ANNALES

DE

L'INSTITUT FOURIER

Veronique FISCHER & Michael RUZHANSKY

Sobolev spaces on graded Lie groups

Tome 67, nº 4 (2017), p. 1671-1723.

<http://aif.cedram.org/item?id=AIF_2017__67_4_1671_0>

Ann. Inst. Fourier, Grenoble
67, 4 (2017) 1671-1723

SOBOLEV SPACES ON GRADED LIE GROUPS

by Veronique FISCHER & Michael RUZHANSKY (*)

Abstract. — In this article, we study the L^p-properties of powers of positive Rockland operators and define Sobolev spaces on general graded Lie groups. We establish that the defined Sobolev spaces are independent of the choice of a positive Rockland operator, and that they are interpolation spaces. Although this generalises the case of sub-Laplacians on stratified groups studied by G. Folland in [12], many arguments have to be different since Rockland operators are usually of higher degree than two. We also prove results regarding duality and Sobolev embeddings, together with inequalities of Hardy–Littlewood–Sobolev type and of Gagliardo–Nirenberg type.

Résumé. — Dans cet article, nous étudions les propriétés L^p des puissances des opérateurs de Rockland positifs et nous définissons les espaces de Sobolev sur tous les groupes de Lie nilpotents gradués. Nous montrons que les espaces de Sobolev ainsi définis sont indépendants du choix de l'opérateur de Rockland positif et qu'ils sont des espaces d'interpolation. Quoique cela généralise le cas des sous-laplaciens sur les groupes stratifiés étudiés par G. Folland dans [12], plusieurs arguments sont différents car les opérateurs de Rockland sont souvent de degrée plus haut que deux. Nous montrons aussi des résultats concernant la dualité et les injections de Sobolev, ainsi que des inégalités de type Littlewood–Sobolev et de type Gagliardo–Nirenberg.

1. Introduction

Sobolev spaces on $\mathbb{R}^n$ may be defined in various equivalent ways, e.g., from the elementary definition in the L^2 case which relies on the Euclidean Fourier transform to the Bessel potential L^p spaces using the properties

Keywords: Harmonic analysis on nilpotent Lie groups, Sobolev spaces, graded Lie groups, Rockland operators, heat semigroup.
Math. classification: 13A50, 43A32, 43A85, 43A90.
(*) The first author acknowledges the support of the London Mathematical Society via the Grace Chisholm Fellowship held at King's College London in 2011 and of the University of Padua (Italy). The second author was supported in part by the EPSRC Leadership Fellowship EP/G007233/1 and by EPSRC Grant EP/K039407/1. No new data was collected or generated during research.

of the Laplace operator or via Littlewood–Paley decomposition. It is quite natural to try and generalise these spaces to a non-Euclidean setting, for instance on Lie groups. The development of analysis on nilpotent Lie groups was initiated by G. Folland and E. Stein in [13], and G. Folland was the first to define and study Sobolev spaces on stratified (nilpotent Lie) groups [12], see also [23]. Using Littlewood–Paley decompositions as well as heat kernel estimates for sub-Laplacians [1, 25], this was generalised to Besov space on Lie groups of polynomial growth, see e.g. [15, 16]. Other properties of Sobolev spaces defined via sub-Laplacians on Lie groups have also been studied, see e.g. [7].

Our main purpose here is to define Sobolev spaces adapted to graded (nilpotent Lie) groups. These groups form a class containing the class of stratified groups, they are also endowed with a homogeneous structure and have polynomial growth of the volume. Although our analysis is closely related to Folland's [12], our results do not follow from consequences or direct adaptations of techniques in the cases of sub-Laplacians on stratified Lie groups or on Lie groups of polynomial volume growth [1, 7, 12, 15, 16, 25]. Indeed, on the one hand, our main object is not a sub-Laplacian but a positive Rockland operators $\mathcal{R}$, which is the natural sub-elliptic operators appearing on graded Lie groups. As in [12] (see also [7]), the Sobolev spaces are defined via the powers of $\mathrm{I}+\mathcal{R}$ using the theory of fractional operators mainly due to Komatsu and Balakrishnan (here we will use its exposition given in [21]), and the properties of the heat semigroup generated by $\mathcal{R}$. On the other hand, the operators $\mathcal{R}$ may be of high degree in contrast with the case of sub-Laplacians which are of degree two. This fact requires for example to give a different proof of the continuity of homogeneous left-invariant differential operators on the Sobolev spaces than in the stratified case, see Remark 4.15. The degree being possibly larger than two has also deeper implications: the operator may not have a unique homogeneous fundamental solution, see Remarks 3.5, 3.12 and 4.19, although it is already known that a parametrix can be constructed, see [5], and that the operator is injective on the tempered distributions modulo polynomials, see Corollary 2.11. Furthermore the corresponding heat semigroup may neither be a contraction on L^p-spaces nor preserve positivity, see Remarks 2.9, 3.5 and 3.10; this shows that the heat semi-group of $\mathcal{R}$ is not sub-markovian and, as suggested above, this rules out many of the techniques used in the case of sub-Laplacians.

