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Loïc GAILLARD & Pascal LEFÈVRE

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LACUNARY MÜNTZ SPACES: ISOMORPHISMS AND CARLESON EMBEDDINGS

by Loïc GAILLARD & Pascal LEFÈVRE

Abstract. — In this paper we prove that M_Λ^p is almost isometric to ℓ^p in the canonical way when Λ is lacunary with a large ratio. On the other hand, our approach can be used to study also the Carleson measures for Müntz spaces M_Λ^p when Λ is lacunary. We give some necessary and some sufficient conditions ensuring that a Carleson embedding is bounded or compact. In the hilbertian case, the membership to Schatten classes is also studied. When Λ behaves like a geometric sequence the results are sharp, and we get some characterizations.

Résumé. — Dans cet article, nous montrons que M_Λ^p est presque isométrique à ℓ^p, et ce de façon naturelle, lorsque Λ est lacunaire avec une raison grande. Par ailleurs, notre approche permet aussi d'étudier les mesures de Carleson pour les espaces Müntz M_Λ^p lorsque Λ est lacunaire. Nous donnons des conditions nécessaires et des conditions suffisantes qui permettent d'assurer qu'un plongement de Carleson est borné ou compact. Dans le cadre hilbertien, nous étudions aussi l'appartenance de ce plongement aux classes de Schatten. Nous obtenons des caractérisations complètes lorsque Λ se comporte comme une suite géométrique.

1. Introduction

Let m be the Lebesgue measure on $[0,1]$. For $p \in [1,+\infty)$, $L^p(m) = L^p([0,1],m)$ (sometimes denoted simply by L^p when there is no ambiguity) denotes the space of complex-valued measurable functions on $[0,1]$, equipped with the norm $\|f\|_p = (\int_0^1 |f(t)|^p \mathrm{d}t)^{\frac{1}{p}}$. In the same way, $\mathcal{C} = C([0,1])$ is the space of continuous functions on $[0,1]$ equipped with the usual sup-norm. We shall also consider some positive and finite measures μ on $[0,1)$ (see the remark at the beginning of Section 2), and the associated $L^p(\mu)$ space. For a sequence $w = (w_n)_n$ of positive weights, we denote $\ell^p(w)$ the Banach space of complex sequences $(b_n)_n$ equipped with

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the norm $\|b\|_{\ell^p(w)} = (\sum_n |b_n|^p w_n)^{\frac{1}{p}}$ and the vector space c_{00} consisting on complex sequences with a finite number of non-zero terms. All along the paper, when $p \in (1,+\infty)$, we denote as usual $p' = \frac{p}{p-1}$ its conjugate exponent.

The famous Müntz theorem ([3, p. 172],[7, p. 77]) states that if $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ is an increasing sequence of non-negative real numbers, then the linear span of the monomials t^{λ_n} is dense in L^p (resp. in $\mathcal{C}$) if and only if $\sum_{n\geqslant 1} \frac{1}{\lambda_n} = +\infty$ (resp. and $\lambda_0 = 0$). We shall assume that the Müntz condition $\sum_{n\geqslant 1} \frac{1}{\lambda_n} < +\infty$ is fulfilled and we define the Müntz space M^p_Λ as the closed linear space spanned by the monomials t^{λ_n}, where $n \in \mathbb{N}$. We shall moreover assume that Λ satisfies the gap condition: $\inf_n \big(\lambda_{n+1} - \lambda_n\big) > 0$. Under this later assumption Clarkson–Erdös theorem holds [7, Thm. 6.2.3]: the functions in M^p_Λ are the functions f in L^p such that $f(t) = \sum a_n t^{\lambda_n}$ (pointwise on $[0,1)$). This gives a class of Banach spaces $M^p_\Lambda \subsetneq L^p$ of analytic functions on $(0,1)$.

In full generality, the Müntz spaces are difficult to study, but for some particular sequences Λ, we can find some interesting properties of the spaces M^p_Λ. Let us mention that lately these spaces received an increasing attention from the point of view of their geometry and operators: the monograph of Gurariy–Lusky [7], and various more or less recent papers (see for instance [1, 2, 4, 9, 10]).

We shall focus on two different questions on Müntz spaces. The first one is related to an old result: Gurariy and Macaev proved in [8] that, in L^p, the normalized sequence $((p\lambda_n + 1)^{\frac{1}{p}} t^{\lambda_n})_n$ is equivalent to the canonical basis of ℓ^p if and only if Λ is lacunary (see Theorem 2.3 below). More recently, the monograph [7] introduces the notion of quasi-lacunary sequence (see Definition 2.1 below), and states that M^p_Λ is still isomorphic to ℓ^p when Λ is quasi-lacunary. On the other hand, some recent papers discuss about Carleson measures for Müntz spaces. In [4], the authors introduced the class of sublinear measures on $[0,1)$, and proved that when Λ is quasi-lacunary, the sublinear measures are Carleson embeddings for M^1_Λ. In [10], the authors extended this result to the case $p = 2$ but only when the sequence Λ is lacunary.

In this paper, we introduce another method to study the lacunary Müntz spaces: for a weight w and a measure μ on $[0,1)$, we define $T^w_{\Lambda,\mu} : \ell^p(w) \to L^p(\mu)$ by $T^w_{\Lambda,\mu}(b) = \sum_n b_n t^{\lambda_n}$ for $b = (b_n) \in \ell^p(w)$. The operator $T^w_{\Lambda,\mu}$ depends on w, μ, p and Λ, and when it is bounded we shall denote its norm by $\big\|T^w_{\Lambda,\mu}\big\|_p$. We shall see that an estimate of $\big\|T^w_{\Lambda,\mu}\big\|_p$ can improve

Gurariy–Macaev theorem, and allows to generalize former Carleson embedding results to lacunary Müntz spaces M^p_Λ for any $p \geqslant 1$.

The paper is organized as follows: in Section 2, we specify the missing notation and some useful lemmas. The main result gives an upper bound for the approximation numbers of $T^w_{\Lambda,\mu}$ (see Proposition 2.9). In Section 3, we focus on the classical case: we fix the weight $w(p)$ defined by $w_n(p) = (p\lambda_n + 1)^{-1}$ and consider $T^{w(p)}_\Lambda = T^{w(p)}_{\Lambda,m} : \ell^p(w(p)) \to M^p_\Lambda$, the isomorphism occuring in Gurariy–Macaev theorem. For $p > 1$, we prove that $T^{w(p)}_\Lambda$ is bounded exactly when Λ is quasi-lacunary. On the other hand, when Λ is lacunary with a large ratio, we also get a sharp bound for $\big\|\big(T^{w(p)}_\Lambda\big)^{-1}\big\|_p$ (see Theorem 3.5 below). Our approach leads to an asymptotically orthogonal version of Gurariy–Macaev theorem exactly for the super-lacunary sequences. In Section 4, we apply the results of Section 2 for a positive and finite measure μ on $[0,1)$ with the weight $w_n = \lambda_n^{-1}$, in order to treat the Carleson embedding problem. When Λ is lacunary, we give an estimate of the approximation numbers of the embedding operator $i^p_\mu : M^p_\Lambda \to L^p(\mu)$. In Section 5, we focus on the compactness of i^p_μ using the same tools as in Section 4. In the case $p = 2$, this leads to some control of the Schatten norm of the Carleson embedding and some characterizations when Λ behaves like a geometric sequence.

As usual the notation $A \lesssim B$ means that there exists a constant $c > 0$ such that $A \leqslant cB$. This constant c may depend on Λ (or sometimes only on its ratio of lacunarity), on $p \dots$. We shall specify this dependence to avoid any ambiguous statement. In the same way, we shall use the notation $A \approx B$ or $A \gtrsim B$.

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2. Preliminary results

Let us first give a few words of explanation about our choice of measures on $[0,1)$. The measures involved (if considered on $[0,1]$) must satisfy $\mu(\{1\}) = 0$. Indeed, we focus either on the Lebesgue measure m (satisfying of course $m(\{1\}) = 0$) or on measures such that the Carleson embedding $M^p_\Lambda \to L^p(\mu)$ is (defined and) bounded, so that testing a sequence of monomials $g_n(t) = t^{\lambda_n}$ we must have

$$\mu(\{1\}) = \lim \|g_n\|^p_{L^p(\mu)} \lesssim \lim \|g_n\|^p_{L^p(m)} = 0.$$

Moreover, by Clarkson–Erdös theorem, the value at any point of $[0,1)$ of any function in M_Λ^p can be defined without ambiguity.

We shall need several notions of growth for increasing sequences.

Definition 2.1.

- *A sequence $u=(u_n)_n$ of positive numbers is called* lacunary *if there exists $r>1$ such that $u_{n+1}\geqslant ru_n$, for every $n\in\mathbb{N}$. We shall say that such a sequence is r-lacunary and that r is a ratio of lacunarity of this sequence.*
- *A sequence u is called* quasi-lacunary *if there is an extraction $(n_k)_k$ such that $\sup_{k\in\mathbb{N}}(n_{k+1}-n_k)<+\infty$, and $(u_{n_k})_k$ is lacunary.*
- *A sequence u is called* quasi-geometric *if there are two constants r and R such that $1<r\leqslant\frac{u_{n+1}}{u_n}\leqslant R<+\infty$, for every $n\in\mathbb{N}$. Such a sequence is lacunary.*
- *A sequence u is called* super-lacunary *if $\frac{u_{n+1}}{u_n}\longrightarrow+\infty$.*

Remark 2.2. — It is proved in [7, Prop. 7.1.3, p. 94] that a sequence is quasi-lacunary if and only if it is a finite union of lacunary sequences.

The following result is due to Gurariy and Macaev.

Theorem 2.3 ([7, Cor. 9.3.4, p. 132]). — *Let $p\in[1,+\infty)$. The following assertions are equivalent*

(1) *The sequence Λ is lacunary.*
(2) *The sequence $\left(\frac{t^{\lambda_n}}{\|t^{\lambda_n}\|_p}\right)$ in L^p is equivalent to the canonical basis of ℓ^p.*

In particular, when Λ is lacunary, we have for any $b\in c_{00}$

$$\Big\|\sum b_n t^{\lambda_n}\Big\|_p\approx\left(\sum\frac{|b_n|^p}{p\lambda_n+1}\right)^{\frac1p}$$

and the underlying constants depend on p and Λ only.

We shall recover and generalize partially this result: for a given sequence of weights $(w_n)_n$ and a positive finite measure μ on $[0,1)$, we study the boundedness of the operator

$$T^w_{\Lambda,\mu}:\left\{\begin{array}{ccc}\ell^p(w) & \longrightarrow & L^p(\mu)\\ b & \longmapsto & \sum b_n t^{\lambda_n}.\end{array}\right.$$

Example 2.4. — In the case of the Lebesgue measure $\mu=m$ and when the weights are $w_n(p)=(p\lambda_n+1)^{-1}$ or in a simpler way (if we do not care about the value of the constants) $w_n=\lambda_n^{-1}$, Theorem 2.3 states in particular that T^w_Λ and $T^{w(p)}_\Lambda$ are bounded on $\ell^p(w)$ or $\ell^p(w(p))$, when Λ is lacunary.

Remark 2.5. — In the case $p > 1$, a (rough) sufficient condition ensuring the boundedness of $T^w_{\Lambda,\mu}$ is

$$\int_{[0,1)} \left(\sum_n w_n^{-\frac{p'}{p}} t^{p'\lambda_n} \right)^{\frac{p}{p'}} \mathrm{d}\mu < \infty.$$

Indeed, this comes from the estimate

$$\begin{aligned} \left\| T^w_{\Lambda,\mu} \right\|_p &= \sup_{\substack{a \in B_{\ell^p} \\ a \in c_{00}}} \sup_{g \in B_{L^{p'}(\mu)}} \left| \int_{[0,1)} \sum_n a_n w_n^{-\frac{1}{p}} t^{\lambda_n} g(t)\, \mathrm{d}\mu \right| \\ &\leqslant \sup_{g \in B_{L^{p'}(\mu)}} \int_{[0,1)} |g(t)| \sup_{\substack{a \in B_{\ell^p} \\ a \in c_{00}}} \left| \sum_n a_n w_n^{-\frac{1}{p}} t^{\lambda_n} \right| \mathrm{d}\mu. \end{aligned}$$

For standard weights, $w_n \approx \lambda_n^{-1}$ and for a quasi-geometric sequence Λ, this condition can be reformulated with the help of Lemma 2.10 below as

$$\int_{[0,1)} \frac{1}{1-t}\, \mathrm{d}\mu \approx \int_{[0,1)} \frac{1}{1-t^{p'}}\, \mathrm{d}\mu < \infty\,.$$

Such a condition will be considered later (see Proposition 5.5 below for instance).

To get a sharper estimate, we introduce the sequence $\left(D^{w,p}_{\Lambda,\mu}(n)\right)_n$ defined for $n \in \mathbb{N}$ and $p \geqslant 1$, with a priori values in $\mathbb{R}_+ \cup \{+\infty\}$ by

$$D^{w,p}_{\Lambda,\mu}(n) = \left(\int_{[0,1)} w_n^{-\frac{1}{p}} t^{\lambda_n} \left(\sum_{k \geqslant 0} w_k^{-\frac{1}{p}} t^{\lambda_k} \right)^{p-1} \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

Proposition 2.6. — *Let $p \in [1, +\infty)$. Assume that $\left(D^{w,p}_{\Lambda,\mu}(n)\right)_n$ is a bounded sequence of real numbers. Then we have for every $b \in \ell^p(w)$,*

$$\Big\| \sum_{n \geqslant 0} b_n t^{\lambda_n} \Big\|_{L^p(\mu)} \leqslant \left(\sum_{n \geqslant 0} |b_n|^p w_n \left(D^{w,p}_{\Lambda,\mu}(n) \right)^p \right)^{\frac{1}{p}}.$$

Proof. — If $p = 1$ the result is obvious. Assume now that $p > 1$. For any $t \in [0,1)$ and $n \in \mathbb{N}$, we have

$$b_n t^{\lambda_n} = b_n w_n^{\frac{1}{pp'}} t^{\frac{\lambda_n}{p}} \times w_n^{-\frac{1}{pp'}} t^{\frac{\lambda_n}{p'}},$$

By Hölder's inequality, we get

$$\left| \sum b_n t^{\lambda_n} \right| \leqslant \left(\sum_n |b_n|^p w_n^{\frac{1}{p'}} t^{\lambda_n} \right)^{\frac{1}{p}} \left(\sum_k w_k^{-\frac{1}{p}} t^{\lambda_k} \right)^{\frac{1}{p'}}.$$

We conclude

$$\int_{[0,1)} \Big|\sum b_n t^{\lambda_n}\Big|^p \mathrm{d}\mu \leqslant \int_{[0,1)} \sum |b_n|^p w_n . w_n^{-\frac{1}{p}} t^{\lambda_n} \left(\sum_k w_k^{-\frac{1}{p}} t^{\lambda_k}\right)^{p-1} \mathrm{d}\mu$$
$$= \sum_n |b_n|^p w_n D^{w,p}_{\Lambda,\mu}(n)^p. \qquad \square$$

Assume that $\left(D^{w,p}_{\Lambda,\mu}(n)\right)_n$ is a bounded sequence of real numbers. We may define the bounded diagonal operator

$$\mathcal{D}^{w,p}_{\Lambda,\mu} : \ell^p(w) \to \ell^p(w)$$

acting on the canonical basis of $\ell^p(w)$ whose diagonal entries are the numbers $D^{w,p}_{\Lambda,\mu}(n)$. In other words $T^w_{\Lambda,\mu}$ and $\mathcal{D}^{w,p}_{\Lambda,\mu}$ are bounded, and we have

$$\forall\, b \in \ell^p(w),\ \|T^w_{\Lambda,\mu}(b)\|_{L^p(\mu)} \leqslant \|\mathcal{D}^{w,p}_{\Lambda,\mu}(b)\|_{\ell^p(w)}.$$

This gives informations about the approximation numbers of $T^w_{\Lambda,\mu}$. Let us recall some definitions.

