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TWISTED EIGENVARIETIES AND SELF-DUAL REPRESENTATIONS

by Zhengyu XIANG

Abstract. — For a reductive group G and a finite order Cartan-type automorphism ι of G, we construct an eigenvariety parameterizing ι-invariant cuspidal Hecke eigensystems of G. In particular, for $G = Gl_n$, we prove, any self-dual cuspidal Hecke eigensystem can be deformed in a p-adic family of self-dual cuspidal Hecke eigensystems containing a Zariski dense subset of classical points.

Résumé. — Pour un groupe réductif G et un automorphisme d'ordre fini ι de type Cartan de G nous construisons une variété propre paramétrant les systèmes propres de Hecke automorphes cuspidaux ι-invariants de G. En particulier, pour $G = Gl_n$, on prouve que chaque système propre de Hecke cuspidale autoduale de pente finie peut être déformé dans une famille p-adique de sytèmes propres de Hecke cuspidaux autoduaux contenant un sous-ensemble Zariski-dense de points classiques.

1. Introduction

Let G be a reductive group over $\mathbb{Q}$, consider the locally symmetric space $S_G(K_f)$ associated to G and a neat open compact subgroup K_f of $G(\mathbb{A}_f)$, the finite adelic points of G. If T is a maximal torus of G and λ a regular dominant algebraic weight of G with respect to T, consider $\mathbb{V}_\lambda$, the finite dimensional irreducible algebraic representation of G with highest weight λ, and its dual $\mathbb{V}_\lambda^\vee$. There is a standard action of the Hecke algebra $\mathcal{H}_G$ on the cohomology spaces $H^*(S_G(K_f), \mathbb{V}_\lambda^\vee(\mathbb{C}))$. An automorphic representation that can be realized in $H^*(S_G(K_f), \mathbb{V}_\lambda^\vee(\mathbb{C}))$ is said of level K_f and of cohomological weight λ.

Once fixed a prime number p and an embedding $i_p : \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$, we are interested in the behavior of automorphic representations when their weights

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varying p-adically. This leads to the study of p-adic automorphic representations. For simplicity, assume G splits over $\mathbb{Q}_p$. Let B be a Borel subgroup of $G_{/\mathbb{Q}_p}$ containing $T_{/\mathbb{Q}_p}$, consider the situation $K_f = K^p I_m$, where K^p is open compact in $G(\mathbb{A}_f^p)$ and I_m is an Iwahori subgroup of $G(\mathbb{Q}_p)$ in a good position with respect to the pair (B,T). Let $\mathcal{H}_p$ be the p-adic Hecke algebras of G under this setting (see Section 2.1). If π is a cuspidal automorphic representation of G whose cohomological weight λ^{alg} is algebraic, its p-stabilizations are irreducible representations of $\mathcal{H}_p$ that can be realized in the cohomology space $H^*(S_G(K_f), \mathcal{V}_\lambda^\vee(\overline{\mathbb{Q}}_p))$ (refer [18, Section 4.1.9]), where $\lambda = \lambda^{\mathrm{alg}}\epsilon$ is a p-adic arithmetic weight obtained by twisting λ^{alg} with some finite order character ϵ of $T(\mathbb{Z}_p)$, and $\mathcal{V}_\lambda$ is the locally algebraic induced representation of a p-adic cell of $G(\mathbb{Q}_p)$ from λ (see Section 2.3). Those representations of $\mathcal{H}_p$ from p-stabilization are most important examples of p-adic automorphic representations, and are called classical. Let $\mathcal{S}$ be the finite subset of "bad" places defined in Section 2.1, further removing the information over $\mathcal{S}$ form $\mathcal{H}_p$, we obtain a commutative algebra $R_{\mathcal{S},p}$, which can be identified in the center of $\mathcal{H}_p$. The central character of a classical p-adic automorphic representation defines a character of $R_{\mathcal{S},p}$ appearing in $H^*(S_G(K_f), \mathcal{V}_\lambda^\vee(\overline{\mathbb{Q}}_p))$ for some arithmetic p-adic weight λ. It is called a p-adic arithmetic Hecke eigensystem of weight λ.

One is interested in interpolating the arithmetic Hecke eigensystems for weight λ over the p-adic weight space $\mathfrak{X}$. For this, Ash and Stevens developped the notion of "overconvergent" cohomology, which played the role of "overconvergent modular forms" in the classical theory of p-adic modular forms ([4]). Concretely, for a p-adic weight $\lambda \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, there is a $\mathbb{Q}_p$-Fréchet space $\mathcal{D}_\lambda$, on which the U_p operators acting as compact operaters (see Section 2.3). This gives an action of $\mathcal{H}_p$ on the "overconvergent" cohomology spaces $H^*(S_G(K_f), \mathcal{D}_\lambda)$. We call an irreducible representation of $\mathcal{H}_p$ (resp. a character of $R_{\mathcal{S},p}$) appearing in $H^*(S_G(K_f), \mathcal{D}_\lambda(\overline{\mathbb{Q}}_p))$ a p-adic overconvergent automorphic representation (resp. Hecke eigensystem). According to [4], [18, Theorem 5.4.4] and [20, Corollary 8.6], that every finite slope arithmetic cuspidal Hecke eigensystem θ lies in a family of finite slope overconvergent cuspidal Hecke eigensystems whose weights vary p-adical analytically. This result can be a consequence of the theory of "eigenvariety". An eigenvariety for group G is a rigid analytic space whose points parametrize finite slope overconvergent Hecke eigensystems. A large part of the work in [4], [18] and [20] mentioned above are devoted to the construction of eigenvarieties for different groups.

There are two motivations for this paper. The first one is about the arithmeticity of a family of overconvergent Hecke eigensystems, that is, if the family contains enough arithmetic Hecke eigensystems. In the language of eigenvariety, it asks an irreducible component of an eigenvareity containing a Zariski dense subset of arithmetic points (such a component is called arithmetic. If (modulo twisting) a Hecke eigensystem θ is *not* in *any* arithmetic component, it is called arithmetically rigid. For a concrete definition, see [3] or Section 7.3 below). In [3], Ash, Pollack and Stevens show that the answer is not always positive, in particular, for $G = Gl_3$, they make the next conjecture:

Conjecture 1.1 (Ash–Pollack–Stevens). — *Let θ be a finite slope cuspidal Hecke eigensystem of Gl_3. If θ is not arithmetically rigid, then θ is essentially self-dual.*

In this paper, we obtain the inverse of its statement for Gl_n:

Theorem 1.2. — *Every essentially self-dual finite slope cuspidal Hecke eigensystem of Gl_n is not arithmetically rigid.*

We actually work on a more general situation. Let ι be Cartan-type automorphism of G such that ι stabilizes $(B,T)_{/\mathbb{Q}_p}$, and consider the ι-invariant automorphic representations (resp. overconvergent representations, Hecke eigensystems, etc.). Let $\mathfrak{X}^\iota$ be the subspace of $\mathfrak{X}$ consisting of ι-invariant p-adic weights (see Section 2.2), to study families of ι-invariant Hecke eigensystems with weights varying in $\mathfrak{X}^\iota$, we construct twisted eigenvarieties over $\mathfrak{X}^\iota$ parametrizing ι-invariant finite slope overconvergent Hecke eigensystems (see Section 6):

Theorem 1.3 (twisted eigenvarities). — *There is an eigenvariety $\mathfrak{K}^\iota_{K^p}$ parameterizing ι-invariant finite slope overconvergent Hecke eigensystems of G, that is, $\mathfrak{K}^\iota_{K^p}$ is a rigid analytic space such that every point $y \in \mathfrak{K}^\iota_{K^p}(\overline{\mathbb{Q}}_p)$ can be viewed as a pair (λ,θ), where θ is a ι-invariant finite slope overconvergent Hecke eigensystem of weight $\lambda \in \mathfrak{X}^\iota(\overline{\mathbb{Q}}_p)$. There is a subvariety $\mathfrak{E}^\iota_{K^p}$ of $\mathfrak{K}^\iota_{K^p}$, satisfying:*

1. *For any arithmetic $(\lambda,\theta) \in \mathfrak{K}^\iota_{K^p}(\overline{\mathbb{Q}}_p)$, (λ,θ) is in $\mathfrak{E}^\iota_{K^p}(\overline{\mathbb{Q}}_p)$ if and only if θ is cuspidal and has a non-trivial ι-twisted Euler–Poincare characteristic.*
2. *Every irreducible component of $\mathfrak{E}^\iota_{K^p}$ is arithmetic, equipped with a projection onto a Zariski dense subset of $\mathfrak{X}^\iota$.*
3. *$\mathfrak{E}^\iota_{K^p}$ is equidimensional with the same dimension to $\mathfrak{X}^\iota$.*

In particular, if $G = Gl_n$ and ι is of Cartan-type, the ι-invariance is same to the self-duality. Then the twisted Euler–Poincare characteristic of an (essentially) self-dual Hecke eigensystem is always non-trivial. So $\mathfrak{E}^{\iota}_{K^p}$ parameterizes all self-dual finite slope cuspidal Hecke eigensystems (see Section 7). In case that $n = 3$, this also gives some hint for Ash–Pollack–Stevens' conjecture, as in Theorem 7.8 and Remark 7.9 below.

Our second motivation is to develop a twisted version of Urban's theory of finite slope character distribution [18, Section 4.5]. A finite slope character distribution is a morphism $J : \mathcal{H}_p \to \overline{\mathbb{Q}}_p$ which is a linear combination of the traces of finite slope overconvergent representations. Urban proves that, there is an eigenvariety associated to every analytic family of effective finite slope character distributions, [18, Section 5]. This eigenvariety parameterizes the finite slope overconvergent Hecke eigensystems appearing in the character distributions. However, Urban's theory excludes many interesting cases, like Gl_n with $n > 2$. The reason is, the coefficients of Urban's distributions are essentially given by the Euler–Poincare characteristics. So for a group G such that $G(\mathbb{R})$ does not satisfy the Harish–Chandra condition, they are trivial. To avoids this issue, we introduce the notion of "twisted" finite slope character distributions (see Section 5). Concretely, we construct a distribution which is a linear combination of the twisted traces of ι-invariant finite slope overconvergent representations. We show this distribution has similar properties as Urban's character distributions, in particular, it gives a construction of the twisted eigenvariety $\mathfrak{E}^{\iota}$ in the theorem above (see Section 6).

In practice, there are two new difficulties. The first one is the lack of a twisted version of Franke's trace formula as in [18, Theorem 1.4.2], which plays an essential role to cut out the cuspidal representations from the whole cohomology. To cure this, we have to go through Franke's theory of Eisenstein spectral sequence ([11]), and study carefully how ι acting on each step of Franke's theory. This is done in Section 4, where we proves a twisted version of Franke's trace formula (Theorem 4.1). The second difficulty appears during the construction of the twisted eigenvariety. Since we consider the twisted traces, locally our twisted distributions are no longer pseudo-representations as in [18, Section 5.3.1], so we do not have the "second construction" as Urban did ([18, Section 5.3]). We bypass this difficulty by borrowing the construction of the full eigenvariety in [20] to construct a "bigger twisted eigenvariety" first and then working in this bigger space. This is done in Section 6.3.

One can view Urban's finite slope character distribution as a p-adic analogue to Selberg's trace formula, then our theory gives an analogue to the twisted trace formula. In [18, Section 6], Urban gives a simplified geometric expansion of his distribution following the work of Franke [11] and Arthur [1]. A complete expansion as [1, (3)] can also be given. In a consequent paper [21], we will establish a geometric expansion of our twisted distributions as well. One can then expect a p-adic family version of Arthur–Clozel's comparison theory ([2]). we hope this comparison will give a relation between eigenvarieties.

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2. Preliminary

2.1. Notation

Throughout this paper, we fix p a rational prime number and an identification $\hat{\overline{\mathbb{Q}}}_p \cong \mathbb{C}$. Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the adelic ring of $\mathbb{Q}$, $\mathbb{A}_\infty$ and $\mathbb{A}_f$ its archimedean and finite part respectively. For any algebraic group H over $\mathbb{Q}$, put $H_\infty = H(\mathbb{A}_\infty)$ and $H_f = H(\mathbb{A}_f)$. We also denote by $H(\mathbb{A})^1 \subset H(\mathbb{A})$ the subgroup of all $h \in H(\mathbb{A})$ with $\prod_v |\xi(h)|_v = 1$ for all characters of H defined over $\mathbb{Q}$, where the product is running over all places of $\mathbb{Q}$.

Let G be a quasi-split[(1)] reductive group over $\mathbb{Q}$, denote by $Z = Z_G$ its center. Let K_∞ be a fixed maximal compact subgroup of G_∞, and fix a good maximal compact subgroup $\mathbb{K} \subset G(\mathbb{A})$ whose archimedean component is K_∞. For every prime number l, denote by K_l an open compact subgroup of $G(\mathbb{Q}_l)$. Put $K_f = \prod_l K_l$ such that for almost all $l \neq p$, K_l to be maximal. Denote by $K^p = K_f^p = \prod_{l \neq p} K_l$ and $K = K_\infty K_f$. Consider the locally symmetric space of G associated to K_f:

$$S_G(K_f) := G(\mathbb{Q})\backslash G(\mathbb{A})/KZ_\infty. \tag{2.1}$$

Properly choose a finite set of representatives $\{g_i\}_i$ in $G(\mathbb{A})$ such that

$$G(\mathbb{A}) = \bigsqcup_i G(\mathbb{Q}) \times G_\infty^+ \times g_i K_f, \tag{2.2}$$

(1) This assumption is not necessary but for the convenience of discussion only. Otherwise, one has to use the notation as in [18, Section 1.3.1].

where G_∞^+ is the identity component of G_∞. We then have

$$S_G(K_f) \cong \bigsqcup_i \Gamma_i \setminus \mathcal{H}_G, \tag{2.3}$$

where $\Gamma_i = \Gamma(g_i, K)$ is the image of $g_i K g_i^{-1} \cap G(\mathbb{Q})^+$ in $G^{ad}(\mathbb{Q})$ and $\mathcal{H}_G = G_\infty^+/K_\infty Z_\infty \cap G_\infty^+$. We further assume K is neat (that is, Γ_i contains no element of finite order), then $S_G(K_f)$ is a smooth real analytic variety of a finite dimension, say, d. We also write

$$S_G := \varinjlim_{K_f} S_G(K_f). \tag{2.4}$$

Let T be a maximal torus of G and B a Borel subgroup of G containing T. Let N be the unipotent radical of B, and N^- its opposite. At p, we fix a Iwahori subgroup I of $G(\mathbb{Q}_p)$ with respect to B, this means that we have fixed compatible integral models $\mathcal{G}, \mathcal{B}, \mathcal{T}, \mathcal{N}, \mathcal{N}^-$ for G, B, T, N, N^- over $\mathbb{Z}_p$ (according to a fixed chamber C_I of the Bruhat–Tits building $\mathcal{BL}$ of $G_{\mathbb{Q}_p}$), such that $I = I_1$, where for any integer $m \geqslant 1$,

$$I_m = \{g \in \mathcal{G}(\mathbb{Z}_p) \,|\, g \in \mathcal{B}(\mathbb{Z}/p^m\mathbb{Z}) \mod p^m\} \tag{2.5}$$

is the Iwahori subgroup of depth m. By Iwahori decomposition,

$$I_m = (I_m \cap N^-(\mathbb{Q}_p))\mathcal{T}(\mathbb{Z}_p)\mathcal{N}(\mathbb{Z}_p). \tag{2.6}$$

We normalize the Haar measure on $G(\mathbb{Q}_p)$ such that the measure of I is 1. Once fixing the Iwahori level at p, we write

$$\tilde{S}_{G,m} := \varinjlim_{K_f^p} S_G(K_f^p I_m). \tag{2.7}$$

Now put

$$T^+ := \{t \in T(\mathbb{Q}_p) \,|\, t\mathcal{N}(\mathbb{Z}_p)t^{-1} \subset \mathcal{N}(\mathbb{Z}_p)\} \tag{2.8}$$

$$T^{++} := \left\{ t \in T^+ \;\middle|\; \bigcap_{i \geqslant 1} t^i \mathcal{N}(\mathbb{Z}_p) t^{-i} = \{1\} \right\}, \tag{2.9}$$

$$\Delta_m^+ := I_m T^+ I_m,\ \Delta_m^{++} := I_m T^{++} I_m, \tag{2.10}$$

and consider the p-adic cells

$$\Omega_m = I_m \cap N^-(\mathbb{Q}_p)\backslash I_m \subseteq N^-(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p). \tag{2.11}$$

For any $g \in \Delta_m^+$, write $g = n_g^- t_g n_g^+$ by Iwahori decomposition, then the $*$-action of Δ_m^+ on Ω_m is defined as follow (see [4, Section 5.2] and [18, Section 3.1.3]): Fixing a splitting ξ of the exact sequence

$$1 \to \mathcal{T}(\mathbb{Z}_p) \to T(\mathbb{Q}_p) \to T(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) \to 1, \tag{2.12}$$

for any $[x] \in \Omega$, define

$$[x] * g = [\xi(t_g)^{-1} x g]. \tag{2.13}$$

As in [18, Section 3.1.2 (11)], we choose ξ so that for any algebraic character $\lambda^{\mathrm{alg}} \in X^*(T)$ and $t \in T(\mathbb{Q}_p)$

$$\lambda^{\mathrm{alg}}(\xi(t)) = |\lambda^{\mathrm{alg}}(t)|_p^{-1}. \tag{2.14}$$

The Atkin–Lehner algebra of G at p is defined by:

$$\mathcal{U}_p = C_c^\infty(\Delta_m^+ // I_m, \mathbb{Z}_p) \simeq \mathbb{Z}_p[T^+/\mathcal{T}(\mathbb{Z}_p)], \tag{2.15}$$

which does not depend on the depth m. We then define the global p-adic Hecke algebras:

$$\mathcal{H}_p := \mathcal{H}_p(G) = C_c^\infty(G(\mathbb{A}_f^p)) \otimes \mathcal{U}_p, \tag{2.16}$$

and for any open compact subgroup K^p of $G(\mathbb{A}_f^p)$, define its subalgebra of K^p-bi-invariant functions by:

$$\mathcal{H}_p(K^p) = C_c^\infty(K^p \backslash G(\mathbb{A}_f^p)/K^p) \otimes \mathcal{U}_p \tag{2.17}$$

Given $t \in T^+$, denote by u_t the element in $\mathcal{U}_p$ whose image in $\mathbb{Z}_p[T^+/T(\mathbb{Z}_p)]$ is t. The operator u_t can be viewed as the double coset operator $I_m t I_m$ as well. A Hecke operator f is called admissible, if $f = f^p \otimes u_t$ and $t \in T^{++}$. We denote by $\mathcal{H}'_p$ the subalgebra of $\mathcal{H}_p$ generated by admissible operators. For fixed K^p, let $\mathcal{S}$ be the finite set of primes l such that K_l is not maximal, define

$$R_{\mathcal{S},p} := C_c^\infty\left(G\left(\mathbb{A}_f^{\mathcal{S}\cup\{p\}}\right) // K^{\mathcal{S}\cup\{p\}}\right) \otimes \mathcal{U}_p \tag{2.18}$$

$R_{\mathcal{S},p}$ is commutative and can be identified in the center of $\mathcal{H}_p(K^p)$.

Throughout this paper, we assume that G has a finite order automorphism ι of Cartan-type, that is, at ∞, ι is of the form $\mathrm{ad}(g_\infty) \circ \theta$, for some $g_\infty \in G_\infty$ and the Cartan involution θ (with respect to K_∞). It is innocuous to assume that the triples (B, T, I_m) are stable under ι. Indeed, let (B, T, I_m) be such a triple and ψ_0 the based root datum associated to it, consider the splitting exact sequence [17, 2.14]:

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \xrightarrow{\beta} \mathrm{Aut}(\psi_0) \to 1. \tag{2.19}$$

If (B, T, I_m) is not stable under ι, we fix a splitting

$$\gamma : \mathrm{Aut}(\psi_0) \xrightarrow{\cong} \mathrm{Aut}(G, B, T, \{u_\alpha\}) \hookrightarrow \mathrm{Aut}(G) \tag{2.20}$$

and replace ι by its image ι' under $\gamma\beta$, then ι' fixes the pair (B, T). Since ι' has the same image under β as ι, it differs ι by a conjugation. So ι' is also of Cartan-type. Since $\mathrm{Aut}(\psi_0)$ is finite, that ι' is of finite order. Finally, noticing that $\{u_\alpha\}$ is the set of an arbitrary choice of nontrivial $u_\alpha \in U_\alpha$ in each unipotent root subgroup U_α associated to the basis $\{\alpha\}$ in ψ_0, we can properly choose $\{u_\alpha\}$ such that each u_α corresponds to a wall of a same chambre $\mathcal{C}$ in $\mathcal{BL}$. $\mathcal{C}$ gives an Iwahori subgroup which is stable under ι'.

Further assuming that K_f^p is stable under ι, we define ι acting on the Hecke algebra $\mathcal{H}_p(K^p)$ by sending f to f^ι, where $f^\iota(g) := f(g^{\iota^{-1}})$ for any $g \in G$. This is well defined since that T^+ and T^{++} are stable under ι by (2.8) and (2.9). Moreover, for $u_t \in \mathcal{U}_p$, it can be verified directly that

$$u_t^\iota = u_{t^\iota}. \tag{2.21}$$

2.2. Weight spaces

2.2.1. Classical weight and co-weight

Let $X^*(T)$ be the set of algebraic weights of T, and $X_*(T)$ the set of algebraic co-weights. There is a canonical duality pairing

$$(\,\cdot\,,\cdot\,) : X^*(T) \times X_*(T) \to \mathbb{Z} \tag{2.22}$$

such that for any $\lambda \in X^*(T)$, $\mu^\vee \in X_*(T)$ and $a \in \mathbb{G}_m$,

$$\lambda \circ \mu^\vee(a) = a^{(\lambda,\mu^\vee)} \tag{2.23}$$

We define ι acting on $X^*(T)$ by sending λ to λ^ι such that $\lambda^\iota(t) = \lambda(t^{\iota^{-1}})$ for any $t \in T$, and define ι acting on $X_*(T)$ by sending $\mu^\vee$ to $(\mu^\vee)^\iota$ such that $(\mu^\vee)^\iota(a) = (\mu^\vee(a))^{\iota^{-1}}$ for any $a \in \mathbb{G}_m$. One can verify directly that

$$(\lambda^\iota, (\mu^\vee)^\iota) = (\lambda, \mu^\vee). \tag{2.24}$$

2.2.2. p-adic weight space

There is a rigid space $\mathfrak{X}_T$ associated to $T_{/\mathbb{Q}_p}$, such that for any field $L \subset \overline{\mathbb{Q}}_p$,

$$\mathfrak{X}_T(L) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{T}(\mathbb{Z}_p), L^\times). \tag{2.25}$$

Since $\mathcal{T}(\mathbb{Z}_p) \cong \mathbb{Z}_p^r \times \Pi$, with some finite group Π, that

$$\mathfrak{X}_T(\overline{\mathbb{Q}}_p) \cong \mathrm{Hom}_{qp}(\Pi, \overline{\mathbb{Q}}_p^\times) \times (B_{1,1}(\overline{\mathbb{Q}}_p)^\circ)^r. \tag{2.26}$$

So the underlying space of $\mathfrak{X}_T$ is finite many copies of the r-tuple open unit ball, whose points are (continuous) p-adic weights. Put $Z_{K^p} = Z(\mathbb{Q}) \bigcap K^p I$ and let $\mathfrak{X} := \mathfrak{X}_{K^p} \subseteq \mathfrak{X}_T$ be the Zariski closure of the subset of p-adic weights which are trivial on Z_{K^p}. The automorphism ι induces a operator on $\mathfrak{X}$ which sends λ to λ^ι, where $\lambda^\iota(t) := \lambda(t^{\iota^{-1}})$ for any $t \in \mathcal{T}(\mathbb{Z}_p)$. Denote by $\mathfrak{X}^\iota$ the subspace of $\mathfrak{X}$ consisting of ι-invariant weights.

Recall, for any n, there is a rigid space $\mathfrak{T}_n$ such that for any field $L \subset \overline{\mathbb{Q}}_p$,

$$\mathcal{O}(\mathfrak{T}_{\mathfrak{n}/L}) = \mathcal{A}_n(\mathcal{T}(\mathbb{Z}_p), L), \tag{2.27}$$

where $\mathcal{A}_n(\mathcal{T}(\mathbb{Z}_p), L)$ is the space of locally n-analytic L-valued functions on $\mathcal{T}(\mathbb{Z}_p)$. The natural pairing

$$\mathfrak{X}_T(L) \times \mathcal{T}(\mathbb{Z}_p) \to L^\times, (\lambda, t) \mapsto \lambda(t) \tag{2.28}$$

induces a continuous injective homomorphism $\mathcal{T}(\mathbb{Z}_p) \hookrightarrow \mathcal{O}(\mathfrak{X}_T)^\times$.

LEMMA 2.1. — *For any affinoid subdomain $\mathfrak{U} \subseteq \mathfrak{X}$ or $\mathfrak{X}^\iota$, there exist a smallest integer $n(\mathfrak{U})$, such that for any finite extension L of $\mathbb{Q}_p$, every element $\lambda \in \mathfrak{U}(L)$ is $n(\mathfrak{U})$-locally analytic. Moreover, there is a rigid analytic map $\mathfrak{U} \times \mathfrak{T}_{n(\mathfrak{U})} \to B_{1,1}$, such that for any L, its realization at L-points is the pairing* (2.28).

It follows immediately from [18, Lemma 3.4.6].

2.3. Analytic induced modules and distribution spaces

2.3.1. Induced modules

We firstly recall necessary definitions from [18, Section 3.2] and introduce varies induced modules. Let F be the splitting field for G and assume $(B, T)_{/F}$ is a Borel pair contained in some minimal p-pair. For $\lambda^{\mathrm{alg}} \in X^*(T_{/F})$, let $\mathbb{V}_{\lambda^{\mathrm{alg}}}$ be the finite dimensional irreducible algebraic representation of G with highest weight λ^{alg} over F. Concretely speaking, for any subfield $F' \subset \mathbb{C}$ containing F, it can be viewed as the algebraic induced representation:

$$\mathbb{V}_{\lambda^{\mathrm{alg}}}(F') = \mathrm{ind}_{B(F')}^{G(F')}(\lambda^{\mathrm{alg}})^{\mathrm{alg}}. \tag{2.29}$$

We can identify λ^{alg} with the p-adic weight obtained by the composition

$$\mathcal{T}(\mathbb{Z}_p) \hookrightarrow T(F) \xrightarrow{\lambda^{\mathrm{alg}}} F^\times \hookrightarrow \overline{\mathbb{Q}}_p^\times. \tag{2.30}$$

Given any finite extension $L/\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ such that $F \subset L$ (under the fixed embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$), let $\epsilon : \mathcal{T}(\mathbb{Z}_p) \to L^\times$ be a finite character factoring through $\mathcal{T}(\mathbb{Z}_p/p^m\mathbb{Z}_p)$, one can consider the p-adic weight $\lambda = \lambda^{\mathrm{alg}}\epsilon$, and its m-locally analytic induction

$$\mathcal{V}_\lambda(L) = \mathrm{ind}_{B(L)}^{G(L)}(\lambda)^{m-an}. \tag{2.31}$$

There is a natural map

$$\mathbb{V}_{\lambda^{\mathrm{alg}}}(L)(\epsilon) \hookrightarrow \mathcal{V}_\lambda(L). \tag{2.32}$$

The $*$-action described in Section 2.1 induces an action of Δ^+ on $\mathcal{V}_\lambda(L)$, via the right $*$-translation.