These difficulties are part of the motivation to study Rockland operators on a deeper level. In fact the analysis of sub-Laplacians on stratified groups such as in [1, 12, 25] may be considered as born out of studying operators based on sums of squares of vector fields on manifolds, while the present analysis might help understanding the case of more general operators of higher order.

Naturally, when we consider a graded Lie group which is stratified, we recover the Sobolev spaces defined by Folland in [12] which then coincide with the Sobolev spaces obtained in [15] on any Lie group of polynomial growth. However, for a general graded (non-stratified) Lie group, our Sobolev spaces may differ from the ones in [15] built out of sub-Laplacians on Lie group of polynomial growth, see Section 4.8 in this paper. They are also slightly different from the Goodman–Sobolev spaces defined by Goodman on graded Lie groups for integer exponents only in [18, Section III.5.4], see again Section 4.8. Note that the Goodman–Sobolev spaces are not interpolation spaces while our spaces are.

Sobolev spaces associated with positive Rockland operators on graded Lie groups are also considered in [3] with the aim of obtaining Gagliardo–Nirenberg, refined Sobolev and Hardy inequalities in this context; the proofs are sketched by indicating modifications of the case studied in [4] of the canonical sub-Laplacian on the Heisenberg group. In our view, the definition and properties of Sobolev spaces used in [3] need the justifications which constitute one purpose for this article. We think that these justifications go beyond the addition of some (natural but non-trivial) properties of the fractional powers of positive Rockland operators and that, together with the other results of this paper, they shed light on this part of analysis. Beside obtaining the Sobolev embeddings and the Gagliardo–Nirenberg inequalities with a proof different from [3], we show for instance that the Sobolev spaces are independent of the positive Rockland operators considered to build the Sobolev spaces.

Our main tool is the heat semigroup associated with a positive Rockland operator. In order to make the paper as clear and self-contained as possible, we choose to rely only on the fundamental result due to Folland and Stein that their associated heat kernels are Schwartz functions together with its proof in [14, Chapter 4.B], and not on other properties of positive Rockland operators. Naturally we also use other general results in analysis on homogeneous Lie groups and in functional analysis, see e.g. [11]. We make no use of, for instance, the more precise Gaussian estimates for these heat kernels obtained later in [8, 9] and [2]; note that the proofs for sub-Laplacians [1, 12, 15, 25] also do not make use of those more precise estimates. We prefer to recover using classical methods the fact that the heat semigroup of a positive Rockland operator $\mathcal{R}$ is continuous on L^p-spaces, although this fact is a particular consequence of the more general (but more involved) result proved in [2]. This approach makes the paper as complete and accessible as possible, and has also the advantage of showing almost simultaneously other properties of the heat semigroup.

Our study of the heat semigroup enables us to apply the general theory of fractional powers of a generator of a semigroup and shows that the powers of $\mathcal{R}$ and of $\mathrm{I}+\mathcal{R}$ make sense as unbounded operators on L^p-spaces. This allows us to define the spaces $L^p_s(G)$ and $\dot{L}^p_s(G)$ on a graded Lie group G as the closure of the domains $\mathrm{Dom}(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ and $\mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$ for the Sobolev norms

$$\|f\|_{L^p_s} \equiv \|f\|_{L^p_s(G)} := \left\|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\right\|_{L^p(G)}$$

and

$$\|f\|_{\dot{L}^p_s} \equiv \|f\|_{\dot{L}^p_s(G)} := \left\|\mathcal{R}_p^{\frac{s}{\nu}} f\right\|_{L^p(G)},$$

respectively. Here ν is the homogeneous degree of $\mathcal{R}$. The most important properties proved in this paper are summarised in the following theorem. In view of these properties, we call the $L^p_s(G)$ spaces the (inhomogeneous) Sobolev spaces on G and $\dot{L}^p_s(G)$ the homogeneous Sobolev spaces on G. The notions used in Theorem 1.1 below, e.g. homogeneity, the notation for X^α, $[\alpha]$, etc... are explained in Section 2.

