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WHITNEY STRATIFICATIONS AND THE CONTINUITY OF LOCAL LIPSCHITZ–KILLING CURVATURES

by Nhan NGUYEN & Guillaume VALETTE (*)

Abstract. — In the paper we prove that the local Lipschitz–Killing curvatures of a definable set in a polynomially bounded o-minimal structure are continuous along the strata of a Whitney stratification. Moreover, if the stratification is (w)-regular the local Lipschitz–Killing curvatures are locally Lipschitz in any o-minimal structure.

Résumé. — On montre que les courbures Lipschitz–Killing locales d'un ensemble définissable dans une structure o-minimale polynomialement bornée sont continues le long des strates d'une stratification de Whitney. De plus, si la stratification est (w)-régulière les courbures Lipschitz–Killing locales sont localement lipschitziennes dans une structure o-minimale arbitraire.

1. Introduction

Given a compact definable set $A \subset \mathbb{R}^n$, we can associate a sequence of curvatures $(\Lambda_0(A), \dots, \Lambda_n(A))$ called Lipschitz–Killing curvatures. The definition according to Comte and Merle [6] is as follows:

For $0 \leqslant k \leqslant n$, *the k-th Lipschitz–Killing curvature of A* is

$$\Lambda_k(A) := c(n,k) \int_{P \in \mathbb{G}_n^k} \int_{x \in P} \chi(A \cap \pi_P^{-1}(x)) \, \mathrm{d}\mathcal{H}^k(x) \, \mathrm{d}P, \tag{1.1}$$

where χ denotes the Euler characteristic, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure, $\mathrm{d}P$ denotes the standard probability measure of $\mathbb{G}_n^k$,

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π_P denotes the orthogonal projection from $\mathbb{R}^n$ onto P and $c(n,k)$ is a constant that depends only on n and k.

The study of the Lipschitz–Killing curvatures of subanalytic sets was initiated by Fu [14] using geometric measure theory and then by Bröker and Kuppe [2] in the more general setting of definable sets by using stratified Morse theory. The formula (1.1) is not the original definition in the sense of Fu and Bröcker–Kuppe but it is the linear kinematic formula (see [2, 14]).

For $x \in A$, Comte and Merle [6] proved that the litmit

$$\Lambda_k^{\mathrm{loc}}(A,x) := \lim_{r\to 0} \frac{1}{\mu_k r^k} \Lambda_k(A \cap \overline{\mathbf{B}}^n_{(x,r)}) \tag{1.2}$$

exists for any $k = 0, \dots, n$. They called the sequence

$$\Lambda_*^{\mathrm{loc}}(A,x) := (\Lambda_0^{\mathrm{loc}}(A,x), \dots, \Lambda_n^{\mathrm{loc}}(A,x))$$

the *local Lipschitz–Killing curvatures of A at x*.

The notion of localization of Lipschitz–Killing curvatures has also been introduced independently by Bernig and Bröker in [1]. Their definition is based on the normalization of the trace of the germ on small spheres instead of the trace of small balls as used in the article of Comte–Merle. The two resulting invariants are the same up to a linear combination. It has been shown in [6] that Λ_*^{loc} is a linear combination of the finite sequence $\sigma_* = (\sigma_1, \dots, \sigma_n)$ of polar invariants. It is proved that along the strata of a (w)-regular stratification of a subanalytic set the sequence σ_*, and hence Λ_*^{loc}, is continuous. Many other interesting results about Lipschitz–Killing curvatures, real polar varieties and equisingularity can be found in [4, 6, 10, 11, 12, 13].

Observe that $\Lambda_{d_A}^{\mathrm{loc}}(A,x)$, where d_A denotes the dimension of A, is also called the density of the set A at x. The existence of the density of a subanalytic set was first proved by Kurdyka and Raby [19]. Soon after, Trotman conjectured that the density of a subanalytic set is continuous along the strata of a Whitney stratification. Comte, in his thesis (1998), proved the conjecture for Verdier (w)-regular stratifications (see also [4]). And, the second author completely resolved the conjecture in [30]. Moreover, he proved that the density is locally Lipschitz if the stratification is (w)-regular.

In the case of complex analytic sets, it is known by a result of Teissier [27] that the Whitney condition (b) and the Verdier condition (w) are equivalent. It is also well-known by a result of Draper [7] that the density is equal to the multiplicity, hence by the continuity, it is constant along the strata of a Whitney stratification. Notice that the condition (w) is strictly stronger than the condition (b) in the real case (see [3, 16, 17, 20, 28]).

In this paper we show that the local Lipschitz–Killing curvatures are continuous along the strata of a Whitney stratification of a definable set in a polynomially bounded o-minimal structure. This result is a real version of Teissier's theorem [27] which says that (b)-regularity implies the constancy of the multiplicities of polar varieties. The result does not hold for non-polynomially bounded o-minimal structures. A counterexample can be found in [29]. We show furthermore that if the stratification is (w)-regular then Λ_*^{loc} is locally Lipschitz in any o-minimal structure, thus strengthening the theorem of Comte–Merle.

The idea of the proof is to improve the technique developed in [30] to study the invariance of the density. Given a closed definable set $A \subset \mathbb{R}^n$ endowed with a Whitney stratification Σ. Assume Y is a stratum of Σ. In order to prove the continuity of $\Lambda_*^{\mathrm{loc}}(A, \cdot\,)$ along stratum Y, in Theorem 5.1, we first reduce the problem to proving the continuity of the local Lipschitz–Killing curvatures of the fibres of a set, denoted $\mathcal{A}$, along a parameter stratum of a Whitney stratification satisfying the conditions $(\star)$ and $(\star\star)$ in Section 4. On this stratification, we integrate a good lifting of a vector field to obtain geometric control along the flow (see Lemma 4.2). Using this control, we show in Proposition 4.3 that for each t in the parameter stratum one can control the volume of the set of points in l-dimensional linear spaces P, such that out of these sets, the fibers of the orthogonal projection of the germs $\mathcal{A}_t$ and $\mathcal{A}_0$ onto P have the same Euler characteristic. Then by the fact that the global Lipschitz–Killing curvatures are the mean values of such Euler characteristics, with respect to the projections on the vector spaces P, we obtain the desired continuity. For the Verdier regularity, the control on the flow is better, and one obtains in the same way locally Lipschitz continuity.

The following notations will be used throughout the paper besides some already given in the introduction. For $k \leqslant n$ positive integers, $\mathbf{B}^k(x,r)$, $\overline{\mathbf{B}}^k(x,r)$, $\mathbf{S}^{k-1}(x,r)$ respectively are the open ball, the closed ball and the sphere in $\mathbb{R}^k$ of radius r centered at x, μ_k is the volume of the k-dimensional unit ball; $\mathbb{G}_n^k$ is the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$; if $A \subset \mathbb{R}^n$, $\overline{A}$ denotes the closure of A in $\mathbb{R}^n$; $\|\cdot\|$ denotes the Euclidean norm, $|\cdot|$ denotes the absolute value; for two functions $f, g : A \to \mathbb{R}_{\geqslant 0}$, we write $f \lesssim g$ (or $g \gtrsim f$) if there is $C > 0$ such that $f(x) \leqslant Cg(x)$ for all $x \in A$, and write $f \sim g$ if $f \lesssim g$ and $g \lesssim f$.

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2. Definable stratifications

2.1. O-minimal structures

A *structure* on the real closed field $(\mathbb{R}, +, \cdot)$ is a family $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ satisfying the following properties:

(1) $\mathcal{D}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$,
(2) If $A \in \mathcal{D}_n$ then $\mathbb{R} \times A \in \mathcal{D}_{n+1}$ and $A \times \mathbb{R} \in \mathcal{D}_{n+1}$,
(3) $\mathcal{D}_n$ contains the zero sets of all polynomials in n variables,
(4) If $A \in \mathcal{D}_n$ then $\pi(A) \in \mathcal{D}_{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection onto the first $n-1$ coordinates.

A structure $\mathcal{D}$ is said to be *o-minimal* if any element of $\mathcal{D}_1$ is a finite union of open intervals and points.

A set belonging to $\mathcal{D}_n$ for some n is called a *$\mathcal{D}$-sets* (or a definable set), a map whose graph is a $\mathcal{D}$-set is called *a $\mathcal{D}$-map* (or a definable map).

A structure $\mathcal{D}$ is said to be *polynomially bounded* if for every $\mathcal{D}$-function $f : \mathbb{R} \to \mathbb{R}$, there exist $a > 0$ and $N \in \mathbb{N}$ such that $|f(x)| \leqslant x^N$ for all $x > a$.

The class of semialgebraic sets and the class of globally subanalytic sets are examples of polynomially bounded o-minimal structures. For more details about o-minimal structures we refer the reader to [5, 8, 9, 21].

2.2. Stratifications

Let A be a $\mathcal{D}$-subset of $\mathbb{R}^n$. A *$\mathcal{D}$-stratification* (or a stratification for simplicity) of A is a partition $\Sigma = \{S_\alpha\}_\alpha$ of A into finitely many connected C^2 $\mathcal{D}$-submanifolds of $\mathbb{R}^n$, called *strata*, satisfying the *frontier condition*: if $S_\alpha, S_\beta \in \Sigma$, $S_\alpha \neq S_\beta$, $\overline{S_\alpha} \cap S_\beta \neq \emptyset$, then $S_\beta \subset \overline{S_\alpha}$. We call such (S_α, S_β) a pair of *adjacent strata*.