Definition 2.7. — *For a bounded operator $S : X \to Y$ between two separable Banach spaces X, Y, the* approximation numbers *$(a_n(S))_n$ of S are defined for $n \geqslant 1$ by*

$$a_n(S) = \inf\{\|S - R\|, \mathrm{rank}(R) < n\}.$$

The essential norm *of S is defined by*

$$\|S\|_e = \inf\{\|S - K\|, K \textit{ compact}\}.$$

It is the distance from S to the compact operators.

We shall use in the sequel the following notions of operator ideals.

Definition 2.8.

- *An operator $S : X \to Y$ is* nuclear *if there is a sequence of rank-one operators (R_n) satisfying $S(x) = \sum_n R_n(x)$ for every $x \in X$ with $\sum_n \|R_n\| < +\infty$. The* nuclear norm *of S is defined as*

$$\|S\|_{\mathcal{N}} = \inf\left\{\sum_n \|R_n\|, \mathrm{rank}(R_n) = 1, \sum_n R_n = S\right\}.$$

- *An operator $S : X \to L^p(\mu)$ is* order bounded *if there exists a positive function $h \in L^p(\mu)$ such that for every $x \in B_X$ and for $\mu-$almost every $t \in \Omega$ we have $|S(x)(t)| \leqslant h(t)$.*

- *For $r > 0$ and when X, Y are Hilbert spaces, we say that a (compact) operator $S : X \to Y$ belongs to the* Schatten class $\mathcal{S}^r$ *if*
$$\sum_n \big(a_n(S)\big)^r < +\infty.$$
In this case, we define its Schatten norm *by* $\|S\|_{\mathcal{S}^r} = \big(\sum_n \big(a_n(S)\big)^r\big)^{\frac{1}{r}}$.

Recall that nuclear and Schatten class operators are always compact.

Of course, the Schatten norm is really a norm when $r \geqslant 1$. The $\mathcal{S}^2$ class is also called the class of *Hilbert–Schmidt* operators.

We shall be interested in how far from compact (the essential norm) or, on the contrary, how strongly compact (possibly Schatten in the Hilbert framework) are the Carleson embeddings.

For technical reasons, we introduce the following notation: for a bounded sequence $(u_n)_n$ in $\mathbb{R}_+$, we define $(u_N^*)_N$ the *decreasing rearrangement* of $(u_n)_n$ by
$$u_N^* = \inf_{\substack{A \subset \mathbb{N} \\ |A| = N}} \sup\{u_n, n \notin A\}.$$
We have $\lim_{N\to+\infty} u_N^* = \limsup_{n\to+\infty} u_n$.

Now, we can state,

Proposition 2.9. — *If $\big(D^{w,p}_{\Lambda,\mu}(n)\big)_n$ is a bounded sequence of real numbers, then we have*

1. $a_{N+1}(T^w_{\Lambda,\mu}) \leqslant \big(D^{w,p}_{\Lambda,\mu}\big)^*(N)$.
2. $\|T^w_{\Lambda,\mu}\|_p \leqslant \sup_{n\in\mathbb{N}} D^{w,p}_{\Lambda,\mu}(n)$.
3. $\|T^w_{\Lambda,\mu}\|_e \leqslant \limsup_{n\to+\infty} D^{w,p}_{\Lambda,\mu}(n)$.
4. $\forall\, p \geqslant 1$, $\|T^w_{\Lambda,\mu}\|_{\mathcal{N}} \leqslant \sum_{n\geqslant 0} w_n^{-\frac{1}{p}} \big\|t^{\lambda_n}\big\|_{L^p(\mu)}$.
5. *If $p = 2$, for any $r > 0$,* $\|T^w_{\Lambda,\mu}\|_{\mathcal{S}^r} \leqslant \big(\sum_{n\geqslant 0} \big(D^{w,2}_{\Lambda,\mu}(n)\big)^r\big)^{\frac{1}{r}}$.

Proof. — We first prove (1). For $n \in \mathbb{N}$, we denote $\varphi_n^* : \ell^p(w) \to \mathbb{C}$ the functional on $\ell^p(w)$ defined by $\varphi_n^*(b) = b_n$ for a sequence $b = (b_n)_n \in \ell^p(w)$. We define also $g_n \in L^p(\mu)$ by $g_n(t) = t^{\lambda_n}$. For any integer N and $A \subset \mathbb{N}$ with $|A| = N$, we have
$$a_{N+1}(T^w_{\Lambda,\mu}) \leqslant \Big\| T^w_{\Lambda,\mu} - \sum_{n\in A} \varphi_n^* \otimes g_n \Big\|.$$
By Proposition 2.6, for $b \in \ell^p(w)$,
$$\Big\| T^w_{\Lambda,\mu}(b) - \sum_{n\in A} \varphi_n^*(b) g_n \Big\| = \Big\| \sum_{n\notin A} b_n t^{\lambda_n} \Big\|_{L^p(\mu)} \leqslant \sup_{n\notin A} \big(D^{w,p}_{\Lambda,\mu}(n)\big) \|b\|_{\ell^p(w)}$$

and so (1) holds.

Assertions (2) and (3) are direct consequences of (1).

Assertion (4) follows easily from the natural decomposition $T^w_{\Lambda,\mu}(b) = \sum_n \varphi_n^*(b) g_n$ and the fact that $\|\varphi_n^*\| = w_n^{-\frac{1}{p}}$.

For (5): if $(D^{w,2}_{\Lambda,\mu}(n))_n \notin \ell^r$ then the result is obvious. If $(D^{w,2}_{\Lambda,\mu}(n))_n \in \ell^r$, we have in particular $D^{w,2}_{\Lambda,\mu}(n) \to 0$ when $n \to +\infty$. Since for all $\varepsilon > 0$, the set $\{n, D^{w,2}_{\Lambda,\mu}(n) \geqslant \varepsilon\}$ is finite, there exists a bijection $\beta : \mathbb{N} \to \mathbb{N}$ such that for any $n \in \mathbb{N}$, $D^{w,2}_{\Lambda,\mu}(n)^* = D^{w,2}_{\Lambda,\mu}(\beta(n))$. We have

$$\begin{aligned}\sum_N a_{N+1}(T^w_{\Lambda,\mu})^r \leqslant \sum_N \left(D^{w,2}_{\Lambda,\mu}(N)^*\right)^r &= \sum_n \left(D^{w,2}_{\Lambda,\mu}(\beta(n))\right)^r \\ &= \sum_n \left(D^{w,2}_{\Lambda,\mu}(n)\right)^r. \qquad \square\end{aligned}$$

Lemma 2.10. — *Let $\alpha \in \mathbb{R}_+^*$. Assume that Λ is a quasi-geometric sequence. Then there are two constants $C_1, C_2 \in \mathbb{R}_+^*$ such that for any $t \in [0,1)$ we have*

$$C_1 \left(\frac{1}{1-t}\right)^\alpha \leqslant \sum_n \lambda_n^\alpha t^{\lambda_n} \leqslant C_2 \left(\frac{1}{1-t}\right)^\alpha.$$

Proof. — Since Λ is quasi-geometric, it is r-lacunary for some $r > 1$, so there exists a constant $C = (r-1)^{-1}$ such that for any $n \in \mathbb{N}, \lambda_n \leqslant C(\lambda_{n+1} - \lambda_n)$. Moreover, there is a constant $R > 1$ such that $\lambda_{n+1} \leqslant R\lambda_n$ and hence we have

$$\lambda_n^\alpha \approx (\lambda_{n+1} - \lambda_n)^\alpha \approx \lambda_{n+1}^\alpha$$

where the underlying constants do not depend on n. We obtain

$$\begin{aligned}\sum_n \lambda_n^\alpha t^{\lambda_n} \approx \sum_n (\lambda_{n+1} - \lambda_n)^\alpha t^{\lambda_n} &\approx \sum_n \sum_{\lambda_n \leqslant m < \lambda_{n+1}} (\lambda_{n+1} - \lambda_n)^{\alpha - 1} t^{\lambda_n} \\ &\approx \sum_n \sum_{\lambda_n \leqslant m < \lambda_{n+1}} m^{\alpha-1} t^{\lambda_n}\end{aligned}$$

For m such that $\lambda_n \leqslant m < \lambda_{n+1}$, we have $t^m \leqslant t^{\lambda_n} \leqslant t^{\frac{m}{R}}$ and so we obtain

$$\sum_n \lambda_n^\alpha t^{\lambda_n} \lesssim \sum_{m \geqslant 0} m^{\alpha-1} t^{\frac{m}{R}} \lesssim \left(\frac{1}{1 - t^{\frac{1}{R}}}\right)^\alpha \lesssim \left(\frac{1}{1-t}\right)^\alpha.$$

On the other hand we have

$$\sum_n \lambda_n^\alpha t^{\lambda_n} \gtrsim \sum_{m \geqslant 1} m^{\alpha-1} t^m \gtrsim \left(\frac{1}{1-t}\right)^\alpha. \qquad \square$$

Remark 2.11. — If Λ is only lacunary, the right inequality in Lemma 2.10 still holds. Indeed, the above proof can be easily adapted. We can also notice that there exists a quasi-geometric sequence $\Lambda' = (\lambda'_n)_n$ which contains Λ, and we have

$$\sum_{n\in\mathbb{N}} \lambda_n^\alpha t^{\lambda_n} \leqslant \sum_{n\in\mathbb{N}} \lambda_n'^\alpha t^{\lambda'_n} \leqslant C_2 \frac{1}{(1-t)^\alpha}\cdot$$

A new proof of the upper bound part in Gurariy–Macaev theorem (Theorem 2.3) follows from the next proposition.

Proposition 2.12. — *Let $p \in [1, +\infty)$. Assume that the weights are given by $w_n = \lambda_n^{-1}$ or $(p\lambda_n + 1)^{-1}$. If Λ is lacunary and μ is the Lebesgue measure, then $\big(D^{w,p}_{\Lambda,\mu}(n)\big)_n$ is a bounded sequence.*

Proof. — From Lemma 2.10 and Remark 2.11 we get

$$\begin{aligned}
\big(D^{w,p}_{\Lambda,\mu}(n)\big)^p &= \lambda_n^{\frac{1}{p}} \int t^{\lambda_n} \left(\sum_{k\in\mathbb{N}} \lambda_k^{\frac{1}{p}} t^{\lambda_k}\right)^{p-1} \mathrm{d}t \\
&\lesssim \lambda_n^{\frac{1}{p}} \int_0^1 t^{\lambda_n} \left(\frac{1}{1-t}\right)^{\frac{1}{p'}} \mathrm{d}t \\
&= \lambda_n^{\frac{1}{p}} \int_0^{1-\frac{1}{\lambda_n}} t^{\lambda_n} \left(\frac{1}{1-t}\right)^{\frac{1}{p'}} \mathrm{d}t + \lambda_n^{\frac{1}{p}} \int_{1-\frac{1}{\lambda_n}}^1 t^{\lambda_n} \left(\frac{1}{1-t}\right)^{\frac{1}{p'}} \mathrm{d}t \\
&\leqslant \lambda_n^{\frac{1}{p}} \lambda_n^{\frac{1}{p'}} \int_0^1 t^{\lambda_n}\,\mathrm{d}t + \lambda_n^{\frac{1}{p}} \int_{1-\frac{1}{\lambda_n}}^1 (1-t)^{-\frac{1}{p'}} \mathrm{d}t \\
&\leqslant \frac{\lambda_n}{\lambda_n+1} + \lambda_n^{\frac{1}{p}} \frac{p}{\lambda_n^{\frac{1}{p}}} \leqslant p+1.
\end{aligned}$$

□

From Proposition 2.6, we obtain as claimed

$$\left\|\sum_{n\in\mathbb{N}} b_n t^{\lambda_n}\right\|_p \lesssim \left(\sum_{n\in\mathbb{N}} \frac{|b_n|^p}{\lambda_n}\right)^{\frac{1}{p}},$$

for any $b \in c_{00}$, when Λ is lacunary.

From Lemma 2.10 and Gurariy–Macaev's Theorem, one can easily get an estimate of point evaluations on M^p_Λ.

Proposition 2.13. — *Let Λ be a quasi-geometric sequence and $p \geqslant 1$. For any $t \in [0, 1)$, the point evaluation $f \in M^p_\Lambda \longmapsto \delta_t(f) = f(t)$ satisfies*

$$\left\|\delta_t\right\|_{(M^p_\Lambda)^*} \approx \frac{1}{(1-t)^{\frac{1}{p}}}\,.$$

When Λ is only lacunary, we only have $\left\|\delta_t\right\|_{(M^p_\Lambda)^*} \lesssim \frac{1}{(1-t)^{\frac{1}{p}}}$.