Now for any $\lambda \in \mathfrak{X}(L)$, let $\mathcal{A}_\lambda(L)$ be the space of locally L-analytic functions f on I such that

$$f(n^- tg) = \lambda(t) f(g), \tag{2.33}$$

where, as in (2.6), $n^- \in I \cap N^-(\mathbb{Q}_p)$, $t \in \mathcal{T}(\mathbb{Z}_p)$ and $g \in I$. $\mathcal{A}_\lambda(L)$ can be viewed as a subspace of $\mathcal{A}(\Omega_1, L)$, the space of locally L-analytic functions on Ω_1: let $\mathcal{T}(\mathbb{Z}_p)$ act on $\mathcal{A}(\Omega_1, L)$ by the natural translation, then

$$\mathcal{A}_\lambda(L) = \mathcal{A}(\Omega_1, L)[\lambda] := \{\phi \in \mathcal{A}(\Omega_1, L) \,|\, t\phi = \lambda(t)\phi\} \tag{2.34}$$

The $*$-action Δ^+ on $\mathcal{A}(\Omega_1, L)$ is naturally defined, it commutes with the translation of $\mathcal{T}(\mathbb{Z}_p)$. So the $*$-action of Δ^+ is well defined on $\mathcal{A}_\lambda(L)$. For $g \in \Delta^+$ and $\phi \in \mathcal{A}_\lambda(L)$, we define

$$g * \phi([x]) := \phi([x] * g). \tag{2.35}$$

Now define the L-valued distribution space

$$\mathcal{D}_\lambda(L) := \mathrm{Hom}_{\mathrm{cont}}(\mathcal{A}_\lambda(L), L), \tag{2.36}$$

the continuous dual of $\mathcal{A}_\lambda(L)$. The $*$-action of Δ^+ on $\mathcal{D}_\lambda(L)$ is naturally defined. A deatiled study of $\mathcal{A}_\lambda(L)$ and $\mathcal{D}_\lambda(L)$ can be found in [18, Lemma 3.2.8], in particular, we have next proposition:

Proposition 2.2. — *$\mathcal{D}_\lambda(L)$ is a compact Fréchet space over L. If $\delta \in \Delta^{++}$, then the $*$-action of δ defines a compact operator on $\mathcal{D}_\lambda(L)$.*

Remark 2.3. — The theory of compact operators on orthonormalizable (p-adic) Banach spaces is originally due to Serre and generalized by Coleman [10]. The theory is generalized to compact Fréchet spaces by Urban in [18, section 2], where he shows that most results of compact operators on Banach spaces still hold for compact Fréchet spaces.

For $\lambda \in \mathfrak{X}^\iota$, ι acts on $\mathbb{V}_\lambda$, $\mathcal{V}_\lambda$ and $\mathcal{A}_\lambda$. Concretely, let f be a function on $N^-(L)\backslash G(L)$, define $f^\iota(g) = f(g^{\iota^{-1}})$. If f is in one of those induced modules, $f^\iota(bg) = f(b^{\iota^{-1}} g^{\iota^{-1}}) = \lambda(t^{\iota^{-1}}) f(g^{\iota^{-1}}) = \lambda(t) f^\iota(g)$, for any $b = tn \in B$. So $\mathbb{V}_\lambda$, $\mathcal{V}_\lambda$ and $\mathcal{A}_\lambda$ are stable under ι. We let ι act on $\mathcal{D}_\lambda$ via duality.

2.3.2. Analytic family of induced modules

Let $\mathfrak{U}$ be an affinoid subdomain of $\mathfrak{X}$ or $\mathfrak{X}^\iota$. Fix $n \geqslant n(\mathfrak{U})$. There is a rigid space $(\Omega_m)_n^{\mathrm{rig}}$ such that $\mathcal{O}((\Omega_m)_n^{\mathrm{rig}}{}_{/L}) = \mathcal{A}_n(\Omega_m, L)$ for any $L \subset \overline{\mathbb{Q}}_p$, where $\mathcal{A}_n(\Omega_m, L)$ is the space of locally L-analytic functions on Ω_m with local analytic radius p^{-n}. Keeping this identity, let $\mathcal{A}_{\mathfrak{U},n}(L)$ be the ring of rigid analytic L-valued functions on $\mathfrak{U} \times (\Omega_1)_n^{\mathrm{rig}}$ such that

$$f(\lambda, [tn]) = \lambda(t) f(\lambda, [n]) \tag{2.37}$$

for any $\lambda \in \mathfrak{U}(L)$, $t \in \mathcal{T}(\mathbb{Z}_p)$ and $n \in \mathcal{N}(\mathbb{Z}_p)$. Here we view $f(\lambda, -)$ as a function in $\mathcal{A}_n(\Omega_m, L)$. This implies that

$$\mathcal{A}_{\mathfrak{U},n} = \mathcal{O}(\mathfrak{U}) \hat{\otimes} \mathcal{A}_n(\mathcal{N}(\mathbb{Z}_p)). \tag{2.38}$$

In particular, $\mathcal{A}_{\mathfrak{U},n}$ is an $\mathcal{O}(\mathfrak{U})$-orthonormalizable Banach space. Similar to (2.34), since

$$\mathcal{A}_{\mathfrak{U},n} = \{f \in \mathcal{O}((\Omega_1)_n^{\mathrm{rig}}) \hat{\otimes} \mathcal{O}(\mathfrak{U}) \,|\, t(f \otimes 1) = f \otimes t, t \in \mathcal{T}(\mathbb{Z}_p)_n^{\mathrm{rig}}\}, \tag{2.39}$$

that the $*$-action of Δ^+ is well defined on $\mathcal{A}_{\mathfrak{U},n}$.

Now define

$$\mathcal{A}_{\mathfrak{U}} := \bigcup_{n \geqslant n(\mathfrak{U})} \mathcal{A}_{\mathfrak{U},n}, \tag{2.40}$$

and let $\mathcal{D}'_{\mathfrak{U},n} := \mathrm{Hom}_{\mathcal{O}(\mathfrak{U})}(\mathcal{A}_{\mathfrak{U},n}, \mathcal{O}(\mathfrak{U}))$ be the continuous $\mathcal{O}(\mathfrak{U})$-dual of $\mathcal{A}_{\mathfrak{U},n}$. There is a canonical injective map

$$\mathcal{O}(\mathfrak{U}) \hat{\otimes}_L \mathcal{D}_n(\mathcal{N}(\mathbb{Z}_p), L) \to \mathcal{D}'_{\mathfrak{U},n}. \tag{2.41}$$

Let $\mathcal{D}_{\mathfrak{U},n}$ be the image of this map, define

$$\mathcal{D}_{\mathfrak{U}} := \varprojlim \mathcal{D}_{\mathfrak{U},n}. \tag{2.42}$$

$\mathcal{A}_{\mathfrak{U}}$ and $\mathcal{D}_{\mathfrak{U}}$ are Δ^+-modules with the $*$-action. Since the inclusions $\mathcal{A}_{\mathfrak{U},n} \subset \mathcal{A}_{\mathfrak{U},n+1}$ are completely continuous, $\mathcal{D}_{\mathfrak{U}}$ is a Fréchet space over $\mathcal{O}(\mathfrak{U})$.

Proposition 2.4. — *Notation as above, we have*

(1) $\mathcal{A}_{\mathfrak{U}} \otimes_\lambda L \cong \mathcal{A}_\lambda(L)$ *and* $\mathcal{D}_{\mathfrak{U}} \otimes_\lambda L \cong \mathcal{D}_\lambda(L)$ *via specialization.*
(2) *If* $\delta \in \Delta^{++}$, *the* $*$-*action of* δ *gives a compact operator on the* $\mathcal{O}(\mathfrak{U})$-*projective compact Fréchet space* $\mathcal{D}_{\mathfrak{U}}$.

All of these results can be found in [18, section 3.4].

Remark 2.5. — We make some remarks here:

(1) The $*$-action of Δ^+ on $\mathcal{D}$ is compatible with the natural action of I on it.

(2) The $*$-actions of Δ^+ on $\mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(L)$, $\mathcal{V}^\vee_\lambda(L)$, $\mathcal{D}_\lambda(L)$ and $\mathcal{D}_\mathfrak{U}(L)$ are right action, we translate it into a left action by defining for every $\delta \in \Delta^+$

$$\delta * := *\delta^{-1}$$

(3) For $K_f = K^p I$, we view $\mathcal{D}$ as a K_f-module via the projection $K_f \to I$.

3. Twisted actions on resolutions and cohomology spaces

3.1. Cohomology spaces and resolutions

We firstly recall some standard results of the cohomology spaces on which we work later. Let M be a $(G(\mathbb{Q}), K)$-module on which Z_K acts trivially. M defines a local system on $S_G(K_f)$, which is denoted by M as well. One is interested in the cohomology space $H^*(S_G(K_f), M)$. In this paper, M is one of $\mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(L)$, $\mathcal{V}^\vee_\lambda(L)$, $\mathcal{D}_\lambda(L)$ and $\mathcal{D}_\mathfrak{U}(L)$, where the upper index $^\vee$ indicates the continuous dual space.

There are two equivalent ways to define the cohomology. Let $\overline{S}_G(K_f) = \overline{S}_G/K_f$ be the Borel–Serre compactification of the real manifold $S_G(K_f)$, where $\overline{S}_G = G(\mathbb{Q})\backslash G(\mathbb{A}_f) \times \overline{\mathcal{H}}_G$ and $\overline{\mathcal{H}}_G$ is a contractible real manifold with corners. There is a canonical projection:

$$\pi : \overline{S}_G \to \overline{S}_G(K_f), \tag{3.1}$$

which extends the natural projection $\pi : S_G \to S_G(K_f)$.

Fix a finite triangulation of $\overline{S}_G(K_f)$ and pull it back to $\overline{S}_G$. Let $C_*(K_f)$ be the corresponding chain complex, that is, $C_q(K_f)$ is the free $\mathbb{Z}$-module over the set of q-dimensional simplexes of the pull-back triangulation. $C_*(K_f)$ admits a right K_f-action, and $C_q(K_f)$ is a free right $\mathbb{Z}[K_f]$-module of finite rank. We define

$$R\Gamma^*(K_f, M) := \mathrm{Hom}_{K_f}(C_*(K_f), M), \tag{3.2}$$

then $R\Gamma^j(K_f, M)$ is isomorphic to finitely many copies of M and

$$h^j(R\Gamma^*(K_f, M)) = H^j(S_G(K_f), M). \tag{3.3}$$

Another way to define the cohomology is using the M-valued de Rham complex $\Omega^*(S_G(K_f), M)$. The natural duality between $\Omega^*(S_G(K_f))$ and $C_*(K_f)$ implies that the two definitions are coincident.

Remark 3.1. — As summarized in [18, Section 1.2], for a $(G(\mathbb{Q}), K)$-module M, there are two equivalent ways to define the local system M on $S_G(K_f)$, with respect to the K_f-module structure and to the $G(\mathbb{Q})$-module structure respectively. So are the two definitions of cohomology space above.

3.1.1. Functoriality

There is a functoriality for $R\Gamma^*(K_f, M)$. Let $\varphi : K'_f \to K_f$ be a group homomorphism and $\varphi^\# : M \to M'$ a homomorphism between a K_f-module M and a K'_f-module M', such that $\varphi^\#(\varphi(k')m) = k'\varphi^\#(m)$ for any $k' \in K'_f$ and $m \in M$. The pair $(\varphi, \varphi^\#)$ then induces a morphism $\varphi^* : R\Gamma^*(K_f, M) \to R\Gamma^*(K'_f, M')$ up to homotopy, see [18, Section 4.2.5].

3.1.2. Hecke operators on resolution and cohomology

Apply the functoriality, as in [18, Section 4.2], $f = f^p \otimes u_t \in \mathcal{H}_p(K^p)$ defines a morphism $R\Gamma(t) : R\Gamma^*(K_f, M) \to R\Gamma^*(K_f, M)$ by the composition:

$$R\Gamma^*(K_f, M) \to R\Gamma^*(tK_ft^{-1}, M) \to R\Gamma^*(K_f \cap tK_ft^{-1}, M) \to R\Gamma^*(K_f, M)$$

where the first map is given by the pair $(\mathrm{ad}(t^{-1}), m \mapsto t * m)$, the second is by the restriction map from $K_f \cap tK_ft^{-1}$ to tK_ft^{-1}, and the last one is given by the corestriction as writing

$$K_f = \sqcup_j k_j(K_f \cap tK_ft^{-1}). \tag{3.4}$$

It is easy to see that $R\Gamma(t_1) \circ R\Gamma(t_2) = R\Gamma(t_1t_2)$. This defines an action of $\mathcal{H}_p(K^p)$ on $R\Gamma^*(K_f, M)$ and therefore defines an action on the cohomology spaces $H^*(S_G(K_f), M)$. We denote this action by $*$ as well.

If $M = \mathcal{D}_\lambda(L)$ and $t \in T^{++}$, by the fact that $R\Gamma^q(K_f, M)$ is a finite copy of M, Proposition 2.2 implies that f is a compact operator on $R\Gamma^*(K^pI, \mathcal{D}_\lambda(L))$. If $\mathfrak{U}$ is an open affinoid of $\mathfrak{X}$ and $\lambda \in \mathfrak{U}$, by Proposition 2.4, $R\Gamma^*(K^pI, \mathcal{D}_\lambda(L))$ can be obtained by the specialization of $R\Gamma^*(K^pI, \mathcal{D}_{\mathfrak{U}})$ at λ. Moreover, for affinoids $\mathfrak{U}' \subset \mathfrak{U}$, $R\Gamma^*(K^pI, \mathcal{D}_{\mathfrak{U}'})$ can be obtained via the natural restriction morphism $\mathcal{O}(\mathfrak{U}) \to \mathcal{O}(\mathfrak{U}')$. The Hecke action is compatible with specialization and restriction.

3.1.3. ι action on resolution and cohomology

Assume $\lambda \in \mathfrak{X}^\iota$ and $\mathfrak{U}$ is an open affinoid of $\mathfrak{X}^\iota$. Let M be one of $\mathbb{V}_\lambda^\vee(L)$, $\mathcal{V}_\lambda^\vee(L)$, $\mathcal{D}_\lambda(L)$ and $\mathcal{D}_\mathfrak{U}(L)$. Choose $K_f = K^p I \subset G(\mathbb{A}_f)$, such that K^p is stable under ι. Consider morphisms $\iota : K_f \to K_f$ and $\iota : M \to M$ defined as in Section 2.

Lemma 3.2. — *Assume $M = \mathbb{V}_\lambda^\vee(L), \mathcal{V}_\lambda^\vee(L), \mathcal{D}_\lambda(L)$ or $\mathcal{D}_\mathfrak{U}(L)$. For $g \in I$, $x \in M$,*

$$g * x^\iota = (g^\iota * x)^\iota \tag{3.5}$$

Therefore, by the functoriality, ι defines an morphism on $R\Gamma^*(K_f, M)$ up to homotopy. In particular, ι acts on the cohomology $H^*(S_G(K_f), M)$.

The lemma follows immediately from a computation by definition.

3.1.4. Action of ${}^\iota\mathcal{H}_p(K^p)$

For ι-invariant K^p, define the ι-twisted Hecke algebra:

$${}^\iota\mathcal{H}_p(K^p) := \mathcal{H}_p(K^p) \rtimes \langle \iota \rangle \,, \tag{3.6}$$

where $\langle \iota \rangle$ is the finite group generated by ι and the semi-product $\rtimes$ is understood as a crossed product, since at every place the local Hecke algebra can be viewed as a group algebra of double cosets. We write $f \times \iota$ and $\iota \times f$ for the products of $f \in \mathcal{H}_p(K^p)$ and ι. We similarly define ${}^\iota\mathcal{H}_p$, then ${}^\iota\mathcal{H}_p$ is the inductive limit of ${}^\iota\mathcal{H}_p(K^p)$.

Lemma 3.3. — *There is an action of ${}^\iota\mathcal{H}_p(K^p)$ on $R\Gamma^*(K_f, M)$, extending the $*$-action of $\mathcal{H}_p(K^p)$ and ι.*

Proof. — We have to check that the $*$-actions of $\mathcal{H}_p(K^p)$ and ι on $R\Gamma^*(K, M)$ are compatible in the sense that $\iota \times f = f^\iota \times \iota$. So we only have to check:

$$\iota \circ R\Gamma(t) \circ \iota^{-1} = R\Gamma(t^\iota), \tag{3.7}$$

which is again directly from the definition. □

3.1.5. Comparison with the standard sheaf-theoretic action

Assume $M = \mathbb{V}_\lambda^\vee(\mathbb{C})$, we compute the cohomology by de Rham complex:

$$H^q(S_G(K_f), M) = h^q(\Omega^*(S_G(K_f), M)), \tag{3.8}$$

and we have the standard sheaf-theoretic definion of $\mathcal{H}_p$ action on it. By (2.13), for any $\phi \in \mathbb{V}_\lambda$ and $\delta \in \Delta^+$,

$$\delta * \phi = \lambda(\xi(t_\delta))^{-1}(\delta \cdot \phi). \tag{3.9}$$

This implies, for any $f = f^p \otimes u_t \in \mathcal{H}_p$, that the $*$-action of f on $H^q(S_G(K_p), M)$ is the twist of the standard action of f by $\lambda(\xi(t))$.

The ι-action on $\Omega^*(S_G(K_f), M) = \Omega^*(S_G(K_f)) \otimes M$ is define by $(\iota^{-1})^* \otimes \iota$, where $(\iota^{-1})^*$ means the pull-back on differential forms induced by the map

$$\iota^{-1} : S_G(K_f) \leftarrow S_G(K_f) \tag{3.10}$$

This ι-action can be described explicitly as follow. Let $T(S_G)$ and $T(S_G(K_f))$ be the sheaves of left invariant vector fields on S_G and $S_G(K_f)$ respectively, the projection π induces a push-forward surjection:

$$\pi_* : T(S_G) \to T(S_G(K_f)). \tag{3.11}$$

One views an q differential form τ in $\Omega^*(S_G(K_f), M)$ as a map

$$\tau : \wedge^q T(S_G(K_f) \to \mathcal{O}(S_G(K_f)) \otimes M \tag{3.12}$$

then τ^ι is defined as

$$\tau^\iota(\pi_*\bar{v}_1 \wedge \cdots \wedge \pi_*\bar{v}_q)([g]) := (\tau((\pi_*\iota_*^{-1}\bar{v}_1 \wedge \cdots \wedge \pi_*\iota_*^{-1}\bar{v}_q))([g]^\iota))^\iota \tag{3.13}$$

where $\bar{v}$ is a left invariant vector field on S_G, $g \in G(\mathbb{A})^1$ and $[g]$ indicates the class of g in S_G or $S_G(K_f)$. It is easy to check that ι is well defined on $H^*(S_G(K_f), M)$ under this definition. The duality between $\Omega^*(S_G(K_f))$ and $C_*(K_f)$ implies that this action coincides with the one defined by functoriality.

3.2. Twisted action on finite slope cohomology

We need a lemma on slope decompositions of a compact projective Fréchet space according to compact operators.

Lemma 3.4. — *Let A be a $\mathbb{Q}_p$-Banach algebra, M a compact projective Fréchet A-module, and f a compact A-linear operator of M. Then the Fredholm determinant $R(f, X)$ of f is entire over A. If $R(f, X) = Q(X)S(X)$ over A, such that Q and S are relatively prime and Q is a Fredholm polynomial with invertible leading coefficient, then there is a decomposition of M:*

$$M = N_f(Q) \oplus F_f(Q) \tag{3.14}$$

into f-stable close submodules satisfing:

(1) *$Q^*(f)$ annihilates $N_f(Q)$ and is invertible on $F_f(Q)$;*
(2) *the projector on $N_f(Q)$ is given by $E_Q(f)$ with $E_Q(X) \in XA\{\{X\}\}$ whose coefficients are polynomials in the coefficients of Q and S.*

Moreover, if A is noetherian, then $N_f(Q)$ is of finite rank, and the characteristic polynomial of f on $N_f(Q)$ is Q. In particular, for $h \in \mathbb{Q}_{\geqslant 0}$, we may choose $Q(x)$ such that $N_f(Q) = M^{\leqslant h}$, the $\leqslant h$-slope decomposition of M.

Proof. — The lemma is known if M is a projective Banach module by Serre [15] and Coleman [10]. Now M is a projective compact Fréchet space, there are projective A-Banach modules M_n with compact operators f_n, such that

$$M = \varprojlim M_n, \ \ f = \varprojlim f_n \tag{3.15}$$

with $f_n = f|M_n$. Now $R(f,X) = \det(1-Xf|M_n)$ for n sufficiently large, so $R(f,X) = Q(X)S(X)$ gives the expected decomposition $M_n = N_{n,f}(Q) \oplus F_{n,f}(Q)$. Let p_n be the projector of M_n onto $N_{n,f}(Q)$, by [8, Theorem 3.3], there is a power series $\phi \in A[\![T]\!]$ depending only on Q, such that $p_n = \phi(f_n)$. Moreover, $N_{n,f}(Q)$ and $F_{n,f}(Q)$ are given by the image and kernel of p_n respectively. Denote by $u_{n+1,n}$ the transation map from M_{n+1} to M_n. By definition, we have a commutative diagram:

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & M_{n+1} \arrow[r, "u_{n+1,n}"] \arrow[d, "f_{n+1}"'] & M_n \arrow[d, "f_n"] \arrow[dl] \\ M \arrow[r] & M_{n+1} \arrow[r, "u_{n+1,n}"'] & M_n \end{tikzcd} \tag{3.16}$$

Taking projective limit, we have a projector $p = \varprojlim \phi(f_n)$ on M and the decomposition $M = N_f(Q) \oplus F_f(Q)$. Indeed, by the definition of compact operator, $N_{n,f}(Q)$ are isomorphic for n sufficiently large. So the last statement follows. □

Considering admissible $f = f^p \otimes u_t \in R_{\mathcal{S},p}$, f defines a compact operator on the complex $R\Gamma(K_f, \mathcal{D}_\lambda(L))$. For any $h \in \mathbb{Q}_{\geqslant 0}$, we define the $\leqslant h$-slope part $H^*(S_G(K^pI), \mathcal{D}_\lambda(L))^{\leqslant h}$ of $H^*(S_G(K^pI), \mathcal{D}_\lambda(L))$ with respect to f according to the previous lemma. Then the finite slope part of $H^*(S_G(K^pI), \mathcal{D}_\lambda(L))$ is defined by:

$$H^*_{fs}(S_G(K^pI), \mathcal{D}_\lambda(L)) := \varinjlim_h H^*(S_G(K^pI), \mathcal{D}_\lambda(L))^{\leqslant h}. \tag{3.17}$$

Since $R_{\mathcal{S},p}$ is in the center of $\mathcal{H}_p(K^p)$, that $H^*_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L))$ is independent of f, and endowed with the $*$-action of $\mathcal{H}_p(K^p)$. We also define

$$H^*_{fs}(\tilde{S}_G,\mathcal{D}_\lambda(L)) := \varinjlim_{K^p} H^*_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L)). \tag{3.18}$$

Proposition 3.5. — *Assume $\lambda \in \mathfrak{X}^\iota$. The $*$-action of $\mathcal{H}_p(K^p)$ on the finite slope cohomology $H^*_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L))$ extends to an action of ${}^\iota\mathcal{H}_p(K^p)$. Therefore the action of $\mathcal{H}_p$ on $H^*_{fs}(\tilde{S}_G,\mathcal{D}_\lambda(L))$ extends to an action of ${}^\iota\mathcal{H}_p$.*

Proof. — We only have to prove the first statement. For this we need the next result from [18, Lemma 2.3.2]:

Lemma 3.6. — *Let M, M' be two L-Banach (or Fréchet) spaces, u and u' endomorphism of M and M', and $M = M_u^{\leqslant h} \oplus M_1$ and $M' = {M'}_{u'}^{\leqslant h} \oplus M'_1$ their $\leqslant h$-slope decompositions respectively. Assume f is a continuous linear map from M to M' such that $f \circ u = u' \circ f$, then f respects the slope decompositions.*

Since $\iota \times f = f^\iota \times \iota$, the lemma implies that

$$\iota : H^q_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L))^{\leqslant h}_{f^\iota} \to H^q_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L))^{\leqslant h}_{f} \tag{3.19}$$

is well defined. Let l be the order of ι, define

$$H^q_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L))^{\leqslant h}_{\iota} := \bigcap_{i=1}^{l} H^q_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L))^{\leqslant h}_{f^{\iota^i}}. \tag{3.20}$$

Then

$$\iota : H^q_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L))^{\leqslant h}_{\iota} \to H^q_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L))^{\leqslant h}_{\iota}. \tag{3.21}$$

Since the finite slope part is independent of f, the proposition is obtained by taking the inductive limit on h. □

3.3. ι-invariant finite slope representations

In this section, we introduce the ι-invariant finite slope automorphic representations, which are the main objects we concern in this paper. We first recall a well-known result for admissible representations of a locally profinite group. Let G be a locally profinite group, and K an open compact subgroup of G. Write $\mathcal{H}(G)$ the Hecke algebra of compact supported smooth functions of G and $\mathcal{H}(G,K)$ its subalgebra of K bi-invariant functions. Then

Proposition 3.7. — *The map $\pi \mapsto \pi^K$ gives a bijection between equivalence classes of irreducible smooth representations (π, V) of $\mathcal{H}(G)$ such that $V^K \neq 0$ and equivalence classes of irreducible $\mathcal{H}(G,K)$-representations.*

3.3.1. Finite slope representations

let (π, V) be an irreducible representation of $\mathcal{H}_p$ defined over a p-adic field L. We say π is admissible overconvergent of weight $\lambda \in \mathfrak{X}(L)$ if it is admissible and a subqoutient of $H^q(\tilde{S}_G, \mathcal{D}_\lambda(L))$. Since π is admissible, that for any K^p, an element in $\mathcal{H}_p(K^p)$ acts on V as an endomorphism of finite rank. By the fact that $\mathcal{H}_p = \varinjlim_{K^p} \mathcal{H}_p(K^p)$, there is a well-defined trace map:

$$J_\pi(f) := \operatorname{tr}(\pi(f)) \tag{3.22}$$

for any $f \in \mathcal{H}_p$. We say π is of level K^p if π^{K^p} as a representation of $\mathcal{H}_p(K^p)$ is not trivial. Let σ be an irreducible representation of $\mathcal{H}_p(K^p)$. We say σ is overconvergent of level K^p and weight λ if it is of the form π^{K^p} for some admissible overconvergent π with level K^p and weight λ. Then σ is finite dimensional and can be realized in the cohomology space $H^q(S_G(K^pI), \mathcal{D}_\lambda(L))$. For fixed K^p, the Hecke algebra $R_{\mathcal{S},p}$ is included in the center of $\mathcal{H}_p$. So the restriction of σ to $R_{\mathcal{S},p}$ is a character, which is denoted by θ_σ. We call θ_σ an overconvergent Hecke eigensystem of level K^p and weight λ. For such θ, obviously the generalized eigenspace $H^q(S_G(K^pI), \mathcal{D}_\lambda(L))[\theta]$ of θ is non-zero.

Let θ be a $\overline{\mathbb{Q}}_p$-valued character of $\mathcal{U}_p$. To recall the definition of the slope of θ, we assume at the moment that G is split at p (refer [18, Section 4.1.2] for general situation). If $\theta(u_t) = 0$ for some $t \in T^+$, then we say that θ is of infinite slope. Otherwise, we say θ is of finite slope. It is easy to check that θ is of finite slope if and only if there is $t \in T^{++}$ such that $\theta(u_t) \neq 0$. In this case, θ induces a homomorphism from $T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ to $\overline{\mathbb{Q}}_p^\times$. We then define the slope of θ to be the element $\mu_\theta \in X^*(T_{/\mathbb{Q}_p})$, such that for any $\mu^\vee \in X_*(T_{/\mathbb{Q}_p})^+$

$$(\mu_\theta, \mu^\vee) = v_p(\theta(u_{\mu^\vee(p)})). \tag{3.23}$$

So we define the slope of an overconvergent Hecke eigensystem θ to be the slope of its restriction to $\mathcal{U}_p$. For a overconvergent representation π or σ, define its slope to be the slope of the Hecke eigensystem associated to it. It is easy to see, π, σ or θ is of finite slope if and only if it is realized in

$H^*_{fs}(S_G, \mathcal{D}_\lambda(L))$. Moreover, for any $\mu \in X^*(T_{/\mathbb{Q}_p})$, we define

$$H^*(S_G(K_f), \mathcal{D}_\lambda(L))^{\leqslant\mu} = \bigoplus_{\theta \,|\, \mu_\theta \leqslant \mu} H^*(S_G(K_f), \mathcal{D}_\lambda(L))[\theta]. \tag{3.24}$$

It is easy to see that

$$\varinjlim_{\mu} H^*(S_G(K_f), \mathcal{D}_\lambda(L))^{\leqslant\mu} = H^*_{fs}(S_G(K_f), \mathcal{D}_\lambda(L)). \tag{3.25}$$

Let θ be a $\overline{\mathbb{Q}}_p$-valued character of $\mathcal{U}_p$ and λ an algebraic weight. We say that θ is non-critical with respect to λ, if its slope μ_θ is. This means that for any $w \neq id$ in $\mathcal{W}_G$, the Weyl group of G, $\mu_\theta \notin w \cdot \lambda - \lambda + X^*(T)^+$.