Let us denote by S^k the union of strata of Σ of dimension less or equal to k. The stratification can be described as a filtration of skeletons as follows

$$A = S^{d_A} \supseteq S^{d_A - 1} \supseteq \cdots \supseteq S^l \neq \emptyset$$

where d_A is the dimension of A. Set $\mathring{S}^k := S^k \setminus S^{k-1}$, so it is the union of k-dimensional strata of Σ.

A vector field v defined on A is said to be a *stratified vector field* with respect to the stratification Σ if the restriction of v to each stratum of Σ is a tangent vector field of class C^1.

Let v^α denote the restriction of v to the stratum S_α of Σ. By integrating the vector field v^α we get a continuous flow $\theta^\alpha : D^\alpha \to S_\alpha$ where D^α is the maximal domain induced by v^α. Note that $D^\alpha \subset S_\alpha \times \mathbb{R}$ is an open neighborhood of $S_\alpha \times \{0\}$ and for each $x \in A$ there is a unique D^α containing x and the intersection $D^\alpha \cap (\{x\} \times \mathbb{R})$ is an open interval $\{x\} \times (s_x^-, s_x^+)$ containing $(x, 0)$. Let $\theta_x = \theta_x^\alpha : (s_x^-, s_x^+) \to A$ denote the integral curve through the point x. Set $D := \bigcup_\alpha D^\alpha$. We obtain the map

$$\theta : D \to A, \quad \theta(x, s) := \theta_x(s),$$

and call it *the flow generated by the vector field* v. The vector field v is said to be *locally integrable* (or integrable) if D contains a neighborhood of $A \times \{0\}$ on which θ is continuous.

The reader may find in [15, 22, 26] the definitions of tubular neighborhood, vector field controlled by a tubular neighborhood and more details of the theory of controlled vector fields. By a *controlled vector field* we mean a stratified vector field controlled by a tubular neighborhood. It is well-known that controlled vector fields are integrable, and every Whitney stratification admits a continuous controlled vector field (see [25, 26]).

Now let us recall the definitions of the regularity conditions which we shall deal with later on. Let X, Y be C^2 $\mathcal{D}$-submanifolds of $\mathbb{R}^n$ and let $z \in \overline{X} \cap Y$.

The pair (X, Y) is said to be (a)*-regular* at the point z if for any sequence $\{x_k\}_{k\in\mathbb{N}}$ in X converging to z such that the sequence of tangent spaces $\{T_{x_k}X\}_{k\in\mathbb{N}}$ converges to $\tau \in \mathbb{G}_n^{\dim X}$, then $T_zY \subset \tau$.

The pair (X, Y) is said to be (b)*-regular* at the point z if for any sequence $\{x_k\}_{k\in\mathbb{N}}$ in X and any sequence $\{y_k\}_{k\in\mathbb{N}}$ in Y, both converging to z such that the sequence of tangent spaces $\{T_{x_k}X\}_{k\in\mathbb{N}}$ converges to $\tau \in \mathbb{G}_n^{\dim X}$ and the sequence of vectors $\frac{x_k - y_k}{\|x_k - y_k\|}$ converges to a unit vector v, then $v \in \tau$.

The pair (X, Y) is said to be (r)*-regular* at the point z if

$$\lim_{X \ni x \to z} \frac{\delta(T_{\pi_Y(x)}Y, T_xX)\|x - z\|}{\|x - \pi_Y(x)\|} = 0,$$

where π_Y denotes the locally orthogonal projection onto Y and

$$\delta(M, N) := \sup_{x \in M, \|x\|=1} \|x - P_N(x)\|$$

where M, N are vector subspaces of $\mathbb{R}^n$, P_N is the orthogonal projection from $\mathbb{R}^n$ onto N.

The pair (X, Y) is said to be (w)-*regular* at the point z if there exist a neighborhood U_z of z in $\mathbb{R}^n$ and a constant $C > 0$ such that

$$\delta(T_y Y, T_x X) \leqslant C\|x - y\|, \quad \forall\, x \in U_z \cap X, \forall\, y \in U_z \cap Y.$$

Suppose that (γ) is a regularity condition defined on (X, Y). The pair (X, Y) is called (γ)-*regular* if (X, Y) is (γ)-regular at every point $z \in Y$. A stratification is said to be *(γ)-regular* if every pair of adjacent strata of the stratification is (γ)-regular. We call a (b)-regular stratification a *Whitney stratification*.

Remark 2.1. — In the o-minimal setting, we have $(w) \Rightarrow (r) \Rightarrow (b)$. If the structure is polynomially bounded and $\dim Y = 1$ then $(r) \Leftrightarrow (b)$ (see [23, 29]). It is also known that every definable set admits a (w)-regular stratification (see [20]).

3. Preliminary results of $\mathcal{D}$-sets

Let A be a $\mathcal{D}$-subset of $\mathbb{R}^n$ (consider A as a family of $\mathcal{D}$-subsets of $\mathbb{R}^{n-k}$ parametrized by $\mathbb{R}^k$). For $U \subset \mathbb{R}^k$, the restriction of A to U is the set

$$A|_U := \{x = (q, t) \in \mathbb{R}^{n-k} \times \mathbb{R}^k : x \in A, t \in U\}.$$

Given $t \in \mathbb{R}^k$ we call the set $A_t := \{q \in \mathbb{R}^{n-k} : (q, t) \in A\}$ the *fibre of A at the point t*. If $\mathfrak{S} = \{S^\alpha\}_\alpha$ is a collection of definable sets in $\mathbb{R}^n$ then we denote $\mathfrak{S}_t := \{S^\alpha_t\}_\alpha$.

Let B be a $\mathcal{D}$-set in $\mathbb{R}^n$ such that $A \subset B$. Given $\varepsilon > 0$, the *neighborhood of A in B of radius ε* is defined as follows

$$\mathcal{N}(A, B, \varepsilon) := \{x \in B : d(x, A) \leqslant \varepsilon\},$$

where $d(\,\cdot\,,\cdot\,)$ denotes the Euclidean distance in $\mathbb{R}^n$.

For $r > 0$, we define

$$\psi(A, r) := \mathcal{H}^{d_A}\left(A \cap \overline{\mathbf{B}}^n_{(0,r)}\right).$$

where $d_A := \dim A$.

Proposition 3.1. — *Let $A \subset \mathbb{R}^n \times \mathbb{R}^m$ be a $\mathcal{D}$-set. Suppose that A is a family of $\mathcal{D}$-sets of dimension l in $\mathbb{R}^n$ parametrized by $\mathbb{R}^m$. Then, there exists a constant $C > 0$ such that for any $r > 0$ and any $t \in \mathbb{R}^m$, we have*

(1) $\psi(A_t, r) \leqslant C r^l$.

(2) *If there is $k > l$ such that $A_t \subset \mathbb{R}^k$ for every t, then for every $\varepsilon > 0$,*

$$\psi(\mathcal{N}(A, \mathbb{R}^k, \varepsilon), r) \leqslant Cr^{k-1}\varepsilon.$$

Proof. — We follow closely the proof of Propositions 3.06 and 3.07 in [30].

(1). — In the case $l = n$, the set $A_t \cap \overline{\mathbf{B}}^n_{(0,r)}$ is included in the ball $\overline{\mathbf{B}}^n_{(0,r)}$ for all $t \in \mathbb{R}^m$, the result is obvious (the constant C is the volume of $\overline{\mathbf{B}}^l_{(0,1)}$). If $l < n$, by removing a $\mathcal{D}$-subset of dimension less than l, we can consider A_t as a finitely disjoint union of graphs of Lipschitz mappings after a possible change of coordinates (the number of these graphs are bounded by a constant independent of t (see [18, Prop. 1.4])). The volume of such a graph is equivalent to the volume of its image under the projection onto $\mathbb{R}^l$. The conclusion then follows from the case $l = n$. Notice that the constant C depends only on the set A, it does not depend on the parameter t.