Proof. — Since Λ is in particular lacunary, Gurariy–Macaev theorem gives

$$\big\|\delta_t\big\|_{(M^p_\Lambda)^*} = \sup_{f\in B_{M^p_\Lambda}} |f(t)| \approx \sup_{a\in B_{\ell^p}} \Big|\sum_{n\geqslant 0} \lambda_n^{\frac{1}{p}} a_n t^{\lambda_n}\Big|$$

where the underlying constants depend on p and Λ. If $p>1$, the last term is $\Big(\sum_{n\geqslant 0} \lambda_n^{\frac{p'}{p}} t^{p'\lambda_n}\Big)^{\frac{1}{p'}}$ and we conclude with Lemma 2.10.

If $p=1$, we have for t close to 1 (say $t \geqslant \exp(-1/\lambda_1)$)

$$\begin{aligned}\big\|\delta_t\big\|_{(M^1_\Lambda)^*} \approx \sup_{a\in B_{\ell^1}} \Big|\sum_{n\geqslant 0} \lambda_n a_n t^{\lambda_n}\Big| &= \sup_{n\geqslant 0} \lambda_n t^{\lambda_n} \\ &\leqslant \sup_{s\geqslant \lambda_0} st^s = \frac{1}{\mathrm{e}|\ln(\mathrm{t})|} \approx \frac{1}{(1-t)}\,\cdot\end{aligned}$$

Moreover, if Λ is quasi-geometric then there exists some $\delta\in(0,1)$ (depending on Λ only) and some integer n_t such that $\lambda_{n_t} \in \big(\delta/|\ln(t)|, 1/|\ln(t)|\big)$ so that

$$\sup_{n\geqslant 0} \lambda_n t^{\lambda_n} \geqslant \lambda_{n_t} t^{\lambda_{n_t}} \geqslant \frac{\delta}{\mathrm{e}|\ln(\mathrm{t})|} \approx \frac{1}{(1-t)}\,\cdot \qquad \square$$

3. Revisiting the classical case

We consider the Lebesgue measure $\mu = m$ on $[0,1]$. We define the operator

$$T_\Lambda^{w(p)} : \left\{ \begin{array}{ccc} \ell^p(w(p)) & \longrightarrow & M^p_\Lambda \\ b & \longmapsto & \sum_n b_n t^{\lambda_n} \end{array}\right.$$

where the weights $w(p) = (w_n(p))_{n\in\mathbb{N}}$ are given by $w_n(p) = (p\lambda_n+1)^{-1} = \|t^{\lambda_n}\|_p^p$. In particular, if we denote by $(e_k)_k$ the canonical basis of $\ell^p(w(p))$, we have

$$\forall\, k\in\mathbb{N},\ \|T_\Lambda^{w(p)}(e_k)\|_p = \|e_k\|_{\ell^p(w(p))}.$$

Gurariy–Macaev theorem says that $T_\Lambda^{w(p)}$ is an isomorphism if and only if Λ is lacunary. By our Propositions 2.6 and 2.12, we recover that $T_\Lambda^{w(p)}$ is bounded when Λ is lacunary.

Since $\big\|T_\Lambda^{w(1)}\big\|_1 = 1$, we focus mainly on the case $p>1$. We shall also prove that $T_\Lambda^{w(p)}$ is bounded if and only if Λ is quasi-lacunary (for $p>1$). We shall refine the method used in Proposition 2.12 and get a sharper estimate of the norm. Our approach is different from the original one (which was based on some slicing of the interval $(0,1)$). We control the norm with explicit quantities depending only on the ratio of lacunarity (and p). As a consequence, we shall get that for $p \in (1,+\infty)$, the operator $T_\Lambda^{w(p)}$ is an asymptotical isometry if and only if Λ is super-lacunary.

Lemma 3.1. — *Let $\alpha \in (0,+\infty)$, $p \in (1,+\infty)$ and $(q_n)_n$ be an r-lacunary sequence. We have*

$$\sup_{n\in\mathbb{N}} \sum_{\substack{k\in\mathbb{N}\\ k\neq n}} \left(\frac{q_n^{\frac{1}{p}} q_k^{\frac{1}{p'}}}{\frac{q_n}{p} + \frac{q_k}{p'}} \right)^\alpha \leqslant \frac{p'^\alpha}{r^{\frac{\alpha}{p}} - 1} + \frac{p^\alpha}{r^{\frac{\alpha}{p'}} - 1}.$$

Proof. — Let $n \in \mathbb{N}$. For $k < n$, we have $\frac{q_n^{\frac{1}{p}} q_k^{\frac{1}{p'}}}{\frac{q_n}{p} + \frac{q_k}{p'}} \leqslant p \left(\frac{q_k}{q_n}\right)^{\frac{1}{p'}} \leqslant pr^{-\frac{n-k}{p'}}$. We obtain

$$\sum_{k=0}^{n-1} \left(\frac{q_n^{\frac{1}{p}} q_k^{\frac{1}{p'}}}{\frac{q_n}{p} + \frac{q_k}{p'}} \right)^\alpha \leqslant p^\alpha \sum_{k=0}^{n-1} \frac{1}{r^{\frac{(n-k)\alpha}{p'}}} \leqslant \frac{p^\alpha}{r^{\frac{\alpha}{p'}} - 1}.$$

Similarly, when $k > n$, we use $\frac{q_n^{\frac{1}{p}} q_k^{\frac{1}{p'}}}{\frac{q_n}{p} + \frac{q_k}{p'}} \leqslant p' \left(\frac{q_n}{q_k}\right)^{\frac{1}{p}} \leqslant p' r^{-\frac{k-n}{p}}$. □

For $p \in [1,+\infty)$ we consider the sequence $\left(D_{\Lambda,\mu}^{w,p}(n)\right)_n$ defined in Section 2, but since we focus on the case $\mu = m$ and $w = w(p)$, we lighten the notation and write

$$D_\Lambda^{(p)}(n) = \left(\int_0^1 (p\lambda_n + 1)^{\frac{1}{p}} t^{\lambda_n} \left(\sum_k (p\lambda_k + 1)^{\frac{1}{p}} t^{\lambda_k} \right)^{p-1} \mathrm{d}t \right)^{\frac{1}{p}}.$$

Proposition 3.2. — *Let $p \geqslant 2$ and Λ be a (lacunary) sequence such that $(p\lambda_n + 1)_n$ is r-lacunary. Then we have*

$$\|T_\Lambda^{w(p)}\|_p \leqslant \left(1 + \frac{2p^{\frac{1}{p-1}}}{r^{\frac{1}{p(p-1)}} - 1} \right)^{\frac{1}{p'}}.$$

Proof. — For $j \in \mathbb{N}$, we denote $q_j = p\lambda_j + 1$ and $f_j(t) = q_j^{\frac{1}{p}} t^{\lambda_j} = \frac{t^{\lambda_j}}{\|t^{\lambda_j}\|_p}$. We have

$$\left(D_\Lambda^{(p)}(n)\right)^p = \int_0^1 f_n \left(\sum_k f_k \right)^{p-1} \mathrm{d}t = \left\| \sum_k f_k \right\|_{L^{p-1}(f_n dt)}^{p-1}.$$

Since $p - 1 \geqslant 1$, the triangle inequality gives

$$\left(D_\Lambda^{(p)}(n)\right)^{p'} \leqslant \sum_k \|f_k\|_{L^{p-1}(f_n dt)} = \sum_k \left(q_n^{\frac{1}{p}} q_k^{\frac{1}{p'}} \int_0^1 t^{\lambda_n + (p-1)\lambda_k} \mathrm{d}t \right)^{\frac{1}{p-1}}.$$

For $n,k\in\mathbb{N}$, we have

$$q_n^{\frac{1}{p}}q_k^{\frac{1}{p'}}\int_0^1 t^{\lambda_n+(p-1)\lambda_k}\,\mathrm{d}t=\frac{q_n^{\frac{1}{p}}q_k^{\frac{1}{p'}}}{\lambda_n+(p-1)\lambda_k+1}=\frac{q_n^{\frac{1}{p}}q_k^{\frac{1}{p'}}}{\frac{q_n}{p}+\frac{q_k}{p'}}\cdot$$

By Lemma 3.1, we obtain for any $n\in\mathbb{N}$

$$\left(D_\Lambda^{(p)}(n)\right)^{p'}\leqslant\sum_{k\in\mathbb{N}}\left(\frac{q_n^{\frac{1}{p}}q_k^{\frac{1}{p'}}}{\frac{q_n}{p}+\frac{q_k}{p'}}\right)^{\frac{1}{p-1}}\leqslant 1+\frac{2p^{\frac{1}{p-1}}}{r^{\frac{1}{p(p-1)}}-1}$$

since $p\geqslant p'$ and the term indexed by $n=k$ is 1. Thanks to Proposition 2.6, we have

$$\|T_\Lambda^{w(p)}\|_p\leqslant\sup_n D_\Lambda^{(p)}(n).\qquad\square$$

Let us point out that the operators $T_\Lambda^{w(p)}:\ell^p(w(p))\to M_\Lambda^p\subset L^p(m)$ are not defined on the same scale of L^p-spaces, since the weight $w(p)$ depends on p. We cannot apply directly the Riesz–Thorin theorem with $T_\Lambda^{w(1)}$ and $T_\Lambda^{w(2)}$ to estimate the norm of $T_\Lambda^{w(p)}$ when $p\in(1,2)$, even not its weighted version. The next result gives a bound different from the one in Proposition 3.2; they coincide when $p=2$.

Proposition 3.3. — *Let $p\in[1,2]$ and Λ be a (lacunary) sequence such that $(p\lambda_n+1)_n$ is r-lacunary. Then we have*

$$\|T_\Lambda^{w(p)}\|_p\leqslant\left(1+\frac{4}{r^{\frac{1}{2}}-1}\right)^{\frac{1}{p'}}.$$

Proof. — Our proof is adapted from the classical proof of Riesz–Thorin theorem, with an additional trick.

Let $\theta=\frac{2}{p'}\in(0,1)$. We have $\frac{1}{p}=1-\frac{\theta}{2}\cdot$ As usual, for $z\in\mathbb{C}$ such that $0\leqslant\mathrm{Re}(z)\leqslant 1$, we define $\frac{1}{p(z)}=1-\frac{z}{2}$ and $\frac{1}{p'(z)}=\frac{z}{2}\cdot$ We have $p(\theta)=p$ and $p'(\theta)=p'$. We fix $a=(a_n)_n$ a sequence in $\mathbb{R}_+$ with a finite number of non-zero terms and a positive function $g\in L^{p'}$, such that $\|a\|_{\ell^p(w(p))}=\|g\|_{p'}=1$. Finally we define

$$F(z)=\sum_{n\in\mathbb{N}}a_n^{\frac{p}{p(z)}}\int_0^1 t^{\frac{p}{p(z)}\lambda_n}g(t)^{\frac{p'}{p'(z)}}\,\mathrm{d}t\ .$$

This is actually a finite sum, and F is a holomorphic function on the band $\{z\in\mathbb{C}\mid\mathrm{Re}(z)\in(0,1)\}$. For $x\in\mathbb{R}$, we have

$$|F(ix)|\leqslant\sum_{n\in\mathbb{N}}a_n^p\int_0^1 t^{p\lambda_n}\,\mathrm{d}t=\sum_{n\in\mathbb{N}}\frac{a_n^p}{p\lambda_n+1}=1\ .$$

On the other hand, for every real number x,

$$|F(1+ix)| \leqslant \sum_{n\in\mathbb{N}} a_n^{p(1-\frac{1}{2})} \int_0^1 t^{p(1-\frac{1}{2})\lambda_n} g(t)^{\frac{p'}{2}} \mathrm{d}t$$
$$= \int_0^1 g(t)^{\frac{p'}{2}} \sum_{n\in\mathbb{N}} b_n t^{\psi_n} \mathrm{d}t$$

where $b_n = a_n^{\frac{p}{2}}$ and $\Psi = (\psi_n)_n = \left(\frac{p\lambda_n}{2}\right)_n$. Since $(2\psi_n+1)_n$ is also r-lacunary we can apply Proposition 3.2. in the hilbertian case

$$\Big\|\sum_{n\in\mathbb{N}} b_n t^{\psi_n}\Big\|_2 \leqslant \left(1+\frac{4}{r^{\frac{1}{2}}-1}\right)^{\frac{1}{2}} \left(\sum_n \frac{|b_n|^2}{2\psi_n+1}\right)^{\frac{1}{2}}.$$

By Cauchy–Schwarz inequality, we get

$$|F(1+ix)| \leqslant \|g^{\frac{p'}{2}}\|_2 \times \Big\|\sum_n b_n t^{\psi_n}\Big\|_2 \leqslant \left(1+\frac{4}{r^{\frac{1}{2}}-1}\right)^{\frac{1}{2}}.$$

Now, the proof ends in a standard way and the three lines theorem gives

$$|F(\theta)| \leqslant \left(1+\frac{4}{r^{\frac{1}{2}}-1}\right)^{\frac{\theta}{2}}.$$

From this, we conclude easily that for arbitrary $a \in \ell^p(w(p))$, we have

$$\|T_\Lambda^{w(p)}(a)\|_p \leqslant \left(1+\frac{4}{r^{\frac{1}{2}}-1}\right)^{\frac{1}{p'}} \|a\|_{\ell^p(w(p))}. \qquad \square$$

Now we can give a characterization of the boundedness of $T_\Lambda^{w(p)}$.

Theorem 3.4. — *Let $p \in (1,+\infty)$. The following are equivalent*

(1) *The sequence Λ is quasi-lacunary ;*
(2) *The operator $T_\Lambda^{w(p)} : \ell^p(w(p)) \to M_\Lambda^p$ is bounded.*

Proof. — Assume that Λ is a quasi-lacunary sequence. Using Remark 2.2, there exist $K \geqslant 1$ and lacunary sets $\Lambda_j \subset \Lambda$ (with $j \in \{1,\dots,K\}$) such that $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_K$. We define the operators

$$T^{(j)} : \begin{cases} \ell^p(w(p)) & \longrightarrow & M_\Lambda^p \\ b & \longmapsto & \sum_n b_n t^{\lambda_n} \mathbb{1}_{\Lambda_j}(\lambda_n) \end{cases}$$

where $\mathbb{1}_{\Lambda_j}$ is the indicator function of the set Λ_j.

We have $T_\Lambda^{w(p)} = \sum_{j=1}^K T^{(j)}$. Since each Λ_j is lacunary, Proposition 3.2 and Proposition 3.3 (or Gurariy–Macaev theorem) imply that $T_\Lambda^{w(p)}$ is bounded.