3.3.2. ι-invariant finite slope representations

Let ρ be a finite slope overconvergent representation of $\mathcal{H}$, where $\mathcal{H}$ can be $\mathcal{H}_p$, $\mathcal{H}_p(K^p)$ or $R_{\mathcal{S},p}$. We denote by ρ^ι the ι-twist of ρ, that is, the representation of $\mathcal{H}$ on V_ρ, which sends $f \in \mathcal{H}$ to $\rho^\iota(f) := \rho(f^\iota)$. We say ρ is ι-invariant if $\rho^\iota \cong \rho$. By [6, Appendix], ρ is ι-invariant if and only if it can be extended to ${}^\iota\mathcal{H}$. Indeed, if $\rho^\iota \cong \rho$, then there is a linear operator $A : V_\rho \to V_\rho$ of order l such that $A \circ \rho = \rho \circ A$. One can extend ρ by setting ι^i acting on V_σ via A^i. Generally, we use capital letters Π, Σ and Θ for an representation of ${}^\iota\mathcal{H}_p$, ${}^\iota\mathcal{H}_p(K^p)$ and ${}^\iota R_{\mathcal{S},p}$ respectively. The next lemma summarizes results in [6, Appendix]:

Lemma 3.8. — *Notations as above*

(1) *Assume Σ (resp. Π, Θ) is irreducible, then its restriction to $\mathcal{H}_p(K^p)$ (resp. $\mathcal{H}_p$, $R_{\mathcal{S},p}$) is irreducible if and only if the trace of Σ (resp. Π, Θ) is not trivial on $\iota \times \mathcal{H}_p(K^p)$ (resp. $\iota \times \mathcal{H}_p$, $\iota \times R_{\mathcal{S},p}$).*
(2) *Assume σ (resp. π, θ) is irreducible ι-invariant, then there are exactly l extensions of σ (resp. π, θ) to ${}^\iota\mathcal{H}_p(k^p)$ (resp. ${}^\iota\mathcal{H}_p$, ${}^\iota R_{\mathcal{S},p}$), say $\tilde{\sigma}_1, \dots, \tilde{\sigma}_l$ (resp. $\tilde{\pi}_1, \dots, \tilde{\pi}_l$; $\tilde{\theta}_1, \dots, \tilde{\theta}_l$). Each two of them are differed by a character of order l and are non-isomorphic.*

Remark 3.9. — By Proposition 3.7, if $\sigma = \pi^{K^P}$, we can assume that

$$\tilde{\pi}_i^{K^p} = \tilde{\sigma}_i. \tag{3.26}$$

Throughout this paper, we use this convention.

Let $\tilde{\sigma}$ be a representation of ${}^\iota\mathcal{H}_p(K^p)$ which extends a ι-invariant overconvergent representation σ of $\mathcal{H}_p(K^p)$. We write

$$J_{\tilde{\sigma}}(f) := \mathrm{tr}(\tilde{\sigma}(\iota \times f)). \tag{3.27}$$

Last lemma tells that $J_{\tilde{\sigma}}$ is not trivial. It is also easy to see the Hecke eigensystem θ_σ satisfies

$$\theta(f) = \theta(f^\iota) := \theta^\iota(f) \tag{3.28}$$

for any $f \in R_{\mathcal{S},p}$. We say such a Hecke eigensystem ι-invariant. Let $\tilde{\theta}_{\tilde{\sigma}}$ be the restriction of $\tilde{\sigma}$ to ${}^\iota R_{\mathcal{S},p}$. Since ι is of finite order and $\tilde{\theta}_{\tilde{\sigma}}$ is finite dimensional, that $\tilde{\theta}_{\tilde{\sigma}}$ is diagonalizable under ι. So $\tilde{\theta}_{\tilde{\sigma}}$ is a direct sum of one dimensional representations of ${}^\iota R_{\mathcal{S},p}$, which must be of the form $(\tilde{\theta_\sigma})_i$. So we have

$$\tilde{\theta}_{\tilde{\sigma}} = \bigoplus_{i=1}^{l} ((\tilde{\theta_\sigma})_i)^{m((\tilde{\theta_\sigma})_i, \tilde{\theta}_{\tilde{\sigma}})} \tag{3.29}$$

where, $m((\tilde{\theta_\sigma})_i, \tilde{\theta}_{\tilde{\sigma}})$ is the multiplicity of $(\tilde{\theta_\sigma})_i$ in $\tilde{\theta}_{\tilde{\sigma}}$ and

$$\sum_{i=1}^{l} m((\tilde{\theta_\sigma})_i, \tilde{\theta}_{\tilde{\sigma}}) = \dim \sigma. \tag{3.30}$$

If $f \in R_{\mathcal{S},p}$, then

$$J_{\tilde{\sigma}}(f) = \mathrm{tr}(\iota \,|\, \tilde{\sigma})\theta_\sigma(f). \tag{3.31}$$

Since ι is of finite order, all its eigenvalues must be p-adic units. This implies a simple but important observation that

$$v_p(\mathrm{tr}(\iota \,|\, \tilde{\sigma})) \geqslant 0. \tag{3.32}$$

Let θ be a finite slope overconvergent Hecke eigensystem with slope μ_θ. It is easy to verify that

$$\mu_{\theta^\iota} = \mu_\theta^\iota \tag{3.33}$$

So we conclude:

Lemma 3.10. — *If a finite slope overconvergent representation (resp. Hecke eigensystem) is ι-invariant, then its slope μ_θ is ι-invariant.*

3.3.3. Classicity

Now we state a twisted analogue to [18, Proposition 4.3.10], which is the classicity theorem in the cohomological setting. For a dominant weight $\lambda \in X^*(T)$ and $t \in T^+$, define

$$N(\lambda, t) := \inf_{w \neq id} |t^{w\cdot\lambda - \lambda}|_p \tag{3.34}$$

and

$$N^\iota(\lambda, t) := \inf_{i=1}^{l} N(\lambda, t^{\iota^i}). \tag{3.35}$$

Proposition 3.11. — *Let $\lambda = \lambda^{\mathrm{alg}}\epsilon$ be an arithmetic weight of conductor p^{n_λ}, and μ a slope which is non-critical with respect to λ^{alg}. Then for any positive integer $m \geqslant n_\lambda$, we have the canonical isomorphism*

$$H^*(S_G(K^pI), \mathcal{D}_\lambda(L))^{\leqslant\mu} \cong H^*(S_G(K^pI_m), \mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(L,\epsilon))^{\leqslant\mu} \tag{3.36}$$

Similarly, with respect to $f = f^p \otimes u_t$, $t \in T^{++}$, for any $h \leqslant v_p(N^\iota(\lambda, t))$,

$$H^*(S_G(K^pI), \mathcal{D}_\lambda(L))^{\leqslant h}_\iota \cong H^*(S_G(K^pI_m), \mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(L,\epsilon))^{\leqslant h}_\iota \tag{3.37}$$

Here $\mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(L,\epsilon) := (\mathbb{V}_{\lambda^{\mathrm{alg}}}(L)(\epsilon))^\vee$.

The proof is same to [18, Proposition 4.3.10].

3.3.4. Spectral expansion of twisted overconvergent traces

Proposition 3.12. — *Let $\lambda \in \mathfrak{X}^\iota(L)$. For any ι-invariant finite slope overconvergent representation π of $\mathcal{H}_p$, there are l integers $\{m^q_i(\pi,\lambda)\}^l_{i=i}$, such that for all $f \in \mathcal{H}'_p$,*

$$\mathrm{tr}(\iota \times f \mid H^q_{fs}(\tilde{S}_G, \mathcal{D}_\lambda(L))) = \sum_{\pi\,|\,\pi^\iota\cong\pi}\sum_{i=1}^{l} m^q_i(\pi,\lambda) J_{\tilde{\pi}_i}(f) \tag{3.38}$$

Proof. — Since f is admissible, the trace is convergent. Fix $t \in T^{++}$, for any $h \in \mathbb{Q}_{\geqslant 0}$, consider the ι-stable $\leqslant h$-slope part $H^q(\tilde{S}_G, \mathcal{D}_\lambda(L))^{\leqslant h}_\iota$ of $H^q_{fs}(\tilde{S}_G, \mathcal{D}_\lambda(L))$ with respect to u_t. Here we define

$$H^q(\tilde{S}_G, \mathcal{D}_\lambda(L))^{\leqslant h}_\iota := \varinjlim_{K^p} H^q(S_G(K^pI), \mathcal{D}_\lambda(L))^{\leqslant h}_\iota. \tag{3.39}$$

It is equipped with an admissible $*$-action of $\mathcal{H}_p$ since u_t is in the center of $\mathcal{H}_p$. As the proof of Proposition 3.5, this action extends to ${}^\iota\mathcal{H}_p$. Let (π, V) be a finite slope overconvergent $\mathcal{H}_p$-submodule of $H^q(\tilde{S}_G, \mathcal{D}_\lambda(L))^{\leqslant h}_\iota$, and $\iota * (V)$ its image under the $*$-action of ι. Consider the next three sets of finite slope overconvergent representations A, B and C whose elements are counted with multiplicity: $A = \{\pi \mid \iota * (V_\pi) = V_\pi\}$, $B = \{\pi \mid \pi^\iota \cong \pi, \iota * (V_\pi) \cap V_\pi = \emptyset\}$ and $C = \{\pi \mid \pi^\iota \ncong \pi\}$.

If $\pi \in B$ or C, write

$$W_\pi = \bigoplus_{i=1}^{l} \iota^i * (V_\sigma) \tag{3.40}$$

as a ${}^\iota\mathcal{H}_p$-submodule of $H^q(\tilde{S}_G, \mathcal{D}_\lambda(L))^{\leqslant h}_\iota$. Then $\iota*$ permutes the components of W_π, and $\mathrm{tr}(\iota \times f|W_\sigma)$ is trivial. If $\pi \in A$ (this implies that σ is ι-invariant), then V_π itself is an irreducible ${}^\iota\mathcal{H}_p$-submodule of

$H^q(\tilde{S}_G, \mathcal{D}_\lambda(L))_\iota^{\leqslant h}$. In particular, $V_\pi = \tilde{\pi}_i$ for some i. Now for any ι-invariant π, let $m^q_{i,h}(\pi, \lambda)$ be the mulitplicity of $\tilde{\pi}_i$ appearing in A. We have

$$\operatorname{tr}(\iota \times f; H^q(\tilde{S}_G, \mathcal{D}_\lambda(L)))_\iota^{\leqslant h} = \sum_{\pi | \pi^\iota \cong \pi} \sum_{i=1}^{l} m^q_{i,h}(\pi, \lambda) J_{\tilde{\pi}_i}(f). \tag{3.41}$$

Let h go to infinite, the multiplicity $m^q_{i,h}$ will stay same as h large enough. So it converges to some $m^q_i \in \mathbb{Z}$, and the proposition follows. □

Remark 3.13. — If G admits multiplicity one theorem, then $B = \emptyset$ in the proof above (otherwise, since σ is self-dual, that $V_\sigma \cong V_{\sigma^\iota} \cong V_\sigma^{*\iota}$ would appear with multiplicity at least two).

4. Twisted Franke's trace formula

In this section, we prove a twisted version of Franke's trace formula [11, Section 7.7].

4.1. Notation

In this section, we consider more general situation as studied in [11]. Let G be a connected reductive group over $\mathbb{Q}$ which is not necessarily quasi-split. We fix a minimal parabolic subgroup P_0 of G and write its Langlands decomposition $P_0 = M_0 A_0 N_0$. Generally, for a standard parabolic subgroup P of G, we consider its Langlands decomposition $P = M_P A_P N_P$ such that $A_P \subset A_0$ and $M_0 \subset M_P$, where $L_P = M_P A_P$ is the standard Levi subgroup of P and A_P is a maximal spit torus in the center of L_P. If group G is quasi-split as in the other sections of this paper, we assume that the minimal p-pair (P_0, A_0) is chosen such that $P_0 = B$ and $A_0 \subset T$. Write $\check{\mathfrak{a}}_P = X^*(P) \otimes \mathbb{R}$ and $\check{\mathfrak{a}}_P^G$ its subspace of elements whose restriction to A_G are trivial. If $P = P_0$, we denote $\check{\mathfrak{a}}_0^G := \check{\mathfrak{a}}_{P_0}^G$. Let $\check{\mathfrak{a}}_0^{G+} \subset \check{\mathfrak{a}}_0^G$ be the open positive Weyl chamber, and ${}^+\check{\mathfrak{a}}_0^G$ the open positive cone dual to it.

In general, we use small gothics letters for real Lie algebras, for example, $\mathfrak{g}$ is the Lie algebra of G_∞ and for any parabolic subgroup P, $\mathfrak{a} = \mathfrak{a}_P$ is the lie algebra of $A_{P\infty}$ and $\mathfrak{a}_0 := \mathfrak{a}_{P_0}$. Since $\mathfrak{a}_P = X_*(A_P) \otimes \mathbb{R}$, there is a natural duality between $\mathfrak{a}_P$ and $\check{\mathfrak{a}}_P$, which has been denoted by $\langle \cdot, \cdot \rangle$. Now the restriction map $\check{\mathfrak{a}}_P \to \check{\mathfrak{a}}_0$ gives an inverse of the dual of natrual inclusion $\mathfrak{a}_0 \to \mathfrak{a}_P$. So we have decompositions

$$\mathfrak{a}_0 = \mathfrak{a}_P \oplus \mathfrak{a}_0^P, \quad \check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_P \oplus \check{\mathfrak{a}}_0^P. \tag{4.1}$$

If there is another parabolic subgroup $Q \supset P$, we use Franke's notation $\mathfrak{a}_P^Q$ for $\mathfrak{a}_P \cap \mathfrak{a}_0^Q$ in $\mathfrak{a}_0$, and similarly $\check{\mathfrak{a}}_P^Q$.

Let $\mathfrak{m}_G \subset \mathfrak{g}$ be the intersection of kernals for all rational characters of G, and $Z(\mathfrak{m}_G)$ the center of the universal enveloping algebra of $\mathfrak{m}_G$. For any standard parabolic subgroup P, define the height function $H_P : P(\mathbb{A}) \to \mathfrak{a}_P$ such that for any $x \in P(\mathbb{A})$ and any character ξ of P,

$$\prod_v |\xi(x)|_v = e^{\langle \xi, H_P(x) \rangle}. \tag{4.2}$$

H_P is then considered as a function on $G(\mathbb{A})$ by the Iwasawa decomposition. Let $\mathbb{V}_\lambda^\vee$ be as in the last section. Assume A_G acting on $\mathbb{V}_\lambda^\vee$ by a character ξ_λ, let $\mathcal{I}_\lambda \subset Z(\mathfrak{m}_G)$ be the annihilator of ξ_λ. For any $G(\mathbb{A}_f)$-module M and $\mu \in \check{\mathfrak{a}}_G$, we denote by $M(\mu)$ the twist of M in which the action of $g \in G(\mathbb{A}_f)$ on M is multiplied by the factor $e^{\langle \mu, H_G(g) \rangle}$.

Let $R = R_G \subset X^*(A_0)$ (resp. $R^+ = R_G^+$, $\Delta = \Delta_G$) be the set of roots (resp. positive roots, simple positive roots) of A_0 in $\mathfrak{g}$. If L is a Levi subgroup of G, let ρ_L be the half sum of all positive roots of A_0 in L, in particular, write $\rho = \rho_G$. For any parabolic subgroup $P = LN$, let ρ_P be the modulus function on $\mathfrak{l}$ associated to P. Without confusion, we also consider ρ_P as in $\check{\mathfrak{a}}_P^G$ to be the half sum of all the positive roots of $\mathfrak{a}_P$ in $\mathfrak{n}_P$. For parabolic subgroups $P \subset Q$, under the natural projection $X^*(T)^+ \otimes \mathbb{R} \to \check{\mathfrak{a}}_P^G \to \check{\mathfrak{a}}_Q^G$, the images of ρ_G are ρ_P and ρ_Q. Let $\mathcal{W}_G$ be the Weyl group of G and fix $w_0^G \in \mathcal{W}_G$ a longest element. We define

$$\mathcal{W}^L := \{w \in \mathcal{W}_G \,|\, w^{-1}(\alpha) \in R^+,\ \forall\, \alpha \in R^+ \cap R_L\}. \tag{4.3}$$

As [18, (7)], for any $w \in \mathcal{W}^L$,

$$w \cdot \lambda = w(\lambda + \rho) - \rho = w(\lambda + \rho_P) - \rho_P. \tag{4.4}$$

Recall the Kostant decomposition:

$$H^q(\mathfrak{n}, \mathbb{V}_\lambda^\vee(\mathbb{C})) = \bigoplus_{\substack{w \in \mathcal{W}^L \\ l(w) = n-q}} \mathbb{V}^{L,\vee}_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C}), \tag{4.5}$$

where $n = \dim(\mathfrak{n})$ and $\mathbb{V}_\mu^L$ is the finite dimensional irreducible algebraic representation of L with highest weight μ. Define

$$\mathcal{W}^L_{\mathrm{Eis}} := \{w \in \mathcal{W}^L \,|\, w^{-1}(\beta^\vee) > 0,\ \forall\, \beta \in R_P\}. \tag{4.6}$$

If λ is regular and $w \in \mathcal{W}^L_{\mathrm{Eis}}$, then the Eisenstein series associated to a class in $H^*(S_L, \mathbb{V}^{L,\vee}_{w\cdot\lambda})$ defines an Eisenstein class in $H^*(S_G, \mathbb{V}_\lambda^\vee)$.

4.2. The Eisenstein spectral sequence

Let $V_G = C^\infty_{umg}(G(\mathbb{Q})A(\mathbb{R})^0\backslash G(\mathbb{A}))$ be the space of C^∞-functions of uniform moderate growth, and R_g the natural right translation action of $g \in G(\mathbb{A})$ on V_G. Let A_λ be the subspace of V_G consisting of the functions that are annihilated by some power of $\mathcal{I}_\lambda$. In [11], Franke proved that the cohomology

$$H^*(S_G, \mathbb{V}^\vee_\lambda(\mathbb{C})) \tag{4.7}$$

can be computed by $(\mathfrak{m}_G, K_\infty)$ cohomology with coefficients in A_λ:

$$\begin{aligned} &H^*(\mathfrak{m}_G, K_\infty; C^\infty(G(\mathbb{Q})A(\mathbb{R})^0\backslash G(\mathbb{A})) \otimes \mathbb{V}^\vee_\lambda(\mathbb{C}))(\xi_\lambda) \\ &\qquad\qquad = H^*(\mathfrak{m}_G, K_\infty; A_\lambda \otimes \mathbb{V}^\vee_\lambda(\mathbb{C}))(\xi_\lambda). \end{aligned} \tag{4.8}$$

Indeed, Franke proved that the last cohomology space is the limit of a spectral sequence, which is compiled via the Laurent coefficients of Eisenstein series.

Let $\mathcal{C} := \mathcal{C}_G$ be the set of associate classes of $\mathbb{Q}$-parabolic subgroups of G. For $\{P\} \in \mathcal{C}_G$, let $V_G(\{P\})$ be the subspace of V_G consists of functions which are negligible along all $Q \notin \{P\}$, that is, the space of functions $\phi \in V_G$, such that for every parabolic subgroup $Q \notin \{P\}$ and for every $g \in A_Q(\mathbb{A})\mathbb{K}$, $R_g\phi_{N_Q}$ is orthogonal to the space of cuspidal functions on $L_Q(\mathbb{Q})\backslash L_Q(\mathbb{A})^1$, where ϕ_{N_Q} is the constant term of ϕ along N_Q, defined by

$$\phi_{N_Q} = \int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} \phi(ng)\mathrm{d}n \tag{4.9}$$

with respect to the normalized Haar measure:

$$\int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} \mathrm{d}n = 1. \tag{4.10}$$

Set $A_{\lambda,\{P\}} := A_\lambda \cap V_G(\{P\})$, then as $(\mathfrak{g}, K_\infty, G(\mathbb{A}_f))$-modules,

$$A_\lambda = \bigoplus_{\{P\}\in\mathcal{C}} A_{\lambda,\{P\}}, \tag{4.11}$$

and the cohomology (4.7) equals

$$\bigoplus_{\{P\}} H^*(\mathfrak{m}_G, K_\infty; A_{\lambda,\{P\}} \otimes \mathbb{V}^\vee_\lambda(\mathbb{C}))(\xi_\lambda). \tag{4.12}$$

For each $\{P\}$, there is a descending filtration (of finite length) $A^{T,p}_{\lambda,\{P\}}, p = 0, 1, \dots$ in the space $A_{\lambda,\{P\}}$. This filtration depends on a finite supported $\mathbb{Z}$-valued function T which is defined on the closure $\overline{\check{\mathfrak{a}}_0^+}$, such that:

$$T(\mu) < T(\nu) \text{ if } \mu \neq \nu \text{ and } \mu - \nu \in \overline{{}^+\check{\mathfrak{a}}_0}. \tag{T}$$

The successive quotients $A^{T,p}_{\lambda,\{P\}}/A^{T,p+1}_{\lambda,\{P\}}$ can be described in term of Eisenstein series. Following [11, Section 6], define $\mathcal{M}^{T,p}_{\lambda,\{P\}}$ to be the set of triples $t = (P, \Lambda, \chi) := (P_t, \Lambda_t, \chi_t)$ with the following properties:

(1) $P \in \{P\}$ is a standard parabolic subgroup;
(2) $\Lambda : A_P(\mathbb{A})/A(\mathbb{R})^0 A_P(\mathbb{Q}) \to \mathbb{C}^\times$ is a continuous character. Let $\lambda_t \in (\check{\mathfrak{a}}^G_P)_\mathbb{C}$ be the differential of the archimedean component of Λ, we assume $Re(\lambda_t) \in \overline{\check{\mathfrak{a}}^+_P}$ and $T(Re(\lambda_t)) = p$.
(3) $\chi : \mathfrak{Z}(\mathfrak{m}_G) \to \mathbb{C}^\times$ is a character, such that:
(4) $\lambda_t \in \mathrm{supp}_t(\mathcal{I}_\lambda)$, i.e. for any $x \in \mathcal{I}_\lambda$, $\xi(x)(\lambda_t + \chi_t) = 0$, where $\xi : \mathfrak{Z}(\mathfrak{m}_G) \to S(\mathfrak{t} \cap \mathfrak{m}_G)^{\mathcal{W}_G}$ is the Harish–Chandra isomorphism.

For $t, t' \in \mathcal{M}^{T,p}_{\lambda,\{P\}}$, define a morphism from t to t' to be an element of the Weyl set $\Omega(\mathfrak{a}_t, \mathfrak{a}_{t'})$ which maps Λ_t to $\Lambda_{t'}$ and χ_t to $\chi_{t'}$. So $\mathcal{M}^{T,p}_{\lambda,\{P\}}$ is a groupoid. Let $\mathcal{C}^{T,p}_{\lambda,\{P\}}$ be a set of representatives for the isomorphism classes of objects of $\mathcal{M}^{T,p}_{\lambda,\{P\}}$.

For $t \in \mathcal{M}^{T,p}_{\lambda,\{P\}}$, define $V(t)$ to be the space of square integrable $\mathbb{K} \cap P(\mathbb{A})$-finite functions f on $P(\mathbb{Q})A_P(\mathbb{R})^0 N_P(\mathbb{A})\backslash P(\mathbb{A})$ with the following properties

(1) For any parabolic subgroup $Q \subsetneq P$, f_{N_P} is orthogonal to the space of cuspidal forms on $M_Q(\mathbb{Q})\backslash M_Q(\mathbb{A})$.
(2) $f(ag) = e^{-\langle \lambda_t, H_P(a)\rangle}\Lambda(a) f(g)$ for any $a \in A_P(\mathbb{A})$.
(3) f is a χ-eigenvector of $\mathfrak{Z}(\mathfrak{m}_G)$.

We let $W(t) = \mathrm{ind}^G_P V(t)$ be the space of $\mathbb{K}$-finite functions on the space $P(\mathbb{Q})A_P(\mathbb{R})^0 N_P(\mathbb{A})\backslash G(\mathbb{A})$ such that for any $g \in G(\mathbb{A})$, the function $f(xg)$ of $x \in P(\mathbb{Q})A_P(\mathbb{R})^0 N_P(\mathbb{A})\backslash P(\mathbb{A})$ belongs to $V(t)$.

Let $S(t)$ be the symmetric algebra $S((\mathfrak{a}^G_P)_\mathbb{C})$, which is the space of polynomials on $(\mathfrak{a}^G_P)_\mathbb{C}$, and also viewed as the algebra of finite sums of iterated derivatives at λ_t. $S(t)$ is equipped with the structure of $\mathfrak{a}_P$-module defined by the rule:

(A) For $\xi \in \mathfrak{a}_P$, $\delta \in S(t)$ and any $\eta \in \mathfrak{a}^G_P$

$$(\xi\delta)(\eta) := e^{\langle \xi, \lambda_t + \rho_{P_t}\rangle}\delta(\eta + \xi). \tag{4.13}$$

It is then extended to a $\mathfrak{p}$-module structure by letting $\mathfrak{m}$ and $\mathfrak{n}$ act trivially. $S(t)$ is also equipped with a $P(\mathbb{A}_f)$-module structure by the rule:

(B) For any $x \in P(\mathbb{A}_f)$ and $\eta \in \mathfrak{a}^G_P$

$$(x\delta)(\eta) = e^{\langle \eta^\vee + \rho_{P_t}, H_P(x)\rangle}\delta(\eta). \tag{4.14}$$

So we get a functor from the groupoid $\mathcal{M}^{T,p}_{\lambda,\{P\}}$ to the category of $(\mathfrak{g}, K_\infty, G(\mathbb{A}_f))$-modules, it assigns to t a module

$$M(t) := W(t) \otimes S(t) = \operatorname{ind}_P^G V(t) \otimes S(t), \tag{4.15}$$

For $f \in W(t)$ and $\mu \in (\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$, define the Eisenstein series

$$E(f,\mu) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \mu+\rho_P, H_P(\gamma g)\rangle} f(\gamma g). \tag{4.16}$$

Moreover, for $f \otimes \delta \in W(t) \otimes S(t)$, let $\mathbf{MW}\delta E(f,\mu) \in V_G(\{P\})$ be the main value of the Laurent expansion of $\delta E(f,\mu)$ at λ_∞ (refer [11, Section 6]). For $\{P\} \in \mathcal{C}$, let $\mathcal{C}(\{P\}) \subset \mathcal{C}$ be the subset defined by the property:

$$\text{(P)}\quad \{Q\} \in \mathcal{C}(\{P\}),\ \text{if there is a parabolic } Q \in \{Q\}$$
$$\text{such that } Q \text{ contains some parabolic subgroup in } \{P\}.$$

[11, Theorem 14] asserts that the quotient $A^{T,p}_{\lambda,\{P\}}/A^{T,p+1}_{\lambda,\{P\}}$ is spanned by the main values $\mathbf{MW}\delta E(f,\mu)$ for all $f \otimes \delta$ in $M(t)$ when t is running over all $\mathcal{M}^{T,p}_{\lambda,\{Q\}}$ and $\{Q\}$ in $\mathcal{C}(\{P\})$.

We can now state Franke's theorem ([11, Theorem 19]). It gives a spectral squence to compute the cohomology space (4.7):

Theorem 4.1 (Franke's Eisenstein Spectral Sequence (ESS)). — *Let λ be a regular algebraic weight, then there is a spectral sequence:*

$$E_1^{p,q} = \bigoplus_{t \in \mathcal{C}^{T,p}_{\lambda,\{P\}}} H^{p+q}(\mathfrak{m}_G, K_\infty; M(t) \otimes \mathbb{V}_\lambda^\vee) \Rightarrow H^{p+q}(\mathfrak{m}_G, K_\infty; A_{\lambda,\{P\}} \otimes \mathbb{V}_\lambda^\vee).$$

Moreover, this spectral sequence degenerates.