(2). — Condider the $\mathcal{D}$-set

$$A' := \{(x, t, \alpha) \in \mathbb{R}^k \times \mathbb{R}^{m+1} : \mathrm{dist}(x, A_t) = \alpha\}$$

Since $\dim A_t < k$, $A'_{(t,\alpha)}$ is a $\mathcal{D}$-set of dimension $k - 1$. By the case (1), there exists $C > 0$ independent of the parameter (t, α) such that

$$\psi(A'_{(t,\alpha)}, r) \leqslant Cr^{k-1}.$$

Then,

$$\begin{aligned}\psi(\mathcal{N}(A_t, \mathbb{R}^k, \varepsilon), r) &= \int_{\mathcal{N}(A_t,\mathbb{R}^k,\varepsilon)\cap\overline{\mathbf{B}}^k_{(0,r)}} \mathrm{d}\mathcal{H}^k \\ &\leqslant \int_0^\varepsilon \psi(A'_{(t,\alpha)}, r)\, \mathrm{d}\mathcal{H}^1(\alpha) \\ &\leqslant Cr^{k-1}\varepsilon.\end{aligned}$$

□

LEMMA 3.2. — *Let A be a closed $\mathcal{D}$-subset of $\mathbb{R}^n$ and x_0 be a point in A. Let Σ be a Whitney stratification of A such that $\{x_0\} \in \Sigma$. Then, there exists an $r_0 > 0$ such that for every $0 < r < r' < r_0$, there is a strong deformation retract from $A \cap \overline{\mathbf{B}}^n_{(0,r')}$ onto $A \cap \overline{\mathbf{B}}^n_{(0,r)}$ which preserves the strata of Σ, i.e. there is a continuous map*

$$F : A \cap \overline{\mathbf{B}}^n_{(0,r')} \times [0,1] \to A \cap \overline{\mathbf{B}}^n_{(0,r')}$$

such that $F(x,0) = x$, $F(x,1) \in A \cap \overline{\mathbf{B}}^n_{(0,r)}$, $F|_{x\in A\cap\overline{\mathbf{B}}^n_{(0,r)}} = x$ and $F|_{S\times[0,1]} \subset S$, $\forall\, S \in \Sigma$.

Moreover,

$$\|F(x,t) - x\| \leqslant 2t|r' - r|.$$

Proof. — Without loss of generality we can assume x_0 is the origin.

Let $\rho : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \|x\|$ be the distance function to the origin. Choose $r_0 > 0$ such that for every $0 < r \leqslant r_0$, $\mathbf{S}^{n-1}_{(0,r)}$ is transverse to every stratum of Σ. Put $X := A \cap \mathbf{B}^n_{(0,r_0)} \setminus \{0\}$. It easy to see that the collection $\Sigma' := \{\mathbf{B}^n_{(0,r_0)} \cap S, S \in \Sigma \setminus \{0\}\}$ is a Whitney stratification of X. Moreover, the restriction of ρ to each stratum of Σ' is a submersion.

For $x \in S$, $S \in \Sigma'$, we define $v(x) := P_x(\nabla_x \rho)$ where $\nabla_x \rho$ is the gradient of ρ and $P_x : \mathbb{R}^n \to T_x S$ is the orthogonal projection from $\mathbb{R}^n$ onto the tangent space at x of S. Since $\rho|_S$ is a submersion, $v(x) \neq 0$ for every $x \neq 0$. Since $\nabla_x \rho$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$ and Σ' is a C^2 stratification, v is a C^2 stratified vector field on Σ'. We claim that given an $\varepsilon > 0$, shrinking r_0 if necessary, we have $\|v(x) - \nabla_x \rho\| \leqslant \varepsilon$, $\forall\, x \in X$. Indeed, if the claim is not true, by Curve Selection ([8, Chap. 5]), there exist an $\varepsilon > 0$, a stratum $S \in \Sigma'$ and a C^1 $\mathcal{D}$-curve $\gamma : (0,1) \to S$, $\lim_{t\to 0} \gamma(t) = 0$ such that for every $t \in (0,1)$,

$$\|v(\gamma(t)) - \nabla_{\gamma(t)} \rho\| > \varepsilon.$$

Since $\nabla_x \rho = \frac{x}{\|x\|}$ and γ is a C^1 curve through the origin, the angle between $\nabla_{\gamma(t)} \rho$ and the tangent line at $\gamma(t)$ of γ, denoted $(\widehat{\nabla_{\gamma(t)} \rho, T_{\gamma(t)} \gamma})$, tends to 0 when t tends to 0. This implies that the angle $(\widehat{\nabla_{\gamma(t)} \rho, T_{\gamma(t)} S})$, and hence $\|v(\gamma(t)) - \nabla_{\gamma(t)} \rho\|$, tends to 0 when t tends to 0, which is a contradiction.

Now we consider Σ' as a filtration of skeletons $X = S^{d_X} \supseteq S^{d_X - 1} \supseteq \cdots \supseteq S^l$, where d_X is the dimension of X. Notice that in general the vector field v is not continuous though its restriction to each stratum of Σ' is continuous. First, we construct a continuous controlled vector field w on X by induction on skeletons such that

$$\|w(x) - \nabla_x \rho\| \leqslant c\varepsilon,$$

where c stands for some constant.

Fix a tubular neighborhood for the stratification Σ'. On the smallest skeleton S^l we take $\mu := v$ which is a C^1 vector field satisfying $\|v(x) - \nabla_x \rho\| \leqslant \varepsilon$. Write $S^{d_X} = \mathring{S}^{d_X} \cup S^{d_X - 1}$ where $\mathring{S}^{d_X}$ is the union of the strata of dimension d_X of the stratification. By the inductive hypothesis, μ is a continuous stratified vector field on S^{d_X-1} controlled by the given tubular neighborhood and satisfying $\|\mu(x) - \nabla_x \rho\| \leqslant c\varepsilon$. It was proved in [25] (see also [26]) that μ can be extended to a continuous stratified vector field on S^{d_X} controlled by the tubular neighborhood. We use the same notation μ for this extension. Since μ and $\nabla_x \rho$ both are continuous on S^{d_X}, for each point $y \in S^{d_X-1}$ we can choose a neighborhood U_y in $\mathbb{R}^n$ such that for any $x \in S^{d_X} \cap U_y$, we have $\|\mu(x) - \mu(y)\| \leqslant \varepsilon$ and

$\|\nabla_x\rho - \nabla_y\rho\| \leqslant \varepsilon$. The union $\bigcup_{y\in S^{d_X-1}} U_y$ is an open neighborhood of S^{d_X-1} in $\mathbb{R}^n$. Put $T := \bigcup_{y\in S^{d_X-1}}(U_y \cap \mathring{S}^{d_X})$. Clearly, T is an open subset of $\mathring{S}^{d_X}$ and $\overline{T} \supset S^{d_X-1}$. We call such a set an *open neighborhood of S^{d_X-1} in $\mathring{S}^{d_X}$*. Let T' be another open neighborhood of S^{d_X-1} in $\mathring{S}^{d_X}$ such that the closure of T' in $\mathring{S}^{d_X}$ is contained in T. Let $\{g_1, g_2\}$ be a C^2 partition of unity subordinate to $\{\mathring{S}^{d_X} \setminus T', T\}$ and define

$$w(x) := \begin{cases} \mu(x), & x \in S^{d_X-1} \\ g_1(x)v(x) + g_2(x)\mu(x), & x \in \mathring{S}^{d_X}. \end{cases}$$

It is clear that w is a continuous controlled vector field (shrinking the given tubular neighborhood a little bit if necessary). Thus, it is integrable.

Now we show that $\|w(x) - \nabla_x\rho\| \leqslant c\varepsilon$. It suffices to check that the formula holds for every $x \in T$ since, outside T, $w(x) = \mu(x)$ if $x \in S^{d_X-1}$ and $w(x) = v(x)$ if $x \in \mathring{S}^{d_X} \setminus T$, and then the formula holds obviously.

Let $x \in T$. By the construction of T, there is $y \in S^{d_X-1}$ such that $\|\mu(x) - \mu(y)\| \leqslant \varepsilon$ and $\|\nabla_x\rho - \nabla_y\rho\| \leqslant \varepsilon$. Hence,

$$\begin{aligned} \|w(x) - \nabla_x\rho\| &= \|g_1(v(x) - \nabla_x\rho) + g_2(\mu(x) - \nabla_x\rho)\| \leqslant \varepsilon + \|\mu(x) - \nabla_x\rho\| \\ &\leqslant \varepsilon + \|\mu(x) - \mu(y)\| + \|\mu(y) - \nabla_y\rho\| + \|\nabla_y\rho - \nabla_x\rho\| \\ &\leqslant \varepsilon + \varepsilon + c\varepsilon + \varepsilon = (3 + c)\varepsilon. \end{aligned}$$

Since w and $\nabla_x\rho$ are continuous and $\|w(x) - \nabla_x\rho\| \leqslant c\varepsilon$ and $\|\nabla_x\rho\| = 1$, we have

$$\xi(x) := \frac{-w(x)}{\langle \nabla_x\rho, w(x)\rangle}$$

is well-defined on X. Moreover, we can choose ε small enough such that $\|\xi(x)\| < 2$.

Because w is integrable, so is ξ. Write $\Phi(x,t)$ as the flow generated by the vector field ξ and denote by $d\rho$ the tangent map of ρ. Notice that $d\rho(\xi(x)) = -1$. Therefore, if $x \in X \cap \overline{\mathbf{B}}^n_{(0,r)}$ then $\Phi(x,s) \in X \cap \overline{\mathbf{B}}^n_{(0,r-s)}$ for all $r < r_0$. The map defined as follows

$$F(x,t) := \begin{cases} \Phi(x, t(\|x\| - r)), & \text{if } x \in A \cap \left(\overline{\mathbf{B}}^n_{(0,r')} \setminus \overline{\mathbf{B}}^n_{(0,r)}\right) \\ x, & \text{if } x \in A \cap \overline{\mathbf{B}}^n_{(0,r)} \end{cases}$$

is the desired deformation retract. □

Remark 3.3. — As in the proof, for any constant $C > 1$, we can choose ε small enough such that $\|\xi\| < C$.