For the converse, we assume that Λ is not quasi-lacunary. We denote $q_k = p\lambda_k + 1$. For an arbitrarily large $N \in \mathbb{N}$, the sequence q_{Nk} is not lacunary since $(Nk)_{k\in\mathbb{N}}$ has bounded gaps. This implies $\liminf_{k\to+\infty} \frac{q_{(k+1)N}}{q_{kN}} = 1$, so there exists k_0 such that it is less than 2. For $n_0 = k_0 N$ we have

$$q_{n_0+N} \leqslant 2q_{n_0}.$$

Let $A = \{n_0, \dots, n_0 + N - 1\}$. Thanks to the inequality between arithmetic and geometric means, we have

$$\|T_\Lambda^{w(p)}(\mathbb{1}_A)\|_p^p = \int_0^1 \left|\sum_{j\in A} t^{\lambda_j}\right|^p \mathrm{d}t \geqslant \int_0^1 N^p \prod_{j\in A} t^{\frac{p\lambda_j}{N}} \mathrm{d}t.$$

We obtain

$$\|T_\Lambda^{w(p)}(\mathbb{1}_A)\|_p^p \geqslant \frac{N^p}{\sum_{j\in A}\frac{q_j}{N}} \geqslant \frac{N^p}{q_{n_0+N}} \geqslant \frac{N^p}{2q_{n_0}}\cdot$$

On the other hand, $\|\mathbb{1}_A\|_{\ell^p(w(p))}^p = \sum_{j\in A}\frac{1}{q_j} \leqslant \frac{N}{q_{n_0}}\cdot$ Since N is arbitrarily large and $p > 1$, $T_\Lambda^{w(p)}$ is not bounded. ☐

The following is a refinement of Gurariy–Macaev theorem for lacunary sequences with a large ratio.

Theorem 3.5. — *Let $p > 1$. For any $\varepsilon \in (0,1)$, there exists $r_\varepsilon > 1$ with the following property:*

For any Λ such that $(p\lambda_n + 1)_n$ is r_ε-lacunary, we have

$$\forall\, a \in \ell^p(w(p)), \quad (1-\varepsilon)\|a\|_{\ell^p(w(p))} \leqslant \left\|T_\Lambda^{w(p)}(a)\right\|_p \leqslant (1+\varepsilon)\|a\|_{\ell^p(w(p))}.$$

Remark 3.6. — If we denote $q = \max\{p, p'\}$, the parameter $r_\varepsilon = \left(1 + \frac{4q^{\frac{1}{q-1}}}{\varepsilon}\right)^{q(q-1)}$ is suitable for Theorem 3.5.

Proof. — Let $q = \max\{p, p'\} \geqslant 2$ and $r_\varepsilon = \left(1 + \frac{4q^{\frac{1}{q-1}}}{\varepsilon}\right)^{q(q-1)}$. In order to lighten the computation below, we shall write ω instead of $w(p)$ so that $\omega_n = w_n(p) = \frac{1}{p\lambda_n+1}$.

Let a be a sequence with $\|a\|_{\ell^p(\omega)} = 1$. Thanks to the above choice of r_ε, we have that $\left\|T_\Lambda^\omega\right\|_p \leqslant \left(1+\frac{\varepsilon}{2}\right)^{\frac{1}{p'}} \leqslant 1 + \frac{\varepsilon}{2}$ either by Proposition 3.2 if $p \geqslant 2$, or by Proposition 3.3 if $p \leqslant 2$.

For the lower estimate, we consider a sequence $b \in \ell^{p'}(\omega)$ such that $\|b\|_{\ell^{p'}(\omega)} = 1$. We define $\Psi = (\psi_n)_n$ by $\psi_n = \frac{p\lambda_n}{p'} = (p-1)\lambda_n$. We have

$$\begin{aligned}\left\|T_\Lambda^\omega(a).T_\Psi^\omega(b)\right\|_1 &= \int_0^1 \left|\sum_{n,k} a_n b_k t^{\lambda_n+(p-1)\lambda_k}\right| \mathrm{d}t \\ &\geqslant \left|\sum_{n=0}^{+\infty} \frac{a_n b_n}{p\lambda_n+1}\right| - \sum_{\substack{n,k\in\mathbb{N}\\ k\neq n}} \frac{|a_n|.|b_k|}{\lambda_n+(p-1)\lambda_k+1}\cdot\end{aligned}$$

Since $\|a\|_{\ell^p(\omega)} = 1$, we have by duality

$$\sup\left\{\sum_n \frac{a_n b_n}{p\lambda_n+1},\ \|b\|_{\ell^{p'}(\omega)} = 1\right\} = 1.$$

Denoting $(q_n)_n = (p\lambda_n+1)_n$, Young's inequality gives for any n, k,

$$\begin{aligned}|a_n b_k| &= \left|a_n \omega_n^{\frac{1}{p}} b_k \omega_k^{\frac{1}{p'}}\right| \times q_n^{\frac{1}{p}} q_k^{\frac{1}{p'}} \\ &\leqslant \left(\frac{1}{p}|a_n|^p \omega_n + \frac{1}{p'}|b_k|^{p'}\omega_k\right) \times q_n^{\frac{1}{p}} q_k^{\frac{1}{p'}}.\end{aligned}$$

We sum over n and k, and we obtain

$$\begin{aligned}\sum_{\substack{n,k\in\mathbb{N}\\ k\neq n}} \frac{|a_n|.|b_k|}{\frac{q_n}{p}+\frac{q_k}{p'}} \leqslant \frac{1}{p}\|a\|^p_{\ell^p(\omega)} \sup_n \sum_{\substack{k\in\mathbb{N}\\ k\neq n}} \frac{q_n^{\frac{1}{p}} q_k^{\frac{1}{p'}}}{\frac{q_n}{p}+\frac{q_k}{p'}} \\ + \frac{1}{p'}\|b\|^{p'}_{\ell^{p'}(\omega)} \sup_k \sum_{\substack{n\in\mathbb{N}\\ n\neq k}} \frac{q_n^{\frac{1}{p}} q_k^{\frac{1}{p'}}}{\frac{q_n}{p}+\frac{q_k}{p'}}\cdot\end{aligned}$$

Applying Lemma 3.1, this quantity is less than $\frac{2q}{r_\varepsilon^{\frac{1}{q}}-1} \leqslant \frac{\varepsilon}{2}$ thanks to the choice of r_ε again. Finally

$$\sup\left\{\left\|T_\Lambda^\omega(a).T_\Psi^\omega(b)\right\|_1;\ \|b\|_{\ell^{p'}(\omega)} = 1\right\} \geqslant 1 - \frac{\varepsilon}{2}\cdot$$

On the other hand, Hölder's inequality gives

$$\left\|T_\Lambda^\omega(a).T_\Psi^\omega(b)\right\|_1 \leqslant \left\|T_\Lambda^\omega(a)\right\|_p.\left\|T_\Psi^\omega(b)\right\|_{p'} \leqslant \left(1+\frac{\varepsilon}{2}\right)\left\|T_\Lambda^\omega(a)\right\|_p$$

because $p'\psi_n + 1 = p\lambda_n + 1$ (hence is also an r_ε-lacunary sequence), so we may apply the upper bound part for $\left\|T_\Psi^\omega\right\|_{p'}$. Considering the supremum over the sequences b, we finally obtain, for any r_ε-lacunary Λ and for any a in the unit sphere of $\ell^p(\omega)$,

$$(1-\varepsilon) \leqslant \frac{1-\frac{1}{2}\varepsilon}{1+\frac{1}{2}\varepsilon} \leqslant \left\|T_\Lambda^\omega(a)\right\|_p \leqslant (1+\varepsilon). \qquad \square$$

Before stating the next corollary, let us recall that a (normalized) sequence (x_n) in a Banach space X is asymptotically isometric to the canonical basis of ℓ^p if for every $\varepsilon \in (0,1)$, there exists an integer N such that

$$(1-\varepsilon)\left(\sum_{n\geqslant N}|a_n|^p\right)^{\frac{1}{p}} \leqslant \Big\|\sum_{n\geqslant N} a_n x_n\Big\|_X \leqslant (1+\varepsilon)\left(\sum_{n\geqslant N}|a_n|^p\right)^{\frac{1}{p}}$$

for any $a=(a_n)_n \in c_{00}$.

Equivalently there exists a null sequence (ε_n) of positive numbers such that for every N, we have for any $a=(a_n)_n \in c_{00}$

$$(1-\varepsilon_N)\left(\sum_{n\geqslant N}|a_n|^p\right)^{\frac{1}{p}} \leqslant \Big\|\sum_{n\geqslant N} a_n x_n\Big\|_X \leqslant (1+\varepsilon_N)\left(\sum_{n\geqslant N}|a_n|^p\right)^{\frac{1}{p}}.$$

When $p=2$, we can also say that such a sequence (x_n) is asymptotically orthonormal.

We can now prove

Corollary 3.7. — *Let $p \in (1,+\infty)$. The following are equivalent*

(1) *Λ is super-lacunary.*
(2) *The sequence $\left(\frac{t^{\lambda_n}}{\|t^{\lambda_n}\|_p}\right)_n$ in L^p is asymptotically isometric to the canonical basis of ℓ^p.*

Proof. — Assume that Λ is super-lacunary: $\lim_{n\to+\infty}\frac{\lambda_{n+1}}{\lambda_n}=+\infty$. As usual, we denote $q_n = p\lambda_n+1$, and $f_n(t) = q_n^{\frac{1}{p}} t^{\lambda_n} = \frac{t^{\lambda_n}}{\|t^{\lambda_n}\|_p}$. We need to prove that for any $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that

$$(1-\varepsilon)\left(\sum_{n\geqslant N}|a_n|^p\right)^{\frac{1}{p}} \leqslant \Big\|\sum_{n\geqslant N} a_n f_n\Big\|_p \leqslant (1+\varepsilon)\left(\sum_{n\geqslant N}|a_n|^p\right)^{\frac{1}{p}} \tag{3.1}$$

for any $a=(a_n)_n\in c_{00}$. For a given $\varepsilon\in(0,1)$ we consider the number r_ε given by Theorem 3.5. Since $(q_n)_n$ is also super-lacunary, there is an integer N large enough to insure that $q_{k+1}\geqslant r_\varepsilon q_k$ when $k\geqslant N$ and so the sequence $(p\lambda_{n+N}+1)_n$ is $r_\varepsilon-$lacunary. We apply the estimate of $\|T_\Lambda^{w(p)}(\widetilde{a})\|_p$ given by Theorem 3.5 to the sequence $\widetilde{a}=\left(a_n q_n^{\frac{1}{p}}\right)_{n\geqslant N}$ and we get the result.

For the converse, let $\varepsilon \in (0,1)$. From the right hand inequality of (3.1), we get the existence of an integer $N \in \mathbb{N}$ such that for any integer $n \geqslant N$, for any $u \in (0,1)$,

$$\|f_n + uf_{n+1}\|_p \leqslant (1+\varepsilon)(1+u^p)^{\frac{1}{p}} \leqslant (1+\varepsilon)(1+u^p).$$

On the other hand, Hölder's inequality and $\|f_n^{p-1}\|_{p'} = 1$ give

$$\begin{aligned}\|f_n + uf_{n+1}\|_p &\geqslant \int_0^1 (f_n + uf_{n+1}) f_n^{p-1} \mathrm{d}t \\ &= 1 + u \int_0^1 f_{n+1} f_n^{p-1} \mathrm{d}t\end{aligned}$$

Applying this for $u = \varepsilon^{\frac{1}{p}}$, we finally get

$$\int_0^1 f_{n+1} f_n^{p-1} \mathrm{d}t \leqslant 3\varepsilon^{1-\frac{1}{p}}$$

and since $p > 1$, we obtain $\int_0^1 f_{n+1} f_n^{p-1} \mathrm{d}t \to 0$ when $n \to +\infty$.

But

$$\int_0^1 f_{n+1} f_n^{p-1} \mathrm{d}t = \int_0^1 q_{n+1}^{\frac{1}{p}} q_n^{\frac{1}{p'}} t^{(p-1)\lambda_n + \lambda_{n+1}} \mathrm{d}t \geqslant q_n \int_0^1 t^{p\lambda_{n+1}} \mathrm{d}t = \frac{q_n}{q_{n+1}}\cdot$$

Thus, $\frac{p\lambda_n+1}{p\lambda_{n+1}+1} \to 0$ when $n \to +\infty$, and Λ is super-lacunary. $\square$

4. Carleson measures

In this section, μ denotes a positive and finite measure on $[0,1)$ and Λ is a fixed lacunary sequence. We shall generalize some results of [4] and [10] using the estimates from Section 2. In particular, we give a positive answer to a question asked in [10]: if μ is a sublinear measure on $[0,1)$ and Λ is lacunary, then the embedding operator $i_\mu^p : M_\Lambda^p \to L^p(\mu)$ is bounded.

Definition 4.1. — *Let $p \in [1, +\infty)$. We say that*

(1) *μ is* sublinear *if there exists a constant $C > 0$ such that*

$$\forall\, \varepsilon \in (0,1), \qquad \mu([1-\varepsilon, 1]) \leqslant C\varepsilon.$$

The smallest admissible constant C above is denoted $\|\mu\|_S$.

(2) *μ* satisfies (B_p) *if there exists a constant C (depending only on Λ and p) such that*

$$(B_p) \qquad \forall\, n \in \mathbb{N}, \qquad \int_{[0,1)} t^{p\lambda_n} \mathrm{d}\mu \leqslant \frac{C}{\lambda_n}\cdot$$

(3) μ *is a* Carleson measure for M_Λ^p *if there exists a constant* C *(depending only on* Λ *and* p*) such that, for any Müntz polynomial* $f(t) = \sum_n a_n t^{\lambda_n}$,

$$\|f\|_{L^p(\mu)} \leqslant C\|f\|_p \ .$$

In this case we may define the following bounded embedding

$$i_\mu^p : \left\{ \begin{array}{ccc} M_\Lambda^p & \longrightarrow & L^p(\mu) \\ f & \longmapsto & f. \end{array} \right.$$

Remark 4.2. — The notions defined above are related as follows:

(1) Since condition (B_p) is equivalent to $\sup_{n\geqslant 0} \frac{\|t^{\lambda_n}\|_{L^p(\mu)}}{\|t^{\lambda_n}\|_p} < +\infty$, any Carleson measure for M_Λ^p satisfies (B_p).