Writing $K_\infty^t := K_\infty \cap P_t(\mathbb{R})$, we compute the summand of $E_1^{p,q}$ by:

$$\begin{aligned} &H^{p+q}(\mathfrak{m}_G, K_\infty; M(t) \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})) \\ &\quad = H^{p+q}(\mathfrak{m}_G, K_\infty; \operatorname{ind}_{P_t}^G V(t) \otimes S(t) \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})) \\ &\quad = \operatorname{ind}_{P_t(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^{p+q}(\mathfrak{m}_G \cap \mathfrak{p}_t, K_\infty^t; V(t) \otimes S(t) \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})) \\ &\quad = \bigoplus_{i+j=p+q} \operatorname{ind}_{P_t(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^i(\mathfrak{l}_t, K_\infty^t; H^j(\mathfrak{n}_t; \mathbb{V}_\lambda^\vee(\mathbb{C})) \otimes V(t) \otimes S(t)). \end{aligned} \tag{4.17}$$

Apply the Kostant decomposition (4.5), we have

$$\begin{aligned} &H^i(\mathfrak{l}_t, K_\infty^t; H^j(\mathfrak{n}_t; \mathbb{V}_\lambda^\vee(\mathbb{C})) \otimes V(t) \otimes S(t)) \\ &\qquad = \bigoplus_{\substack{w \in \mathcal{W}^{L_t} \\ l(w)=n_t-j}} H^i(\mathfrak{l}_t, K_\infty^t; \mathbb{V}^{L,\vee}_{w(\lambda+\rho_{P_t})+\rho_{P_t}}(\mathbb{C}) \otimes V(t) \otimes S(t)). \end{aligned} \tag{4.18}$$

Using the notation of [11, Section 6], for any $\Theta \in \check{\mathfrak{a}}_t$, let $\mathbb{C}_\Theta$ be the one dimensional vector space $\mathbb{C}$ on which $x \in \mathfrak{a}_t$ acts by muliplication of $e^{\langle x,\Theta\rangle}$. Twisting $\mathbb{V}^{L,\vee}_{w(\lambda+\rho_{P_t})+\rho_{P_t}} \otimes V(t)$ by a proper $\mathbb{C}_\Theta$ to make it a trivial $\mathfrak{a}_t$-module, we apply K*ü*nneth theorem for each summand in last equation with respect to $\mathfrak{l} = \mathfrak{m} + \mathfrak{a}$. Then a standard computation shows that

$$\begin{aligned}(4.19)\quad & H^i(\mathfrak{l}_t, K^t_\infty; H^j(\mathfrak{n}_t; \mathbb{V}^\vee_\lambda(\mathbb{C})) \otimes V(t) \otimes S(t)) \\ &= \bigoplus_{w\in\mathcal{W}^t_j} H^i(\mathfrak{m}_t, K^t_\infty; \mathbb{V}^{L_t,\vee}_{w(\lambda+\rho_{P_t})+\rho_{P_t}}(\mathbb{C}) \otimes V(t)) \otimes \mathbb{C}_{\rho_{R_t}+\lambda_t}\end{aligned}$$

where $\mathcal{W}^t_j$ is the subset of $w \in \mathcal{W}^L$, such that $l(w) = n - j$ and the natual projection of $w(\lambda + \rho_{P_t})$ to $\check{\mathfrak{a}}^G_{P_t}$ is λ_t. Combing all the results above, for λ regular, the E_1-term, $E^{p,q}_1$, of (ESS) can be computed by

$$\begin{aligned}(4.20)\quad & \bigoplus_{t\in\mathcal{C}^{T,p}_{\lambda,\{P\}}} \bigoplus_{i+j=p+q} \bigoplus_{w\in\mathcal{W}^t_j} \operatorname{ind}^{G(\mathbb{A}_f)}_{P_t(\mathbb{A}_f)} H^i(\mathfrak{m}_t, K^t_\infty; \\ & \qquad \mathbb{V}^{L_t,\vee}_{w(\lambda+\rho_{P_t})+\rho_{P_t}}(\mathbb{C}) \otimes V(t)) \otimes \mathbb{C}_{\rho_{R_t}+\lambda_t}\end{aligned}$$

Now Franke's theorem implies $H^r(S_G, \mathbb{V}^\vee_\lambda(\mathbb{C}))(\xi^{-1}_\lambda)$ equals:

$$\begin{aligned}(4.21)\quad & \bigoplus_{\{P\}} \bigoplus_{p} \bigoplus_{t\in\mathcal{C}^{T,p}_{\lambda,\{P\}}} H^r(\mathfrak{m}_G, K_\infty; M(t) \otimes \mathbb{V}^\vee_\lambda(\mathbb{C})) \\ &= \bigoplus_{\{P\}} \bigoplus_{p} \bigoplus_{t\in\mathcal{C}^{T,p}_{\lambda,\{P\}}} \bigoplus_{w\in\mathcal{W}^t} \operatorname{ind}^{G(\mathbb{A}_f)}_{P_t(\mathbb{A}_f)} H^{r+l(w)-n_t}(\mathfrak{m}_t, K^t_\infty; \\ & \qquad \mathbb{V}^{L_t,\vee}_{w(\lambda+\rho_{P_t})+\rho_{P_t}}(\mathbb{C}) \otimes V(t)) \otimes \mathbb{C}_{\rho_{P_t}+\lambda_t}\end{aligned}$$

where $\mathcal{W}^t$ is the subset of $w \in \mathcal{W}^L$, such that the natual projection of $w(\lambda + \rho_{P_t})$ to $\check{\mathfrak{a}}^G_{P_t}$ is λ_t.

4.3. Twisted Franke's trace formula

Let $\lambda \in X^*(T)$ be a regular dominant ι-invariant weight, $f \in \mathcal{H}_p \subset C^\infty_c(G(\mathbb{A}_f))$ an admissible p-adic Hecke operator. In this section, we use (4.21) to compute the alternating trace

$$(4.22)\quad \operatorname{tr}(\iota \times f \,|\, H^*(S_G, \mathbb{V}^\vee_\lambda(\mathbb{C}))) := \sum_r (-1)^r \operatorname{tr}(\iota \times f \,|\, H^r(S_G, \mathbb{V}^\vee_\lambda(\mathbb{C}))).$$

The alternating trace $\operatorname{tr}(f \,|\, H^*(S_G, \mathbb{V}^\vee_\lambda(\mathbb{C})))$ without twisting was computed by Franke and Urban in [11, Section 7.7] and [18, Theorem 1.4.2]. Here we have to study how ι acts on each step from (4.11) to (4.21).

4.3.1. ι-action on $(\mathfrak{m}_G, K_\infty)$-cohomology

Consider the complex

$$C^*(\mathfrak{m}_G, K_\infty; \mathbb{V}_\lambda^\vee(\mathbb{C})) := \mathrm{Hom}_{K_\infty}(\wedge^*(\mathfrak{m}_G/\mathfrak{k}_\infty), V_G \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})), \tag{4.23}$$

which computes the $(\mathfrak{m}_G, K_\infty)$-cohomology $H^*(\mathfrak{m}_G, K_\infty; V_G \otimes \mathbb{V}_\lambda^\vee(\mathbb{C}))$ by definition. Let $\iota_* : \mathfrak{m}_G \to \mathfrak{m}_G$ be the push-forward map induced from $\iota : G(\mathbb{R}) \to G(\mathbb{R})$. Define $\iota : V_G \to V_G$ by sending φ to φ^ι, such that for any $[g] \in G(\mathbb{Q})A(\mathbb{R})^0\backslash G(\mathbb{A})$, $\varphi^\iota([g]) = \varphi([g]^\iota)$. Then ι acts on $\mathrm{Hom}_{K_\infty}(\wedge^q(\mathfrak{m}_G/\mathfrak{k}_\infty), V_G \otimes \mathbb{V}_\lambda^\vee(\mathbb{C}))$ by sending ϕ to ϕ^ι, such that

$$\phi^\iota : X \mapsto (\phi((\iota^{-1})_* X))^\iota \tag{4.24}$$

for any $X \in \wedge^q(\mathfrak{m}_G/\mathfrak{k}_\infty)$. It is straightforward to verify that the action is well defined up to homotopy.

Now consider the morphism of complexes $\alpha : C^*(\mathfrak{m}_G, K_\infty; \mathbb{V}_\lambda^\vee(\mathbb{C})) \to \Omega^*(S_G, \mathbb{V}_\lambda^\vee(\mathbb{C}))$ that induces the isomorphism $H^q(\mathfrak{m}_G, K_\infty; V_G \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})) \cong H^q(S_G, \mathbb{V}_\lambda^\vee(\mathbb{C}))$. Concretely, for $\phi \in C^q(\mathfrak{m}_G, K_\infty; \mathbb{V}_\lambda^\vee(\mathbb{C}))$, α assigns it a q-th differential form τ_ϕ, such that for any $g \in G(\mathbb{A})^1$,

$$\tau_\phi(\bar{v}_1 \wedge \cdots \wedge \bar{v}_q)([g]) = \phi(\bar{v}_1([e]), \ldots, \bar{v}_q([e]))([g]), \tag{4.25}$$

where $\bar{v}$ indicates a left invariant vector field on S_G and $\bar{v}([e])$ its value at $[e] \in S_G$. Since $\phi(\bar{v}_1([e]), \ldots, \bar{v}_q([e])) \in V_G \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})$, $\phi(\bar{v}_1([e]), \ldots, \bar{v}_q([e]))([g])$ means to evaluate its first component at $[g]$. Compare with Section 3.1, it is easy to see that the ι-action on $H^q(\mathfrak{m}_G, K_\infty; V_G \otimes \mathbb{V}_\lambda^\vee(\mathbb{C}))$ is compatible with the ι-action on $H^q(S_G, \mathbb{V}_\lambda^\vee(\mathbb{C}))$ defined in Section 3.1.

4.3.2. Image of $A_{\lambda,\{P\}}$ under ι

Since that A_λ is stable under ι, we now study the behavior of decomposition (4.11) under ι. Given an associate class $\{P\}$, let $\{P^\iota\}$ be the class whose elements are P^ι for all $P \in \{P\}$. Apparently the map $\{P\} \mapsto \{P^\iota\}$ permutes the associate classes. Write $\{P\}^\iota := \{P^\iota\}$.

Lemma 4.2. — *Let $A^\iota_{\lambda,\{P\}}$ be the image of $A_{\lambda,\{P\}}$ under ι,*

$$A^\iota_{\lambda,\{P\}} = A_{\lambda,\{P^{\iota^{-1}}\}} \tag{4.26}$$

Proof. — Given $\phi \in A_{\lambda,\{P\}}$, we have to show that, for any parabolic $Q \notin \{P^{\iota^{-1}}\}$ and $g \in A_Q(\mathbb{A})\mathbb{K}$, $R_g(\phi^\iota)_{N_Q} \perp L^2_{\mathrm{cusp}}(L_Q(\mathbb{Q})\backslash L_Q(\mathbb{A})^1)$. Let dn_Q be the normalized Haar measure on N_Q, then dn_{Q^ι} is same to the

Haar measure on $N_Q^\iota = N_{Q^\iota}$ induced by the map $\iota : N_Q \to N_Q^\iota = N_{Q^\iota}$. Now a direct computation shows that

$$(\phi^\iota)_{N_Q} = (\phi_{N_{Q^\iota}})^\iota. \tag{4.27}$$

Then $g^\iota \in A_{Q^\iota}(\mathbb{A})\mathbb{K}$, and

$$R_g(\phi^\iota)_{N_Q} = R_g((\phi_{N_{Q^\iota}})^\iota) = (R_{g^\iota}\phi_{N_{Q^\iota}})^\iota. \tag{4.28}$$

Noting that the restriction of ι on $L^2_{\mathrm{cusp}}(L_Q(\mathbb{Q})\backslash L_Q(\mathbb{A})^1)$ identify its image with $L^2_{\mathrm{cusp}}(L_{Q^{\iota^{-1}}}(\mathbb{Q})\backslash L_{Q^{\iota^{-1}}}(\mathbb{A})^1)$, we have

$$L^2_{\mathrm{cusp}}(L_{Q^\iota}(\mathbb{Q})\backslash L_{Q^\iota}(\mathbb{A})^1)^\iota = L^2_{\mathrm{cusp}}(L_Q(\mathbb{Q})\backslash L_Q(\mathbb{A})^1). \tag{4.29}$$

Now the conclusion follows. □

This lemma implies that, in the formula (4.21), only those summand parameterized by $\{P\} = \{P\}^\iota$ will contribute to the twisted alternating trace.

4.3.3. On Eisenstein series

Consider an associate class $\{P\}$ and $t = (P, \Lambda, \chi) \in \mathcal{M}^{T,p}_{\lambda,\{P\}}$ for some p. For $\varphi \in W(t)$ and $\mu \in (\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$, let $E_t(\varphi,\mu)(g) := E(\varphi,\mu)(g)$ be the Eisenstein series defined in (4.16), then A_λ is spanned by the principal values of derivatives of all such $E(\varphi,\mu)$.

Lemma 4.3.

$$E_t(\varphi,\mu)^\iota = E_{t^\iota}(\varphi^\iota, \mu^{\iota^{-1}}) \tag{4.30}$$

where we define

1. $\Lambda^{\iota^{-1}} : A_{P^{\iota^{-1}}}(\mathbb{A})/A(\mathbb{R})^0 A_{P^{\iota^{-1}}}(\mathbb{Q}) \to \mathbb{C}^\times$ *as* $\Lambda^{\iota^{-1}}(a) := \Lambda(a^\iota)$;
2. $\chi^{\iota^{-1}} : \mathfrak{m}_G \to \mathbb{C}^\times$ *as* $\chi^{\iota^{-1}}(x) := \chi(x^\iota)$;
3. $t^\iota := (P^{\iota^{-1}}, \Lambda^{\iota^{-1}}, \chi^{\iota^{-1}})$
4. $\varphi^\iota(g) := \varphi(g^\iota)$. *Then* $\varphi^\iota \in W(t^\iota)$.
5. $\mu^{\iota^{-1}} \in (\check{\mathfrak{a}}^G_{P^{\iota^{-1}}})_{\mathbb{C}}$, *as a character, it is defined by* $\mu^{\iota^{-1}}(a) := \mu(a^\iota)$. *In particular, there is a natural homomorphism* $\iota : S((\check{\mathfrak{a}}_P^G)_{\mathbb{C}}) \to S((\check{\mathfrak{a}}^G_{P^{\iota^{-1}}})_{\mathbb{C}})$.

So we have a homomorphism between vector spaces

$$\iota : M(t) \to M(t^\iota) \tag{4.31}$$

such that for $\varphi \otimes \delta \in M(t) = W(t) \otimes S(t)$

$$\mathbf{MW}_{\lambda_t}\delta E_t(\varphi,\mu)^\iota = \mathbf{MW}_{\lambda_{t^\iota}}\delta^\iota E_{t^\iota}(\varphi^\iota, \mu^{\iota^{-1}}). \tag{4.32}$$

Moreover, ι induces an homomorphism between the $(\mathfrak{m}_G, K_\infty)$-cohomology group,

$$\iota : H^*(\mathfrak{m}_G, K_\infty; M(t) \otimes \mathbb{V}^\vee_\lambda(\mathbb{C})) \to H^*(\mathfrak{m}_G, K_\infty; M(t^\iota) \otimes \mathbb{V}^\vee_\lambda(\mathbb{C})) \tag{4.33}$$

Proof. — (4.30) and (4.32) are from definition directly. To show (4.33), one only has to notice that ι is compatible with (4.24). □

Remark 4.4. — The definition of $\mathbf{MW}_{\lambda_t}$ actually depends on the choice of a regular element $\xi_t \in \check{\mathfrak{a}}^G_P$, so here (4.3.11) depends on the choice $\xi_{t^\iota} = \xi_t^{\iota^{-1}}$. However, just as the situation in [11, Section 7], it does not matter our goal.

Recall that the quotient $A^{T,p}_{\lambda,\{P\}}/A^{T,p+1}_{\lambda,\{P\}}$ is spanned by the elements of form $\mathbf{MW}_{\lambda_t}\delta E(f,\mu)$. As observed by Franke and Schwermer in [12], the only relations between these vectors are the relations provided by the functional equation of the Eisenstein series for singular λ_t. So if $t \cong t^\iota$ in $\cup_p \mathcal{M}^{T,p}_{\lambda,\{P\}}$, the image of $H^*(\mathfrak{m}_G, K_\infty; M(t) \otimes \mathbb{V}^\vee_\lambda(\mathbb{C}))$ and $H^*(\mathfrak{m}_G, K_\infty; M(t^\iota) \otimes \mathbb{V}^\vee_\lambda(\mathbb{C}))$ in $H^*(\mathfrak{m}_G, K_\infty; A_{\lambda,\{P\}} \otimes \mathbb{V}^\vee_\lambda(\mathbb{C}))$ coincide. This implies that, in the first step of (4.21), only those terms with $t \cong t^\iota$ will contribute to the twisted alternating trace.

4.3.4.

For a standard parabolic subgroup P and the associate class $\{P\}$ containing P, consider a triple $(P,\mu,\chi) \in \mathcal{M}^{T,T(\mu)}_{\lambda,\{P\}}$. Let $n_P(\mu)$ be the cardinality of the isomorphism class containing (P,μ,χ), then $n_P(\mu)$ is the number of Weyl chambres to which μ belongs, in particular, $n_P(\mu) = 1$ if μ is regular. Define:

$$\Upsilon := \left\{\mu \in \check{\mathfrak{a}}_0 \,\middle|\, pr_{\check{\mathfrak{a}}_0 \to \check{\mathfrak{a}}^G_P}(\mu+\rho) \text{ is regular in } \check{\mathfrak{a}}^G_P,\ \forall\ P\right\}, \tag{4.34}$$

Υ is dense in $\check{\mathfrak{a}}_0$. From now on, assume that

$$\lambda \in \Upsilon. \tag{R}$$

With the assumption (R), $\mathcal{W}^t = \emptyset$ for any $t \in \mathcal{M}^{T,p}_{\lambda,\{P\}}$, unless λ_t is regular. In particular, in the alternating trace (4.22), only those t with λ_t regular will contribute to the trace. In this case, $n_p(\lambda_t) = 1$, and the condition $t \cong t^\iota$ equals the condidtion $t = t^\iota$.

Combing the discussion in last several sections, (4.22) is computed by:

$$\begin{aligned}(4.35)\quad &\sum_{t\cong t^\iota} \frac{1}{n_P(\lambda_t)} \operatorname{tr}\left(\iota\times f \,\middle|\, \left(\operatorname{ind}_{P_t(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \bigoplus_{w\in\mathcal{W}^t} H^{*+l(w)-n_t}(\mathfrak{m}_t, K_\infty^t;\right.\right.\\ &\qquad\left.\left.\mathbb{V}^{L_t,\vee}_{w(\lambda+\rho_{P_t})+\rho_{P_t}}(\mathbb{C})\otimes V(t))\otimes\mathbb{C}_{\rho_{R_t}+\lambda_t}\right)(\xi_\lambda)\right)\\ &= \sum_{P=P^\iota}\sum_{\Lambda=\Lambda^\iota}\sum_{\chi=\chi^\iota}(-1)^{n_P}\operatorname{tr}\left(\iota\times f_L\,\middle|\,\left(\bigoplus_{w\in\mathcal{W}^t} H^{*+l(w)}(\mathfrak{m}_P, K_\infty^t;\right.\right.\\ &\qquad\left.\left.\mathbb{V}^{L,\vee}_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C})\otimes V(t))\right)(\xi_{w(\lambda+\rho_P)+\rho_P})\right),\end{aligned}$$

where, t is a triple (P,Λ,χ); on the right side of this equation, μ is the differential of the Archimedean part of Λ and $P=LN=MAN$ the Langlands decomposition. For a Levi subgroup L, f_L is defined by

$$(4.36)\qquad f_L(l)=e^{\langle\rho_P,H_P(l)\rangle}\int_{\mathbb{K}_f}\int_{N_P(\mathbb{A}_f)} f(klnk^{-1})\mathrm{d}n\mathrm{d}k,$$

where the Haar measures are normalized with respect to the Iwasawa decomposition as in [11, Section 7.7]. The twisting factor appears in (4.36) because we have twisted the character ξ_λ in the $(\mathfrak{m}_G,K_\infty)$-cohomology. The twisting term $\mathbb{C}_{\rho_P+\mu}$ disappears because, as a $\mathfrak{a}_P$-module, it does not contribute to the trace. Now combing all χ and all the finite parts Λ_f of Λ in the summation, by the definition (1) of $V(t)$, (4.35) equals

$$\begin{aligned}(4.37)\quad &\sum_{P=P^\iota}\sum_{\mu=\mu^\iota}(-1)^{n_P}\operatorname{tr}\left(\iota\times f_L\,\middle|\,\left(\bigoplus_{w\in\mathcal{W}^t} H^{*+l(w)}(\mathfrak{m}_P, K_\infty^L;\right.\right.\\ &\left.\left.\mathbb{V}^{L,\vee}_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C})\otimes L^2_{\mathrm{disc}}(A_P(\mathbb{R})^0L_P(\mathbb{Q})\backslash L_P(\mathbb{A}))\right)(\xi_{w(\lambda+\rho_P)+\rho_P})\right)\end{aligned}$$

4.3.5.

Now we study the action of ι on the direct sum over Weyl elements in the last formula. For every $\varphi\in\mathbb{V}^L_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C})\subset C(L_P(\mathbb{Q})\backslash L_P(\mathbb{A}))$, write $\varphi^\iota(x):=\varphi(x^{\iota^{-1}})$. It is easy to see that

$$(4.38)\qquad (\mathbb{V}^L_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C}))^\iota=\mathbb{V}^L_{(w(\lambda+\rho_P)+\rho_P)^\iota}(\mathbb{C}).$$

Lemma 4.5. — *Let ι act on the Weyl group $\mathcal{W}=N_G(T)/T$ via $[x]\mapsto[x^\iota]$ for any $x\in N_G(T)$. Then*

(1) *Let* S_α *be a simple reflection in* $\mathcal{W}$ *corresponding to a simple root* α*, then* $(S_\alpha)^\iota = S_{\alpha^\iota}$*. In particular,* ι *preserves the length of a Weyl element.*
(2) *For any* $w \in \mathcal{W}$*,* $(w(\alpha))^\iota = w^\iota(\alpha^\iota)$*.*
(3) *Let* $P \in \mathcal{P}$*, then* $\rho_P^\iota = \rho_P$*.*

In particular

$$(\mathbb{V}^L_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C}))^\iota = \mathbb{V}^L_{w^\iota(\lambda+\rho_P)+\rho_P}(\mathbb{C}). \tag{4.39}$$

Moreover, if λ *is regular,* $(\mathbb{V}^L_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C}))^\iota = \mathbb{V}^L_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C})$ *if and only if* $w^\iota = w$*.*

Proof. — The proof is straightforward. Concretely speaking, (1) follows from the fact that S_α is the only nontrivial element in $N_{G_\alpha}(T)/T$ (see, e.g. [14, IV]) and $(S_\alpha)^\iota$ is a non-trivial element in $N_{G_{\alpha^\iota}}(T)/T$. (2) follows from a direct computation: let $[x]$ be a representative of w, for any $t \in T$, $(w(\alpha))^\iota(t) = w(\alpha)(t^{\iota^{-1}}) = \alpha(x^{-1}t^{\iota^{-1}}x)$ and $w^\iota(\alpha^\iota)(t) = \alpha^\iota((x^\iota)^{-1}tx^\iota) = \alpha(x^{-1}t^{\iota^{-1}}x)$. (3) follows from the definition that $\rho_P(t) := \det(\mathrm{Ad}(t)|_{\mathfrak{n}_P})^{1/2}$ (see, e.g. [7, III]) and the commutative diagram $\mathrm{ad}(t)\circ\iota = \iota\circ\mathrm{ad}(t^\iota)$. Finally, one deduces (4.39) from (1)–(3) directly. □

Now let $\mathcal{W}^{t,\iota}$ be the subset of $\mathcal{W}^t$ consisting of elements which are invariant under ι. The previous lemma implies that (4.37) equals

$$\sum_{P=P^\iota}\sum_{\mu=\mu^\iota}\sum_{w\in\mathcal{W}^{t,\iota}}(-1)^{n_P-l(w)}\,\mathrm{tr}(\iota\times f_L\,|\,(H^*(\mathfrak{m}_P,K^L_\infty;\\ \mathbb{V}^{L,\vee}_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C})\otimes L^2_{\mathrm{disc}}(A_P(\mathbb{R})^0L_P(\mathbb{Q})\backslash L_P(\mathbb{A}))))(\xi_{w(\lambda+\rho_P)+\rho_P})) \tag{4.40}$$

Recall that for any $w \in \mathcal{W}^t$, $w(\lambda+\rho) = \mu$. So when μ is running over all classes $t \in \mathcal{M}^T_{\lambda,\{P\}}$, w is running over $\mathcal{W}^L_{\mathrm{Eis}}$. Let $\mathcal{W}^{L,\iota}_{\mathrm{Eis}}$ be the subset of $\mathcal{W}^L_{\mathrm{Eis}}$ consisting of elements which are invariant under ι, the previous formula equals

$$\begin{aligned}&\sum_{P=P^\iota}\sum_{w\in\mathcal{W}^{L,\iota}_{\mathrm{Eis}}}(-1)^{n_P-l(w)}\,\mathrm{tr}(\iota\times f_L\,|\,(H^*(\mathfrak{m}_P,K^L_\infty;\\ &\quad\mathbb{V}^{L,\vee}_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C})\otimes L^2_{\mathrm{disc}}(A_P(\mathbb{R})^0L_P(\mathbb{Q})\backslash L_P(\mathbb{A}))))(\xi_{w(\lambda+\rho_P)+\rho_P}))\\ &=\sum_{P=P^\iota}\sum_{w\in\mathcal{W}^{L,\iota}_{\mathrm{Eis}}}(-1)^{n_P-l(w)}\,\mathrm{tr}(\iota\times f_L\,|\,(H^*(\mathfrak{m}_P,K^L_\infty;\\ &\quad\mathbb{V}^{L,\vee}_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C})\otimes L^2_{\mathrm{cusp}}(A_P(\mathbb{R})^0L_P(\mathbb{Q})\backslash L_P(\mathbb{A}))))(\xi_{w(\lambda+\rho_P)+\rho_P}))\end{aligned} \tag{4.41}$$

where the equality holds since $w(\lambda+\rho_P)+\rho_P$ is regular for $\lambda \in \Upsilon$.

Finally, we summarize all the computation above together by:

THEOREM 4.6 (Twisted Franke's trace formula). — *Assume λ is regular in Υ, then for any $f \in C_c^\infty(G(\mathbb{A}_f))$,*

$$\begin{aligned}(4.42)\quad & \mathrm{tr}^{st}(\iota \times f \mid H^*(S_G, \mathbb{V}_\lambda^\vee(\mathbb{C}))) \\ &= \sum_{P=P^\iota} \sum_{w \in \mathcal{W}_{Ein}^{L,\iota}} (-1)^{l(w)+n_P} \, \mathrm{tr}^{st}(\iota \times f_L \mid H^*_{\mathrm{cusp}}(S_L, \mathbb{V}^{L,\vee}_{w(\lambda+\rho_P)+\rho_P}(\mathbb{C}))),\end{aligned}$$

where the notation st indicates that we are using the standard Hecke action.

4.4. Cuspidal decomposition formula

Let L be a finite field extension of $\mathbb{Q}_p$ as in Section 2, we have fixed an embedding $L \hookrightarrow \mathbb{C}$. Let $\lambda = \lambda^{\mathrm{alg}}\epsilon$ be an arithmetic regular dominant weight in $\mathfrak{X}^\iota(L)$, define the alternating trace:

$$I_G^{\mathrm{cl}}(\iota \times f, \lambda) := \mathrm{tr}^*(\iota \times f | H^*(S_G, \mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(L, \epsilon))), \tag{4.43}$$

where $*$ indicates that we are using the $*$-action defined in Section 3.1.

LEMMA 4.7. — *Assume $f = f^p \otimes u_t \in \mathcal{H}_p'$, then*

$$I_G^{\mathrm{cl}}(\iota \times f, \lambda) = \lambda(\xi(t)) \, \mathrm{tr}^{st}(\iota \times f \mid H^*(S_G, \mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(\mathbb{C}, \varepsilon))). \tag{4.44}$$

Just like [18, 4.5.1 (29)], this is a direct consequence of (3.9) and [18, Lemma 4.3.8].

For any $f \in \mathcal{H}_p$, define the twisted cuspidal alternating trace by:

$$\begin{aligned}(4.45)\quad & I_{G,0}^{\mathrm{cl}}(\iota \times f, \lambda) \\ & := \mathrm{meas}(K^p)\lambda(\xi(t)) \, \mathrm{tr}^{st}(\iota \times f \mid H^*_{\mathrm{cusp}}(S_G(K_f), \mathbb{V}^\vee_\lambda(\mathbb{C}, \epsilon)))\end{aligned}$$

if $f \in \mathcal{H}_p(K^p)$. This is well defined since ι is well defined on the cuspidal cohomology.