4. Lipschitz–Killing curvatures on fibres

Let $A \subset \mathbb{R}^n$ be a closed $\mathcal{D}$-set. For $0 \leqslant k \leqslant n$ and $0 \leqslant l \leqslant n-k$, we denote

$$E := \Big\{(z,t,x,P,r) \in \mathbb{R}^{n-k} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{G}^l_{n-k} \times \mathbb{R} : \\ x \in P, z \in \pi_P^{-1}(x) \cap A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big\},$$

where π_P denotes the orthogonal projection from $\mathbb{R}^{n-k}$ to P.

It is obvious that E is a $\mathcal{D}$-set. Consider E as a family of $\mathcal{D}$-sets parametrized in the variable (t,x,P,r). Let p be the orthogonal projection from $\mathbb{R}^{n-k} \times \mathbb{R}^k \times \mathbb{R}^{n-k} \times \mathbb{G}^l_{n-k} \times \mathbb{R}$ onto the parameter space. By Hardt's triviality theorem ([8, Chap. 9]), there is a finite definable partition $\mathfrak{F}$ of the parameter space such that (p,E) is definably trivial over elements of $\mathfrak{F}$. This implies that for each $\sigma_i \in \mathfrak{F}$ and for (t,x,P,r) and (t',x',P',r') in σ_i, $E_{t,x,P,r}$ and $E_{t',x',P',r'}$ are homeomorphic. Therefore, $\chi(E_{t,x,P,r}) = \chi(\pi_P^{-1}(x) \cap A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)})$ is a finite constant on each $\sigma_i \in \mathfrak{F}$. Since $\mathfrak{F}$ has finite many elements, there exists $N \in \mathbb{N}$ such that $|\chi(\pi_P^{-1}(x) \cap A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)})| \leqslant N$ for every (t,x,P,r).

For $P \in \mathbb{G}^l_{n-k}$, we decompose P into $\mathcal{D}$-sets as follows

$$K^P_{l,j}\Big(A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big) := \Big\{x \in P : \chi\Big(\pi_P^{-1}(x) \cap A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big) = j\Big\}, \tag{4.1}$$

where $j \in \{-N, \dots, N\}$. The formula (1.1) then becomes

$$\Lambda_l\Big(A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big) = c(n-k,l) \sum_{j=-N}^{N} j \int_{P \in \mathbb{G}^l_{n-k}} \mathcal{H}^l\Big(K^P_{l,j}\Big(A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big)\Big) \mathrm{d}P. \tag{4.2}$$

Throughout this section, we assume that A is a $\mathcal{D}$-set, Σ is a stratification of A, Y is a stratum of Σ. We also suppose that

($\star$) $0 \in Y \subset \{0\}^{n-k} \times \mathbb{R}^k$ for some $k \leqslant n$,

($\star\star$) $A = A|_Y$ (consider $\mathbb{R}^k$ as the parameter set).

By abuse of notation we identity $\mathbb{R}^k$ with $\{0\}^{n-k} \times \mathbb{R}^k$. We denote by $\pi : \mathbb{R}^n \to \mathbb{R}^k$ the orthogonal projection onto the last k coordinates.

Lemma 4.1. — *Let V be a C^2 submanifold of Y. If Σ is a Whitney (resp. (w)-regular) stratification, then*

$$\Sigma' := \{X|_V\}_{X \in \Sigma}$$

is a Whitney (resp. (w)-regular) stratification of $A|_V$ in a neighborhood of the origin.

Proof. — We may write $\Sigma = \{Y, X_1, \dots, X_m\}$ with $Y \subset \overline{X_i} \setminus X_i$, $i = 1, \dots, m$.

Case 1. — Assume $Y = \mathbb{R}^l \subset \mathbb{R}^k$.

Since $A = A|_Y$, we have $A|_V = A \cap \pi^{-1}(\mathbb{R}^{k-l} \oplus V)$. Similarly, for $i = 1, \dots, m$, $X_i|_V = X_i \cap \pi^{-1}(\mathbb{R}^{k-l} \oplus V)$.

Notice that Y is transverse to $\pi^{-1}(\mathbb{R}^{k-l} \oplus V)$. Since Σ satisfies Whitney (b) condition, X_i is transverse to $\pi^{-1}(\mathbb{R}^{k-l} \oplus V)$ and their intersection $X_i \cap \pi^{-1}(\mathbb{R}^{k-l} \oplus V)$ is nonempty in a neighborhood of V. This shows that Σ' is a Whitney (resp. (w)-regular) stratification (see [24]).

General case. — By a linear change of coordinates, we may assume that $T_0 Y = \mathbb{R}^l$ where $l = \dim Y$. In a neighborhood of the origin we can consider Y as the graph of a C^2 map $\varphi : \mathbb{R}^l \to \mathbb{R}^{k-l}$ with $D\varphi(0) = 0$.

Now we write $(x_1, x_2, x_3) \in \mathbb{R}^{n-k} \times \mathbb{R}^{k-l} \times \mathbb{R}^l$ and define a map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ by $\psi(x_1, x_2, x_3) := (x_1, x_2 - \varphi(x_3), x_3)$. It is obvious that ψ is a C^2 diffeomorphism at the origin and $\psi(X_i|_V) = \psi(X_i)|_{\psi(V)}$. It is known that the conditions (b) and (w) are invariant under C^2 diffeomorphisms. Thus, to prove that Σ' is a Whitney (resp. (w)) stratification it is enough to show that

$$\psi(\Sigma') = \{\psi(V), \psi(X_1)|_{\psi(V)}, \dots, \psi(X_m)|_{\psi(V)}\}$$

is a Whitney (resp. (w)-regular) stratification.

Since $\psi(\Sigma) = \{\psi(Y) = \mathbb{R}^l, \psi(X_1), \dots, \psi(X_m)\}$ is a Whitney (resp. (w)-regular) stratification and $\psi(\Sigma')$ is the restriction of $\psi(\Sigma)$ onto the C^2 manifold $\psi(V)$, it must be a Whitney (resp. (w)-regular) stratification by the case 1. □

4.1. Whitney condition (b)

In this part we assume that Σ is a Whitney stratification and $\mathcal{D}$ is a polynomially bounded o-minimal structure.

Lemma 4.2. — *Let $\gamma(-\varepsilon, \varepsilon)$ be a C^2 $\mathcal{D}$-curve in Y such that $\gamma(0) = 0$. Then, there exist $\nu > 0$, $0 \leqslant a < 1$, $c > 0$, $r^* > 0$ and a germ of homeomorphism*

$$h : A|_{\gamma([0,\nu])} \to A_0 \times \gamma([0, \nu]), \quad h(q, t) = (h_t(q), t),$$

such that $\forall\, t \in \gamma([0,\nu])$, *the map* h_t *and* h_t^{-1} *are well-defined inside* $\overline{\mathbf{B}}^{n-k}_{(0,r^*)}$, *and*

$$\|h_t(q) - q\| \leqslant c\|t\|^{1-a} r$$
$$\|h_t^{-1}(q) - q\| \leqslant c\|t\|^{1-a} r$$

$\forall\, r < r^*$, $\forall\, q \in \overline{\mathbf{B}}^{n-k}_{(0,r)}$.

Proof. — We may write $\Sigma = \{Y, X_1, \dots, X_m\}$ with $Y \subset \overline{X_i} \setminus X_i$, $\forall\, i \in \{1, \dots, m\}$. Set $Y' := \gamma(-\varepsilon, \varepsilon)$. It follows from Lemma 4.1 that in a neighborhood of the origin

$$\Sigma' := \{Y', X_1|_{Y'}, \dots, X_m|_{Y'}\}$$

is a Whitney stratification of $A|_{Y'}$.

Since $\mathcal{D}$ is polynomially bounded and $\dim Y' = 1$, Σ' is (r)-regular along Y' (see [24, 29]). As proven in [24, Prop. 2.6], there are an open neighborhood $\mathbb{R}^n \supset U$ of the origin and an $0 \leqslant a < 1$ such that

$$\delta(T_{\pi(x)}Y', T_x(X_i|_{Y'}))\|\pi(x)\|^a \lesssim \|x - \pi(x)\|, \tag{4.3}$$

for every $x \in X_i|_{Y'} \cap U$. Recall that $Y \subset \mathbb{R}^k$ and π is the orthogonal projection from $\mathbb{R}^n$ onto $\mathbb{R}^k$.