(2) For $p \in [1, +\infty)$, we have

$$\mu \text{ is sublinear } \implies \mu \text{ satisfies } (B_1) \implies \mu \text{ satisfies } (B_p).$$

Indeed, since $t \in [0,1) \mapsto t^{p\lambda_n}$ is an increasing function, [4, Lem. 2.2] gives

$$\int_{[0,1)} t^{p\lambda_n} \mathrm{d}\mu \leqslant \|\mu\|_S \int_0^1 t^{p\lambda_n} \mathrm{d}t \leqslant \frac{p^{-1}\|\mu\|_S}{\lambda_n}.$$

(3) Moreover, if Λ is a quasi-geometric sequence, and μ satisfies (B_p) for some $p \in [1, +\infty)$ then μ is sublinear. It is essentially proved in [4] in the case $p = 1$. More precisely, there exists a constant $C > 0$ depending only on Λ and p such that

$$\|\mu\|_S \leqslant C \left(\sup_{n\in\mathbb{N}} \lambda_n \int_{[0,1)} t^{p\lambda_n} \mathrm{d}\mu \right),$$

The previous remarks suggest a natural question: does (B_p) imply that μ is a Carleson measure for M_Λ^p ?

The answer is not clear in general. In [4, Ex. 6.2], the authors build a sublinear measure (so it satisfies (B_1)) and a sequence Λ such that μ is not a Carleson measure for M_Λ^1. When Λ is lacunary, we shall see that condition (B_p) is almost sufficient for μ to be a Carleson measure for M_Λ^p (see Theorem 4.5 below) and even sufficient when $p = 1$ (see Proposition 4.4).

The cornerstone of our approach is the following remark.

Remark 4.3. — For a lacunary sequence Λ, we can factorize i_μ^p through $\ell^p(w)$ as follows:

$$\begin{tikzcd}
M_\Lambda^p \arrow[rr, "i_\mu^p"] \arrow[dr, "\left(T_\Lambda^w\right)^{-1}"'] & & L^p(\mu) \\
& \ell^p(w) \arrow[ur, "T^w_{\Lambda,\mu}"'] &
\end{tikzcd}$$

where $w = (w_n)_n$ is a weight satisfying $w_n \approx \lambda_n^{-1}$. With such a weight, the operator T_Λ^w realizes an isomorphism between $\ell^p(w)$ and M_Λ^p: this is a rewording of Gurariy–Macaev Theorem (Theorem 2.3). The most natural weight is $w_n = w_n(p) = (p\lambda_n + 1)^{-1}$ but in this section, we are interested in estimates up to constants (possibly depending on p and Λ). Of course, the results are the same with equivalent weights. So, we choose (in order to simplify) to fix the weight $w_n = \lambda_n^{-1}$.

In particular, by Proposition 2.6 we obtain

$$\|i_\mu^p\| \lesssim \left\|T^w_{\Lambda,\mu}\right\|_p \leqslant \sup_n D_\mu^{(p)}(n),$$

and for $n \in \mathbb{N}$, by Proposition 2.9 we have

$$a_{n+1}(i_\mu^p) \lesssim a_{n+1}(T^w_{\Lambda,\mu}) \leqslant \left(D_\mu^{(p)}\right)^*(n)$$

where the sequence $\left(D_\mu^{(p)}(n)\right)_n$ is defined as in Section 2 by the formula (here with our specified weight and since there is no ambiguity relatively to Λ in the sequel):

$$D_\mu^{(p)}(n) = D_{\Lambda,\mu}^{w,p}(n) = \left(\int_{[0,1)} \lambda_n^{\frac{1}{p}} t^{\lambda_n} \left(\sum_{k\in\mathbb{N}} \lambda_k^{\frac{1}{p}} t^{\lambda_k}\right)^{p-1} \mathrm{d}\mu\right)^{\frac{1}{p}}.$$

We first treat the case $p = 1$.

Proposition 4.4. — *Let $\Lambda = (\lambda_n)_n$ be a lacunary sequence. The following are equivalent:*

(1) *μ satisfies (B_1) ;*

(2) *μ is a Carleson measure for M_Λ^1.*

In this case there exists a constant C depending only on Λ such that

$$\|i_\mu^1\| \leqslant C\left(\sup_{n\in\mathbb{N}} \lambda_n \int_{[0,1)} t^{\lambda_n} \mathrm{d}\mu\right).$$

Proof. — (2) $\Rightarrow$ (1) is obvious. For the converse, we apply the factorization described in Remark 4.3 : this gives $\|i_\mu^1\| \leqslant \left\|T^w_{\Lambda,\mu}\right\|_1 . \left\|\left(T_\Lambda^w\right)^{-1}\right\|_1$. On the other hand, Proposition 2.9 gives $\|T^w_{\Lambda,\mu}\|_1 \leqslant \sup_n D^{(1)}_\mu(n)$ and we get the result. □

As a corollary, we recover quickly [4, Thm. 5.5] in the lacunary case: the sublinear measures satisfy (B_1), and so any sublinear measure is a Carleson measure for M^1_Λ. For the general lacunary case, we have the following theorem.

Theorem 4.5. — *Let $\Lambda = (\lambda_n)_n$ be a r-lacunary sequence. Let μ be a positive measure on $[0,1)$ and $p \in [1,+\infty)$. We assume that μ satisfies (B_p).*

Then μ is a Carleson measure for M^q_Λ for any $q > p$. Moreover, we have

$$\|i^q_\mu\| \leqslant C\left(\sup_{n\in\mathbb{N}} \lambda_n \int_{[0,1)} t^{p\lambda_n}\,\mathrm{d}\mu\right)^{\frac{1}{q}}$$

where C depends only on p, q and Λ.

Proof. — Since Λ is lacunary, we can factorize i^q_μ through $\ell^q(w)$ as in Remark 4.3. We obtain

$$\|i^q_\mu\| \lesssim \left\|T^w_{\Lambda,\mu}\right\|_q \leqslant \sup_n D^{(q)}_\mu(n)$$

The following lemma 4.6 gives the result. □

Lemma 4.6. — *Under the assumptions of Theorem 4.5, we have*

$$\begin{aligned}\left(D^{(q)}_\mu(n)\right)^q &\leqslant C\left(\sup_{k\geqslant n} \lambda_k \int_{[0,1)} t^{p\lambda_k}\,\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\sup_{k\in\mathbb{N}} \lambda_k \int_{[0,1)} t^{p\lambda_k}\,\mathrm{d}\mu\right)^{\frac{1}{p'}}\\ &\leqslant C\sup_{k\in\mathbb{N}} \lambda_k \int_{[0,1)} t^{p\lambda_k}\,\mathrm{d}\mu,\end{aligned}$$

where C is a constant depending only on p, q and r.

Proof. — Since $(\lambda_k)_k$ is r-lacunary, for any $\beta \in \mathbb{R}^*_+$ we have

$$\sum_{k\leqslant n} \lambda_k^\beta \leqslant \frac{1}{1-r^{-\beta}}\lambda_n^\beta \quad \text{and} \quad \sum_{k>n} \lambda_k^{-\beta} \leqslant \frac{1}{r^\beta - 1}\lambda_n^{-\beta}.$$

For any $j \in \mathbb{N}$, we denote $M_j = \lambda_j \int_{[0,1)} t^{p\lambda_j}\,\mathrm{d}\mu$ and $M = \sup_j M_j < +\infty$. Since $q > 1$, we have for any $A, B \in \mathbb{R}_+$, $(A+B)^{q-1} \leqslant 2^{q-1}(A^{q-1} + B^{q-1})$.

This gives:

$$
\begin{aligned}
&\left(D_\mu^{(q)}(n)\right)^q \\
&\quad = \int_{[0,1)} \lambda_n^{\frac{1}{q}} t^{\lambda_n} \left(\sum_{k\in\mathbb{N}} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{q-1} \mathrm{d}\mu \\
&\quad \lesssim \int_{[0,1)} \lambda_n^{\frac{1}{q}} t^{\lambda_n} \left(\sum_{k\leqslant n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{q-1} \mathrm{d}\mu + \int_{[0,1)} \lambda_n^{\frac{1}{q}} t^{\lambda_n} \left(\sum_{k> n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{q-1} \mathrm{d}\mu
\end{aligned}
$$

We estimate the first term above. If $p>1$, Hölder's inequality gives

$$
\begin{aligned}
&\int_{[0,1)} \lambda_n^{\frac{1}{q}} t^{\lambda_n} \left(\sum_{k\leqslant n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{q-1} \mathrm{d}\mu \\
&\qquad\qquad \leqslant \lambda_n^{\frac{1}{q}} \left(\int t^{p\lambda_n} \mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int \left(\sum_{k\leqslant n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{p'(q-1)} \mathrm{d}\mu\right)^{\frac{1}{p'}} \\
&\qquad\qquad \leqslant M_n^{\frac{1}{p}} \lambda_n^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\leqslant n} \lambda_k^{\frac{1}{q}} \|t^{\lambda_k}\|_{L^{p'(q-1)}(\mu)}\right)^{q-1}
\end{aligned}
$$

where we used the triangle inequality since $p'(q-1) \geqslant p \geqslant 1$. For any $k \leqslant n$ we have $\int_{[0,1)} t^{p'(q-1)\lambda_k} \mathrm{d}\mu \leqslant \int_{[0,1)} t^{p\lambda_k} \mathrm{d}\mu \leqslant M_k \lambda_k^{-1}$. This gives

$$
\begin{aligned}
&\int_{[0,1)} \lambda_n^{\frac{1}{q}} t^{\lambda_n} \left(\sum_{k\leqslant n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{q-1} \mathrm{d}\mu \\
&\qquad\qquad \leqslant \left(\sup_{k\leqslant n} M_k^{\frac{1}{p'}}\right) M_n^{\frac{1}{p}} \lambda_n^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\leqslant n} \lambda_k^{\frac{1}{q}-\frac{1}{p'(q-1)}}\right)^{q-1} \\
&\qquad\qquad \lesssim \left(\sup_{k\leqslant n} M_k^{\frac{1}{p'}}\right) M_n^{\frac{1}{p}} \lambda_n^{\frac{1}{q}-\frac{1}{p}} \lambda_n^{\frac{1}{q'}-\frac{1}{p'}} \\
&\qquad\qquad = \left(\sup_{k\leqslant n} M_k^{\frac{1}{p'}}\right) M_n^{\frac{1}{p}}.
\end{aligned}
$$

If $p=1$, the inequality $t^{\lambda_k} \leqslant 1$ gives directly

$$
\int_{[0,1)} \lambda_n^{\frac{1}{q}} t^{\lambda_n} \left(\sum_{k\leqslant n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{q-1} \mathrm{d}\mu \leqslant M_n \lambda_n^{-1} \lambda_n^{\frac{1}{q}} \left(\sum_{k\leqslant n} \lambda_k^{\frac{1}{q}}\right)^{q-1} \lesssim M_n.
$$

For the second term we treat two cases. First if $q-1 \geqslant p$, the triangle inequality gives

$$\begin{aligned}
&\int_{[0,1)} \lambda_n^{\frac{1}{q}} t^{\lambda_n} \left(\sum_{k>n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{q-1} \mathrm{d}\mu \\
&\qquad \leqslant \lambda_n^{\frac{1}{q}} \left(\sum_{k>n} \|\lambda_k^{\frac{1}{q}} t^{\lambda_k}\|_{L^{q-1}(t^{\lambda_n}\mu)}\right)^{q-1} \\
&\qquad = \lambda_n^{\frac{1}{q}} \left(\sum_{k>n} \lambda_k^{\frac{1}{q}} \left(\int_{[0,1)} t^{(q-1)\lambda_k+\lambda_n} \mathrm{d}\mu\right)^{\frac{1}{q-1}}\right)^{q-1} \\
&\qquad \leqslant \lambda_n^{\frac{1}{q}} \left(\sum_{k>n} \lambda_k^{\frac{1}{q}} \left(\int_{[0,1)} t^{p\lambda_k} \mathrm{d}\mu\right)^{\frac{1}{q-1}}\right)^{q-1} \\
&\qquad \leqslant \left(\sup_{k>n} M_k\right) \lambda_n^{\frac{1}{q}} \left(\sum_{k>n} \lambda_k^{\frac{1}{q}-\frac{1}{q-1}}\right)^{q-1} \\
&\qquad \lesssim \left(\sup_{k>n} M_k\right) \lambda_n^{\frac{1}{q}} \big(\lambda_n^{\frac{-1}{q(q-1)}}\big)^{q-1} = \sup_{k>n} M_k.
\end{aligned}$$

If $q-1 < p$, let $\alpha = \frac{p}{p-(q-1)}\cdot$ It satisfies $\alpha > q$ and $(q-1)\alpha' = p$. We apply Hölder's inequality:

$$\begin{aligned}
&\int_{[0,1)} \lambda_n^{\frac{1}{q}} t^{\lambda_n} \left(\sum_{k>n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{q-1} \mathrm{d}\mu \\
&\qquad \leqslant \lambda_n^{\frac{1}{q}} \left(\int_{[0,1)} t^{\alpha\lambda_n} \mathrm{d}\mu\right)^{\frac{1}{\alpha}} \left(\int_{[0,1)} \big(\sum_{k>n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\big)^p \mathrm{d}\mu\right)^{\frac{1}{\alpha'}} \\
&\qquad \leqslant M_n^{\frac{1}{\alpha}} \lambda_n^{\frac{1}{q}-\frac{1}{\alpha}} \left(\sum_{k>n} \lambda_k^{\frac{1}{q}} \left(\int_{[0,1)} t^{p\lambda_n} \mathrm{d}\mu\right)^{\frac{1}{p}}\right)^{\frac{p}{\alpha'}}
\end{aligned}$$

where we applied again the triangle inequality. We obtain

$$\begin{aligned}
\int_{[0,1)} \lambda_n^{\frac{1}{q}} t^{\lambda_n} \left(\sum_{k>n} \lambda_k^{\frac{1}{q}} t^{\lambda_k}\right)^{q-1} \mathrm{d}\mu &\leqslant M_n^{\frac{1}{\alpha}} \left(\sup_{k>n} M_k^{\frac{1}{\alpha'}}\right) \lambda_n^{\frac{1}{q}-\frac{1}{\alpha}} \left(\sum_{k>n} \lambda_k^{\frac{1}{q}-\frac{1}{p}}\right)^{q-1} \\
&\lesssim M_n^{\frac{1}{\alpha}} \left(\sup_{k>n} M_k^{\frac{1}{\alpha'}}\right).
\end{aligned}$$

We finally get

$$\left(D_\mu^{(q)}(n)\right)^q \lesssim M_n^{\frac{1}{p}} \left(\sup_{k\leqslant n} M_k^{\frac{1}{p'}}\right) + \sup_{k\geqslant n} M_k. \qquad \square$$

Corollary 4.7. — *If μ is sublinear and Λ is lacunary, then μ is a Carleson measure for M_Λ^q, for any $q \in [1,+\infty)$.*

Proof. — Remark 4.2 implies that the sublinear measures satisfy (B_1), and we obtain

$$\|i_\mu^q\| \lesssim \|\mu\|_S^{\frac{1}{q}}. \qquad \square$$

The previous fact was proved for $p = 2$ in [10, Thm. 4.3], and the authors announced the result for $p \in (1,2)$ (see [10, Cor. 5.2]). Unfortunately there is a gap in the proof of their interpolation result [10, Thm. 5.1] : interpolation is not easy to handle in Müntz spaces because $f \in M_\Lambda^p$ does not imply that $|f| \in M_\Lambda^p$ in general.