For any $w \in \mathcal{W}^L$ and $f \in \mathcal{H}_p(G)$, define $f_{L,w}^{\mathrm{reg}} \in \mathcal{H}_p(L)$ as below: For $f = f^p \otimes u_t \in \mathcal{H}_p(G)$, define:

$$f_{L,w}^{\mathrm{reg}} := e^{\langle \rho_P, H_P(l) \rangle} \epsilon_{\xi,w}(t) (f^p)'_L \otimes u_{wtw^{-1}}, \tag{4.46}$$

where the factor $\epsilon_{\xi,w}(t) := \xi(t)^{w^{-1}(\rho_P)+\rho_P} |t^{w^{-1}(\rho_P)+\rho_P}|_p$, which is trivial according to our choice of ξ as (2.14), $(f^p)'_L$ is the usual non-normalized constant term

$$(f^p)'_L(l^p) = \int_{\mathbb{K}_f^p} \int_{N(\mathbb{A}_f^p)} f_p(k^p l^p n^p (k^p)^{-1}) \mathrm{d}k^p \mathrm{d}n^p. \tag{4.47}$$

For general f, the definition is given by linear extension.

Theorem 4.8. — *Let λ be a regular arithmetic weight such that $\lambda^{\mathrm{alg}} \in \Upsilon$, then for any f as above, $I_G^{\mathrm{cl}}(\iota \times f, \lambda)$ equals*

$$\sum_{P=P^\iota} \sum_{v\in\mathcal{W}_{Ein}^{L,\iota}} \sum_{w\in\mathcal{W}^{L,\iota}} (-1)^{l(v)+n_P} \xi(t)^{\lambda - w^{-1}v*\lambda} I_{L,0}^{\mathrm{cl}}(\iota \times f_{L,w}^{\mathrm{reg}}, v(\lambda+\rho_P)+\rho_P).$$

Proof. — The proof is essentially same to [18, Lemma 4.6.2]. Since both sides of the equation are linear on f, it is innocuous to assume that $f = f^p \otimes u_t$ and $f^p = 1_{K^p}$. If $\lambda = \lambda^{\mathrm{alg}}\varepsilon$, the finite order character ε will simply appear in every step of the proof by multiplying a twisting factor, so we only have to deal with the case that $\lambda = \lambda^{\mathrm{alg}}$ algebraic. By the twisted Franke's trace formula and Lemma 4.7, $I_G^{\mathrm{cl}}(\iota \times f, \lambda)$ equals:

$$\lambda(\xi(t)) \sum_{P=P^\iota} \sum_{v\in\mathcal{W}_{Ein}^{L,\iota}} (-1)^{l(v)+n_P} \operatorname{tr}^{st}(\iota \times f_L \mid H^*_{\mathrm{cusp}}(S_L, \mathbb{V}^{L,\vee}_{v\cdot\lambda+2\rho_P})). \tag{4.48}$$

For group $H = L, N$ or P, write $K_H^p := K^p \cap H(\mathbb{A}_f^p)$. Then for $f = 1_{K^p}\otimes u_t$, we have

$$f_L(l) = \operatorname{meas}(K^p)\operatorname{meas}(K_N^p) 1_{K_L^p} \otimes (u_t)_L. \tag{4.49}$$

So (4.48) equals

$$\begin{aligned}\lambda(\xi(t)) &\sum_{P=P^\iota} \sum_{v\in\mathcal{W}_{Ein}^{L,\iota}} (-1)^{l(v)+n_P} \operatorname{meas}(K^p)\operatorname{meas}(K_N^p) \\ &\qquad\cdot \operatorname{tr}^{st}(\iota \times 1_{K_L^p} \otimes (u_t)_L \mid H^*_{\mathrm{cusp}}(S_L, \mathbb{V}^{L,\vee}_{v\cdot\lambda+2\rho_P})) \\ = \lambda(\xi(t)) &\sum_{P=P^\iota} \sum_{v\in\mathcal{W}_{Ein}^{L,\iota}} (-1)^{l(v)+n_P} \operatorname{meas}(K^p)\operatorname{meas}(K_P^p) \\ &\qquad\cdot \operatorname{tr}^{st}(\iota \times (u_t)_L \mid H^*_{\mathrm{cusp}}(S_L, \mathbb{V}^{L,\vee}_{v\cdot\lambda+2\rho_P})^{K_L^p}).\end{aligned} \tag{4.50}$$

Write $\mu := v\cdot\lambda + 2\rho_P$ and $\sigma_\mu := H^*_{\mathrm{cusp}}(S_L, \mathbb{V}^{L,\vee}_\mu(\mathbb{C}))^{K_L^p}$. Since ι is well defined on σ_μ, viewing $\operatorname{ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \sigma_\mu$ as the restriction of $\operatorname{ind}_{P(\mathbb{Q}_p)\rtimes\langle\iota\rangle}^{G(\mathbb{Q}_p)\rtimes\langle\iota\rangle} \sigma_\mu$ to $G(\mathbb{Q}_p)$, we have

$$\operatorname{tr}^{st}(\iota \times (u_t)'_L \mid \sigma_\mu) = \operatorname{tr}^{st}\left(\iota \times u_t \,\middle|\, \operatorname{ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \sigma_\mu\right). \tag{4.51}$$

According to the decomposition

$$G(\mathbb{Q}_p) = \bigsqcup_{w\in\mathcal{W}^L} P(\mathbb{Q}_p) w I, \tag{4.52}$$

there is

$$\left(\operatorname{ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \sigma_\mu\right)^I \cong (\sigma_\mu^{I_L})^{\mathcal{W}^L}, \tag{4.53}$$

where the isomorphism is given by $\phi \mapsto (\phi(w))_{w\in\mathcal{W}^L}$. In particular, ι acts on the right side by sending $(\phi(w))$ to $(\phi(w^\iota))$. Let $\mathcal{W}^{L,\iota}$ be the subset of $\mathcal{W}^L$ consisting of elements which are invariant under ι. Write $N_w := N \cap w^{-1}Nw$ and $I_L := I \cap L(\mathbb{Q}_p) = wIw^{-1} \cap L(\mathbb{Q}_p)$, (4.4.8) equals

$$\begin{aligned}(4.54)\quad & \lambda(\xi(t)) \sum_{P=P^\iota} \sum_{v\in\mathcal{W}^{L,\iota}_{Ein}} (-1)^{l(v)+n_P} \operatorname{meas}(K^p) \operatorname{meas}(K^p_P) \\ &\qquad \cdot |\rho_P(t)|_p \operatorname{tr}^{st}\left(\iota \times ItI \,\middle|\, \left(\operatorname{ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} \sigma_\mu\right)^I\right) \\ &= \lambda(\xi(t)) \sum_{P=P^\iota} \sum_{v\in\mathcal{W}^{L,\iota}_{Ein}} (-1)^{l(v)+n_P} \operatorname{meas}(K^p) \operatorname{meas}(K^p_P) |\rho_P(t)|_p \\ &\qquad \sum_{w\in\mathcal{W}^{L,\iota}} [\mathcal{N}_w(\mathbb{Z}_p) : t\mathcal{N}_w(\mathbb{Z}_p)t^{-1}] \operatorname{tr}^{st}(\iota \times I_L wtw^{-1} I_L \mid \sigma^{I_L}_\mu)\end{aligned}$$

Noting that

$$\begin{aligned}(4.55)\quad & \operatorname{tr}^{st}(\iota \times I_L wtw^{-1} I_L \mid \sigma^{I_L}_\mu) \\ &= \operatorname{tr}^{st}(\iota \times I_L wtw^{-1} I_L \mid H^*_{\mathrm{cusp}}(S_L(K^p_L I_L), \mathbb{V}^{L,\vee}_\mu)) \\ &= \frac{\operatorname{tr}^{st}(\iota \times (1_{K^p})_L \otimes u_{wtw^{-1}} \mid H^*_{\mathrm{cusp}}(S_L(K^p_L I_L), \mathbb{V}^{L,\vee}_\mu))}{\operatorname{meas}(K^p) \operatorname{meas}(K^p_P)},\end{aligned}$$

and

$$(4.56)\qquad [\mathcal{N}_w(\mathbb{Z}_p) : t\mathcal{N}_w(\mathbb{Z}_p)t^{-1}] = |(w^{-1}(\rho_P) + \rho_P)(t)|_p^{-1},$$

(4.54) equals

$$\begin{aligned}(4.57)\quad & \\ & \lambda(\xi(t)) \sum_{P=P^\iota} \sum_{v\in\mathcal{W}^{L,\iota}_{Ein}} (-1)^{l(v)+n_P} |\rho_P(t)|_p \\ &\quad \sum_{w\in\mathcal{W}^{L,\iota}} [\mathcal{N}_w(\mathbb{Z}_p) : t\mathcal{N}_w(\mathbb{Z}_p)t^{-1}] \operatorname{tr}^{st}(\iota \times (1_{K^p})_L \otimes u_{wtw^{-1}} \mid \sigma^{I_L}_\mu) \\ &= \sum_{P=P^\iota} \sum_{v\in\mathcal{W}^{L,\iota}_{Ein}} (-1)^{l(v)+n_P} |\rho_P(t)|_p \sum_{w\in\mathcal{W}^{L,\iota}} \lambda(\xi(t)) \\ &\quad w^{-1}(\mu)(\xi(t))^{-1} |(w^{-1}(\rho_P)+\rho_P)(t)|_p^{-1} I^{\mathrm{cl}}_{L,0}(\iota\times(1_{K^p})_L\otimes u_{wtw^{-1}}, \mu) \\ &= \sum_{P=P^\iota} \sum_{v\in\mathcal{W}^{L,\iota}_{Ein}} \sum_{w\in\mathcal{W}^{L,\iota}} (-1)^{l(v)+n_P} \xi(t)^{\lambda - w^{-1}(\mu) + w^{-1}(\rho_P) + \rho_P} \\ &\quad \cdot |\rho_P(t)|_p I^{\mathrm{cl}}_{L,0}(\iota \times \epsilon_{\xi,w}(t)(1_{K^p})_L \otimes u_{wtw^{-1}}, \mu) \\ &= \sum_{P=P^\iota} \sum_{v\in\mathcal{W}^{L,\iota}_{Ein}} \sum_{w\in\mathcal{W}^{L,\iota}} (-1)^{l(v)+n_P} \xi(t)^{\lambda - w^{-1}v*\lambda} I^{\mathrm{cl}}_{L,0}(\iota \times f^{\mathrm{reg}}_{L,w}, \mu).\end{aligned}$$

Here in the last two equations, we used (2.14) again and substituted $\mu = v \cdot \lambda + 2\rho_P$. This completes the proof. $\square$

5. Twisted finite slope character distributions

5.1. Twisted finite slope character distributions

In this section, we define the notion of twisted finite slope character distribution, which is a twisted version of Urban's finite slope character distributions in [18, Section 4.1.10].

Definition 5.1. — *Let ι be an automorphism of G with finite order l, L a finite extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. An L-valued ι-twisted finite slope character distribution (ι-twisted FSCD) is a $\mathbb{Q}_p$-linear map $J : \mathcal{H}'_p \to L$, such that for any ι-invariant finite slope overconvergent representation π of $\mathcal{H}_p$, there is a set of l integers $\bar{m}_J(\pi) := \{m_{J,i}(\pi) \mid i = 1, \ldots, l\}$, satisfying:*

1. *for any $t \in T^{++}$, $h \in \mathbb{Q}$ and K^p, there are finitely many π of slope less than or equal to h, such that $\bar{m}_J(\pi) \neq 0$ and $\pi^{K^p} \neq 0$.*
2. *for any $f \in \mathcal{H}'_p$,*

$$J(f) = \sum_{\pi} \sum_{i=1}^{l} m_{J,i}(\pi) J_{\tilde{\pi}_i}(f)$$

For any irreducible ι-invariant finite slope representation π, we define the multiplicity of π in J by

$$m_J(\pi) := \sum_{i} m_{J,i}(\pi). \tag{5.1}$$

We say J is effective if it is non-trivial and all its coefficients $m_{J,i}(\pi)$ are non-negative. Given a twisted FSCD J, for any $f \in \mathcal{H}'_p$, define the Fredholm determinant of f associated to J by

$$P_J(X, f) := \prod_{\pi} \prod_{i} \det(1 - X\tilde{\pi}_i(\iota \times f))^{m_{J,i}(\pi)} \tag{5.2}$$

As [18, Lemma 4.1.12], $P_J(X, f)$ is an entire power series for all $f = f^p \otimes u_t \in \mathcal{H}'_p$, if and only if J is effective.

If J is effective, for any ι-invariant K^p, define $V_J(K^p)$ to be the completion of

$$\bigoplus_{\pi} \bigoplus_{i} (V_{\tilde{\pi}_i}^{K^p})^{m_{J,i}(\pi)} \tag{5.3}$$

under the super norm $\|\sum_i v_i\| := \sup_i \|v_i\|$. $V_J(K^p)$ is a p-adic Banach space over $\mathbb{C}_p$ with an action of ${}^\iota\mathcal{H}_p(K^p)$. Moreover, if f in $\mathcal{H}'_p(K^p)$ is admissible, then it is a completely continuous operator and

$$J(f) = \mathrm{meas}(K^p)\,\mathrm{tr}(\iota \times f \,|\, V_J(K^p)). \tag{5.4}$$

This observation leads us to give the next definition:

Definition 5.2. — *Fix K^p, an L-valued ι-twisted finite slope character distribution of level K^p is a $\mathbb{Q}_p$-linear map $J' : \mathcal{H}'_p(K^p) \to L$, such that for any ι-invariant finite slope overconvergent representation σ of $\mathcal{H}_p(K^p)$, there is a set of l integers $\bar{m}_{J'}(\sigma) := \{m_{J',i}(\sigma) \,|\, i = 1, \dots, l\}$, satisfying:*

1. *for any $t \in T^{++}$ and $h \in \mathbb{Q}$, there are finitely many σ of slope less than or equal to h, such that $\bar{m}_{J'}(\sigma) \neq 0$.*
2. *for any $f \in \mathcal{H}'_p(K^p)$,*

$$J'(f) = \sum_\sigma \sum_i m_{J',i}(\sigma) J_{\tilde{\sigma}_i}(f)$$

Lemma 5.3. — *If J is twisted finite slope character distribution, then $J'_{K^p} := \mathrm{meas}(K^p)^{-1} J$ is a twisted finite slope character distribution of level K^p.*

Proof. — Let π be a ι-invariant finite slope overconvergent representation of $\mathcal{H}_p$ and $\tilde{\pi}$ an extension of π to ${}^\iota\mathcal{H}_p$. Since K^p is ι-invariant, for $f \in \mathcal{H}_p(K^p)$,

$$\mathrm{meas}(K^p)^{-1} J_{\tilde{\pi}}(f) = \mathrm{tr}(\iota \times f \,|\, \tilde{\pi}^{K^p}) \tag{5.5}$$

(if $\tilde{\pi}^{K^p} = 0$, both sides are 0). Let $\tilde{\sigma}$ be an irreducible constitute of ${}^\iota\mathcal{H}_p(K^p)$ acting on $\tilde{\pi}^{K^p}$. If the restriction of $\tilde{\sigma}$ on $\mathcal{H}_p(K^p)$ is reducible, by Lemma 3.8, $\mathrm{tr}(\iota \times f \,|\, \tilde{\sigma}) = 0$. So we have

$$\begin{aligned} \mathrm{tr}(\iota \times f \,|\, \tilde{\pi}^{K^p}) &= \sum_{\tilde{\sigma}} m(\tilde{\sigma}, \tilde{\pi}^{K^p})\,\mathrm{tr}(\iota \times f \,|\, \tilde{\sigma}) \\ &= \sum_\sigma \sum_{j=1}^{l} m(\tilde{\sigma}_j, \tilde{\pi}^{K^p}) J_{\tilde{\sigma}_j}(f), \end{aligned} \tag{5.6}$$

where, in the first equality, the sum of $\tilde{\sigma}$ is running over all irreducible constitute of ${}^\iota\mathcal{H}_p(K^p)$ in $\tilde{\pi}^{K^p}$ and $m(\tilde{\sigma}, \tilde{\pi}^{K^p})$ is its multiplicity, which equals 1 by Proposition 3.7; in the second equality, the sum of σ is running over all irreducible constitute of $\mathcal{H}_p(K^p)$ in π^{K^p} such that σ is ι-invariant and

$m(\tilde{\sigma}_j, \tilde{\pi}^{K^p})$, the multiplicities of $\tilde{\sigma}_j$ in $\tilde{\pi}^{K^p}$, are all 0 except for one j. Now

$$\begin{aligned}\operatorname{meas}(K^p)^{-1}J(f) &= \sum_{\sigma^\iota\cong\sigma}\sum_j\left(\sum_i m_{J,i}(\pi)m(\tilde{\sigma}_j,\tilde{\pi}_i^{K^p})\right)J_{\tilde{\sigma}_j}(f)\\ &= \sum_{\sigma^\iota\cong\sigma}\sum_j m_{J,j}(\pi)J_{\tilde{\sigma}_j}(f).\end{aligned}\tag{5.7}$$

This verifies the condition (2) in the definition. Condition (1) is a direct consequence of Definition 5.1 (1). □

Corollary 5.4. — *If $\sigma = \pi^{K^p}$ is a finite slope overconvergent representation of $\mathcal{H}_p(K^p)$, then*

$$m_{J'_{K^p},i}(\sigma) = m_{J,i}(\pi).\tag{5.8}$$

Similarly, we define the Fredholm determinant for any $f \in \mathcal{H}'_p$ associated to J' by

$$P_{J'}(X,f) := \prod_\sigma\prod_i \det(1 - X\tilde{\sigma}_i(\iota\times f))^{m_{J',i}(\sigma)}\tag{5.9}$$

We say that J' is effective if it is non-trivial and all its coefficients $m_{J',i}(\sigma)$ are non-negative. In this case, we define $V_{J'}$ as completion of

$$\bigoplus_\sigma\bigoplus_i (V_{\tilde{\sigma}_i})^{m_{J',i}(\sigma)}\tag{5.10}$$

under the super norm $\|\sum_i v_i\|$. Then

$$J'(f) = \operatorname{tr}(\iota\times f\mid V_{J'}).\tag{5.11}$$

If $J' = J'_{K^p}$ for some effective J, then it is obvious that J' is effective, $V_{J'} = V_J(K^p)$ and for any $f \in \mathcal{H}'_p(K^p)$

$$P_{J'_{K^p}}(X,f)\mid P_J(X,f).\tag{5.12}$$

5.2. Some ι-twisted distributions

For $\lambda \in \mathfrak{X}^\iota(L)$ and $f \in \mathcal{H}_p$, define

$$I_G^\dagger(\iota\times f,\lambda) := \operatorname{tr}(\iota\times f\mid H^*_{fs}(\tilde{S}_G,\mathcal{D}_\lambda(L))).\tag{5.13}$$

Once we fix K^p and if $f\in\mathcal{H}_p(K^p)$, then

$$\begin{aligned}I_G^\dagger(\iota\times f,\lambda) &= \operatorname{meas}(K^p)\times\operatorname{tr}(\iota\times f\mid H^*_{fs}(S_G(K^pI),\mathcal{D}_\lambda(L)))\\ &= \operatorname{meas}(K^p)\times\operatorname{tr}(\iota\times f\mid R\Gamma^*(S_G(K^pI),\mathcal{D}_\lambda(L))),\end{aligned}\tag{5.14}$$

and we also define

$$I_G^{'\dagger}(\iota \times f, \lambda, K^p) := \mathrm{tr}(\iota \times f \mid H^*_{fs}(S_G(K^pI), \mathcal{D}_\lambda(L))). \tag{5.15}$$

Let $\mathcal{P}_G^\iota$ (resp. $\mathcal{L}_G^\iota$) be the set of standard parabolic (resp. Levi) subgroups which are invariant under ι. For $L \in \mathcal{L}_G^\iota$ and $w \in \mathcal{W}_{\mathrm{Eis}}^{L,\iota}$, we now define distributions $I^\dagger_{G,0}(\iota\times f, \lambda)$ and $I^\dagger_{G,L,w}(\iota\times f, \lambda)$ by induction on the unipotent rank of G:

If $rk(G) = 0$, define

$$I^\dagger_{G,0}(\iota \times f, \lambda) = I^\dagger_{G,G}(\iota \times f, \lambda) := I^\dagger_G(\iota \times f, \lambda). \tag{5.16}$$

Given a positive integer r and assume the distributions have been defined for cases that $rk(G)$ is less than r, then for proper $L \in \mathcal{L}_G^\iota$ and $f = f^p \otimes u_t$, define

$$I^{\mathrm{cl}}_{G,L,w}(\iota \times f, \lambda) := I^{\mathrm{cl}}_{L,0}(\iota \times f^{\mathrm{reg}}_{L,w}, w \cdot \lambda + 2\rho_P) \tag{5.17}$$

for regular dominant weight $\lambda \in \Upsilon$, and define

$$I^\dagger_{G,L,w}(\iota \times f, \lambda) := I^\dagger_{L,0}(\iota \times f^{\mathrm{reg}}_{L,w}, w \cdot \lambda + 2\rho_P), \tag{5.18}$$

$$I^\dagger_{G,L}(\iota \times f, \lambda) := \sum_{w \in \mathcal{W}_{\mathrm{Eis}}^{L,\iota}} (-1)^{l(w)+\dim \mathfrak{n}_L} I^\dagger_{G,L,w}(\iota \times f, \lambda) \tag{5.19}$$

for general p-adic weights. Then define:

$$I^\dagger_{G,0}(\iota \times f, \lambda) := I^\dagger_G(\iota \times f, \lambda) - \sum_{\textit{proper } L \in \mathcal{L}_G^\iota} I^\dagger_{G,L}(\iota \times f, \lambda).$$

Proposition 5.5. — *For $? = L, \{L, w\}$ or 0, $I^\dagger_{G,?}(\iota\times f, \lambda)$ is a ι-twisted FSCD.*

Proof. — For $L \in \mathcal{L}_G^\iota$, let σ^L be an irreducible finite slope representation of $\mathcal{H}_p^L$, define

$$I^G_{L,w} := \mathrm{ind}_{L(\mathbb{A}_f^p)}^{G(\mathbb{A}_f^p)}(\sigma_f^p) \otimes \theta_{\sigma,w}, \tag{5.20}$$

where $\mathrm{ind}_{L(\mathbb{A}_f^p)}^{G(\mathbb{A}_f^p)}$ is the induction normalized by multiplying the factor $e^{\langle \rho_P, H_P(g)\rangle}$, $\theta_{\sigma,w}$ is the character of $\mathcal{U}_p$ defined by

$$u_t \mapsto |\rho_P(t)|_p \theta_\sigma(u_{wtw^{-1}}). \tag{5.21}$$

If σ^L is ι-invariant, let $\tilde{\sigma}_i^L$ be one of its irreducible extensions to ${}^\iota\mathcal{H}_p^L$. It is easy to see

$$J_{\tilde{\sigma}_i^L}(f^{\mathrm{reg}}_{L,w}) = \mathrm{tr}(\iota \times f \mid I^G_{L,w}). \tag{5.22}$$

So by the induction process in the definition above, we only have to verify the proposition for $I_G^\dagger(\iota\times f,\lambda)$. This follows from Proposition 3.12. □

5.2.1. Classical distributions

The distributions $I_?^\dagger$ defined above will be p-adic interpolations of the classical distributions $I_?^{\mathrm{cl}}$, which are defined from (twisted) alternating traces on classical cohomology. For an arithmetic regular dominant weight $\lambda\in\Upsilon$ and $f\in\mathcal{H}_p(K^p)$, $\mathrm{meas}(K^p)^{-1}I_{G,0}^{\mathrm{cl}}(\iota\times f,\lambda)$ equals

$$\begin{aligned}(5.23)\quad &\lambda(\xi(t))\,\mathrm{tr}^{st}(\iota\times f\mid H^*_{\mathrm{cusp}}(S_G(K^pI),\mathbb{V}_\lambda^\vee(\mathbb{C})))\\ &=\lambda(\xi(t))\,\mathrm{tr}^{st}(\iota\times f\mid H^*(\mathfrak{m}_G,K_\infty;\\ &\qquad\qquad L^2_{\mathrm{cusp}}(G(\mathbb{Q})A_G(\mathbb{R})^0\backslash G(\mathbb{A}))^K\otimes\mathbb{V}_\lambda^\vee(\mathbb{C}))(\xi_\lambda))\\ &=\sum_\pi\lambda(\xi(t))m_{\mathrm{cusp}}(\pi)\,\mathrm{tr}^{st}(\iota\times f\mid H^*(\mathfrak{m}_G,K_\infty;(\pi^{\mathrm{fin}})^{K_f}\otimes\mathbb{V}_\lambda^\vee(\mathbb{C})))\\ &=\sum_\pi m_{\mathrm{cusp}}(\pi)\,\mathrm{tr}^{st}(\iota\mid H^*(\mathfrak{m}_G,K_\infty;\pi_\infty^{\mathrm{fin}}\otimes\mathbb{V}_\lambda^\vee(\mathbb{C})))\,\mathrm{tr}(\iota\times f\mid\pi_f^{K_f})\\ &=\mathrm{meas}(K^p)^{-1}\sum_\pi m_{\mathrm{cusp}}(\pi)\,\mathrm{tr}^{st}(\iota\mid H^*(\mathfrak{m}_G,K_\infty;\pi_\infty^{\mathrm{fin}}\otimes\mathbb{V}_\lambda^\vee(\mathbb{C})))\\ &\qquad\qquad\mathrm{tr}(\iota\times f\mid\pi_f^I)\end{aligned}$$

where, as Proposition 3.12, the summation is running over all cuspidal representations $\pi\subset L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}),\xi_\lambda)$ such that $\pi^\iota=\pi$; π^{fin} is the Harish–Chandra module of π, i.e., the subspace of π consisting of smooth vectors that generate a finite dimensional vector space under K_∞; and $m_{\mathrm{cusp}}(\pi)$ is the multiplicity of π in L^2_{cusp}, given by the decomposition

$$L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}),\xi_\lambda)=\bigoplus_\pi\pi^{m_{\mathrm{cusp}}(\pi)}.\tag{5.24}$$

By Proposition 3.7, if $\pi_f^I\neq0$ then it is ι-invariant and irreducible as a $C_c^\infty(I\backslash G(\mathbb{A}_f)/I,\overline{\mathbb{Q}}_p)$-module. A constitute of the restriction of π_f^I on $\mathcal{H}_p$ gives a p-stabilization of π_f, and there are only finitely many such p-stabilizations (see [18, Section 4.1.9]). So

$$\mathrm{tr}(\iota\times f\mid\pi_f^I)=\sum_\rho m(\rho,\pi_f^I)\,\mathrm{tr}(\iota\times f\mid\rho),\tag{5.25}$$

where ρ is running over all irreducible $\mathcal{H}_p$-submodules of π_f^I such that $\rho^\iota=\rho$, and $m(\rho,\pi_f^I)$ is the multiplicity of ρ in π_f^I.

By the last equality of (5.23), we have to compute the Lefschetz number

$$\begin{aligned} L(\iota,\pi,\lambda) &:= \operatorname{tr}^{st}(\iota \mid H^*(\mathfrak{m}_G, K_\infty; \pi_\infty^{\mathrm{fin}} \otimes \mathbb{V}_\lambda^\vee(\mathbb{C}))) \\ &= \sum_i (-1)^i \operatorname{tr}^{st}(\iota \mid H^i(\mathfrak{m}_G, K_\infty; \pi_\infty^{\mathrm{fin}} \otimes \mathbb{V}_\lambda^\vee(\mathbb{C}))). \end{aligned} \tag{5.26}$$

To do so, we need the next theorem, which is easily deduced from Proposition 5.4, 5.5 and Theorem 5.6 in the original paper of Vogan and Zuckerman [19, Section 5]:

Theorem 5.6. — *Let π be a cuspidal representation (so it is essentially unitary) such that $H^*(\mathfrak{m}_G, K_\infty; \pi_\infty^{\mathrm{fin}} \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})) \neq 0$, then there exists a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that*

$$\dim \mathbb{V}_\lambda^\vee(\mathbb{C})/\mathfrak{u}\mathbb{V}_\lambda^\vee(\mathbb{C}) = 1 \tag{5.27}$$

and π_∞ is of the form $A_{\mathfrak{q}}(w_0\lambda)$. Moreover, one has

$$H^i(\mathfrak{m}_G, K_\infty; \pi_\infty^{\mathrm{fin}} \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})) \cong \operatorname{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^{i-R_{\mathfrak{q}}}(\mathfrak{l}\cap\mathfrak{k}), \mathbb{C}) \tag{5.28}$$

To understand the theorem we have to recall some notation from [19]. Here θ is a usual Cartan involution of G_∞, which gives a Cartan decomposition

$$\mathfrak{g} = \mathfrak{p} + \mathfrak{k}. \tag{5.29}$$

For a θ-stable parabolic algebra $\mathfrak{q}$ defined as in [19, Section 2] and an admissible weight λ defined as in [19, (5.1)], $A_{\mathfrak{q}}(\lambda)$ is an irreducible $\mathfrak{g}$-module whose restriction on $\mathfrak{k}$ contains a represntation $\mu(\mathfrak{q},\lambda)$, which is the representation of K_∞ of highest weight $\lambda + 2\rho(\mathfrak{p}\cap\mathfrak{u})$, as in [19, Theorem 5.3]. Here $R_{\mathfrak{q}} = \dim(\mathfrak{p}\cap\mathfrak{u})$.

