform related to Schrödinger operators on a manifold", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **8** (2009), no. 4, p. 725-765.

[7] D. Bakry, "Étude des Transformations de Riesz dans les variétés Riemanniennes à courbure de Ricci minorée", in *Séminaire de probabilités XIX (1983/84)*, Lecture Notes in Mathematics, vol. 1123, Springer, 1985, p. 145-174.

[8] S. Boutayeb, T. Coulhon & A. Sikora, "A new approach to pointwise heat kernel upper bounds on doubling metric measure spaces", *Adv. Math.* **270** (2015), p. 302-374.

[9] P. Buser, "A note on the isoperimetric constant", *Ann. Sci. Éc. Norm. Supér.* **15** (1982), no. 2, p. 213-230.

[10] G. Carron, "Geometric inequalities for manifolds with Ricci curvature in the Kato class", `https://arxiv.org/abs/1612.03027`, 2016.

[11] ———, "Riesz transform on manifolds with quadratic curvature decay", `https://arxiv.org/abs/1403.6278`, 2016.

[12] G. Carron, T. Coulhon & A. Hassell, "Riesz transform and L^p-cohomology for manifolds with Euclidean ends", *Duke Math. J.* **133** (2006), no. 1, p. 59-93.

[13] J. Cheeger, M. Gromov & M. Taylor, "Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds", *J. Differ. Geom.* **17** (1982), no. 1, p. 15-53.

[14] P. Chen, J. Magniez & E. M. Ouhabaz, "Riesz transforms on non-compact manifolds", `https://arxiv.org/abs/1411.0137`, 2014.

[15] Z.-Q. Chen, "Gaugeability and conditional gaugeability", *Trans. Am. Math. Soc.* **354** (2002), no. 11, p. 4639-4679.

[16] T. Coulhon, "Dimensions at infinity for Riemannian manifolds", *Potential Anal.* **4** (1995), no. 4, p. 335-344.

[17] T. Coulhon, B. Devyver & A. Sikora, "Gaussian heat kernel estimates: from functions to forms", `https://arxiv.org/abs/1606.02423`, 2016.

[18] T. Coulhon & N. Dungey, "Riesz transform and perturbation", *J. Geom. Anal.* **17** (2007), no. 2, p. 213-226.

[19] T. Coulhon & X. T. Duong, "Riesz transforms for $1 \leqslant p \leqslant 2$", *Trans. Am. Math. Soc.* **351** (1999), no. 3, p. 1151-1169.

[20] ———, "Riesz transform and related inequalities on noncompact Riemannian manifolds", *Commun. Pure Appl. Math.* **56** (2003), no. 12, p. 1728-1751.

[21] T. Coulhon, I. Holopainen & L. Saloff-Coste, "Harnack inequality and hyperbolicity for subelliptic p-Laplacians with applications to Picard type theorems", *Geom. Funct. Anal.* **11** (2001), no. 6, p. 1139-1191, erratum in *ibid.* **12** (2002), no. 11, p. 217.

[22] T. Coulhon & L. Saloff-Coste, "Variétés Riemanniennes isométriques à l'infini", *Rev. Mat. Iberoam.* **11** (1995), no. 3, p. 687-726.

[23] T. Coulhon, L. Saloff-Coste & N. Varopoulos, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, 1992.

[24] T. Coulhon & Q. S. Zhang, "Large time behavior of heat kernels on forms", *J. Differ. Geom.* **77** (2007), no. 3, p. 353-384.

[25] E. B. Davies & B. Simon, "L^p norms of noncritical Schrödinger semigroups", *J. Funct. Anal.* **102** (1991), no. 1, p. 95-115.

[26] B. Devyver, "A Gaussian estimate for the heat kernel on differential forms and application to the Riesz transform", *Math. Ann.* **358** (2014), no. 1-2, p. 25-68.

[27] ———, "A perturbation result for the Riesz transform", *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* **14** (2015), no. 3, p. 937-964.

[28] B. Devyver, M. Fraas & Y. Pinchover, "Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon", *J. Funct. Anal.* **266** (2014), no. 7, p. 4422-4489.

[29] D. Ganguly & Y. Pinchover, "On the equivalence of heat kernels of second-order parabolic operators", `https://arxiv.org/abs/1606.08601`, 2016.

[30] A. Grigoryan, "Heat kernel upper bounds on a complete non-compact manifold", *Rev. Mat. Iberoam.* **10** (1994), no. 2, p. 395-452.

[31] ———, "Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds", *Bull. Am. Math. Soc.* **36** (1999), no. 2, p. 135-249.

[32] ———, "Heat kernels on weighted manifolds and applications", in *The ubiquitous heat kernel*, Contemporary Mathematics, vol. 398, American Mathematical Society, 2006, p. 93-191.