Theorem 4.5 has the following interesting consequence.

Corollary 4.8. — *Let Λ be a lacunary sequence and $p, q \in [1,+\infty)$ such that $p < q$.*

(1) *If i_μ^p is bounded, then i_μ^q is bounded.*
(2) *The converse is false in general.*

Proof. — If i_μ^p is bounded, then μ satisfies (B_p). Theorem 4.5 implies that i_μ^q is bounded. Assertion (2) is a consequence of the Examples 5.14 and 5.15 below. $\square$

Corollary 4.9. — *Let $q \in [1,+\infty)$ and let Λ be a quasi-geometric sequence. Then we have*

$$\begin{aligned}\|i_\mu^q\| &\approx \sup_n \left(\int_{[0,1)} \lambda_n t^{q\lambda_n} \mathrm{d}\mu\right)^{\frac{1}{q}} \approx \|\mu\|_S^{\frac{1}{q}} \\ &\approx \sup_n \left(\int_{[0,1)} \lambda_n t^{\lambda_n} \mathrm{d}\mu\right)^{\frac{1}{q}} \approx \sup_n \left(D_\mu^{(q)}(n)\right),\end{aligned}$$

where the underlying constants depend only on q and Λ.

In particular, μ is a Carleson measure if and only if it is sublinear.

Proof. — Since Λ is lacunary, Remark 4.2 (2) and Lemma 4.6 give easily

$$\|i_\mu^q\| \lesssim \sup_n \left(\lambda_n \int_{[0,1)} t^{\lambda_n} \mathrm{d}\mu\right)^{\frac{1}{q}} \lesssim \|\mu\|_S^{\frac{1}{q}}.$$

On the other hand, since Λ quasi-geometric, Remark 4.2 (3) gives

$$\|\mu\|_S \lesssim \sup_n \int_{[0,1)} \lambda_n t^{q\lambda_n} \mathrm{d}\mu \leqslant \|i_\mu^q\|^q. \qquad \square$$

5. Compactness and Schatten classes

In this part we are interested in the compactness of the embedding

$$i_\mu^p : \begin{cases} M_\Lambda^p & \longrightarrow & L^p(\mu) \\ f & \longmapsto & f \end{cases}$$

where μ is a Carleson measure for M_Λ^p.

We now investigate its membership of various classes of operator ideals, in particular Schatten classes (when $p = 2$).

As in Section 4, we work with $w_n = \lambda_n^{-1}$; we consider the operators T_Λ^w and $T_{\Lambda,\mu}^w$ and the sequence $D_\mu^{(p)}(n) = D_{\Lambda,\mu}^{w,p}(n)$ associated to this weight.

Definition 5.1. — *Let $p \in [1, +\infty)$. We say that*

(1) μ *is* vanishing sublinear *when* $\lim_{\varepsilon\to 0} \frac{\mu([1-\varepsilon,1])}{\varepsilon} = 0$;

(2) μ satisfies (B_p) *when we have*

$$\lim_{n\to+\infty} \lambda_n \int_{[0,1)} t^{p\lambda_n} \mathrm{d}\mu = 0. \tag{b_p}$$

Let us point out that (b_p) exactly means that $\lim_{n\to+\infty} \frac{\|t^{\lambda_n}\|_{L^p(\mu)}}{\|t^{\lambda_n}\|_{L^p(m)}} = 0$.

Remark 5.2. — Let μ be a Carleson measure for M_Λ^p.

(1) If i_μ^p is compact and $p > 1$, then μ satisfies (b_p). Indeed, since for any $k \in \mathbb{N}$,

$$\int_0^1 t^{\lambda_n} \lambda_n^{\frac{1}{p}} t^k \mathrm{d}t = \frac{\lambda_n^{\frac{1}{p}}}{\lambda_n + k + 1} \to 0 \quad \text{when } n \to +\infty,$$

for any polynomial g, $\int_0^1 t^{\lambda_n} \lambda_n^{\frac{1}{p}} g(t)\mathrm{d}t \to 0$. Since $p > 1$ the polynomials are dense in $L^{p'}$ and so $(\lambda_n^{\frac{1}{p}} t^{\lambda_n})_n$ converges weakly to 0 in M_Λ^p. Since the embedding i_μ^p is compact, $\|\lambda_n^{\frac{1}{p}} t^{\lambda_n}\|_{L^p(\mu)} \to 0$ when $n \to +\infty$.

(2) For $p, q \in [1, +\infty)$ such that $p < q$, we have

$$\mu \text{ is vanishing sublinear } \Longrightarrow \mu \text{ satisfies } (b_p) \text{ hence } (b_q).$$

Indeed, by assumption, for any $\varepsilon > 0$, there exists $\eta > 0$ such that $\|\mu_{|[1-\eta,1)}\|_S \leqslant \varepsilon$. We have

$$\lambda_n \int_{[0,1)} t^{p\lambda_n} \mathrm{d}\mu \leqslant \lambda_n \eta^{p\lambda_n} \mu([0,1)) + \lambda_n \int_{[1-\eta,1)} t^{p\lambda_n} \mathrm{d}\mu.$$

The first term tends to 0 when $n \to +\infty$ and the second is less than $p^{-1}\|\mu|_{[1-\eta,1)}\|_S \leqslant \frac{\varepsilon}{p}$ thanks to Remark 4.2 (2).

(3) The conditions in (2) are equivalent when Λ is a quasi-geometric sequence. More precisely, for $\varepsilon > 0$ close to 0, we have

$$\frac{\mu([1-\varepsilon,1))}{\varepsilon} \leqslant 3pR\lambda_n \int_{[0,1)} t^{p\lambda_n} \mathrm{d}\mu$$

where n is the index such that $\varepsilon \in \left(\frac{1}{p\lambda_{n+1}}, \frac{1}{p\lambda_n}\right]$, and R is a constant such that $\lambda_{k+1} \leqslant R\lambda_k$ for any $k \in \mathbb{N}$. So if μ satisfies (b_p), then μ is vanishing sublinear.

5.1. The case $p = 1$.

If i^1_μ is compact, μ satisfies (b_1) but the method to prove it is not the same as for $p > 1$.

Proposition 5.3. — *Let Λ be a lacunary sequence. The following are equivalent*

(1) *μ satisfies (b_1) ;*
(2) *i^1_μ is compact ;*
(3) *i^1_μ is weakly compact.*

Proof. — Let us prove that (1) $\Rightarrow$ (2). Since Λ is lacunary, we can factorize i^1_μ through $\ell^1(w)$ as in Remark 4.3: we have $i^1_\mu = T^w_{\Lambda,\mu} \circ \left(T^w_\Lambda\right)^{-1}$. On the other hand, μ satisfies (b_1), so we have $D^{(1)}_\mu(n) = \lambda_n \int_{[0,1)} t^{\lambda_n} \mathrm{d}\mu \to 0$ when $n \to +\infty$. Proposition 2.9 implies that $a_n(T^w_{\Lambda,\mu}) \to 0$ and we get $a_n(i^1_\mu) \to 0$ when $n \to +\infty$.

The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are valid for any Müntz space M^1_Λ, without any assumption of lacunarity for Λ.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). We denote $H = \{\lambda_n t^{\lambda_n}\}$. Since H is bounded in M^1_Λ, H is bounded and by assumption weakly relatively compact in $L^1(\mu)$, hence uniformly integrable in $L^1(\mu)$ (see [12, Thm. III.C.12, p. 137]). This means

that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $n \in \mathbb{N}$ and any measurable set $A \subset [0,1)$ with $\mu(A) \leqslant \delta$, we have

$$\int_A \lambda_n t^{\lambda_n} \mathrm{d}\mu \leqslant \varepsilon.$$

Since $\mu(\{1\}) = 0$, there exists $s \in (0,1)$ such that $\mu([s,1)) \leqslant \eta$. We have

$$\begin{aligned}\int_{[0,1)} \lambda_n t^{\lambda_n} \mathrm{d}\mu &= \int_{[0,s)} \lambda_n t^{\lambda_n} \mathrm{d}\mu + \int_{[s,1]} \lambda_n t^{\lambda_n} \mathrm{d}\mu \\ &\leqslant \lambda_n s^{\lambda_n} \mu([0,1)) + \varepsilon.\end{aligned}$$

Since $\lambda_n s^{\lambda_n} \to 0$ as $n \to +\infty$, we obtain that μ satisfies (b_1). □

Remark. — Without any assumption of lacunarity on Λ, the embedding i^1_μ is a Dunford–Pettis operator (i.e. maps a weakly convergent sequence into a norm-convergent sequence) if and only if i^1_μ is bounded. This is due to the fact that M^1_Λ has the Schur property since it is isomorphic to a subspace of ℓ^1 (see [11], see also [6] for some extensions of this result).

5.2. The case $p > 1$.

Let us mention without proof the next remark (the argument is the same as in Lemma 5.10 below, but we shall not use this result in the general case).

Remark 5.4. — Let Λ be a quasi-geometric sequence. For $p \geqslant 1$, there exist an integer $K \geqslant 1$ and C depending only on Λ such that for any $n \in \mathbb{N}$ we have

$$C\lambda_{n+K} \int_{[0,1)} t^{\lambda_{n+K}} \mathrm{d}\mu \leqslant \lambda_n \int_{[0,1)} t^{p\lambda_n} \mathrm{d}\mu \leqslant \big(D^{(p)}_\mu(n)\big)^p.$$

We first give an easy sufficient condition ensuring compactness. This is closely related to the rough sufficient condition ensuring the boundedness of i^p_μ stated in Remark 2.5

Proposition 5.5. — *Let Λ be a quasi-geometric sequence and $p \geqslant 1$. The Carleson embedding i^p_μ is order bounded if and only if*

$$\int_{[0,1)} \frac{\mathrm{d}\mu}{1-t} < \infty.$$

This condition ensures that i^p_μ is a p-summing operator (see [5, Thm. 5.18]), hence compact from M^p_Λ to $L^p(\mu)$ when $p > 1$. In the case $p = 1$, apply Proposition 5.3.

Proof. — Since the space M^p_Λ is separable, i^p_μ is order bounded if and only if $t \mapsto \sup_{f\in B_{M^p_\Lambda}} |f(t)|$ belongs to $L^p(\mu)$. Now, the estimate on the point evaluation (see Proposition 2.13) gives the conclusion. □

In the same way as for the boundedness problem, we can "almost" characterize the compactness of i^q_μ for $q>1$, by testing the monomials.

Theorem 5.6. — *Let Λ be a lacunary sequence. Assume that μ satisfies (b_p) for some $p \in [1,+\infty)$. Then i^q_μ is compact for any $q>p$.*

Proof. — Since Λ is lacunary, we can factorize i^q_μ through $\ell^q(w)$ as in Remark 4.3: $i^q_\mu = T^w_{\Lambda,\mu}\circ\left(T^w_\Lambda\right)^{-1}$ (recall that T^w_Λ is an isomorphism). Proposition 2.9 gives

$$\|i^q_\mu\|_e \lesssim \|T^w_{\Lambda,\mu}\|_e \leqslant \limsup_{n\to+\infty} D^{(q)}_\mu(n).$$

Since μ satisfies (b_p), Lemma 4.6 implies that $D^{(q)}_\mu(n)\to 0$ when $n\to+\infty$ and so i^q_μ is compact. □

Corollary 5.7. — *Let Λ be a lacunary sequence and $p,q\in[1,+\infty)$ such that $p<q$.*

(1) *If i^p_μ is compact, then i^q_μ is compact.*
(2) *The converse is false in general.*
(3) *If μ is vanishing sublinear, i^p_μ is compact.*

Proof. — If i^p_μ is compact, then μ satisfies (b_p) and i^q_μ is compact by Theorem 5.6. Assertion (2) is a consequence of Example 5.14 or Example 5.15 below. At last (3) holds since any vanishing sublinear measure satisfies (b_1). □

Corollary 5.8. — *Let $q\in[1,+\infty)$ and let Λ be a quasi-geometric sequence. Assume that μ is a Carleson measure for M^q_Λ. Then we have*

$$\begin{aligned}\|i^q_\mu\|_e &\approx \limsup_n \left(\int_{[0,1)} \lambda_n t^{\lambda_n}\,\mathrm{d}\mu\right)^{\frac{1}{q}} \approx \left(\limsup_{\varepsilon\to 0}\frac{\mu([1-\varepsilon,1)}{\varepsilon}\right)^{\frac{1}{q}}\\ &\approx \limsup_{n\to+\infty} D^{(q)}_\mu(n),\end{aligned}$$

where the underlying constants depend only on q and Λ.

In particular, i^q_μ is compact if and only if μ is vanishing sublinear.

Proof. — We already saw in Lemma 4.6 and the proof of Theorem 5.6 that

$$\|i_\mu^q\|_e \lesssim \limsup_{n\to+\infty} D_\mu^{(q)}(n) \lesssim \limsup_n \left(\int_{[0,1)} \lambda_n t^{\lambda_n}\,\mathrm{d}\mu\right)^{\frac{1}{q}}$$
$$\lesssim \left(\limsup_{\varepsilon\to 0} \frac{\mu([1-\varepsilon,1)}{\varepsilon}\right)^{\frac{1}{q}},$$

this part only requires the lacunarity assumption on Λ.