Corollary 5.7. — *Assumptions as last theorem, if λ is regular, then $\mathfrak{u}$ is maximal unipotent and $\mathfrak{q}$ is Borel.*

Proof. — It's a simple observation from the previous theorem. The fact that $\mathbb{V}_\lambda^\vee(\mathbb{C})/\mathfrak{u}\mathbb{V}_\lambda^\vee(\mathbb{C})$ is one dimensional implies that $\mathbb{V}_\lambda^\vee(\mathbb{C})$ can be realized in $\operatorname{ind}_{\mathfrak{q}}^{\mathfrak{g}}(\chi)$ with some character χ of $\mathfrak{l}$ whose restriction to $\mathfrak{t}$ is $-w_0\lambda$. However, since λ is regular, $-w_0\lambda$ is regular too. This means that $-w_0\lambda$ cannot be extended to a character of any Levi subgroup which contains T propoly (see e.g. [13, Section II.1.18, II.2.4]). So $\mathfrak{u}$ is maximal unipotent and $\mathfrak{q}$ is Borel. □

The corollary implies that there are only finitely many cuspidal π such that $H^*(\mathfrak{m}_G, K_\infty; \pi_\infty^{\mathrm{fin}} \otimes \mathbb{V}_\lambda^\vee(\mathbb{C})) \neq 0$, and their infinity parts π_∞ are $A_{\mathfrak{b}}(w_0\lambda)$.

Write $R = R_{\mathfrak{b}}$, then for each such π

$$
\begin{aligned}
L(\iota,\pi,\lambda) &= \sum_i (-1)^i \operatorname{tr}(\iota \mid \operatorname{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^{i-R}(\mathfrak{t}\cap\mathfrak{p}),\mathbb{C})) \\
&= (-1)^R \sum_i (-1)^i \operatorname{tr}(\iota \mid \operatorname{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^{i}(\mathfrak{t}\cap\mathfrak{p}),\mathbb{C})).
\end{aligned}
\tag{5.30}
$$

So we define for any λ arithmetic regular dominant in Υ,

$$
q_{G,\iota} := L(\iota,\pi,\lambda).
\tag{5.31}
$$

It is an integer depends on G and ι only. In particular

$$
\begin{aligned}
I^{\mathrm{cl}}_{G,0}(\iota\times f,\lambda) &= \sum_{\pi=\pi^\iota} q_{G,\iota} m_{\mathrm{cusp}}(\pi)\operatorname{tr}(\iota\times f \mid \pi^I_f) \\
&= \sum_{\rho=\rho^\iota} q_{G,\iota} \sum_{\rho\subset\pi^I_f} m_{\mathrm{cusp}}(\pi) m(\rho,\pi^I_f)\operatorname{tr}(\iota\times f\mid\rho) \\
&= \sum_{\rho\cong\rho^\iota} q_{G,\iota} \sum_{i=1}^{l}\Big(\sum_{\rho\subset\pi^I_f} m_{\mathrm{cusp}}(\pi) m(\tilde{\rho}_i,\pi^I_f)\Big) J_{\tilde{\rho}_i}(f).
\end{aligned}
\tag{5.32}
$$

The discussion above together with (5.22) implies:

Proposition 5.8. — *Let λ be an arithmetic regular dominant weight in Υ. For $? = \emptyset, \{L,w\}$ or 0, $I^{\mathrm{cl}}_{G,?}(\iota\times f,\lambda)$ is a ι-twisted finite slope character distributions.*

Proof. — We only have to show the representations appearing in the twisted traces are of finite slope. Let σ be an classical cuspidal representation of $\mathcal{H}_p(K^p)$, its p-component σ_p must be of dimension one. As we have seen in (5.32), every σ appearing in the distribution is ι-invariant, so for any $f = f^p \otimes u_t \in \mathcal{H}_p(K^p)$

$$
J_{\tilde{\sigma}}(f) = J_{\tilde{\sigma}^p}(f^p)\xi\theta_\sigma(u_t)
\tag{5.33}
$$

with some l-th root of unit ξ. Now as observed in Section 3.3.2, if σ is of infinite slope, $\theta_\sigma(u_t) = 0$. □

Proposition 5.9. — *The character distribution $I^{\mathrm{cl}}_{G,0}(\iota\times f,\lambda) \neq 0$ if and only if $q_{G,\iota} \neq 0$. In this case, define $e_{G,\iota} := q^{-1}_{G,\iota}$, then $e_{G,\iota} I^{\mathrm{cl}}_{G,0}(\iota\times f,\lambda)$ is effective.*

Proof. — We only have to prove the first statement. The "only if" part is obvious. Noting that $\mathcal{U}_p$ is commutative, the component ρ_p of ρ at place p appearing in (5.32) must be a character θ_ρ of $\mathcal{U}_p$. So we only have to show that there exits a one dimensional subspace of π^I_p which is stable under ι. This is true, because π_p^I is finite dimensional and is diagonalizable under ι. $\square$

Remark 5.10. — In last proposition, we have to divide $q_{G,\iota}$ to make sure that the distribution has integral coefficients. If ι is of order 2, the Lefschetz numbers are necessarily integral. In this case, we can define $e_{G,\iota} = sign(q_{G,\iota})$. All the results and discussion will be same but the distribution carries more information.

Remark 5.11. — Throughout this paper we have assumed that ι is of Cartan-type to make sure that the cuspidal cohomology is not always trivial ([6, Theorem 10.6]). However, our results hold for any $\mathbb{Q}$-rational, finite order automorphism such that the Lefschetz number is non-trivial.

Now for any K^p, write

$$V_{G,0}^{\mathrm{cl},\lambda}(K^p) := V_{e_{G,\iota} I_{G,0}^{\mathrm{cl}}(\lambda)}(K^p) \quad \text{and} \quad V_{G,0}^{\mathrm{cl},\lambda,'}(K^p) := V_{(e_{G,\iota} I_{G,0}^{\mathrm{cl},'}(\lambda))_{K^p}} \tag{5.34}$$

Corollary 5.12.

$$V_{G,0}^{\mathrm{cl},\lambda}(K^p) = V_{G,0}^{\mathrm{cl},\lambda,'}(K^p). \tag{5.35}$$

5.2.2. Congruence between overconvergent and classical distributions

Lemma 5.13. — *Let λ be a regular arithmetic weight in Υ and $f = f^p \otimes u_t \in \mathcal{H}_p'(K^p)$ is $\mathbb{Z}_p$-valued, then*

$$I_G^{\dagger}(\iota \times f, \lambda) \equiv I_G^{\mathrm{cl}}(\iota \times f, \lambda) \mod N^{\iota}(\lambda, t)\,\mathrm{Meas}(K^p), \tag{5.36}$$

where $N^{\iota}(\lambda, t)$ is defined in (3.35).

Proof. — Both sides of the congruence are $\mathrm{meas}(K^p) \times \mathbb{Z}_p$-valued by definition. Let $h = v_p(N^{\iota}(\lambda, t))$. Noting that for any $a \in \mathbb{Z}$ and $t \in T^+$,

$$|a\mu_\theta(t)|_p = |\theta(u_{t^a})|_p, \tag{5.37}$$

(3.31) and the observation (3.32) imply that

$$I_G^{\dagger}(\iota \times f, \lambda) \equiv \mathrm{tr}(\iota \times f \mid H_{fs}^{*}(S_G(K^p I), \mathcal{D}_\lambda(L)))_{\iota}^{\leqslant h} \mod N^{\iota}(\lambda, t)\,\mathrm{Meas}(K^p)$$

$$I_G^{\mathrm{cl}}(\iota \times f, \lambda) \equiv \mathrm{tr}(\iota \times f \mid H^{*}(S_G(K^p I_m), \mathbb{V}_\lambda^{\vee}(L)))_{\iota}^{\leqslant h} \mod N^{\iota}(\lambda, t)\,\mathrm{Meas}(K^p).$$

Then the lemma is obtained from Proposition 3.11. $\square$

Proposition 5.14. — *Let $f = f^p \otimes u_t \in \mathcal{H}_p(K^p)$ be $\mathbb{Z}_p$-valued and λ regular arithmetic. Then*

$$I_{G,0}^{\dagger}(\iota \times f, \lambda) \equiv I_{G,0}^{\mathrm{cl}}(\iota \times f, \lambda) \mod N^{\iota}(\lambda, t)\,\mathrm{Meas}(K^p). \tag{5.38}$$

Proof. — We prove the proposition by induction on $rk(G)$. The case $rk(G)=0$ is just the Lemma 5.13 above. Now assume the proposition has been proved for any proper Levi subgroup $L\in\mathcal{L}_G^{\iota}$. Noting that $N^{\iota}(\lambda,t)$ divides $\xi(t)^{\lambda-w^{-1}v*\lambda}$ if $v\neq w$, the cuspidal decomposition formula, Theorem 4.8 implies that, modulo $N^{\iota}(\lambda,t)\,\mathrm{Meas}(K^p)$, $I^{\mathrm{cl}}_{G,0}(\iota\times f,\lambda)$ is congruent to

$$I^{\mathrm{cl}}_{G}(\iota\times f,\lambda)-\sum_{P\in\mathcal{P}^{\iota},\,P\neq G}\ \sum_{w\in\mathcal{W}^{L,\iota}_{\mathrm{Eis}}}(-1)^{l(w)+n_P}I^{\mathrm{cl}}_{L,0}(\iota\times f^{\mathrm{reg}}_{L,w},w\cdot\lambda+2\rho_P). \tag{5.39}$$

Now using the induction hypotheses and Lemma 5.13 again, it is congruent to

$$I^{\dagger}_{G}(\iota\times f,\lambda)-\sum_{P\in\mathcal{P}^{\iota},\,P\neq G}I^{\dagger}_{G,L}(\iota\times f,w\cdot\lambda+2\rho_P), \tag{5.40}$$

which is, by definition, $I^{\dagger}_{G,0}(\iota\times f,\lambda)$. □

Now recall the Definition [18, Definition 4.6.6], let $\{\lambda_n\}$ be a sequence of algebraic dominant weights in $\mathfrak{X}(\overline{\mathbb{Q}}_p)$ which converges p-adically to a weight λ in $\mathfrak{X}(\overline{\mathbb{Q}}_p)$. We say the sequence is highly regular if, for all positive simple root α, we have

$$\lim_{n\to\infty}\lambda_n(H_\alpha)=\infty. \tag{5.41}$$

Parallel to [18, Corollary 4.6.8], we have a direct consequence of Proposition 5.14:

Corollary 5.15. — *let $\{\lambda_n\}$ be a highly regular sequence of ι-invariant dominant weights which converges p-adically to a weight $\lambda\in\mathfrak{X}^{\iota}(L)$. Then for any $f=f^p\otimes u_t\in\mathcal{H}'_p$, there is*

$$\lim_{n\to\infty}I^{\mathrm{cl}}_{G,?}(\iota\times f,\lambda_n)=I^{\dagger}_{G,?}(\iota\times f,\lambda) \tag{5.42}$$

for $?=\emptyset,0$.

5.3. Analyticity with respect to weight

Now we study $I^{\dagger}_{G,?}(\iota\times f,\lambda)$ when weights λ varying over the weight space $\mathfrak{X}^{\iota}$. Let $\mathfrak{U}$ be an open affinoid of $\mathfrak{X}^{\iota}$ and $\mathcal{O}^0(\mathfrak{U})$ the ring of analytic functions on $\mathfrak{U}$ bounded by 1. For any finite extension L of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$, define

$$\Lambda_{\mathfrak{X}^{\iota}}:=\varprojlim_{\mathfrak{U}\subset\mathfrak{X}^{\iota}}\mathcal{O}^0(\mathfrak{U})\subset\mathcal{O}(\mathfrak{X}^{\iota}) \tag{5.43}$$

and

$$\Lambda_{\mathfrak{X}^{\iota},L}:=\Lambda_{\mathfrak{X}^{\iota}}\otimes L. \tag{5.44}$$

Proposition 5.16. — *Fix $f \in \mathcal{H}'_p(K^p)$, then as functions of $\lambda \in \mathfrak{X}^\iota$, $I^\dagger_G(\iota \times f, \lambda)$, $I^\dagger_{G,M,w}(\iota \times f, \lambda)$ and $I^\dagger_{G,0}(\iota \times f, \lambda)$ are all in $\Lambda_{\mathfrak{X}^\iota, \mathbb{Q}_p}$. In particular, they are analytic over $\mathfrak{X}^\iota$.*

Proof. — The proof is same to [18, Theorem 4.7.3], so we only sketch it here. By the induction process in defining $I^\dagger_{G,?}(\iota \times f, \lambda)$, it suffices to prove the proposition for $I^\dagger_G(\iota \times f, \lambda)$. Locally over an open affinoid $\mathfrak{U} \subset \mathfrak{X}^\iota$, for $n \geqslant n(\mathfrak{U})$, Lemma 2.1 and Proposition 2.4 (1) imply that

$$R\Gamma^*(K^p I, \mathcal{D}_{\mathfrak{U},n}) \otimes_\lambda L \cong R\Gamma^*(K^p I, \mathcal{D}_{\lambda,n}(L)). \tag{5.45}$$

Therefore $F_\mathfrak{U} := \mathrm{meas}(K^p)\,\mathrm{tr}(\iota \times f \mid R\Gamma^*(K^p I, \mathcal{D}_{\mathfrak{U},n(\mathfrak{U})}))$ (viewed as a function on $\mathfrak{U}$ via specialization) is in $\mathcal{O}(\mathfrak{U})$, such that, for any $\lambda \in \mathfrak{U}$, $F_\mathfrak{U}(\lambda) = I^\dagger_G(\iota \times f, \lambda)$. Moreover, since ι and $f \in \mathcal{H}'_p$ preserve the $\mathcal{O}^0(\mathfrak{U})$-lattice $R\Gamma^*(K^p I, \mathcal{D}^0_{\mathfrak{U},n})$, that $F_\mathfrak{U}$ is in $\mathcal{O}^0(\mathfrak{U})$, where $\mathcal{D}^0_{\mathfrak{U},n}$ is the $\mathcal{O}^0(\mathfrak{U})$ dual of $\mathcal{A}^0_{\mathfrak{U},n}$. So we have

$$I^\dagger_G(\iota \times f, -) = \varprojlim_{\mathfrak{U} \subset \mathfrak{X}^\iota} F_\mathfrak{U} \in \varprojlim_{\mathfrak{U} \subset \mathfrak{X}^\iota} \mathcal{O}^0(\mathfrak{U}) = \Lambda_{\mathfrak{X}^\iota}. \tag{5.46}$$ □

5.4. Effectivity

Proposition 5.17. — *If $q_{G,\iota} \neq 0$, then $e_{G,\iota} I^\dagger_{G,0}(\iota \times f, \lambda)$ is an effective ι-twisted finite slope character distribution.*

Proof. — Since algebraic regular dominant weights are dense in the weight space, by Proposition 5.16, it suffices to prove the proposition for algebraic regular dominant weights λ in Υ. Since $q_{G,\iota} \neq 0$, by Proposition 5.9, $e_{G,\iota} I^{\mathrm{cl}}_{G,0}(\iota \times f, \lambda)$ is effective. Let $P^{\mathrm{cl}}_{G.0}(\iota \times f, \lambda, X)$ and $P^\dagger_{G.0}(\iota \times f, \lambda, X)$ be the Fredholm determinants associated to $e_{G,\iota} I^{\mathrm{cl}}_{G,0}(\iota \times f, \lambda)$ and $e_{G,\iota} I^\dagger_{G,0}(\iota \times f, \lambda)$ respectively, define

$$P^{\dagger-cl}_{G,0}(\iota \times f, \lambda, X) = \frac{P^\dagger_{G.0}(\iota \times f, \lambda, X)}{P^{\mathrm{cl}}_{G.0}(\iota \times f, \lambda, X)}. \tag{5.47}$$

We need the lemma below, which is a direct consequence of Proposition 5.14.

Lemma 5.18. — *If λ is regular, then $P^{\dagger-cl}_{G,0}(\iota \times f, \lambda, X)$ is a meromorphic function on $\mathbb{A}^1_{\mathrm{rig}}(\mathbb{C}_p)$, its zeros and poles are all lying in*

$$\{x \in \mathbb{C}_p \mid |x|_p \geqslant N^\iota(\lambda, t)\}$$

Proof of the Lemma. — If J_1 and J_2 are two twisted finite slope character distributions, then so is J_1-J_2, and $P_{J_1-J_2}(X,f)=P_{J_1}(X,f)/P_{J_2}(X,f)$. Write $I^{\dagger-cl}_{G,0}(\iota\times f,\lambda):=I^{\dagger}_{G,0}(\iota\times f,\lambda)-I^{\mathrm{cl}}_{G,0}(\iota\times f,\lambda)$. Proposition 5.14 implies that, for $\mathbb{Z}_p$-valued $f=f^p\otimes u_t\in\mathcal{H}_p(K^p)$, $I^{\dagger-cl}_{G,0}(\iota\times f,\lambda)'_{K^p}\equiv 0 \mod N^\iota(\lambda,t)$. So $P^{\dagger-cl}_{G,0}(\iota\times f,\lambda,X)\equiv 1 \mod N^\iota(\lambda,t)$. This proves the lemma. □

Now we can run the same argument as in the proof of [18, Theorem 4.7.3] to show our proposition. Choose a closed affinoid subdomain $\mathfrak{U}\subset\mathfrak{X}^\iota$ which contains one hence dense algebraic weights in Υ. Shrink $\mathfrak{U}$ if necessary so that we can write $P^{\dagger}_{G.0}(\iota\times f,\lambda,X)$ as a quotient of relatively prime Fredholm series over $\mathfrak{U}$, that is,

$$P^{\dagger}_{G.0}(\iota\times f,\lambda,X)=\frac{T(\iota\times f,\lambda,X)}{B(\iota\times f,\lambda,X)},\tag{5.48}$$

with both $T(\iota\times f,\lambda,X)$ and $B(\iota\times f,\lambda,X)$ are in $\mathcal{O}(\mathfrak{U}\times\mathbb{A}^1_{\mathrm{rig}})$. Assume $B(\iota\times f,\lambda,X)\neq 1$, let $\mathfrak{W}$ be the Fredholm subvariety of $\mathfrak{U}\times\mathbb{A}^1_{\mathrm{rig}}$ cut out by $T(\iota\times f,\lambda,X)$ and $B(\iota\times f,\lambda,X)$, that is, $\mathfrak{W}=Z(B)-Z(T)$. Since the projection $pr: Z(B)\to\mathfrak{U}$ is flat, that its image $pr(\mathfrak{W})$ also contains dense algebraic weights. Now let $w=(\lambda,x)\in\mathfrak{W}(\overline{\mathbb{Q}}_p)$ with λ algebraic, we can choose $w'=(\lambda',x')$ p-adically close to w such that λ' is algebraic regular dominant in Υ and $|x'|_p<N^\iota(\lambda',t)$. So by Lemma 5.18, x' must be a pole of $P^{\mathrm{cl}}_{G.0}(\iota\times f,\lambda',X)$. However, since $e_{G,\iota}I^{\mathrm{cl}}_{G,0}(\iota\times f,\lambda')$ is effective, $P^{\mathrm{cl}}_{G.0}(\iota\times f,\lambda',X)$ is entire. So our assumption leads to a contradiction. This implies that $P^{\dagger}_{G.0}(\iota\times f,\lambda,X)$ is entire, therefore, $e_{G,\iota}I^{\dagger}_{G,0}(\iota\times f,\lambda)$ is also effective. □

Corollary 5.19. — *For any ι-invariant standard Levi subgroup of G, there exists a number $e_{L,\iota}$ such that, for any $w\in\mathcal{W}^{L,\iota}_{\mathrm{Eis}}$, $e_{L,\iota}I^{\mathrm{cl}}_{G,L,w}(\iota\times f,\lambda)$ and $e_{L,\iota}I^{\dagger}_{G,L,w}(\iota\times f,\lambda)$ are effective, unless $I^{\mathrm{cl}}_{G,L,w}(\iota\times f,\lambda)$ is trivial for some algrbraic regular weight $\lambda\in\Upsilon$*

Proof. — It follows from the definition of $I^{\dagger}_{G,L,w}(\iota\times f,\lambda)$ and an exactly same argument as the proof of the previous proposition, but for Levi subgroup L. □

5.5. Multiplicities

For ? = cl or †, write the Fredholm determinants associated to $e_{G,\iota}I^{?}_{G,0}(\iota\times f,\lambda)$ by $P^{?}_{G,0}(\iota\times f,\lambda,X)$, to $e_{L,\iota}I^{?}_{G,L,w}(\iota\times f,\lambda)$ by $P^{?}_{G,L,w}(\iota\times f,\lambda,X)$. Let

π be a finite slope overconvergent representation, write the multiplicities:

$$\begin{aligned} \bar{m}^{\iota,?}_{G,0}(\pi,\lambda) &:= \bar{m}_{e_G,\iota I^?_{G,0}(\lambda)}(\pi), \\ m^{\iota,?}_{G,0}(\pi,\lambda) &:= m_{e_G,\iota I^?_{G,0}(\lambda)}(\pi), \end{aligned} \tag{5.49}$$

$$\begin{aligned} \bar{m}^{\iota,?}_{G,L,w}(\pi,\lambda) &:= \bar{m}_{e_L,\iota I^?_{G,L,w}(\lambda)}(\pi), \\ m^{\iota,?}_{G,L,w}(\pi,\lambda) &:= m_{e_L,\iota I^?_{G,L,w}(\lambda)}(\pi), \end{aligned} \tag{5.50}$$

For given K^p, if θ is an overconvergent Hecke eigensystem of level K^p, write its multiplicity in $V^{?,\lambda}_{G,0}(K^p)$ by $m^{?,\iota}_{G,0}(\theta,\lambda)$. By Proposition 5.5, if θ or π is not ι-invariant, its multiplicities are 0.

Lemma 5.20. — *Let λ be an arithmetic regular weight and π an ι-invariant finite slope overconvergent representation, which is non-critical with respect to λ^{alg}, then*

$$m^{\iota,\mathrm{cl}}_{G,0}(\pi,\lambda) = m^{\iota,\dagger}_{G,0}(\pi,\lambda) \tag{5.51}$$

Proof. — Assume π is of level K^p. Since π is non-critical with respect to λ, there is $t \in T^{++}$ such that $v_p(N^\iota(\lambda,t)) > v_p(\theta_\pi(u_t))$. if necessary, we can replace t by t^N for some positive integer N to make $v_p(N^\iota(\lambda,t)) - v_p(\theta_\pi(u_t))$ arbitrarily large. Now consider the finite set of ι-invariant finite slope overconvergent representations:

$$\Sigma^t_{K^p} = \{\rho \,|\, \rho^{K^p} \neq 0, v_p(\rho(u_t)) \leqslant v_p(N^\iota(\lambda,t)), \bar{m}^{\iota,\dagger}_{G,0}(\rho,\lambda) \text{ or } \bar{m}^{\iota,\mathrm{cl}}_{G,0}(\rho,\lambda) \neq 0\}$$

Since $\Sigma^t_{K^p}$ is finite, by Jacobson's Lemma, there is $f \in \mathcal{H}_p(K^p)$ such that $\pi(f) = \mathrm{id}_{\pi^{K^p}}$ and $\rho(f) = 0$ for any $\rho \in \Sigma^t_{K^p}$ that is not isomorphic to π.

Consider $f_0 = (1_{K^p} \otimes u_t)f$, we have for ? = † or *cl*

$$\begin{aligned} &P^?_{G,0}(\iota \times f_0, \lambda, X) \\ &\qquad = \prod_{i=1}^{l} \det\left(1 - \tilde{\pi}_i(\iota \times 1_{K^p} \otimes u_t)X\right)^{m^{\iota,?}_{G,0,i}(\pi,\lambda)} S^?_{G,0}(X), \end{aligned} \tag{5.52}$$

where $S^?_{G,0}(X)$ is the product of the determinants of all representations whose slopes are strictly greater than $v_p(N^\iota(\lambda,t))$. Noticing that $\Sigma^{t^M}_{K^p} = \Sigma^t_{K^p}$ for any positive integer M, we can therefore choose f independently for any M. So if necessary, we can replace t by t^M such that $f_0 = (1_{K^p} \otimes u_t)f$

is $\mathbb{Z}_p$-valued. Then by Proposition 5.14 and Lemma 5.18, we have

$$\begin{aligned}(5.53)\quad &\prod_{i=1}^{l}\det\left(1-\tilde{\pi}_i^{K^p}(\iota\times 1_{K^p}\otimes u_t)X\right)^{m^{\iota,\dagger}_{G,0,i}(\pi,\lambda)}\\ &\equiv\prod_{i=1}^{l}\det\left(1-\tilde{\pi}_i^{K^p}(\iota\times 1_{K^p}\otimes u_t)X\right)^{m^{\iota,\mathrm{cl}}_{G,0,i}(\pi,\lambda)}\mod N^\iota(\lambda,t)\end{aligned}$$

and they share the same zeros (order counted) $\mod N^\iota(\lambda,t)$. If necessary, replace t by t^N as we observed at the beginning of the proof, we can assume that $v_p(N^\iota(\lambda,t))$ is strictly greater than the p-adic valuation of all the coefficients of $\prod_{i=0}^{l}\det\left(1-\tilde{\pi}_i^{K^p}(\iota\times 1_{K^p}\otimes u_t)X\right)^{m^{\iota,?}_{G,0,i}(\pi,\lambda)}$, so we have

$$\begin{aligned}(5.54)\quad &\prod_{i=1}^{l}\det\left(1-\tilde{\pi}_i^{K^p}(\iota\times 1_{K^p}\otimes u_t)X\right)^{m^{\iota,\dagger}_{G,0,i}(\pi,\lambda)}\\ &=\prod_{i=1}^{l}\det\left(1-\tilde{\pi}_i^{K^p}(\iota\times 1_{K^p}\otimes u_t)X\right)^{m^{\iota,\mathrm{cl}}_{G,0,i}(\pi,\lambda)}.\end{aligned}$$

In particular, they have the same degree, which are $\dim(\pi)m^{\iota,\dagger}_{G,0}(\pi,\lambda)$ and $\dim(\pi)m^{\iota,\mathrm{cl}}_{G,0}(\pi,\lambda)$ respectively. So $m^{\iota,\mathrm{cl}}_{G,0}(\pi,\lambda)=m^{\iota,\dagger}_{G,0}(\pi,\lambda)$. □

Noting that $\tilde{\pi}_i^{K^p}(\iota\times 1_{K^p}\otimes u_t)=\tilde{\pi}_i^{K^p}(\iota)\theta_\pi(u_t)$ and π^{K^p} is finite dimensional, we can compute (5.54) more explicitly. Fix ξ a primitive l-th roots of unity, let $k_j^{(i)}$ be the multiplicity of ξ^j as an eigenvalue of ι in $\tilde{\pi}_i^{K^p}$. Then for any $i=1,\dots,l$,

$$(5.55)\qquad \sum_{j=i}^{l}k_j^{(i)}=\dim\pi^{K^p}$$

and

$$\begin{aligned}(5.56)\quad &\prod_{i=1}^{l}\det\left(1-\tilde{\pi}_i^{K^p}(\iota\times 1_{K^p}\otimes u_t)X\right)^{m^{\iota,?}_{G,0,i}(\pi,\lambda)}\\ &=\prod_{i=1}^{l}\left(\prod_{j=1}^{l}(1-\xi^j\theta_\pi(u_t)X)^{k_j^{(i)}}\right)^{m^{\iota,?}_{G,0,i}(\pi,\lambda)}\\ &=\prod_{j=1}^{l}(1-\xi^j\theta_\pi(u_t)X)^{\sum_i k_j^{(i)}m^{\iota,?}_{G,0,i}(\pi,\lambda)}.\end{aligned}$$

This gives us varies identities between the multiplicities. If we compare the coefficients of degree 1, we have

$$\sum_{i=1}^{l} \xi^i m^{\iota,\dagger}_{G,0,i}(\pi,\lambda) = \sum_{i=1}^{l} \xi^i m^{\iota,\mathrm{cl}}_{G,0,i}(\pi,\lambda). \tag{5.57}$$

In particular, if $l=2$, (5.51) and (5.57) imply that

$$\bar{m}^{\iota,\mathrm{cl}}_{G,0}(\pi,\lambda) = \bar{m}^{\iota,\dagger}_{G,0}(\pi,\lambda). \tag{5.58}$$

Let $\lambda = \lambda^{\mathrm{alg}}\epsilon$ be an arithmetic weight. If π is a ι-invariant finite slope cuspidal representation, its ι-twisted Euler–Poincare characteristic $m^{\iota}_{EP}(\pi,\lambda)$ is defined by:

$$\sum_q (-1)^q \operatorname{tr}(\iota \mid \operatorname{Hom}_{\mathcal{H}_p}(\pi, H^q(\tilde{S}_G, \mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(\mathbb{C},\epsilon)))). \tag{5.59}$$

If π is of level K^p, then $m^{\iota}_{EP}(\pi,\lambda)$ equal:

$$\sum_q (-1)^q \operatorname{tr}(\iota \mid \operatorname{Hom}_{\mathcal{H}_p(K^p)}(\pi^{K^p}, H^q(S_G(K^pI), \mathbb{V}^\vee_{\lambda^{\mathrm{alg}}}(\mathbb{C},\epsilon)))). \tag{5.60}$$

Then a computation as in Section 5.2 shows that

$$m^{\iota}_{EP}(\pi,\lambda) = q_{G,\iota} m^{\iota,\mathrm{cl}}_{G,0}(\pi,\lambda). \tag{5.61}$$

Since the distribution $e_{G,\iota} I^{\mathrm{cl}}_{G,0}(\iota\times f,\lambda)$ is effective, $m^{\iota,\mathrm{cl}}_{G,0,i}(\pi,\lambda)$ are all nonnegative. So in case $q_{G,\iota}\neq 0$, $m^{\iota}_{EP}(\pi,\lambda)\neq 0$ if and only if $m^{\iota,\mathrm{cl}}_{G,0}(\pi,\lambda)\neq 0$, if and only if $m^{\iota,\mathrm{cl}}_{G,0,i}(\pi,\lambda)\neq 0$ for some i.