[33] ———, *Heat Kernel and Analysis on Manifolds*, AMS/IP Studies in Advanced Mathematics, vol. 47, American Mathematical Society; International Press, 2009, xvii+482 pages.

[34] A. Grigoryan, J. Hu & K.-S. Lau, "Heat kernels on metric spaces with doubling measure", in *Fractal geometry and stochastics IV*, Progress in Probability, vol. 61, Birkhäuser, 2009, p. 3-44.

[35] C. Guillarmou & A. Hassell, "Resolvent at low energy and Riesz transform for Schrödinger operators on asymptotically conic manifolds. I", *Math. Ann.* **341** (2008), no. 4, p. 859-896.

[36] I. Holopainen, "Volume growth, Green's functions, and parabolicity of ends", *Duke Math. J.* **97** (1999), no. 2, p. 319-346.

[37] C. LeBrun, "Complete Ricci-flat Kähler metrics on $\mathbb{C}^n$ need not be flat", in *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, Proceedings of Symposia in Pure Mathematics, vol. 52, American Mathematical Society, 1991, p. 297-304.

[38] P. Li, *Geometric analysis*, Cambridge Studies in Advanced Mathematics, vol. 134, Cambridge University Press, 2012, x+406 pages.

[39] M. Murata, "Semismall perturbations in the Martin theory for elliptic equations", *Isr. J. Math.* **102** (1997), p. 29-60.

[40] Y. Pinchover, "On positive solutions of second-order elliptic equations, stability results, and classification", *Duke Math. J.* **57** (1988), no. 3, p. 955-980.

[41] ———, "Criticality and ground states for second-order elliptic equations", *J. Differ. Equations* **80** (1989), no. 2, p. 237-250.

[42] ———, "On criticality and ground states of second order elliptic equations. II", *J. Differ. Equations* **87** (1990), no. 2, p. 353-364.

[43] ———, "On the equivalence of Green functions of second order elliptic equations in $\mathbb{R}^n$", *Differ. Integral Equ.* **5** (1992), no. 3, p. 481-493.

[44] ———, "Maximum and anti-maximum principles and eigenfunctions estimates via perturbation theory of positive solutions of elliptic equations", *Math. Ann.* **314** (1999), no. 3, p. 555-590.

[45] ———, "Topics in the theory of positive solutions of second-order elliptic and parabolic partial differential equations", in *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday*, Proceedings of Symposia in Pure Mathematics, vol. 76, American Mathematical Society, 2007, p. 329-355.

[46] Y. Pinchover & K. Tintarev, "A ground state alternative for singular Schrödinger operators", *J. Funct. Anal.* **230** (2006), no. 1, p. 65-77.

[47] ———, "Ground state alternative for p-Laplacian with potential term", *Calc. Var. Partial Differ. Equ.* **28** (2007), no. 2, p. 179-201.

[48] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, vol. 289, Cambridge University Press, 2002, x+190 pages.

[49] Z. W. Shen, "L^p estimates for Schrödinger operators with certain potentials", *Ann. Inst. Fourier* **45** (1995), no. 2, p. 513-546.

[50] A. Sikora, "Riesz transform, Gaussian bounds and the method of wave equation", *Math. Z.* **247** (2004), no. 3, p. 643-662.

[51] B. Simon, "Brownian motion, L^p properties of Schrödinger operators and the localization of binding", *J. Funct. Anal.* **35** (1980), no. 2, p. 215-229.

[52] M. Takeda, "Conditional gaugeability and subcriticality of generalized Schrödinger operators", *J. Funct. Anal.* **191** (2002), no. 2, p. 343-376.

[53] ———, "Gaussian bounds of heat kernels for Schrödinger operators on Riemannian manifolds", *Bull. Lond. Math. Soc.* **39** (2007), no. 1, p. 85-94.

[54] Q. S. Zhang, "On a parabolic equation with a singular lower order term. II. The Gaussian bounds", *Indiana Univ. Math. J.* **46** (1997), no. 3, p. 989-1020.

[55] ———, "Global bounds of Schrödinger heat kernels with negative potentials", *J. Funct. Anal.* **182** (2001), no. 2, p. 344-370.

[56] ———, "A sharp comparison result concerning Schrödinger heat kernels", *Bull. Lond. Math. Soc.* **35** (2003), no. 4, p. 461-472.

[57] Z. Zhao, "Subcriticality and gaugeability of the Schrödinger operator", *Trans. Am. Math. Soc.* **334** (1992), no. 1, p. 75-96.

Manuscrit reçu le 26 mars 2015,
révisé le 11 décembre 2016,
accepté le 24 janvier 2017.

Baptiste DEVYVER
Technion, Israel Institute of Technology
Dept. of mathematics
Haifa (Israel)
devyver@technion.ac.il