To get the minoration of $\|i_\mu^q\|_e$ we use [4, Thm. 3.5] which proves

$$\|i_\mu^1\|_e = \lim_{n\to+\infty} \|i_{\mu_n'}^1\|$$

where μ_n' is the restriction $\mu|_{[1-\frac{1}{n},1)}$. The proof can be easily adapted for $q>1$ as it was noticed in [10, Prop. 2.6] and we have

$$\|i_\mu^q\|_e = \lim_{n\to+\infty} \|i_{\mu_n'}^q\|.$$

Since Λ is quasi-geometric, Corollary 4.9 gives a constant $C>0$ such that for any measure ν on $[0,1)$: $\|i_\nu^q\| \geqslant C\|\nu\|_S^{\frac{1}{q}}$. We have

$$\|i_\mu^q\|_e = \lim_{n\to+\infty} \|i_{\mu_n'}^q\| \geqslant C \lim_{n\to+\infty} \|\mu|_{[1-\frac{1}{n},1)}\|_S^{\frac{1}{q}}$$
$$= \left(\limsup_{\varepsilon\to 0} \frac{\mu([1-\varepsilon,1)}{\varepsilon}\right)^{\frac{1}{q}}. \qquad \square$$

The following result is an improvement of [4, Prop. 3.2]. It requires no assumption on the lacunarity of Λ but a strong assumption on μ.

Proposition 5.9. — *If* $\mathrm{Supp}(\mu)$ *lies in a compact set of* $[0,1)$*, then* i_μ^p *is a nuclear operator.*

Proof. — Assume that $\mathrm{Supp}(\mu) \subset [0,\delta]$ with $\delta<1$. We fix $\varepsilon>0$ such that $(1+\varepsilon)\delta<1$. Since Λ satisfies the gap condition, we have the following classical estimate essentially proved in [7, Prop. 6.2.2]: there exists K_ε such that for any Müntz polynomial $f(t)=\sum_k a_k t^{\lambda_k}$, we have

$$|a_n| \leqslant K_\varepsilon (1+\varepsilon)^{\lambda_n}\|f\|_p.$$

This implies that the functionals

$$e_n^* : \left\{ \begin{array}{ccc} M_\Lambda^p & \longrightarrow & \mathbb{C} \\ \sum_k a_k t^{\lambda_k} & \longmapsto & a_n \end{array} \right.$$

are well defined, bounded, and $\|e_n^*\| \leqslant K_\varepsilon(1+\varepsilon)^{\lambda_n}$.

We define $g_n : [0,1) \to \mathbb{C}$ by $g_n(t)=t^{\lambda_n}$. The functions $(g_n)_n$ belong to $L^p(\mu)$ and we have $\|g_n\|_{L^p(\mu)} \leqslant \mu([0,1))\delta^{\lambda_n}$. On the other hand, for any

Müntz polynomial f, we have $i_\mu^p(f) = \sum_{k\geqslant 0} e_k^*(f) g_k$. So i_μ^p and $\sum_{k\geqslant 0} e_k^* \otimes g_k$ coincide on a dense set of M_Λ^p. Moreover, we have

$$\sum \|e_k^* \otimes g_k\| \leqslant K_\varepsilon \mu([0,1)) \sum_k (\delta(1+\varepsilon))^{\lambda_k} < +\infty$$

Therefore i_μ^p is a nuclear operator. □

5.3. The case $p = 2$.

From now on we focus on the hilbertian setting.

LEMMA 5.10. — *Let Λ be a quasi-geometric sequence and μ such that i_μ^2 is bounded.*

(1) *There exist an integer $K \geqslant 1$ and $C > 0$ depending only on Λ such that for any $n \in \mathbb{N}$ we have*

$$C\lambda_{n+K} \int_{[0,1)} t^{\lambda_{n+K}} \mathrm{d}\mu \leqslant \lambda_n \int_{[0,1)} t^{2\lambda_n} \mathrm{d}\mu \leqslant \left(D_\mu^{(2)}(n)\right)^2.$$

(2) *For any $q \in (0, +\infty)$, we have*

$$\|(D_\mu^{(2)}(n))_n\|_{\ell^q} \approx \left\| \left(\lambda_n \int_{[0,1)} t^{2\lambda_n} \mathrm{d}\mu \right)^{\frac{1}{2}}_n \right\|_{\ell^q} \approx \left\| \left(\lambda_n \int_{[0,1)} t^{\lambda_n} \mathrm{d}\mu \right)^{\frac{1}{2}}_n \right\|_{\ell^q}$$

where the involved constants depend only on Λ and q.

Proof. — We first prove (1). For $n \in \mathbb{N}$ we have

$$\left(D_\mu^{(2)}(n)\right)^2 = \sum_{k\in\mathbb{N}} (\lambda_n \lambda_k)^{\frac{1}{2}} \int_{[0,1)} t^{\lambda_n+\lambda_k} \mathrm{d}\mu \geqslant \lambda_n \int_{[0,1)} t^{2\lambda_n} \mathrm{d}\mu$$

since this last term is the term $n = k$ in the sum. On the other hand, we assume that Λ is r-lacunary. There exists $K \in \mathbb{N}$ such that $r^K \geqslant 2$ and since Λ is quasi-geometric, there exists $R \in \mathbb{R}$ such that $\lambda_{k+1} \leqslant R\lambda_k$ for any k. We obtain

$$\lambda_{n+K} \int_{[0,1)} t^{\lambda_{n+K}} \mathrm{d}\mu \leqslant R^K \lambda_n \int_{[0,1)} t^{r^K \lambda_n} \mathrm{d}\mu \lesssim \lambda_n \int_{[0,1)} t^{2\lambda_n} \mathrm{d}\mu$$

Now, we prove (2). For $k \in \mathbb{N}$ we shall denote $m_k = \lambda_k \int_{[0,1)} t^{\lambda_k} \mathrm{d}\mu$. Assume that the sequence $(m_k^{\frac{1}{2}})_k$ lies in ℓ^q. We shall compare $\|D_\mu^{(2)}(n)\|_{\ell^q}$

and $\|m_n^{\frac{1}{2}}\|_{\ell^q}$ and shall, in some sense, improve the estimate of Lemma 4.6. For $n \in \mathbb{N}$, we have:

$$\begin{aligned}\left(D_\mu^{(2)}(n)\right)^2 &= \sum_{k\leqslant n}(\lambda_n\lambda_k)^{\frac{1}{2}}\int_{[0,1)} t^{\lambda_n+\lambda_k}\mathrm{d}\mu + \sum_{k>n}(\lambda_n\lambda_k)^{\frac{1}{2}}\int_{[0,1)} t^{\lambda_n+\lambda_k}\mathrm{d}\mu \\ &\leqslant \sum_{k\leqslant n}(\lambda_n\lambda_k)^{\frac{1}{2}}\frac{m_n}{\lambda_n} + \sum_{k>n}(\lambda_n\lambda_k)^{\frac{1}{2}}\frac{m_k}{\lambda_k} \\ &\leqslant m_n\frac{1}{1-\frac{1}{\sqrt{r}}} + \sum_{k>n} m_k\frac{1}{\sqrt{r}^{k-n}}.\end{aligned}$$

So the number $\left(D_\mu^{(2)}(n)\right)^2$ is less than the n-th entry of the vector $A[(m_k)_k]$, where $A = (A_{n,k})_{n,k}$ is the matrix defined by

$$A_{n,k} = \begin{cases} 0 & \text{if } k<n \\ (1-r^{-\frac{1}{2}})^{-1} & \text{if } k=n \\ \frac{1}{\sqrt{r}^{k-n}} & \text{if } k>n. \end{cases}$$

Assume first that $q \geqslant 2$. Since A satisfies

$$\sup_n \sum_k A_{n,k} \leqslant \frac{2}{1-\frac{1}{\sqrt{r}}} \quad \text{and} \quad \sup_k \sum_n A_{n,k} \leqslant \frac{2}{1-\frac{1}{\sqrt{r}}},$$

we can apply the Schur test: A defines a bounded operator $A : \ell^{\frac{q}{2}} \to \ell^{\frac{q}{2}}$ and we have $\|A\|_{\frac{q}{2}} \leqslant \frac{2}{1-\frac{1}{\sqrt{r}}}\cdot$ In particular, for $(m_k)_k \in \ell^{\frac{q}{2}}$ we obtain

$$\|(D_\mu^{(2)}(n))\|_{\ell^q} \leqslant \left(\frac{2}{1-\frac{1}{\sqrt{r}}}\right)^{\frac{1}{2}} \|(m_k)^{\frac{1}{2}}\|_{\ell^q}.$$

Now we treat the case $q<2$. Since $\frac{q}{2}<1$, we have

$$\begin{aligned}(D_\mu^{(2)}(n))^q &\leqslant \left(m_n\frac{1}{1-\frac{1}{\sqrt{r}}} + \sum_{k>n} m_k\frac{1}{\sqrt{r}^{k-n}}\right)^{\frac{q}{2}} \\ &\leqslant \left(m_n^{\frac{1}{2}}\right)^q \frac{1}{(1-\frac{1}{\sqrt{r}})^{\frac{q}{2}}} + \sum_{k>n}\left(m_k^{\frac{1}{2}}\right)^q \frac{1}{r^{\frac{q(k-n)}{4}}}\end{aligned}$$

and we get

$$\begin{aligned}\sum_{n\in\mathbb{N}}(D_\mu^{(2)}(n))^q &\leqslant \sum_n \left(m_n^{\frac{1}{2}}\right)^q\left(\frac{1}{1-\frac{1}{\sqrt{r}}}\right)^{\frac{q}{2}} + \sum_{k\in\mathbb{N}}\left(m_k^{\frac{1}{2}}\right)^q \sum_{n=0}^{k-1}\left(\frac{1}{r}\right)^{\frac{q(k-n)}{4}} \\ &\lesssim \|\left(m_n^{\frac{1}{2}}\right)_n\|_{\ell^q}^q\end{aligned}$$

where the underlying constants depend on r and q only.

We have then

$$\|(D_\mu^{(2)}(n))_n\|_{\ell^q} \lesssim \left\| \left(\lambda_n \int_{[0,1)} t^{\lambda_n} \mathrm{d}\mu \right)^{\frac{1}{2}}_n \right\|_{\ell^q}$$

whereas (1) implies

$$\left\| \left(\lambda_n \int_{[0,1)} t^{\lambda_n} \mathrm{d}\mu \right)^{\frac{1}{2}}_n \right\|_{\ell^q} \lesssim \left\| \left(\lambda_n \int_{[0,1)} t^{2\lambda_n} \mathrm{d}\mu \right)^{\frac{1}{2}}_n \right\|_{\ell^q} \lesssim \|(D_\mu^{(2)}(n))_n\|_{\ell^q}$$

The conclusion follows. □

Theorem 5.11. — *Let Λ be a lacunary sequence and $q > 0$. We have*

(1) *If $(D_\mu^{(2)}(n))_n \in \ell^q$ then we have*

$$\|i_\mu^2\|_{\mathcal{S}^q} \lesssim \|D_\mu^{(2)}(n)\|_{\ell^q};$$

(2) *If moreover Λ is quasi-geometric and $q \geqslant 2$, we have*

$$\|i_\mu^2\|_{\mathcal{S}^q} \approx \|D_\mu^{(2)}(n)\|_{\ell^q},$$

where the involved constants depend only on q and Λ.

Proof. — We first prove (1). As in Remark 4.3, since Λ is lacunary we may factorize i_μ^2 through $\ell^2(w)$, and we get $a_n(i_\mu^2) \lesssim a_n(T^w_{\Lambda,\mu})$. Proposition 2.9 gives

$$\sum_n (a_n(i_\mu^2))^q \lesssim \sum_n \left(D_\mu^{(2)}(n)\right)^q.$$

Now, we prove (2). As a direct consequence of [5, Thm. 4.7], for any Riesz basis $(f_n)_n$ of M^2_Λ (i.e. an orthonormal basis up to an invertible operator), there exists a constant $C > 0$ such that

$$\|i_\mu^2\|_{\mathcal{S}^q} \geqslant C \left(\sum_n \|f_n\|^q_{L^2(\mu)} \right)^{\frac{1}{q}}.$$

By Gurariy–Macaev theorem the sequence $(f_n)_n = (\lambda_n^{\frac{1}{2}} t^{\lambda_n})_n$ is a Riesz basis of M^2_Λ, so we obtain:

$$\|i_\mu^2\|^q_{\mathcal{S}^q} \gtrsim \sum_n \left(\lambda_n \int_{[0,1)} t^{2\lambda_n} \mathrm{d}\mu \right)^{\frac{q}{2}}$$

and Lemma 5.10 gives the result. □

We also have an integral expression for $\|i_\mu^2\|_{\mathcal{S}^q}$.

Proposition 5.12. — *Assume that Λ is quasi-geometric and $q \geqslant 2$. We have:*

$$\|i_\mu^2\|_{\mathcal{S}^q} \approx \left(\int_0^1 \left(\int_{[0,1)} \frac{\mathrm{d}\mu(t)}{(1-st)^{\frac{2}{q}+1}}\right)^{\frac{q}{2}} \mathrm{d}s\right)^{\frac{1}{q}}.$$

Proof. — We denote $M_n = \lambda_n \int_{[0,1)} t^{2\lambda_n}\mathrm{d}\mu$. The previous estimate gives:

$$\|i_\mu^2\|_{\mathcal{S}^q} \approx \left(\sum_n M_n^{\frac{q}{2}}\right)^{\frac{1}{q}} = \|(M_n)_n\|_{\ell^{\frac{q}{2}}}^{\frac{1}{2}}.$$

On the other hand we can apply Gurariy–Macaev theorem to estimate an equivalent of $\|(M_n)\|_{\ell^{\frac{q}{2}}}$. We obtain, using Lemma 2.10,

$$\begin{aligned}
\|i_\mu^2\|_{\mathcal{S}^q} &\approx \left\|\sum_n M_n \lambda_n^{\frac{2}{q}} s^{\lambda_n}\right\|_{L^{\frac{q}{2}}(ds)}^{\frac{1}{2}} \\
&= \left(\int_0^1 \left(\sum_n \lambda_n \int_{[0,1)} t^{2\lambda_n}\mathrm{d}\mu(t)\lambda_n^{\frac{2}{q}} s^{\lambda_n}\right)^{\frac{q}{2}} \mathrm{d}s\right)^{\frac{1}{q}} \\
&= \left(\int_0^1 \left(\int_{[0,1)} \sum_n \lambda_n^{\frac{2}{q}+1}(st^2)^{\lambda_n}\mathrm{d}\mu(t)\right)^{\frac{q}{2}} \mathrm{d}s\right)^{\frac{1}{q}} \\
&\approx \left(\int_0^1 \left(\int_{[0,1)} \frac{\mathrm{d}\mu(t)}{(1-st^2)^{\frac{2}{q}+1}}\right)^{\frac{q}{2}} \mathrm{d}s\right)^{\frac{1}{q}}.
\end{aligned}$$

We get the result since $(1-st) \leqslant (1-st^2) \leqslant (1+st)(1-st) \leqslant 2(1-st)$ for $s,t \in [0,1]$. □

Note that the previous estimate shows that, as soon as Λ is quasi-geometric, i_μ^2 belongs to the Schatten class $\mathcal{S}^q$ if and only if a quantity depending only on μ and q is finite.