We close this section by considering the multiplicities of Hecke eigensystems:

Corollary 5.21. — *Let λ be an arithmetic regular weight and θ a finite slope ι-invariant overconvergent Hecke eigensysem, which is noncritical with respect to λ^{alg}, then*

$$m^{\iota,\mathrm{cl}}_{G,0}(\theta,\lambda) = m^{\iota,\dagger}_{G,0}(\theta,\lambda) \tag{5.62}$$

Proof. — This is a direct consequence of Lemma 5.20 and the formula that

$$m^{\iota,?}_{G,0}(\theta,\lambda) = \sum_{\sigma|_{R_{\mathcal{S},p}}=\theta} m^{\iota,?}_{G,0}(\sigma,\lambda)\dim\sigma. \tag{5.63}$$

□

6. Twisted eigenvarieties

In this section, for group G and finite Cartan-type automorphism ι, assuming that $q_{G,\iota} \neq 0$ (so $e_G := e_{G,\iota} \neq 0$), we construct eigenvarieties which parametrize ι-invariant finite slope overconvergent Hecke eigensystems of G. We call such an eigenvariety a ι-twisted eigenvariety of G.

6.1. Twisted spectral varieties

Consider the effective distribution $e_G I^{\dagger}_{G.0}(\iota \times f, \lambda)$, for fixed K^p and $f \in \mathcal{H}_p(K^p)$, write $V^{\dagger,\lambda}_{G,0}(K^p)$ and $P^{\dagger}_{G.0}(\iota \times f, \lambda, X)$ for it as (5.34).

Proposition 6.1 (twisted spectral varieties). — *For any $f = f^p \otimes u_t \in \mathcal{H}'_p(K^p)$ with $t \in T^{++}$, there is a rigid analytic space $\mathfrak{S}^{\iota}(f) \subset \mathfrak{X}^{\iota} \times \mathbb{A}^1_{\mathrm{rig}}$, such that $(\lambda, \alpha) \in \mathfrak{S}^{\iota}(f)(\overline{\mathbb{Q}}_p)$ if and only if α^{-1} is an eigenvalue of $\iota \times f$ on $V^{\dagger,\lambda}_{G,0}(K^p)$.*

Proof. — This is same to [18, Proposition 5.1.6]. $\mathfrak{S}^{\iota}(f)$ is simply defind as the Fredholm hypersurface cut out by $P^{\dagger}_{G.0}(\iota \times f, \lambda, X)$ in $\mathfrak{X}^{\iota} \times \mathbb{A}^1_{\mathrm{rig}}$. □

6.2. Full eigenvariety

For later use, we summarize some results of [20]. Given K^p, let $\widetilde{R}_{\mathcal{S},p}$ be the p-adic completion of $R_{\mathcal{S},p}[u_t^{-1}, t \in T^+]$. Define $\mathfrak{R}_{\mathcal{S},p}$ to be the p-adic analytic space, such that for any $L/\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$,

$$\mathfrak{R}_{\mathcal{S},p}(L) = \mathrm{Hom}_{\mathrm{ct\ alg}}(\widetilde{R}_{\mathcal{S},p}, L). \tag{6.1}$$

By construction, $\theta \in \mathfrak{R}_{\mathcal{S},p}(L)$ is of finite slope. $\mathfrak{R}_{\mathcal{S},p}(L)$ has the canonical p-adic topology induced by the metric $|\theta - \theta'| =: \sup_{f \in R_{\mathcal{S},p}} |\theta(f) - \theta'(f)|_p$.

Set $\mathfrak{Y} = \mathfrak{X} \times \mathfrak{R}_{\mathcal{S},p}$, its L-points $\mathfrak{Y}(L)$ are pairs (λ, θ). The full eigenvariety is a rigid analytic space $\mathfrak{E} := \mathfrak{E}_{K^p}$ over $\mathbb{Q}_p$, which is a p-adic analytic subspace of $\mathfrak{Y}$. The space $\mathfrak{E}$ is equipped with a projection onto $\mathfrak{X}$, such that $(\lambda, \theta) \in \mathfrak{E}_{K^p}(L)$ if and only if $H^*_{fs}(S_G(K^pI), \mathcal{D}_\lambda(L))[\theta] \neq 0$, and, for any $f \in R_{\mathcal{S},p}$, $R_f(\theta) := \theta(f)^{-1}$ is an eigenvalue of f acting on $H^*(S_G(K^pI), \mathcal{D}_\lambda(L))$. Indeed, for any $f \in R_{\mathcal{S},p}$, $(\lambda, \theta) \mapsto (\lambda, R_f(\theta))$ gives a projection from $\mathfrak{E}_{K^p}$ onto a subvariety $\mathfrak{S}_f$ of $\mathfrak{X} \times \mathbb{A}^1_{\mathrm{rig}}$, and $\mathfrak{S}_f$ is a piece of the spectral variety which parameterizes Hecke eigenvalues of f. For detail, refer [20, Section 6, 7, 8].

6.3. A big twisted eigenvariety

In this subsection we construct our first twisted eigenvariety $\mathfrak{K}^\iota$ using the method and notation in [20]. $\mathfrak{K}^\iota$ will give us a bigger rigid space so that we can construct inside it other twisted eigenvarities we really concern. Not like the work of Urban in [18], we have to do so rather than to construct our twisted eigenvarieties directly. The reason is, since our twisted finite slope character distributions are constructed from twisted traces, they do not provide pseudo-repesentations, so we do not have Urban's "second construction" as [18, Section 5.3]. Then if we follow the direct construction, what we obtained is only a p-adic analytic space but not a rigid space. However, $\mathfrak{K}^\iota$ will be a rigid analytic space, so its subspaces inherit rigid structures automatically.

Most results in this subsection are parallel to [20], so we omit the proofs.

6.3.1. Spectral varieties

Let $\mathfrak{U}$ be an open affinoid subdomain of $\mathfrak{X}^\iota$, consider the action of ${}^\iota R_{\mathcal{S},p}$ on $R\Gamma(K^pI,\mathcal{D}_\mathfrak{U}) := \bigoplus R\Gamma^q(K^pI,\mathcal{D}_\mathfrak{U})$ as in Section 3.1. By Proposition 2.4 and the discussion in Section 3.1.3, for any $f \in R_{\mathcal{S},p}$ admissible, there is a power series $P_\mathfrak{U}(f,\lambda,X) \in \mathcal{O}(\mathfrak{U})\{\{X\}\}$, such that for any $\lambda \in \mathfrak{U}$, the specialization of $P_\mathfrak{U}(f,\lambda,X)$ at λ is the Fredholm determinant of f acting on $R\Gamma(K^pI,\mathcal{D}_\lambda)$.

Lemma 6.2 (Urban). — *Let $j : N \hookrightarrow M$ be a continuous injection of L-Banach spaces. Let u_N and u_M be respectively compact endomorphisms of N and M such that $j\circ u_N = u_M\circ j$. Then $M/j(N)$ has slope decomposition with respect to $u_{M/N} = u_M \pmod{j(N)}$, and*

$$\det(1 - Xu_M) = \det(1 - Xu_N)\det(1 - Xu_{M/N})$$

This lemma is [18, Proposition 2.3.9]. Apply Lemma 6.2 to the situation $M = N = R\Gamma(K,\mathcal{D}_\mathfrak{U})$ and $j = \iota*$, we have

$$P_\mathfrak{U}(f,\lambda,X) = P_\mathfrak{U}(f^\iota,\lambda,X). \tag{6.2}$$

Let $P_\mathfrak{U}(f,\lambda,X) = Q_\mathfrak{U}(X)S_\mathfrak{U}(X)$ be a polynomial decomposition as in Lemma 3.4, and

$$R\Gamma(K,\mathcal{D}_\mathfrak{U}) = N_f(Q_\mathfrak{U}) \oplus F_f(Q_\mathfrak{U}) \tag{6.3}$$

the corresponding $\mathcal{O}(\mathfrak{U})$-module decomposition. Apply Lemma 6.2 to the situation $N = N_f(Q_\mathfrak{U})$, $M = R\Gamma(K,\mathcal{D}_\mathfrak{U})$, $j = \iota*$, $u_N = f$ and $u_M = f^\iota$, we have

$$N_{f^\iota}(Q_\mathfrak{U}) = \iota * (N_f(Q_\mathfrak{U})). \tag{6.4}$$

Define

$$N_{f,\iota}(Q_{\mathfrak{U}}) = \bigcap_{i=1}^{l} N_{f^{\iota^i}}(Q_{\mathfrak{U}}), \tag{6.5}$$

which is equipped with a ${}^{\iota}\mathcal{H}_p(K^p)$ action.

Recall that, in [20, Proposition 6.4], we defined a weight space $\mathfrak{W} \subset \mathfrak{X}$, such that for $\lambda \in \mathfrak{X}$, $\lambda \in \mathfrak{W}$ if and only if $H_{fs}(S_G(K), \mathcal{D}_\lambda) \neq 0$. Similarly, define

$$\mathfrak{W}^{\iota}_{Q,\mathfrak{U}} := \operatorname{supp}_{\mathcal{O}(\mathfrak{U})} H(\widetilde{N^*_{f,\iota}(Q_{\mathfrak{U}})}) \subset \mathfrak{U} \tag{6.6}$$

and $\mathfrak{W}^{\iota}$ the subspace of $\mathfrak{X}^{\iota}$ obtained by gluing $\mathfrak{W}^{\iota}_{Q,\mathfrak{U}}$ for all $\mathfrak{U}$ and $Q_{\mathfrak{U}}$. Then $\lambda \in \mathfrak{W}^{\iota}_{Q}(f)(\overline{\mathbb{Q}}_p)$ if and only if $H(N^*_{f,\iota}(Q)) \neq 0$. The next proposition is a direct consequence of the fact that

$$\begin{aligned} H_{fs}(S_G(K), \mathcal{D}_\lambda) &= \varinjlim_h H(S_G(K), \mathcal{D}_\lambda)_f^{\leqslant h} \\ &= \varinjlim_h \bigcap_i H(S_G(K), \mathcal{D}_\lambda)_{f^{\iota^i}}^{\leqslant h} : \end{aligned} \tag{6.7}$$

Proposition 6.3.

$$\mathfrak{W}^{\iota} = \mathfrak{X}^{\iota} \cap \mathfrak{W} \tag{6.8}$$

For an admissible $f \in R_{\mathcal{S},p}$, define set $\{f\}^{\iota} := \{f^{\iota^i} \times \iota^j \mid 1 \leqslant i,j \leqslant l\}$. For any $g \in \{f\}^{\iota}$, define

$$\mathfrak{S}^{\iota}_{Q,\mathfrak{U},g} := \operatorname{supp}_{\mathcal{O}(\mathfrak{U})[g]} \widetilde{H^*_{f,\iota}(Q)} \subset \mathfrak{U} \times \mathbb{A}^1_{\mathrm{rig}} \tag{6.9}$$

Proposition 6.4. — *$\mathfrak{S}^{\iota}_{Q,\mathfrak{U},g}$ is locally finite over $\mathfrak{W}^{\iota}_{Q,\mathfrak{U}}$. A point $s = (\lambda, \alpha)$ of $\mathfrak{U} \times \mathbb{A}^1_{\mathrm{rig}}$ is in $\mathfrak{S}^{\iota}_{Q,\mathfrak{U},g}(\overline{\mathbb{Q}}_p)$ if and only if $\lambda \in \mathfrak{W}^{\iota}_{Q,\mathfrak{U}}(\overline{\mathbb{Q}}_p)$ and α^{-1} is an eigenvalue of g acting on $H^*_{f,\iota}(Q)$.*

The proof is same to [20, Proposition 6.6]. Moreover, discuss as in [20, Section 6], given $\{f\}^{\iota}$ and $g \in \{f\}^{\iota}$, we can glue the local spectral varieties $\mathfrak{S}^{\iota}_{Q,\mathfrak{U},g}$ for polynomials $Q_{\mathfrak{U}}$ and open affinoid domains $\mathfrak{U} \subset \mathfrak{X}^{\iota}$:

Theorem 6.5. — *There is a spectral variety $\mathfrak{S}^{\iota}_g = \mathfrak{S}^{\iota}_{\mathfrak{W}^{\iota},g}$ as a rigid subspace of $\mathfrak{X}^{\iota} \times \mathbb{A}_1^{\mathrm{rig}}$, such that, $s = (\lambda, \alpha) \in \mathfrak{S}^{\iota}_g(\overline{\mathbb{Q}}_p)$ if and only if $\lambda \in \mathfrak{W}^{\iota}(\overline{\mathbb{Q}}_p)$ and α^{-1} is an eigenvalue of g acting on $H^*_{fs}(S_G(K), \mathcal{D}_\lambda)$.*

Corollary 6.6. — *If $g = \iota \times f$, then*

$$\mathfrak{S}^{\iota}(f) \subset \mathfrak{S}^{\iota}_g. \tag{6.10}$$

6.3.2. A big twisted eigenvariety

We build an eigenvariety over the spectral varieties constructed in last subsection as in [20, Section 8]. Let ${}^{\iota}\tilde{R}_{\mathcal{S},p}$ be the p-adic completion of ${}^{\iota}R_{\mathcal{S},p}[u_t^{-1}, t \in T^+]$. Since ${}^{\iota}R_{\mathcal{S},p} = R_{\mathcal{S},p} \rtimes \langle \iota \rangle$ and ι is of finite order l, that ${}^{\iota}\tilde{R}_{\mathcal{S},p}$ is an l pieces union of $\tilde{R}_{\mathcal{S},p}$. Define the p-adic space $\mathfrak{B} = \mathfrak{B}_{\mathcal{S},p}$ be such that for any $L/\mathbb{Q}_p$,

$$\mathfrak{B}(L) = \mathrm{Hom}_{\mathrm{alg\ ct}}({}^{\iota}\tilde{R}_{\mathcal{S},p}, L). \tag{6.11}$$

There is a natural morphism

$$\mathfrak{i} : \mathfrak{B}_{\mathcal{S},p}(L) \longrightarrow \mathfrak{R}_{\mathcal{S},p}(L) \tag{6.12}$$

given by restricting a character $\tilde{\theta}$ of ${}^{\iota}\tilde{R}_{\mathcal{S},p}$ to $\tilde{R}_{\mathcal{S},p}$, i.e. $\theta := \mathfrak{i}(\tilde{\theta}) = \tilde{\theta}|_{\tilde{R}_{\mathcal{S},p}}$. This morphism is finite and continuous.

Write $\mathfrak{Z}^{\iota} = \mathfrak{Z}^{\iota}_{\mathcal{S},p} := \mathfrak{X}^{\iota} \times \mathfrak{B}$. For any admissible f and $g \in \{f\}^{\iota} \subset {}^{\iota}R_{\mathcal{S},p}$, define the morphism of ringed space

$$R_g : \mathfrak{Z}^{\iota} \to \mathfrak{X}^{\iota} \times \mathbb{A}_1^{\mathrm{rig}} \tag{6.13}$$

by $(\lambda, \tilde{\theta}) \mapsto (\lambda, \tilde{\theta}(g)^{-1})$ on L-points, and

$$R_g^* : \mathcal{O}(\mathfrak{X}^{\iota})\{\{X\}\} \longrightarrow \mathcal{O}(\mathfrak{X}^{\iota})\hat{\otimes}\,{}^{\iota}\tilde{R}_{\mathcal{S},p} \tag{6.14}$$

by $\sum a_n X^n \mapsto \sum a_n (g)^{-n}$ on the function rings.

Define the rigid space

$$\tilde{\mathfrak{D}}^{\iota} := \prod_{[f]^{\iota}} \prod_{g \in [f]^{\iota}} R_g^{-1} \mathfrak{S}_g^{\iota} \tag{6.15}$$

as in [20, Section 8], where its G-topology is defined via R_g's. Concretely, an open subset of $\tilde{\mathfrak{D}}^{\iota}$ is admissible if it is a union of open subsets of the form $R_{g,1} \times \cdots \times R_{g_r}^{-1}(\mathfrak{V})$ for $\mathfrak{V}$ an open admissible affinoid of $\mathfrak{S}_{g_1}^{\iota} \times \cdots \times \mathfrak{S}_{g_r}^{\iota}$; and an admissible covering is the inverse images by R_g's of the admissible coverings of the corresponding spectral varieties. Then we have a parallel result to [20, Proposition 8.1]:

Proposition 6.7. — *Assume $\tilde{y} = (\lambda, \tilde{\theta})$ is in $\mathfrak{Z}^{\iota}(\overline{\mathbb{Q}}_p)$, then $\tilde{y} \in \tilde{\mathfrak{D}}^{\iota}(\overline{\mathbb{Q}}_p)$ if and only if $H^*(S_G(K), \mathcal{D}_{\lambda})[\tilde{\theta}] \neq 0$ as a ${}^{\iota}R_{\mathcal{S},p}$-module. Moreover, given $\tilde{y} \in \tilde{\mathfrak{D}}^{\iota}(\overline{\mathbb{Q}}_p)$, there exists an admissible f, such that*

$$\bigcap_{g \in [f]^{\iota}} R_g^{-1}(R_g(\tilde{y})) \bigcap \tilde{\mathfrak{D}}^{\iota}(\overline{\mathbb{Q}}_p) = \{\tilde{y}\} \tag{6.16}$$

For $\mathfrak{U} \subset \mathfrak{X}^\iota$ and $P_{\mathfrak{U}}(f,X) = Q(X)S(X)$ as in Section 6.3.1, let $h_{\mathfrak{U}}$ and $h^\iota_{\mathfrak{U}}$ be the image of $R_{\mathfrak{U}} := \mathcal{O}(\mathfrak{U}) \otimes R_{\mathcal{S},p}$ and $R^\iota_{\mathfrak{U}} := \mathcal{O}(\mathfrak{U}) \otimes {}^\iota R_{\mathcal{S},p}$ in $End^b_{pf}(R\Gamma(K^pI, \mathcal{D}_{\mathfrak{U}}))$ respectively, and let $h_{\mathfrak{U},Q}$ and $h^\iota_{\mathfrak{U},Q}$ be the image of $R_{\mathfrak{U}}$ and $R^\iota_{\mathfrak{U}}$ in $End^b_{pf}(N_{f,\iota}(Q))$ respectively. Define

$$\tilde{\mathfrak{K}}^{\iota\prime}_{\mathfrak{U}} := \mathrm{sp}(h^\iota_{\mathfrak{U}}) \tag{6.17}$$

and

$$\tilde{\mathfrak{K}}^\iota_{\mathfrak{U},Q} := \mathrm{supp}_{h^\iota_{\mathfrak{U},Q}} H(\widetilde{N^*_{f,\iota}(Q)}). \tag{6.18}$$

Proposition 6.8.

$$\tilde{\mathfrak{K}}^\iota_{\mathfrak{U},Q}(\overline{\mathbb{Q}}_p) = \prod_{g\in[f]^\iota} R_g^{-1}\mathfrak{S}^\iota_{Q,\mathfrak{U},g}(\overline{\mathbb{Q}}_p)$$

The proof of proposition is same to [20, Proposition 8.2]. Moreover, an argument as [20, Propositions 8.2, 8.3, 8,4] shows that we can patch $\tilde{\mathfrak{K}}^\iota_{\mathfrak{U},Q}$ with respect to $\mathfrak{U}$ and Q to obtain a rigid space $\tilde{\mathfrak{K}}^\iota_f$. Define

$$\tilde{\mathfrak{K}}^\iota := \prod_f \tilde{\mathfrak{K}}^\iota_f. \tag{6.19}$$

It is a reduced rigid analytic space, and by Proposition 6.7, 6.8

$$\tilde{\mathfrak{K}}^\iota(\overline{\mathbb{Q}}_p) = \tilde{\mathfrak{D}}^\iota(\overline{\mathbb{Q}}_p). \tag{6.20}$$

Now given $\mathfrak{U}$ and Q as above, define $\mathfrak{i} : \tilde{\mathfrak{K}}^\iota \to \mathfrak{E}$ by locally defining

$$\mathfrak{i} : \tilde{\mathfrak{K}}^\iota_{\mathfrak{U},Q} := \mathrm{supp}_{h^\iota_{\mathfrak{U},Q}} H(\widetilde{N^*_{f,\iota}(Q)}) \to \mathfrak{E}_{\mathfrak{U},Q} := \mathrm{supp}_{h_{\mathfrak{U},Q}} H(\widetilde{N^*_f(Q)}). \tag{6.21}$$

This is defined by the inclusions $h_{\mathfrak{U},Q} \hookrightarrow h^\iota_{\mathfrak{U},Q}$ and $H(N^*_{f,\iota}(Q)) \to H(N^*_f(Q))$. In particular, on $\overline{\mathbb{Q}}_p$-points, $\mathfrak{i}$ sends $(\lambda, \tilde{\theta})$ to $(\lambda, \theta := \tilde{\theta}|_{R_{\mathcal{S},p}})$. So it is defined on each fibre $\tilde{\mathfrak{K}}^\iota_f$ coincident with (6.12) and defined locally on points $R_g^{-1}\mathfrak{S}^\iota_{Q,\mathfrak{U},g}(\overline{\mathbb{Q}}_p)$. Finally, define $\mathfrak{K}^\iota$ as the image of $\tilde{\mathfrak{K}}^\iota$ under $\mathfrak{i}$. Its points are described by the theorem below

Theorem 6.9. — *Assume $y = (\lambda, \theta)$ is in $\mathfrak{E}(\overline{\mathbb{Q}}_p)$, then $y \in \mathfrak{K}^\iota(\overline{\mathbb{Q}}_p)$ implies that θ is a ι-invariant finite slope overconvergent Hecke eigensystem of weight λ. For any $f \in R_{\mathcal{S},p}$, R_f maps $\mathfrak{K}^\iota$ to $\mathfrak{S}^\iota_{\mathfrak{W}^\iota,f}$. It is locally finite and surjective. In particular, $\dim \mathfrak{K}^\iota \leqslant \dim \mathfrak{X}^\iota$.*

This is directly follows from the definition, noting that $f \in \{f\}^\iota$.

Remark 6.10. — Generally, $\mathfrak{K}^\iota$ is NOT the eigenvariety parameterizing all ι-invariant Hecke eigensystems. Actually, it parameterizes those θ such that, as a one-dimensional subspace of $H(S_G(K), \mathcal{D}_\lambda)$, $\iota*$ maps V_θ to itself, as the set A in the proof of Proposition 3.12.

6.4. Twisted eigenvariety

Now fix K^p, for any $f \in R_{\mathcal{S},p}$, consider the morphism $R_{\iota\times f} : \mathfrak{Z}^\iota \to \mathfrak{X}^\iota \times \mathbb{A}^1_{\mathrm{rig}}$ defined as in Section 6.3, and define

$$\tilde{\mathfrak{E}}^\iota = \prod_f (R_{\iota\times f})^{-1}(\mathfrak{S}^\iota(f)), \tag{6.22}$$

where the product is running over admissible Hecke operators f.

Theorem 6.11 (twisted eigenvarities). — *Given K^p as above, there is a subvariety $\mathfrak{E}^\iota_{K^p}$ of $\mathfrak{E}_{K^p}$, satisfying:*

1. *For any $(\lambda,\theta) \in \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$, (λ,θ) is in $\mathfrak{E}^\iota_{K^p}(\overline{\mathbb{Q}}_p)$ if and only if θ is ι-invariant and $m^{\dagger,\iota}_{G,0}(\theta,\lambda) \neq 0$.*
2. *Every irreducible component of $\mathfrak{E}^\iota_{K^p}$ projects surjectively onto a Zariski dense subset of $\mathfrak{X}^\iota$.*
3. *$\mathfrak{E}^\iota_{K^p}$ is equidimensional with the same dimension to $\mathfrak{X}^\iota$, and every irreducible component is arithmetic.*

Proof. — Define $\mathfrak{E}^\iota := \mathfrak{E}^\iota_{K^p}$ to be the image of $\tilde{\mathfrak{E}}^\iota$ under $\mathfrak{i}$. Its underlying topological space $\mathfrak{E}^\iota(\overline{\mathbb{Q}}_p)$ is given by the image $\mathfrak{i}(\tilde{\mathfrak{E}}^\iota(\overline{\mathbb{Q}}_p))$. Firstly we show $\mathfrak{E}^\iota(\overline{\mathbb{Q}}_p) \subset \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$ and the part (1) by modifying the proof of [18, Proposition 5.2.3]. As in Section 5.1, write

$$V = V^{\dagger,\lambda}_{G,0}(K^p) = \bigoplus_\sigma \bigoplus_{i=1}^{l} V_{\tilde{\sigma}_i}^{m^{\iota,\dagger}_{G,0,i}(\sigma,\lambda)}, \tag{6.23}$$

where $\sigma = \pi^{K^p}$ is running over all ι-invariant finite slope representations of $\mathcal{H}_p(K^p)$ appearing in the distribution $e_G I^\dagger_{G,0}(\iota\times f,\lambda)'_{K^p}$. Given $(\lambda,\tilde{\theta}) \in \tilde{\mathfrak{E}}^\iota(L)$, fix $t \in T^{++}$, set $h = v_p(\tilde{\theta}(u_t))$ and

$$W = V^{\leqslant h}.$$

$\mathcal{H}_p(K^p)$ acts on W since $R_{\mathcal{S},p}$ is in its center. Since every σ appearing in V is ι-invariant, that ${}^\iota\mathcal{H}_p(K^p)$ acts on W. Let h_W be the image of $R_{S,p} \to \mathrm{End}_L(W)$. It is finitely generated by the image of finitely many elements $\{f_1,\dots f_r\}$ in $R_{\mathcal{S},p}$. Let Ω be the set consisting of $\tilde{\theta}(u_t), \tilde{\theta}(f_i)$, and all eigenvalues of u_t, $\iota\times u_t$, f_i, $\iota\times f_i$ on W. Now let R be a number such that for any $\alpha,\alpha' \in \Omega$, $v_p(\alpha-\alpha') \leqslant v_p(R)$, define operators $h_1 = f_1$, $h_{i+1} = f_{i+1}(1+Rh_i)$ and $f = u_t(1+Rh_r)$.