For a point $y = \gamma(s) \in Y'$, we define $\mu(y) := \gamma'(s)/\|\gamma'(s)\|$. For $x \in U \cap X_i|_{Y'}$, set $w(x) := P_x\left(\mu(\pi(x))\right) \in T_x(X_i|_{Y'})$, where P_x denotes the orthogonal projection from $\mathbb{R}^n$ onto the tangent space $T_x(X_i|_{Y'})$. Notice that $w(x) = \mu(x)$ for every $x \in Y'$. It follows from (4.3) that

$$\|w(x) - w(\pi(x))\|\|\pi(x)\|^a \lesssim \|x - \pi(x)\|. \tag{4.4}$$

Since Σ' satisfies the Whitney condition (a), we can shrink U to make the angle between $T_{\pi(x)}Y'$ and $T_x(X_i|_{Y'})$ arbitrarily small for all $x \in U \cap X_i|_{Y'}$. This implies that for every $x \in U \cap X_i|_{Y'}$, $w(x)$ and $\pi(w(x))$ are bounded below away from 0. Thus, the vector field

$$v(x) := \frac{w(x)}{\|\pi(w(x))\|},$$

is well-defined, satisfying (4.4) and

$$\pi(v(x)) = v(\pi(x)) = \mu(\pi(x)). \tag{4.5}$$

It has been shown in the proof of Theorem 3.1 of [24] that the properties (4.4) and (4.5) keep the integral curves generated by v originating at points outside Y' from touching Y'. However, these curves might touch other strata of the stratification Σ'. In other words, the vector field v is not integrable in general.

We are going to deform v slightly to obtain an integrable vector field such that properties (4.4) and (4.5) still hold. The process is inductive "skeleton by skeleton".

Put $\Sigma'' := \Sigma' \setminus \{Y'\}$, which is a stratification of $A|_{Y'} \setminus Y'$, and fix a tubular neighborhood for this stratification. Write Σ'' as a filtration of skeletons

$$A|_{Y'} \setminus Y' = S^d \supseteq S^{d-1} \supseteq \cdots \supseteq S^l.$$

For S^l we take $v' := v$. We assume inductively that v' is a continuous controlled vector field on S^{d-1} satisfying (4.4) and (4.5). Extend v' to a continuous controlled vector field on S^d (this is possible due to [25, 26]) and use the same notation v' for this extension. Since this vector field is controlled, it is integrable. Set

$$v''(x) := \frac{v'(x)}{\|\pi(v'(x)\|}.$$

Because v' is continuous and integrable so is v'' even though it might not be controlled. For each point $y \in S^{d-1}$ we can easily choose a neighborhood U_y in $\mathbb{R}^n$ such that for every x in $U_y \cap \mathring{S}^d$,

$$\|v''(x) - v''(\pi(x))\| \|\pi(x)\|^a \lesssim \|y - \pi(y)\|$$

and

$$\|y - \pi(y)\| \sim \|x - \pi(x)\|.$$

The above conditions implies that

$$\|v''(x) - v''(\pi(x))\| \|\pi(x)\|^a \lesssim \|x - \pi(x)\|, \quad \forall\, x \in U_y \cap \mathring{S}^d.$$

Set $T := \bigcup_y U_y \cap \mathring{S}^d$. The set T is an open subset of $\mathring{S}^d$ and its closure contains S^{d-1}, we call such a set an open neighborhood of S^{d-1} in $\mathring{S}^d$. Remark that the restriction of v'' to T satisfies properties (4.4) and (4.5).

Now, on $\mathring{S}^d$ we shall glue vector field v'' and v together by using a partition of unity. For simplicity, we assume that $\mathring{S}^d$ has one connected component. Let T' be another open neighborhood of S^{d-1} in $\mathring{S}^d$ such that its closure in $\mathring{S}^d$ is contained in T. Let $\{g_1, g_2\}$ be a C^2 partition of unity subordinate to $\{\mathring{S}^d \setminus T', T\}$ and define

$$\xi(x) := \begin{cases} v''(x), & x \in S^{d-1} \\ g_1(x)v(x) + g_2(x)v''(x), & x \in \mathring{S}^d. \end{cases}$$

Since v and v'' both satisfy properties (4.4) and (4.5), so does ξ. Notice that ξ is a continuous stratified vector field and it is also integrable since it is formed by gluing smoothly integrable stratified vector fields. We then put

$$\vartheta(x) := \begin{cases} \xi(x), & x \in A|_{Y'} \setminus Y' \\ \upsilon(x), & x \in Y'. \end{cases}$$

Next, we show that integral curves generated by ϑ through points outside Y' do not touch Y' by using the same arguments as in [24]. Since ξ is integrable, these curves also do not touch other strata, thus ϑ is integrable, and hence the flow generated by ϑ gives a homeomorphism between fibres of A along Y'.

Let us denote by $\Phi(x,s) := \Phi_x(s)$ the flow generated by the vector field ϑ. Write

$$\Phi_x(s) = (\Phi_x^1(s), \Phi_x^2(s)) \in \mathbb{R}^{n-k} \times \mathbb{R}^k.$$

By shrinking U if necessary, we can reparametrize γ by the distance to the origin, i.e. $\|\gamma(s)\| = s$ for every s. Choose $\nu > 0$ such that $\gamma([0,\nu]) \subset U$. Since U is open and $\gamma([0,\nu])$ is closed, there exists $R > 0$ such that $\mathcal{N}(\gamma([0,\nu]), A|_{Y'}, R) \subset U$.

Let $x := (q,t) \in \mathcal{N}(\gamma([0,\nu]), A|_{Y'}, R)$. For $0 < s \leqslant \nu$, set $f(s) := \|\Phi_x(s) - \pi(\Phi_x(s))\|$. We have

$$f'(s) = \frac{\langle \Phi_x'(s) - \pi(\Phi_x'(s)), \Phi_x(s) - \pi(\Phi_x(s))\rangle}{\|\Phi_x(s) - \pi(\Phi_x(s))\|} \leqslant \|\Phi_x'(s) - \pi(\Phi_x'(s))\|.$$

Here

$$\|\Phi_x'(s) - \pi(\Phi_x'(s))\| = \|\vartheta(\Phi_x(s)) - \vartheta(\pi(\Phi_x(s)))\|.$$

Combining with (4.4), there is a constant $C > 0$ such that

$$|f'/f| \leqslant C\|\pi(\Phi_x(s))\|^{-a}.$$

Since $\pi(\Phi_x(s)) = \Phi_{\pi(x)}(s)$ and $\|\Phi_{\pi(x)}(s)\| = s$, $|f'/f| \leqslant Cs^{-a}$. Integrating with respect to s over $[0,s]$ (note that $f(0) = \|q\|$) we get

$$\exp\left(\frac{-Cs^{(1-a)}}{1-a}\right)\|q\| \leqslant f(s) \leqslant \exp\left(\frac{Cs^{1-a}}{1-a}\right)\|q\|. \tag{4.6}$$

The left inequality of (4.6) shows that the integral curve through x does not touch γ.

Now we construct the desired homeomorphism.

Set $C' := \exp\left(\frac{C\nu^{1-a}}{1-a}\right)$. Since $s \leqslant \nu$, and by the right inequality of (4.6),

$$f(s) \leqslant C'\|q\|.$$

Then,

$$\begin{aligned}
\|\Phi_x^1(s)-q\| &= \left\|\int_0^s \frac{\mathrm{d}}{\mathrm{d}u}(\Phi_x(u)-\pi(\Phi_x(u)))\,\mathrm{d}u\right\| \\
&= \left\|\int_0^s (\Phi_x'(u)-\pi(\Phi_x'(u)))\,\mathrm{d}u\right\| \\
&= \left\|\int_0^s (\vartheta(\Phi_x(u))-\vartheta(\pi(\Phi_x(u))))\,\mathrm{d}u\right\| \\
&\leqslant \int_0^s \frac{\|\Phi_x(u)-\pi(\Phi_x(u))\|}{\|\pi(\Phi_x(u))\|^a}\,\mathrm{d}u = \int_0^s f(u)u^{-a}\,\mathrm{d}u \\
&\leqslant C'\|q\|\int_0^s u^{-a}\,\mathrm{d}u = \frac{C'}{1-a}\|q\|s^{1-a} = C''\|q\|s^{1-a}, \qquad (4.7)
\end{aligned}$$

where $C''=\frac{C'}{1-a}$. This implies that $|\|\Phi_x^1(s)\|-\|q\||<C''\|q\|s^{1-a}$, and then

$$\|q\| < \frac{1}{1-C''s^{1-a}}\|\Phi_x^1(s)\|.$$

By (4.7),

$$\|\Phi_x^1(s)-q\| < \frac{C''}{1-C''s^{1-a}}\|\Phi_x^1(s)\|s^{1-a}. \qquad (4.8)$$

Again shrinking U, we assume that $C''\nu^{1-a}<1$. Set $h_t(q)=(h_t(q),t):=(\Phi^1_{(q,t)}(-\|t\|),t)$, $c:=\max\left\{C'',\frac{C''}{1-C''\nu^{1-a}}\right\}$, $r^*:=\frac{R}{1+c\nu^{1-a}}$.