As a particular case ($q = 2$), we have a characterization of the Hilbert–Schmidt embeddings.

Theorem 5.13. — *Let Λ be a quasi-geometric sequence. The following are equivalent*

(1) *i_μ^2 is a Hilbert–Schmidt operator ;*
(2) *$\int_{[0,1)} \frac{1}{1-t}\mathrm{d}\mu < +\infty$.*

In this case we have

$$\|i_\mu^2\|_{\mathcal{S}^2} \approx \left(\int_{[0,1)} \frac{1}{1-t}\mathrm{d}\mu\right)^{\frac{1}{2}}.$$

Proof 1. — We apply Proposition 5.12 in the case $q=2$. Fubini theorem gives

$$\|i_\mu^2\|_{\mathcal{S}^2}^2 \approx \int_0^1 \int_{t\in[0,1)} \frac{\mathrm{d}\mu(t)}{(1-st)^2}\mathrm{d}s = \int_{[0,1)} \frac{1}{1-t}\mathrm{d}\mu. \qquad \square$$

Proof 2. — Since order bounded and Hilbert–Schmidt operators are the same in an L^2-framework, Proposition 5.5 gives the result. $\square$

5.4. Examples

Now we give two examples inspired by [10], showing that in a strong way, boundedness and compactness of Carleson embeddings on Müntz spaces M_Λ^p depend in general on p and not only on Λ.

Example 5.14. — Let $p\in[1,+\infty)$. We are going to construct a lacunary sequence Λ and a measure μ on $[0,1)$ such that

(1) i_μ^q is not bounded when $q\in[1,p]$;
(2) i_μ^q is compact when $q\in(p,+\infty)$.

Proof. — Note that Λ cannot be a quasi-geometric sequence (see Corollary 4.9 and 5.8). We shall consider a measure μ with the form $\mu=\sum_{k\geqslant 2} c_k\delta_{x_k}$ where $x_k\in(0,1)$ and $c_k>0$.

We define $\lambda_2=1$, $(\lambda_n)_{n\geqslant 2}$ such that for any $n\geqslant 3$, $\lambda_n\geqslant n^{p+1}\lambda_{n-1}$. For $n\geqslant 2$ let $c_n=\frac{n^p\log(n)}{\lambda_n}$ and $x_n=1-\frac{\log(n)}{\lambda_n}$. We have $x_n^{\lambda_n}\sim\frac{1}{n}$ when $n\to+\infty$, and for $n\geqslant k$ we have $x_k^{\lambda_n}\lesssim\left(\frac{1}{k}\right)^{\frac{\lambda_n}{\lambda_k}}$. We check that μ does not satisfy (B_p):

$$\begin{aligned}\lambda_n\int_{[0,1)} t^{p\lambda_n}\mathrm{d}\mu = \sum_k \lambda_n c_k x_k^{p\lambda_n} &\geqslant \lambda_n c_n x_n^{p\lambda_n}\\ &\sim \lambda_n\frac{n^p\log(n)}{\lambda_n}\frac{1}{n^p} = \log(n)\to+\infty.\end{aligned}$$

Hence i_μ^p is not bounded.

On the other hand, for $q>p$, we have

$$\lambda_n\int_{[0,1)} t^{q\lambda_n}\mathrm{d}\mu = \sum_{k<n}\lambda_n c_k x_k^{q\lambda_n} + \lambda_n c_n x_n^{q\lambda_n} + \sum_{k>n}\lambda_n c_k x_k^{q\lambda_n}.$$

We control these three terms. For the first:

$$\sum_{k<n} \lambda_n c_k x_k^{q\lambda_n} \lesssim \sum_{k<n} \log(k) k^p \frac{\lambda_n}{\lambda_k} \left(\frac{1}{k^q}\right)^{\frac{\lambda_n}{\lambda_k}} \lesssim \sum_{k<n} \frac{\lambda_n}{\lambda_k} \left(\frac{1}{k^q}\right)^{\frac{\lambda_n}{\lambda_k}-1}.$$

Since $k \geqslant 2$ and $\frac{\lambda_n}{\lambda_{n-1}} \to +\infty$, this term tends to 0 when $n \to +\infty$. For the term $n = k$ we have $\lambda_n c_n x_n^{q\lambda_n} \sim \lambda_n \frac{n^p \log(n)}{\lambda_n} \frac{1}{n^q} = \frac{\log(n)}{n^{q-p}} \to 0$. For the last sum, $x_k \leqslant 1$ gives

$$\begin{aligned}\sum_{k>n} \lambda_n x_k^{q\lambda_n} c_k &\leqslant \sum_{k=n+1}^{+\infty} \lambda_n \frac{k^p \log(k)}{\lambda_k} \leqslant \sum_{k=n+1}^{+\infty} \frac{\log(k)}{k} \times \frac{\lambda_n}{\lambda_{k-1}} \\ &\lesssim \frac{\log(n)}{n} \sum_{k=n}^{+\infty} \frac{\lambda_n}{\lambda_k} \to 0.\end{aligned}$$

Thus, μ satisfies (b_q), and using Theorem 5.6, i_μ^r is compact for any $r > q$. □

Example 5.15. — Let $p \in (1, +\infty)$. We shall construct a lacunary sequence Λ and a measure μ on $[0,1)$ such that

(1) i_μ^q is not bounded when $q \in [1, p)$;
(2) i_μ^q is compact when $q \in [p, +\infty)$.

Proof. — We consider again a measure μ with the form $\mu = \sum_{k\geqslant 2} c_k \delta_{x_k}$. Let $\Lambda = (\lambda_n)_{n\geqslant 2}$ with $\lambda_2 = 1$, and for all $n \geqslant 3$, $\lambda_n \geqslant n^{p\max\{p,p'\}} \lambda_{n-1}$.

Let $c_n = \frac{n^p}{\lambda_n \log(n)}$ and $x_n = 1 - \frac{\log(n)}{\lambda_n}$. We have $x_n^{\lambda_n} \sim \frac{1}{n}$ when $n \to +\infty$, and for $n \geqslant k$ we have $x_k^{\lambda_n} \lesssim \left(\frac{1}{k}\right)^{\frac{\lambda_n}{\lambda_k}}$.

Let $q \in [1, p)$. We check that μ does not satisfy (B_q):

$$\lambda_n \int_{[0,1)} t^{q\lambda_n} \mathrm{d}\mu \geqslant \lambda_n c_n x_n^{q\lambda_n} \sim \lambda_n \frac{n^p}{\lambda_n \log(n)} \frac{1}{n^q} = \frac{n^{p-q}}{\log(n)} \to +\infty.$$

Hence i_μ^q is not bounded. On the other hand, let us show that the sequence $D_\mu^{(p)}(n)$ tends to 0 when $n \to +\infty$:

$$\begin{aligned}\left(D_\mu^{(p)}(n)\right)^p &= \sum_{j\in\mathbb{N}} \lambda_n^{\frac{1}{p}} c_j x_j^{\lambda_n} \left(\sum_k \lambda_k^{\frac{1}{p}} x_j^{\lambda_k}\right)^{p-1} \\ &\lesssim \lambda_n^{\frac{1}{p}} c_n x_n^{\lambda_n} \left(\sum_k \lambda_k^{\frac{1}{p}} x_n^{\lambda_k}\right)^{p-1} + \sum_{j\neq n} \lambda_n^{\frac{1}{p}} c_j x_j^{\lambda_n} \left(\frac{1}{1-x_j}\right)^{\frac{1}{p'}}\end{aligned}$$

using Lemma 2.10 and Remark 2.11 for the second term. We first control the second term. If $j > n$, $x_j^{\lambda_n} \leqslant 1$ gives

$$\sum_{j>n} \lambda_n^{\frac{1}{p}} c_j x_j^{\lambda_n} \left(\frac{1}{1-x_j}\right)^{\frac{1}{p'}} \leqslant \sum_{j>n} \lambda_n^{\frac{1}{p}} \frac{j^p}{\lambda_j} \frac{\lambda_j^{\frac{1}{p'}}}{\log(j)^{1+\frac{1}{p'}}} \leqslant \sum_{j>n} j^p \left(\frac{\lambda_n}{\lambda_j}\right)^{\frac{1}{p}} \leqslant \sum_{j>n} \frac{1}{j^p}$$

since $\lambda_j \geqslant j^{p^2} \lambda_{j-1}$. Hence this term tends to 0.

For $j < n$, since $x_j^{\lambda_n} \lesssim \left(\frac{1}{j}\right)^{\frac{\lambda_n}{\lambda_j}}$, we obtain:

$$\begin{aligned} \sum_{j<n} \lambda_n^{\frac{1}{p}} c_j x_j^{\lambda_n} \left(\frac{1}{1-x_j}\right)^{\frac{1}{p'}} &\lesssim \sum_{j<n} \lambda_n^{\frac{1}{p}} \frac{j^p}{\lambda_j} \left(\frac{1}{j}\right)^{\frac{\lambda_n}{\lambda_j}} \frac{\lambda_j^{\frac{1}{p'}}}{\log(j)^{1+\frac{1}{p'}}} \\ &\leqslant \sum_{j<n} \left(\frac{\lambda_n}{\lambda_j}\right)^{\frac{1}{p}} \left(\frac{1}{j}\right)^{\frac{\lambda_n}{\lambda_j}-p} \end{aligned}$$

and since $j \geqslant 2$ and $\frac{\lambda_n}{\lambda_{n-1}} \to +\infty$, this term tends to 0 when $n \to +\infty$.

To get an upper bound for the part "$j = n$", we split the sum in three terms:

$$\begin{aligned} \lambda_n^{\frac{1}{p}} c_n x_n^{\lambda_n} \left(\sum_k \lambda_k^{\frac{1}{p}} x_n^{\lambda_k}\right)^{p-1} \lesssim \lambda_n^{\frac{1}{p}} c_n x_n^{\lambda_n} \left(\sum_{k<n} \lambda_k^{\frac{1}{p}} x_n^{\lambda_k}\right)^{p-1} &+ \lambda_n x_n^{p\lambda_n} c_n \\ &+ \lambda_n^{\frac{1}{p}} c_n x_n^{\lambda_n} \left(\sum_{k>n} \lambda_k^{\frac{1}{p}} x_n^{\lambda_k}\right)^{p-1} \end{aligned}$$

For $k < n$, since $x_n \leqslant 1$ and $\lambda_{n-1} \geqslant 2^{p(n-1-k)} \lambda_k$, we obtain

$$\begin{aligned} \lambda_n^{\frac{1}{p}} c_n x_n^{\lambda_n} \left(\sum_{k<n} \lambda_k^{\frac{1}{p}} x_n^{\lambda_k}\right)^{p-1} &\lesssim \frac{\lambda_n^{\frac{1}{p}} n^p}{\log(n)\lambda_n} \frac{1}{n} \left(\sum_{k\leqslant n-1} \lambda_k^{\frac{1}{p}}\right)^{p-1} \\ &\lesssim \frac{n^{p-1}}{\log(n)} \left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{\frac{1}{p'}} \leqslant \frac{1}{n\log(n)} \end{aligned}$$

since $\lambda_n \geqslant \lambda_{n-1} n^{pp'}$.

For the term $n=k$, we have $\lambda_n x_n^{p\lambda_n} c_n \sim \frac{\lambda_n n^p}{n^p \lambda_n \log(n)} = \frac{1}{\log(n)} \to 0$. For $k>n$, since $x_n^{\lambda_k} \lesssim \left(\frac{1}{n}\right)^{\frac{\lambda_k}{\lambda_n}}$, we obtain

$$\lambda_n^{\frac{1}{p}} c_n x_n^{\lambda_n} \left(\sum_{k>n} \lambda_k^{\frac{1}{p}} x_n^{\lambda_k}\right)^{p-1} \lesssim \frac{n^{p-1}}{\log(n)} \lambda_n^{-\frac{1}{p'}} \left(\sum_{k>n} \lambda_k^{\frac{1}{p}} \left(\frac{1}{n}\right)^{\frac{\lambda_k}{\lambda_n}}\right)^{p-1}$$
$$\leqslant \left(\sum_{k>n} \left(\frac{\lambda_k}{\lambda_n}\right)^{\frac{1}{p}} \left(\frac{1}{n}\right)^{\frac{\lambda_k}{\lambda_n}-1}\right)^{p-1}$$

and this term tends to 0 since $\frac{\lambda_{n+1}}{\lambda_n} \to +\infty$. Indeed one might invoke Lebesgue domination theorem. In a simpler way, for n large enough and $k>n$, λ_k/λ_n is large enough to ensure

$$\left(\frac{\lambda_k}{\lambda_n}\right)^{\frac{1}{p}} \left(\frac{1}{n}\right)^{\frac{\lambda_k}{\lambda_n}-1} \leqslant \left(\frac{1}{n}\right)^{\frac{\lambda_k}{2\lambda_n}-1} \leqslant \frac{1}{2^k}$$

since $\lambda_k \geqslant k^4 \lambda_n \geqslant 2(k+1)\lambda_n$; and this suffices to conclude.

Thus, $D_\mu^{(p)}(n) \to 0$ when $n \to +\infty$. Since Λ is lacunary we can factorize i_μ^p as in Remark 4.3. We have $i_\mu^p = T_{\Lambda,\mu}^w \circ \left(T_\Lambda^w\right)^{-1}$ (recall that T_Λ^w is an isomorphism) and $T_{\Lambda,\mu}^w$ is compact thanks to Proposition 2.9. Hence Corollary 5.7 implies that i_μ^q is compact for any $q \geqslant p$. □

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Loïc GAILLARD
Laboratoire de Mathématiques de Lens (LML)
EA 2462
Fédération CNRS Nord-Pas-de-Calais FR 2956
Université d'Artois
rue Jean Souvraz S.P. 18
62307 Lens Cedex, France
loic.gaillard@univ-artois.fr

Pascal LEFÈVRE
Laboratoire de Mathématiques de Lens (LML)
EA 2462
Fédération CNRS Nord-Pas-de-Calais FR 2956
Université d'Artois
rue Jean Souvraz S.P. 18
62307 Lens Cedex, France
pascal.lefevre@univ-artois.fr