Since $(\lambda,\tilde{\theta}) \in \tilde{\mathfrak{E}}^\iota(L)$, there is $0 \neq w_f \in V$ and $\tilde{\sigma}_i$ appeaing in V, such that

$$\tilde{\theta}(\iota\times f)w_f = \sigma(\iota\times f)w_f. \tag{6.24}$$

in particular, w_f is an eigenvector of $\sigma(\iota \times u_t)$ and $\sigma(\iota)$. Denote their eigenvalue by a_f and b_f respectively. Since ι is of finite order, b_f is a unit. Write $\theta = \sigma|_{R_{S,p}}$, then $(\lambda, \theta) \in \mathfrak{E}_{K^p}(\overline{\mathbb{Q}}_p)$, we want to show $\mathfrak{i}(\tilde{\theta}) = \theta$.

Indeed, $v_p(b_f) + v_p(\theta(u_t)) + v_p(\theta(1+Rh_r)) = v_p(\tilde{\theta}(\iota)) + v_p(\tilde{\theta}(u_t)) + v_p(\tilde{\theta}(1+Rh_r))$. Since b_f and $\tilde{\theta}(\iota)$ are units, by the setting of R, $v_p(\theta(u_t)) = v_p(\tilde{\theta}(u_t))$. This implies that σ actually appears in W. On the other hand, $(a_f - \tilde{\theta}(\iota\times u_t))\theta(1+Rh_r) = \tilde{\theta}(\iota\times u_t)(\tilde{\theta}(1+Rh_r) - \theta(1+Rh_r))$. This implies $v_p(a_f - \tilde{\theta}(\iota \times u_t)) > v_p(R)$. So $a_f = \tilde{\theta}(\iota\times u_t)$ and $\theta(h_r) = \tilde{\theta}(h_r)$. Repeating the process, we have $\tilde{\theta}(f_i) = \theta(f_i)$ for all f_i. Therefor $\tilde{\theta}|_{R_{\mathcal{S},p}} = \theta$ and $\mathfrak{i}(\lambda, \tilde{\theta}) = (\lambda, \theta) \in \mathfrak{E}(L)$. In particular θ is ι-invariant. By our construction of θ and formula (5.63), we have $m^{\dagger,\iota}_{G,0}(\theta, \lambda) \neq 0$.

Now we prove the other direction of (1). If $V_i := V_{\tilde{\sigma}_i}$ appearing in (6.23), let $V_i = \bigoplus_\zeta V_i[\zeta]$ be the eigen decomposition of V_i under ι, then ${}^\iota R_{S,p}$ acts on each $V_i[\zeta]$. Let $(\lambda, \theta) \in \mathfrak{E}(\overline{\mathbb{Q}}_p)$ be ι-invariant. If $m^{\iota,\dagger}_{G,0}(\theta, \lambda) \neq 0$, there is some V_i such that $V_i[\theta] \neq 0$ as a $R_{S,p}$-module. In particular, θ appears in some $V_i[\zeta]$. Define $\tilde{\theta}$ be the extension of θ to ${}^\iota R_{S,p}$ by setting $\tilde{\theta}(\iota) = \zeta$. It is then clear that $(\lambda, \tilde{\theta}) \in \tilde{\mathfrak{E}}^\iota(\overline{\mathbb{Q}}_p)$ and $\mathfrak{i}(\lambda, \tilde{\theta}) = (\lambda, \theta)$.

With (1), Theorem 6.9 and Remark 6.10 imply that

$$\mathfrak{E}^\iota(\overline{\mathbb{Q}}_p) \subset \mathfrak{K}^\iota(\overline{\mathbb{Q}}_p) \tag{6.25}$$

By construction, for any $f \in R_{\mathcal{S},p}$, there is a commutative diagram:

$$\begin{tikzcd}
\tilde{\mathfrak{E}}^\iota_{K^p}(\overline{\mathbb{Q}}_p) \arrow[rr, "\mathfrak{i}"] \arrow[d, "R_{\iota\times f}"] & & \mathfrak{E}^\iota_{K^p}(\overline{\mathbb{Q}}_p) \arrow[rr, hook] \arrow[d, "R_f"] & & \mathfrak{K}^\iota_{K^p}(\overline{\mathbb{Q}}_p) \arrow[d, "R_f"] \\
\mathfrak{S}^\iota(f)(\overline{\mathbb{Q}}_p) \arrow[dr] & & (\mathfrak{X}^\iota \times \mathbb{A}_1^{\mathrm{rig}}) \cap \mathfrak{S}_f(\overline{\mathbb{Q}}_p) \arrow[dl] & & \mathfrak{S}^\iota_f(\overline{\mathbb{Q}}_p) \arrow[dl] \\
& \mathfrak{X}^\iota \arrow[rr, hook] \arrow[dr] & & \mathfrak{W}^\iota = \mathfrak{X}^\iota \arrow[dl] & \\
& & \mathfrak{X} & &
\end{tikzcd} \tag{6.26}$$

Consider the first column of the diagram. $R_{\iota\times f}$ is locally finite, and $\mathfrak{S}_f$ is constructed by the Fredholm power series. So by the same argument of [18, Theorem 5.3.7], $R_{\iota\times f}$ is finite surjective. Now Proposition 6.7 and Corollary 6.6 enable us to run an argument as in [18, Corollary 5.3.8], so the composition of the first two arrows in the first column is surjective onto a Zariski dense subset of $\mathfrak{X}^\iota$. Since $\mathfrak{i}$ keeps the first coordinate λ, that the projection from $\mathfrak{E}^\iota(\overline{\mathbb{Q}}_p)$ to $\mathfrak{X}^\iota$ in the second column is also Zariski surjective. So $\dim(\mathfrak{E}^\iota) \geqslant \dim(\mathfrak{X}^\iota)$. However, (6.25) and Theorem 6.9 imply that $\dim(\mathfrak{E}^\iota) \leqslant \dim(\mathfrak{K}^\iota) = \dim(\mathfrak{W}^\iota) \leqslant \dim(\mathfrak{X}^\iota)$. So we have $\dim(\mathfrak{E}^\iota) = \dim(\mathfrak{X}^\iota)$, $\mathfrak{X}^\iota = \mathfrak{W}^\iota$, (which is why we have the third row of the diagram) and, in particular, the R_f in the third column is also surjective. Together with Proposition 6.7, an argument as in [18, Corollary 5.3.8] again shows that R_f in the second column is also locally finite and surjective.

Now since $\dim(\mathfrak{K}^\iota) = \dim(\mathfrak{X}^\iota)$, denote by $\mathfrak{K}^{\iota,c}$ the union of codimension 0 arithmetic components (i.e. irreducible components contains a Zariski dense subset of arithmetic points) of $\mathfrak{K}^\iota$. Then the same argument as [20, Proposition 8.9] shows that

$$\mathfrak{K}^{\iota,c}(\overline{\mathbb{Q}}_p) = \mathfrak{E}^\iota(\overline{\mathbb{Q}}_p) \tag{6.27}$$

This proves (2) and (3) of the theorem. □

Remark 6.12. — By the last observation (6.27) in the proof above, throughout this section we could have worked on the p-adic analytic spaces $\mathfrak{E}^\iota(\overline{\mathbb{Q}}_p)$ and finally define the rigid analytic structure on $\mathfrak{E}^\iota$ by the one induced from $\mathfrak{K}^{\iota,c}$.

7. The case of Gl_n

In this section, we study the case of $G = Gl_n$ over $\mathbb{Q}$. Fix the pair (B, T), where B is the subgroup of upper triangular matrices and T the diagonal subgroup. Then

$$T^+ = \{\mathrm{diag}(t_1, \dots, t_n) \mid v_p(t_1) \geqslant \dots \geqslant v_p(t_n)\} \tag{7.1}$$

$$T^{++} = \{\mathrm{diag}(t_1, \dots, t_n) \mid v_p(t_1) > \dots > v_p(t_n)\} \tag{7.2}$$

Define $g^\iota := j({}^t g^{-1})j^{-1}$ for any $g \in G$, where $j = (\delta_{i,n+1-j})_{1 \leqslant i,j \leqslant n}$ if n is odd; and

$$j = \begin{pmatrix} 0 & (\delta_{i,k+1-j})_{1\leqslant i,j\leqslant k} \\ -(\delta_{i,k+1-j})_{1\leqslant i,j\leqslant k} & 0 \end{pmatrix}$$

if $n = 2k$ is even. It is easy to check that ι is an automorphism of G of order 2, and ι stabilizes (B, T, I_m) and T^+, T^{++}.

Let π be an automorphic representation of G. Apparently, π is ι-invariant if and only if π is self-dual in the usual sense.

7.1. $I^{\dagger}_{G,0}(\iota\times f,\lambda)$ is non-trivial

Consider the ι-twisted distribution $I^{\dagger}_{G,0}(\iota\times f,\lambda)$ defined for G and ι as in Section 5.2, we prove it is non-trivial by computing $q_{G,\iota}$ as in Proposition 5.9.

Proposition 7.1. — *Let $\lambda\in\mathfrak{X}^{\iota}$ be an arithmetic regular dominant weight in Υ. Let π be a self-dual finite slope cuspidal representation of weight λ. Assume that π is non-critical with respect to λ. Then $q_{G,\iota}\neq 0$.*

Proof. — It is a computation given by Barbasch and Speh in [5, VI.3] that

$$\begin{aligned} q_{G,\iota} &= L(\iota, A_{\mathfrak{b}}(\lambda),\lambda)\\ &= (-1)^{R_{\mathfrak{b}}}\sum_i (-1)^i \operatorname{tr}(\iota \mid \operatorname{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^i(\mathfrak{l}\cap\mathfrak{p}),\mathbb{C}))\\ &= (-1)^{\lceil\frac{n}{2}\rceil}2^{\lceil\frac{n}{2}\rceil}\neq 0. \end{aligned}\tag{7.3}$$

□

Since ι is of order 2, for $? =$ cl or $\dagger$, we define $e_{G,\iota}I^{?}_{G,0}(\iota\times f,\lambda)$ as in Remark 5.10. Considering the last paragraph in Section 5.5, we have

Corollary 7.2. — *Let $\lambda\in\mathfrak{X}^{\iota}$ be an arithmetic regular dominant weight in Υ. Let π be a self-dual finite slope cuspidal representation of weight λ. Assume that π is non-critical with respect to λ. Then $m^{\iota}_{EP}(\sigma,\lambda)\neq 0$.*

Proof. — By the definition (5.59), combining (5.51) and (5.61), we compute as in Section 5.2:

$$m^{\iota}_{EP}(\pi,\lambda)=\sum_{\rho} m_{\mathrm{cusp}}(\rho)L(\iota,\rho,\lambda)\operatorname{tr}(\iota\mid \operatorname{Hom}_{\mathcal{H}_p(K^p)}(\pi_f^{K^p},\rho_f)).\tag{7.4}$$

Since $G=Gl_n$, the cohomological packet at infinity for λ has only one element, which is of the form $A_{\mathfrak{b}}(\lambda)$ as in Theorem 5.2.3 (see also [16]). So

$$\begin{aligned} &m^{\iota}_{EP}(\pi,\lambda)\\ &\quad= L(\iota,A_{\mathfrak{b}}(\lambda),\lambda)\sum_{\rho_\infty=A_{\mathfrak{b}}(\lambda)} m_{\mathrm{cusp}}(\rho)\operatorname{tr}(\iota\mid \operatorname{Hom}_{\mathcal{H}_p(K^p)}(\pi_f^{K^p},\rho_f))\\ &\quad= (-1)^{\lceil\frac{n}{2}\rceil}2^{\lceil\frac{n}{2}\rceil}\sum_{\rho_\infty=A_{\mathfrak{b}}(\lambda)}\operatorname{tr}(\iota\mid \operatorname{Hom}_{\mathcal{H}_p(K^p)}(\pi_f^{K^p},\rho_f))\\ &\quad= (-1)^{\lceil\frac{n}{2}\rceil}2^{\lceil\frac{n}{2}\rceil}\neq 0, \end{aligned}\tag{7.5}$$

where the last two lines hold since Gl_n admits the multiplicity one theorem. □

Remark 7.3. — Corollary 7.2 implies that the ι-twisted eigenvariety $\mathfrak{E}^{\iota}_{K^p}$ we constructed in Theorem 6.11 for Gl_n parameterizes all non-critical self-dual finite slope cuspidal Hecke eigensystems of level K^p.

7.2. Essentially self-dual representations

Generally, a representation π of a reductive group is essentially ι-invariant if there exists an algebraic character χ of $\mathbb{G}_m$ such that $\chi \circ \det \otimes \pi^{\iota} \cong \pi$. In our case $G = Gl_n$, a representation π is essentially ι-invariant if and only if it is essentially self-dual in the usual sense.

Now a p-adic weight $\lambda \in \mathfrak{X}(L)$ is determined by n p-adic characters $\chi_1, \dots, \chi_n$ of $\mathbb{G}_m(\mathbb{Z}_p)$ such that

$$\lambda : \mathrm{diag}(t_1, \dots, t_n) \mapsto \chi_1(t_1) \dots \chi_n(t_n) \tag{7.6}$$

An essentially self-dual weight is characterized by $\chi_i \chi_{n+1-i} = \chi_j \chi_{n+1-j}$ for any $1 \leqslant i, j \leqslant n$. Denote by $\mathfrak{X}^e$ the subspace of essentially self-dual weights in $\mathfrak{X}$, then $\dim(\mathfrak{X}^e) = [\frac{n}{2}] + 1$. Given a character χ, denote by $\mathfrak{X}^e_{\chi}$ the subspace of essentially self-dual weights with respect to χ in $\mathfrak{X}^e$. It is cut out by the relation $\chi_i \chi_{n+1-i} = \chi_j \chi_{n+1-j} = \chi$, then $\dim(\mathfrak{X}^e_{\chi}) = [\frac{n}{2}]$.

We now construct an eigenvariety $\mathfrak{E}^e$, which parameterizes all essentially self-dual finite slope overconvergent Hecke eigensystems of G, by applying our method to the group $\tilde{G} = Gl_n \times Gl_1$, with involution $\mu : (g, x) \mapsto (g^{\iota}, \det(g) x)$. We remark here that μ is not of Cartan-type, however, one can verify that our discussion through Sections 2-6 still works.

Let $\mathfrak{B}^1$ be the p-adic weight space of Gl_1, it is the rigid unit ball. Consider the weight space $\tilde{\mathfrak{X}} = \mathfrak{X} \times \mathfrak{B}^1$. Denote by $\tilde{\mathfrak{X}}^{\mu}$ the μ-invariant subspace of $\tilde{\mathfrak{X}}$. It is easy to check that

$$\tilde{\mathfrak{X}}^{\mu} = \{\tilde{\lambda} = (\lambda, \chi) \mid \lambda \in \mathfrak{X}^e_{\chi}, \chi \in \mathfrak{B}^1\}. \tag{7.7}$$

Its first component projects bijectively to $\mathfrak{X}^e$.

Consider an open compact subgroup $\tilde{K}_f = K_f \times K^1_f$ of $\tilde{G}(\mathbb{A}_f)$, where K_f is defined as previous sections and K^1_f is a neat open compact subgroup of $Gl_1(\mathbb{A}_f)$. We simply fix $K^1_p = \hat{\mathbb{Z}}$ and normalize the Haar measure on Gl_1 such that $\mathrm{meas}(\mathbb{Z}_l) = 1$ for any finite place l. Then the associated locally symmetric space for $\tilde{G}$ is

$$S_{\tilde{G}}(\tilde{K}_f) \cong S_G(K_f) \times (\mathbb{Q}^{\times} \backslash \mathbb{A}^{\times}_{\mathbb{Q}} / \hat{\mathbb{Z}} \mathbb{R}^{\times}) \simeq S_G(K_f) \times \{pt\}\,. \tag{7.8}$$

For the group Gl_1, $T^+_{Gl_1} = T^{++}_{Gl_1} = T_{Gl_1}(\mathbb{Q}_p) = \mathbb{Q}^{\times}_p$, so the set of U_p operators $\mathcal{U}_p(Gl_1) = \mathbb{Z}_p[T^+/T(\mathbb{Z}_p)] = \mathbb{Z}_p[\mathbb{Q}^{\times}_p/\mathbb{Z}^{\times}_p]$. This implies that the p-adic

Hecke algebra for $\tilde{G}$ is given by:

$$\begin{aligned}\mathcal{H}_p(\tilde{K}^p) &= \mathcal{H}_p(K^p) \times C_c^\infty(\hat{\mathbb{Z}}^p\backslash \mathbb{A}_f^{\times p}/\hat{\mathbb{Z}}^p) \otimes \mathbb{Z}_p[\mathbb{Q}_p^\times/\mathbb{Z}_p^\times] \\ &= \mathcal{H}_p(K^p) \times C_c^\infty(\mathbb{A}_f^\times/\hat{\mathbb{Z}}, \mathbb{Z}_p).\end{aligned} \tag{7.9}$$

For any $f \in \mathcal{H}'_p(\tilde{K}^p)$ admissible, write $f = (f_G, f_1)$ with $f_G = f_G^p \otimes u_t$, $t \in T^{++}$ and $f_1^p \otimes u_{t'}$ $t' \in \mathbb{Q}_p^\times$.

For a regular dominant weight $\tilde{\lambda} = (\lambda, \chi) \in \tilde{\mathfrak{X}}^\mu$, let $\mathbb{V}_{\tilde{\lambda}}$ be the finite dimensional irreducible algebraic representation of $\tilde{G}$ with highest weight $\tilde{\lambda}$ and $\mathcal{D}_{\tilde{\lambda}}$ the local distribution space defined in Section 2.3. It is not hard to see

$$\mathbb{V}_{\tilde{\lambda}} \cong \mathbb{V}_\lambda \times \chi \tag{7.10}$$

where the right side is understood as the space $\mathbb{V}_\lambda$ together with an action of Gl_1 given by multiplying the value of χ and the isomorphism is given by $\phi \mapsto \phi_1$ such that $\phi_1(g) := \phi(g, 1)$ for any $g \in G$. Generally, let π be an irreducible representation of $\tilde{G}$, since Gl_1 is in the center of $\tilde{G}$, $\pi|_{Gl_1}$ is given by a character χ and $\pi|_G$ is irreducible as well. It is easy to check that $\pi \cong \pi|_G \times \chi$, and π is μ-invariant if and only if $\pi|_G$ is essentially ι-invariant with respect to χ. If π is a μ-invariant representation of $\tilde{G}$, as in Section 3.3, we can extend π to a representation $\tilde{\pi}$ of $\tilde{G} \rtimes \langle\mu\rangle$, and then restrict $\tilde{\pi}$ to $G \rtimes \langle\mu\rangle$. This gives an μ-action on V_π such that for any $g \in G$,

$$\mu \times \chi(\det(g))\pi(g^\iota) = \pi(g) \times \mu. \tag{7.11}$$

Now we can define μ-twisted finite slope character distributions $I_?^{\mathrm{cl}}(\mu \times f, \tilde{\lambda})$ and $I_?^\dagger(\mu \times f, \tilde{\lambda})$ as (4.4.1), (4.4.3) and Section 5.2, where $? = \tilde{G}, (\tilde{G}, 0), (\tilde{G}, \tilde{M}, 0)$. As Remark 5.10 and the discussion in Section 5.4, $I_{\tilde{G},0}^{\mathrm{cl}}(\mu\times f, \tilde{\lambda})$ and $I_{\tilde{G},0}^\dagger(\mu\times f, \tilde{\lambda})$ are essentially effective. We can indeed compute them explicitly and relate them to the distributions $I_{G,0}^{\mathrm{cl}}(\iota\times f_G, \lambda)$ and $I_{G,0}^\dagger(\iota \times f_G, \lambda)$ for G, where $I_{\tilde{G},0}^{\mathrm{cl}}(\mu \times f, \tilde{\lambda})$ equals

$$\begin{aligned}&\mathrm{meas}(K^p)\tilde{\lambda}(\xi(t,t'))\,\mathrm{tr}(\mu \times f \mid H^*_{\mathrm{cusp}}(S_{\tilde{G}}(\tilde{K}_f), \mathbb{V}_{\tilde{\lambda}}^\vee(\mathbb{C}))) \\ &\qquad= \mathrm{meas}(K^p)\lambda(\xi(t))\chi(t')\,\mathrm{tr}(\mu \times f \mid H^*_{\mathrm{cusp}}(S_{\tilde{G}}(\tilde{K}_f), \mathbb{V}_{\tilde{\lambda}}^\vee(\mathbb{C})) \\ &\qquad= \mathrm{meas}(K^p)\lambda(\xi(t))\chi(t')\chi^\vee(f_1)\,\mathrm{tr}(\mu \times f_G \mid H^*_{\mathrm{cusp}}(S_G(K_f), \mathbb{V}_\lambda^\vee)) \\ &\qquad= \chi(t')\chi^\vee(f_1)I_{G,0}^{\mathrm{cl}}(\mu \times f_G, \lambda),\end{aligned}$$

where the last second equation follows from Section 3.1 and (7.10), the action of μ is given as in (7.11). In particular, if χ is trivial,

$$I_{\tilde{G},0}^{\mathrm{cl}}(\mu \times f, \tilde{\lambda}) = I_{G,0}^{\mathrm{cl}}(\iota \times f_G, \lambda). \tag{7.12}$$

Now Corollary 5.15 implies that

$$I^{\dagger}_{\tilde{G},0}(\mu \times f, \tilde{\lambda}) = \chi(t')\chi^{\vee}(f_1) I^{\dagger}_{G,0}(\mu \times f_G, \lambda), \tag{7.13}$$

and if χ is trivial,

$$I^{\dagger}_{\tilde{G},0}(\mu \times f, \tilde{\lambda}) = I^{\dagger}_{G,0}(\iota \times f_G, \lambda). \tag{7.14}$$

Remark 7.4. — Given character χ and $\lambda \in \mathfrak{X}^e$ that is essentially ι-invariant with respect to χ, inspired by the above computation, we can directly define for $f \in \mathcal{H}_p(K^p)$ that

$$I^{\mathrm{cl},\chi}_{G,0}(\iota \times f, \lambda) := I^{\mathrm{cl}}_{G,0}(\mu_\chi \times f, \lambda) \tag{7.15}$$

and

$$I^{\dagger,\chi}_{G,0}(\iota \times f, \lambda) := I^{\dagger}_{G,0}(\mu_\chi \times f, \lambda) \tag{7.16}$$

where the lower index in μ_χ emphsis that the twisted action μ on the cohomology spaces are defined according to χ. Just like Proposition 3.12, one can show that only the traces of representations that are essentially ι-invariant with respect to χ can contribute to these distributions, and the discussion in Sections 4-6 works for them as well.

Now let $\tilde{\lambda} = (\lambda, \chi)$ be an arithmetic weight, π a μ-invariant finite slope cuspidal representation of $\tilde{G}$, such that $\pi|_{Gl_1} = \chi$. Let $\pi(\chi^{-1})$ be π twisting the inverse of χ by Gl_1, and π_χ be the factorization of $\pi(\chi^{-1})$ to $G \cong \tilde{G}/Gl_1$, it is easy to see that π_χ is ι-invariant. Compute by definition, we have

$$m^{\mu}_{EP}(\pi, \tilde{\lambda}) = m^{\iota}_{EP}(\pi_\chi, \lambda) \neq 0. \tag{7.17}$$

Now by Theorem 6.11, we have

Theorem 7.5. — *Assume $G = Gl_n$ there is an eigenvariety $\mathfrak{E}^e \subset \mathfrak{E}$ defined as in Theorem 6.11, such that*

1. *there are two projections $p_1 : \mathfrak{E}^e \to \mathfrak{X}^e$ and $p_2 : \mathfrak{E}^e \to \mathfrak{B}^1$, such that $y = (\lambda_y, \theta_y) \in \mathfrak{E}^e(\bar{\mathbb{Q}}_p)$ if and only if θ is a finite slope overconvergent Heche eigensystem of weight $\lambda_y = p_1(y)$ and is essentially self-dual with respect to $\chi_y = p_2(y)$ with $m^{\dagger,\iota}_{G,0}((\theta \times \chi_y)_{\chi_y}, (\lambda \times \chi_y)_{\chi_y}) \neq 0$.*
2. *$\mathfrak{E}^e$ is equidimensional of dimension $[\frac{n}{2}] + 1$. Its every irreducible component is arithmetic.*
3. *For any $\chi \in \mathfrak{B}^1$, set $\mathfrak{E}^e_\chi = p_2^{-1}(\chi)$, then $\mathfrak{E}^e_\chi$ is the eigenvariety parameterizing essentially self-dual Hecke eigensystems of G with respect to χ. In particular, $\mathfrak{E}^e_0 = \mathfrak{E}^\iota$.*
4. *The projection p_1 maps $\mathfrak{E}^e_\chi$ onto $\mathfrak{X}^e_\chi \subset \mathfrak{X}^e$. $\mathfrak{E}^e_\chi$ is equidimensional over $\mathfrak{X}^e_\chi$ and all its irreducible components are arithmetic.*

Remark 7.6.

(1) As Remark 11, $\mathfrak{E}^e(\overline{\mathbb{Q}}_p)$ contains all non-critical essentially self-dual finite slope cuspidal Hecke eigensysems of G.
(2) Applying our theory to $I^{\dagger,\chi}_{G,0}(\iota \times f, \lambda)$, we can construct $\mathfrak{E}^e_\chi$ directly, and we can obtain parallel results to Theorem 6.11 for $\mathfrak{E}^e_\chi$.

7.3. Ash–Pollack–Stevens Conjecture

Let θ_0 be a classical finite slope cuspidal Hecke eigensystem of Gl_n of regular weight λ_0. Recall, as defined in [3], θ_0 is called p-adic arithmetically rigid, if (modulo twisting) it does not admit a p-adic deformation containing a Zariski dense subset of arithmetic specializations. So if it is contained in any arithmetic irreducible component of $\mathfrak{E}^e_\chi$, it is not arithmetically rigid. Conjecture 1.1 claims that, if θ is not essentially self-dual then it is p-adic arithmetic rigid. Theorem 7.5 gives its inverse:

Corollary 7.7. — *Assume θ_0 is essentially self-dual with respective to χ, then it is lying in an arithmetic component of $\mathfrak{E}^e_\chi$. In particular, it is not p-adic arithmetically rigid,.*

Proof. — By Theorem 7.5, $(\lambda_0, \theta_0) \in \mathfrak{E}^e_\chi(\overline{\mathbb{Q}}_p)$. Consider the subset Σ of $\mathfrak{E}^e_\chi(\bar{\mathbb{Q}}_p)$ consisting of (λ, θ) such that λ is arithmetic and θ is non-critical with respect to λ. Σ is Zariski dense in $\mathfrak{E}^e_\chi(\bar{\mathbb{Q}}_p)$ since its projection to $\mathfrak{X}^e_\chi$ contains an arithmetic point λ_0. By Corollary 5.21, those points in Σ are classical and corresponding to cuspidal Hecke eigensystems. □

The next theorem shows that the smooth hyperthesis on arithmetic points of an eigenvarity may give some hint on the Ash–Pollack–Stevens conjecture.

Theorem 7.8. — *Assume that every arithmetic point in the eigenvariety is smooth. Let θ_0 be a classical Hecke eigensystem which is not p-adic arithmetic rigid. Assume the arithmetic component containing θ_0 also contains an arithmetic, essentially self-dual Hecke eigensystem, then this arithmetic component contains a Zariski dense subset of essentially self-dual Hecke eigensystems. Moreover, θ_0 is essentially self-dual.*

Proof. — By [9], the cuspidal cohomology $H^*_{\mathrm{cusp}}(S_{Gl_n}(K), \mathbb{V}^\vee_\lambda)$ is trivial unless λ is essentially self-dual. So as [3], (λ_0, θ_0) is contained in an arithmetic component over $\mathfrak{X}^e$. Then the component must be of dimension $\leqslant [\frac{n}{2}] + 1$ and, by our assumption, intersects with $\mathfrak{E}^e$ at some smooth point. Since $\mathfrak{E}^e$ is of dimension $[\frac{n}{2}]+1$. So by our smooth condition that this arithmetic component is at the same time an irreducible component of $\mathfrak{E}^e$. Finally, assume that λ_0 is essentially self-dual with respect to χ_0. Then (λ_0, θ_0) is contained in an arithmetic component of $\mathfrak{E}^e_{\chi_0}$, in particular, it is essentially self-dual with respect to χ_0. □

Remark 7.9. — Assume $G = Gl_3$. The Theorem 7.8 assumes that there is an essentially self-dual point in the arithmetic component. This is not surprising if the eigenvarieties have good geometry. By [20], we know the full eigenvariety $\mathfrak{E}$ has dimension $\leqslant 3$. Let $\mathfrak{A}$ an arithmetic component, it also projects onto $\mathfrak{X}^e$ and is of dimension at least 1. Since we know that $\mathfrak{E}^e$ has dimension 2, it should meet $\mathfrak{A}$.

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