It is easy to check that for every $x=(q,t)\in A_{\gamma([0,\nu])}$ such that $\|q\|\leqslant r^*$, $\|h_t(q)\|$ and $\|h_t^{-1}(q)\|$ do not exceed R, i.e. their images are inside U. This shows that the map h_t and h_t^{-1} are well-defined. Moreover, by (4.7) and (4.8) we can conclude that the map h is the desired homeomorphism. □

Proposition 4.3. — *Fix $0\leqslant l\leqslant n-k$. There exists $C>0$ such that for every $\varepsilon>0$, there exists a neighborhood U_ε of 0 in Y such that $\forall\, t\in U_\varepsilon$, $\exists\ r_{t,\varepsilon}>0$ such that $\forall\ P\in\mathbb{G}^l_{n-k}$ and for every $0<r<r_{t,\varepsilon}$, there is a $\mathcal{D}$-subset $\Delta(t,P,r,\varepsilon)$ of P with*

$$\psi\left(\Delta(t,P,r,\varepsilon),r\right)\leqslant C\varepsilon r^l$$

such that for any $x\in(P\cap\overline{\mathbf{B}}^{n-k}_{(0,r)})\setminus\Delta(t,P,r,\varepsilon)$,

$$\chi\left(\pi_P^{-1}(x)\cap A_t\cap\overline{\mathbf{B}}^{n-k}_{(0,r)}\right)=\chi\left(\pi_P^{-1}(x)\cap A_0\cap\overline{\mathbf{B}}^{n-k}_{(0,r)}\right),$$

where π_P is the orthogonal projection form $\mathbb{R}^{n-k}$ onto P.

Proof. — For $r>0$ and $t\in Y$ we denote by $\mathfrak{A}_t^r$ the collection

$$\left\{S_t\cap\mathbf{B}^{n-k}_{(0,r)},S_t\cap\mathbf{S}^{n-k-1}_{(0,r)}\right\}_{S\in\Sigma}.$$

For each stratum $S \in \Sigma$ and for each $\sigma \in \{0,1,2,3\}$ we define

$$\nabla^{0,\sigma}_{S,Int} := \Big\{(z,t,x,P,r) \in \mathbb{R}^{n-k} \times Y \times \mathbb{R}^{n-k} \times \mathbb{G}^l_{n-k} \times (0,1) : \\ x \in P, z \in \pi_P^{-1}(x) \cap A_0 \cap \mathbf{B}^{n-k}_{(0,r+\sigma\varepsilon r)}\Big\},$$

$$\nabla^{1,\sigma}_{S,Int} := \Big\{(z,t,x,P,r) \in \mathbb{R}^{n-k} \times Y \times \mathbb{R}^{n-k} \times \mathbb{G}^l_{n-k} \times (0,1) : \\ x \in P, z \in \pi_P^{-1}(x) \cap A_t \cap \mathbf{B}^{n-k}_{(0,r+\sigma\varepsilon r)}\Big\},$$

$$\nabla^{0,\sigma}_{S,bd} := \Big\{(z,t,x,P,r) \in \mathbb{R}^{n-k} \times Y \times \mathbb{R}^{n-k} \times \mathbb{G}^l_{n-k} \times (0,1) : \\ x \in P, z \in \pi_P^{-1}(x) \cap A_0 \cap \mathbf{S}^{n-k-1}_{(0,r+\sigma\varepsilon r)}\Big\},$$

$$\nabla^{1,\sigma}_{S,bd} := \Big\{(z,t,x,P,r) \in \mathbb{R}^{n-k} \times Y \times \mathbb{R}^{n-k} \times \mathbb{G}^l_{n-k} \times (0,1) : \\ x \in P, z \in \pi_P^{-1}(x) \cap A_t \cap \mathbf{S}^{n-k-1}_{(0,r+\sigma\varepsilon r)}\Big\}.$$

Observe that $\mathfrak{B} := \big\{\nabla^{0,\sigma}_{S,Int}, \nabla^{1,\sigma}_{S,Int}, \nabla^{0,\sigma}_{S,bd}, \nabla^{1,\sigma}_{S,bd}\big\}^{\sigma=0,\ldots,3}_{S\in\Sigma}$ is a finite collection of $\mathcal{D}$-sets. Consider these sets as families of subsets of $\mathbb{R}^{n-k}$ parametrized in the variable (t,x,P,r). Let p denote the orthogonal projection from $\mathbb{R}^{n-k} \times Y \times \mathbb{R}^{n-k} \times \mathbb{G}^l_{n-k} \times (0,1)$ to the parameter space. By Hardt's triviality theorem ([8, Chap. 9]), there exists a finite definable partition $\mathfrak{D}$ of the parameter space such that $(p, \mathfrak{B})$ is definably trivial over each element of $\mathfrak{D}$. For (t,P,r) we put $\mathfrak{C}(t,P,r) := P \cap \mathfrak{D}_{t,P,r}$, the set of intersections between P and elements of $\mathfrak{D}_{t,P,r}$. Then $\mathfrak{C}(t,P,r)$ is a partition of P and $(\pi_P, \{\mathfrak{A}_0^{r+\sigma\varepsilon}, \mathfrak{A}_t^{r+\sigma\varepsilon}\}_{\sigma=0,\ldots,3})$ is definably trivial over each its element. This trivialization is induced from the trivialization over $\mathfrak{D}$.

Denote by $\Delta^1_{r,P,t}, \ldots, \Delta^\nu_{r,P,t}$ (ν depends on (t,P,r)) the elements of dimension l of $\mathfrak{C}(t,P,r)$ and by $\partial\Delta^1_{t,P,r}, \ldots, \partial\Delta^\nu_{t,P,r}$ their corresponding topological boundaries. We may assume that ε is sufficiently small. Set

$$\Delta(t,P,r,\varepsilon) := \mathcal{N}(\cup_{i=1}^\nu \partial\Delta^i_{t,P,r}, P, 18\varepsilon r).$$

Claim 1. — $\psi(\Delta(t,P,r,\varepsilon) \leqslant C\varepsilon r^l$ for some $C>0$ independent of (t,P,r,ε).

Let us give a proof for the claim 1. Consider the set

$$H := \{(t,x,P,r) \in Y \times \mathbb{R}^{n-k} \times \mathbb{G}^l_{n-k} \times (0,1) : x \in P, \dim D_{t,P,r}(x) = l\}$$

where $D_{t,P,r}(x)$ denotes the element of $\mathfrak{D}_{t,P,r}$ containing x. It is clear that H is a $\mathcal{D}$-set and $H_{t,P,r}$ is the union of l-dimensional elements of $\mathfrak{C}(t,P,r)$. Set

$$H' := \{(t,y,P,r) \in Y \times \mathbb{R}^{n-k} \times \mathbb{G}^l_{n-k} \times (0,1) : y \in \partial H_{t,P,r}\}.$$

Obviously, $\dim H'_{t,P,r} < l$ and $\mathcal{N}(H'_{t,P,r}, P, 18\varepsilon r) = \Delta(t, P, r, \varepsilon)$. Applying Proposition 3.1(2) to the family H' (identify P with $\mathbb{R}^l$) we get a constant C independent of (t, P, r, ε) such that

$$\psi(\mathcal{N}(H'_{t,P,r}, P, 18\varepsilon r), r) \leqslant C\varepsilon r^l.$$

This ends the proof of the claim.

Since Σ is a Whitney stratification, for each $t \in Y$ there is a $\bar{r}_t > 0$ such that for all $0 < r \leqslant \bar{r}_t$, $\mathfrak{A}_0^r$ and $\mathfrak{A}_t^r$ are Whitney stratifications of $A_0 \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}$ and $A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}$ respectively.

It has been shown in the proof of Lemma 3.2 that for each $t \in Y$ there is an integrable stratified vector field ξ_t on A_t such that $d\rho(\xi_t) = -1$ where ρ is the distance function to the origin, and for $0 < r < r' < \bar{r}_t$ the flow of this vector field, denoted Φ_t, provides a deformation retract

$$F_t^{r,r'} : A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r')} \times [0,1] \to A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r')}$$

from $A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r')}$ onto $A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}$ satisfying

$$\|F_t^{r,r'}(q,s) - F_t^{r,r'}(q,0)\| \leqslant 2s|r' - r|.$$

For $P \in \mathbb{G}^l_{n-k}$, $x \in P$ and $\lambda \geqslant 0$ we define

$$A_t^r(x,\lambda) := A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)} \cap \pi_P^{-1}(\overline{\mathbf{B}}^{n-k}_{(x,\lambda)} \cap P).$$

Claim 2. — It is possible to choose $\bar{r}_t$ small enough such that for any $r < \bar{r}_t$ and any $x \in \big(\overline{\mathbf{B}}^{n-k}_{(0,r)} \cap P\big) \setminus \Delta(t, P, r, \varepsilon)$, the homomorphisms of the homology groups induced by the following inclusion maps

(I) $U_1 := \pi_P^{-1}(x) \cap A_0 \cap \overline{\mathbf{B}}^{n-k}_{(0,r)} \hookrightarrow U_2 := A_0^{r+2\varepsilon r}(x, 4\varepsilon r)$

(II) $W_1 := A_t^{r+\varepsilon r}(x, \varepsilon r) \hookrightarrow W_2 = A_t^{r+3\varepsilon r}(x, 6\varepsilon r)$

(III) $W_3 := \pi_P^{-1}(x) \cap A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)} \hookrightarrow W_2 := A_t^{r+3\varepsilon r}(x, 6\varepsilon r)$

are isomorphisms.

We shall prove (I). The proofs for (II) and (III) are similar.

Let t be fixed. Take $\bar{r}_t$ sufficiently small such that the deformation retracts $F_0^{r,r'}$ and $F_t^{r,r'}$ are well-defined for every $r < r' < \bar{r}_t + 3\varepsilon r$.

It follows from the definition of $\Delta(t, P, r, \varepsilon)$ that for $x \in \overline{\mathbf{B}}^{n-k}_{(0,r)} \cap P \setminus \Delta(t, P, r, \varepsilon)$, there is a $j \in \{1, \dots, \nu\}$ such that $x \in \Delta^j_{t,P,r}$. Moreover, $\overline{\mathbf{B}}^{n-k}_{(x,18\varepsilon r)} \cap P \subset \Delta^j_{t,P,r}$. Since $(\pi_P, \{\mathfrak{A}_0^{r+\sigma\varepsilon r}, \mathfrak{A}_t^{r+\sigma\varepsilon r}\}_{\sigma=0,\dots,3})$ is definably trivial over $\Delta^j_{t,P,r}$, for $0 \leqslant \lambda < \lambda' \leqslant 18\varepsilon r$ and $\sigma = 0, \dots, 3$, there are two deformation retracts, the first denoted $\Psi_0^{r+\sigma\varepsilon r}(x, \lambda, \lambda')$, from $A_0^{r+\sigma\varepsilon r}(x, \lambda')$ onto $A_0^{r+\sigma\varepsilon r}(x, \lambda)$ that preserves the strata of $\mathfrak{A}_0^{r+\sigma\varepsilon r}$; the second denoted

$\Psi_t^{r+\sigma\varepsilon r}(x,\lambda,\lambda')$, from $A_t^{r+\sigma\varepsilon r}(x,\lambda')$ onto $A_t^{r+\sigma\varepsilon r}(x,\lambda)$ that preserves the strata of $\mathfrak{A}_t^{r+\sigma\varepsilon r}$.

Let $V_1 := A_0^r(x, 8\varepsilon r)$ and $V_2 := A_t^{r+2\varepsilon r}(x, 12\varepsilon r)$. Consider the flow Φ_0 on A_0. Put

$$U' := U_1 \cup \left\{\Phi_0(y,-2\varepsilon r), y \in U_1 \cap \mathbf{S}^{n-k-1}_{(0,r)}\right\} \subset A_0 \cap \overline{\mathbf{B}}^{n-k}_{(0,r+2\varepsilon r)}$$

and

$$V_1' := V_1 \cup \left\{\Phi_0(y,-2\varepsilon r), y \in V_1 \cap \mathbf{S}^{n-k-1}_{(0,r)}\right\} \subset A_0 \cap \overline{\mathbf{B}}^{n-k}_{(0,r+2\varepsilon r)}.$$

Because $\|\xi_0\| \leqslant 2$,

$$U_1' \subset U_2 \subset V_1' \subset V_2.$$

Consider the following commutative diagram of homology maps induced by the inclusion maps

$$\begin{tikzcd} H_*(U_1') \arrow[r] \arrow[dr, "\alpha_*"'] & H_*(U_2) \arrow[d] \arrow[dr, "\beta_*"] & \\ & H_*(V_1') \arrow[r] & H_*(V_2) \end{tikzcd} \tag{4.9}$$

According to the construction, U_1 and V_1 are retracts of U_1' and V_1' respectively given by the deformation retract $F_0^{r,r+2\varepsilon r}$. Moreover, U_1 is a retract of V_1 given by the deformation retract $\Psi_0^r(x,0,8\varepsilon r)$. Therefore, the homology maps induced by the inclusion maps

$$\begin{tikzcd} H_*(U_1) \arrow[r] \arrow[d] \arrow[dr] & H_*(U_1') \arrow[d, "\alpha_*"] \\ H_*(V_1) \arrow[r] & H_*(V_1') \end{tikzcd}$$

are isomorphisms. We also have that U_2 is a retract of V_2 given by the deformation retract $\Psi_0^{r+2\varepsilon r}(x, 4\varepsilon r, 12\varepsilon r)$. Thus β_* is an isomorphism. Since α_* and β_* are isomophisms, so are the homomorphisms in the diagram (4.9). This ends the proof of (I).

Now we are ready to prove the proposition.

Case 1: $\dim Y = 1$. — First, we choose a neighborhood U of 0 sufficiently small so that Lemma 4.2 holds, i.e. there are $0 \leqslant a < 1$, $c > 0$, $r^* > 0$ (all are independent of t) such that for every $t \in U$ there exists a homeomorphism $h_t : (A_t, 0) \to (A_0, 0)$ such that

$$\|h_t(q) - q\| \leqslant c\|t\|^{1-a} r$$

and

$$\|h_t^{-1}(q) - q\| \leqslant c\|t\|^{1-a} r$$

where $\|q\| \leqslant r < r^*$.

We then choose $U_\varepsilon \subset U$ small enough such that $c\|t\|^{1-a} < \varepsilon$ for all $t \in U_\varepsilon$. For $t \in U_\varepsilon$, we have

$$U_1 \subset h_t(W_1) \subset U_2 \subset h_t(W_2).$$

Let us show that $U_1 \subset h_t(W_1)$ (or $h_t^{-1}(U_1) \subset W_1$). The other cases can be done similarly. Let $q \in U_1$ and $q' := h_t^{-1}(q)$. Since $\pi_P(q) = x$ and $\|q - q'\| \leqslant \varepsilon r$, $\pi_P(q') \in \overline{\mathbf{B}}^{n-k}_{(x,\varepsilon r)} \cap P$. This implies $q' \in W_1$.

Consider the following commutative diagram induced by the inclusion maps

$$\begin{tikzcd} H_*(U_1) \arrow[r, "\iota_{1*}"] \arrow[dr, "u_*"'] & H_*(h_t(W_1)) \arrow[d, "\iota_*"] \arrow[dr, "w_*"] & \\ & H_*(U_2) \arrow[r, "\iota_{2*}"] & H_*(h_t(W_2)) \end{tikzcd} \tag{4.10}$$

Take $r_{t,\varepsilon} := \min\{\bar{r}_t, r^*\}$. Combining the Claim 2 ((I) and (II)) and the fact that h_t is a homeomorphism, we have that for every $r < r_{t,\varepsilon}$, the maps u_* and w_* are isomorphisms. Since the diagram (4.10) commutes, $\iota_*, \iota_{1*}, \iota_{2*}$ are also isomorphisms. Together with (III) in the Claim 2, we get

$$H_*(U_1) \cong H_*(h_t(W_1)) \cong H_*(h_t(W_2)) \cong H_*(W_2) \cong H_*(W_3).$$

Case 2: $\dim Y > 1$. — We define

$$\begin{aligned} \Omega := \Big\{ t \in Y :\ & \forall\, \varepsilon > 0, \exists\, b > 0, \forall\, r \in (0,b), \forall\, P \in \mathbb{G}^l_{n-k}, \\ & \forall\, x \in \overline{\mathbf{B}}^{n-k}_{(0,r)} \cap P \setminus \Delta(t,P,r,\varepsilon), \\ & \chi\left(\pi_P^{-1}(x) \cap A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\right) = \chi\Big(\pi_P^{-1}(x) \cap A_0 \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big) \Big\}. \end{aligned}$$

Since $\chi(x,t,P,r,\varepsilon) := \chi(\pi_P^{-1}(x) \cap A_t \cap \mathbf{B}^{n-k}_{(0,r)})$ is a $\mathcal{D}$-function, Ω is a $\mathcal{D}$-set. It suffices to prove that the set Ω contains a neighborhood of 0. This fact follows directly from Curve Selection ([8, Chap. 6]) and the Case 1. $\square$

Proposition 4.4. — *For every $0 \leqslant l \leqslant n-k$, the function $\Lambda^{\mathrm{loc}}_l(A_t, 0)$ is continuous in t along Y.*

Proof. — It suffices to prove that $\Lambda^{\mathrm{loc}}_l(A_t, 0)$ continuous at $t = 0$. By Proposition 4.3, with $0 \leqslant l \leqslant n$ fixed, there is $C > 0$ such that for every $\varepsilon > 0$, there is a neighborhood U_ε of the origin in Y such that for any $t \in U_\varepsilon$, there is $r_{t,\varepsilon} > 0$ such that for $0 < r < r_{t,\varepsilon}$ and $P \in \mathbb{G}^l_{n-k}$, there exists a $\mathcal{D}$-subset $\Delta(t,P,r,\varepsilon)$ of P with

$$\psi(\Delta(t,P,r,\varepsilon), r) \leqslant C\varepsilon r^l$$

such that for any $x \in (\overline{\mathbf{B}}^{n-k}_{(0,r)} \cap P) \setminus \Delta(t,P,r,\varepsilon)$:

$$\chi\Big(\pi_P^{-1}(x) \cap A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big) = \chi\Big(\pi_P^{-1}(x) \cap A_0 \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big).$$

It follows from (4.1) that there is an $N \in \mathbb{N}$ independent of (t,P,r,ε) such that for any $j \in [-N,N]$,

$$\psi\Big(K^P_{l,j}\Big(A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big) \setminus K^P_{l,j}\Big(A_0 \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big), r\Big) \leqslant C\varepsilon r^l$$

and

$$\psi\Big(K^P_{l,j}\Big(A_0 \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big) \setminus K^P_{l,j}\Big(A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big), r\Big) \leqslant C\varepsilon r^l$$

Thus, we get

$$\Big|\psi\Big(K^P_{l,j}(A_t) \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}, r\Big) - \psi\Big(K^P_{l,j}(A_0) \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}, r\Big)\Big| \leqslant C\varepsilon r^l.$$

By formula (4.2),

$$\Big|\Lambda_l\Big(A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big) - \Lambda_l\Big(A_0 \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big)\Big| \leqslant C'\varepsilon r^l.$$

where C' is a constant that depends only on (n,l,k,C).

Dividing by $\mu_l r^l$ and taking the limit when r tends to 0, we obtain

$$|\Lambda^{\mathrm{loc}}_l(A_t,0) - \Lambda^{\mathrm{loc}}_l(A_0,0)| \leqslant \frac{C'}{\mu_l}\varepsilon.$$

The theorem is proved. □

4.2. Kuo–Verdier condition (w)

In this part we assume that Σ is a (w)-regular stratification and $\mathcal{D}$ is an arbitrary o-minimal structure. We shall establish results of the same types as Lemma 4.2 and Proposition 4.3.

Lemma 4.5. — *There are a neighborhood U of 0 in Y and constants $c > 0$, $r^* > 0$ such that for every t, t' in U, there is a germ of homeomorphism*

$$h_{t,t'} : A_t \to A_{t'}$$

such that the maps $h_{t,t'}$ and $h^{-1}_{t,t'}$ are well-defined inside $\overline{\mathbf{B}}^{n-k}_{(0,r^*)}$, and

$$\|h_{t,t'}(q) - q\| \leqslant c\|t-t'\|r$$
$$\|h^{-1}_{t,t'}(q) - q\| \leqslant c\|t-t'\|r$$

$\forall\, r < r^*, \forall\, q \in \overline{\mathbf{B}}^{n-k}_{(0,r)}$.

Proof. — First, since Σ is a (w)-regular stratification, there is an open neighborhood V of 0 in $\mathbb{R}^n$ and a constant $C > 0$ such that

$$\delta(T_{\pi(x)}Y, T_x X_i) \leqslant C\|x - \pi(x)\|.$$

Take U to be an open neighborhood of 0 in Y such that $\overline{U} \subset V \cap Y$. Since Y is a C^2 manifold, shrinking U if necessary, we may assume that for any t, t' in U, there is a C^2 $\mathcal{D}$-curve $\gamma_{t,t'} : [0, \nu_{t,t'}] \to U$ in U joining t and t' such that $\gamma_{t,t'}(0) = t'$ and $\|\gamma_{t,t'}(s) - t'\| \sim s, \forall\, s \in [0, \nu_{t,t'}]$. The same arguments as in the proof of Lemma 4.2 applied to $\gamma_{t,t'}$ (with the constant $a = 0$ and consider t' as the origin), there are constants $c > 0$, $r^* > 0$ (independent of (t, t')), a germ of homeomorphism

$$h : A_{\gamma_{t,t'}} \to A_{t'} \times \gamma_{t,t'}, \quad h(q, u) = (h_u(q), u),$$

such that h_u and h_u^{-1} are well-defined on $\overline{\mathbf{B}}^{n-k}_{(0,r^*)}$, and moreover

$$\|h_u(q) - q\| \leqslant c\|u - t'\|r$$

$$\|h_u^{-1}(q) - q\| \leqslant c\|u - t'\|r,$$

$\forall\, r < r^*, \forall\, q \in \overline{\mathbf{B}}^{n-k}_{(0,r)}$. Then, the map $h_{t,t'} := h_t : A_t \to A_{t'}$ is the desired homeomorphism. □

Proposition 4.6. — *Fix $0 \leqslant l \leqslant n - k$. There exist a constant $C > 0$ and a neighborhood U of 0 in Y such that for every t and t' in U, there exists $r_{t,t'} > 0$ such that for every $P \in \mathbb{G}^l_{n-k}$ and $0 < r \leqslant r_{t,t'}$, there is a $\mathcal{D}$-subset $\Delta(P, r, t, t')$ of P with*

$$\psi\left((\Delta(P, r, t, t')), r\right) \leqslant C\|t - t'\|r^l$$

such that for any $x \in (\overline{\mathbf{B}}^{n-k}_{(0,r)} \cap P) \setminus \Delta(P, r, t, t')$,

$$\chi\Big(\pi_P^{-1}(x) \cap A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big) = \chi\Big(\pi_P^{-1}(x) \cap A_{t'} \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}\Big).$$

Proof. — Choose a neighborhood U of 0 in Y sufficiently small so that Lemma 4.5 holds, i.e. there exist $c > 0$, $r^* > 0$ such that for every t and t' in U, there is a germ of homeomorphism $h_{t,t'} : A_t \to A_{t'}$ such that

$$\|h_{t,t'}(q) - q\| \leqslant c\|t - t'\|r, \quad \forall\, r < r^*, \forall\, q \in A_t \cap \overline{\mathbf{B}}^{n-k}_{(0,r)}.$$

Applying the same arguments as in the proof of case 1 in Proposition 4.3 (just replace ε with $\|t - t'\|$ and consider t' as the origin) we obtain the desired result. □

Using Proposition 4.6 and arguments as in proof of Proposition 4.4 we get:

Proposition 4.7. — *For every $0 \leqslant l \leqslant n-k$, the function $\Lambda_l^{\mathrm{loc}}(A_t, 0)$ is locally Lipschitz in t along Y.*

5. Continuity of local Lipschitz–Killing curvatures

Theorem 5.1 (Main Theorem). — *Let $A \subset \mathbb{R}^n$ be a closed $\mathcal{D}$-set. Let Σ be a stratification of A. We have*

(1) *If $\mathcal{D}$ is polynomially bounded o-minimal structure and Σ is a Whitney stratification, then $\Lambda_*^{\mathrm{loc}}(A, \cdot\,)$ is continuous along the strata of Σ.*

(2) *If Σ is a (w)-regular stratification, then $\Lambda_*^{\mathrm{loc}}(A, \cdot\,)$ is locally Lipschitz along the strata of Σ.*

Proof. — Fix $l \in \{0, \dots, n\}$. Assume that Y is a stratum of Σ. Define

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (x,t) \mapsto \varphi(x,t) := (x-t, t).$$

Set $\mathcal{A} := \varphi(A \times Y)$ and consider $\mathcal{A}$ as a family of $\mathcal{D}$-subsets of $\mathbb{R}^n$ parametrized by $\mathbb{R}^n$. We can consider the germ of A at $t \in Y$ as the germ of $\mathcal{A}_t$ at 0. This implies that

$$\Lambda_l^{\mathrm{loc}}(A, t) = \Lambda_l^{\mathrm{loc}}(\mathcal{A}_t, 0), \quad \forall\, t \in Y.$$

We may write $\Sigma = \{Y, X_1, \dots, X_m\}$ where $Y \subset \overline{X_i} \setminus X_i, i = 1, \dots, m$. If Σ is a Whitney (resp. (w)-regular) stratification, then

$$\Sigma^1 := \{Y \times Y, X_i \times Y\}_{i=1,\dots,m}$$

is a Whitney (resp. (w)-regular) stratification of $A \times Y$.

Set $\Delta(Y) := \{(x,x) \in \mathbb{R}^{2n}, x \in Y\}$. It is obvious that $\Delta(Y)$ is a C^2 submanifold of $Y \times Y$. Hence,

$$\Sigma^2 := \{\Delta(Y), Y \times Y \setminus \Delta(Y), X_i \times Y\}_{i=1,\dots,m}$$

is also a Whitney (resp. (w)-regular) stratification of $A \times Y$.

Since the conditions (b) and (w) are preserved under C^2 diffeomorphisms, the collection

$$\Sigma^3 := \varphi(\Sigma^2) = \{\varphi(\Delta(Y)), \varphi(Y \times Y \setminus \Delta(Y)), \varphi(X_i \times Y)\}_{i=1,\dots,m}$$

is a Whitney (resp. (w)-regular) stratification of $\mathcal{A}$.

Notice that $\varphi(\Delta(Y)) = \{0\}^n \times Y$ and $\mathcal{A}|_{\varphi(\Delta(Y))} = \mathcal{A} \subset \mathbb{R}^n \times \mathbb{R}^n$, i.e. the stratification Σ^3 satisfies the conditions $(\star)$ and $(\star\star)$ in Section 4. Applying

Proposition 4.4 (resp. Proposition 4.7) to Σ^3 we obtain that $\Lambda_l^{\mathrm{loc}}(\mathcal{A}_t, 0)$ is continuous (resp. locally Lipschitz) along $\{0\}^n \times Y$. This ends the proof. □

Remark 5.2. — In the paper we always assume the stratifications to be C^2 (i.e. their strata are C^2 manifolds). We do not know whether the results hold for C^1 stratifications.

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