Theorem 1.1. — *Let G be a graded Lie group.*

(1) *For any $p \in (1,\infty)$ and $a \in \mathbb{R}$, the spaces $L^p_a(G)$ and $\dot{L}^p_a(G)$ are Banach spaces independent of the positive Rockland operators used to define them. Different choices of positive Rockland operators yield equivalent (inhomogeneous or homogeneous) Sobolev norms.*

The Schwartz space $\mathcal{S}(G)$ is dense in $L^p_a(G)$ for any $p \in (1,\infty)$ and $a \in \mathbb{R}$, and in $\dot{L}^p_a(G)$ for any $p \in (1,\infty)$ and $a \geqslant 0$.

(2) *We have the continuous inclusions for any $a, b \in \mathbb{R}$, $a > b$, and $p, q \in (1,\infty)$*

$$\mathcal{S}(G) \subset L^p_a(G) \subset L^p_b(G) \subset \mathcal{S}'(G)$$

$$L^p_a(G) \subset C(G), \qquad a > Q/p, \tag{1.1}$$

$$L^p_b \subset L^q_a \quad \text{and} \quad \dot{L}^p_b \subset \dot{L}^q_a, \qquad \frac{b-a}{Q} = \frac{1}{p} - \frac{1}{q}. \tag{1.2}$$

In particular, for $Q(\frac{1}{p}-1) < s < \frac{Q}{p}$, the members of $\dot{L}^p_s$ and L^p_s are locally integrable.

(3) *Let $a, b, c \in \mathbb{R}$ with $c \in (a,b)$. We define $\theta \in (0,1)$ via $c = \theta b + (1-\theta)a$. For any $p \in (1,\infty)$, there exists a constant $C = C_{a,b,c,p} > 0$*

such that for any $f \in L^p_a(G) \cap L^p_b(G)$ *or* $f \in \dot{L}^p_a(G) \cap \dot{L}^p_b(G)$ *we have, respectively,*

$$\|f\|_{L^p_c} \leqslant C \|f\|^{1-\theta}_{L^p_a} \|f\|^{\theta}_{L^p_b} \quad \text{or} \quad \|f\|_{\dot{L}^p_c} \leqslant C \|f\|^{1-\theta}_{\dot{L}^p_a} \|f\|^{\theta}_{\dot{L}^p_b}.$$

(4) *If* $q, r \in (1, \infty)$ *and* $0 < \sigma < s$ *then there exists* $C > 0$ *such that we have*

$$\forall\, f \in L^q(G) \cap \dot{L}^r_s(G) \qquad \|f\|_{\dot{L}^p_\sigma} \leqslant C \|f\|^{\theta}_{L^q} \|f\|^{1-\theta}_{\dot{L}^r_s},$$

where $\theta := 1 - \frac{\sigma}{s}$ *and* $p \in (1, \infty)$ *is given via* $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$.

(5) *If* $p \in (1, \infty)$ *and* $a > 0$, *then* $L^p_a(G) = L^p(G) \cap \dot{L}^p_a(G)$ *and, after a choice of positive Rockland operators to realise the Sobolev norms, the Sobolev norm* $\|\cdot\|_{L^p_a(G)}$ *is equivalent to the norm given by* $\|\cdot\|_{L^p(G)} + \|\cdot\|_{\dot{L}^p_a(G)}$.

(6) *If* $s = \nu\ell$ *with* $\ell \in \mathbb{N}_0$, ν *being the homogeneous degree of a positive Rockland operator, and if* $p \in (1, \infty)$, *then the space* $L^p_{\nu\ell}(G)$ *is the collection of functions* $f \in L^p(G)$ *such that* $X^\alpha f \in L^p(G)$ *for any* $\alpha \in \mathbb{N}_0^n$ *with* $[\alpha] = \nu\ell$, *and the map* $\|\cdot\|_p + \sum_{[\alpha]=\nu\ell} \|X^\alpha \cdot\|_p$ *is a norm on* $L^p_{\nu\ell}(G)$ *which is equivalent to the Sobolev norm.*

With the same hypotheses, the space $\dot{L}^p_{\nu\ell}(G)$ *coincides with the completion of* $\mathcal{S}(G)$ *for the map* $\sum_{[\alpha]=\nu\ell} \|X^\alpha \cdot\|_p$, *which yields a norm equivalent to the homogeneous Sobolev norm.*

(7) *If* T *is any homogeneous left-invariant differential operator on* G *of homogeneous degree* d, *then* T *maps continuously* $L^p_s(G)$ *to* $L^p_{s-d}(G)$ *and* $\dot{L}^p_s(G)$ *to* $\dot{L}^p_{s-d}(G)$, *for any* $s \in \mathbb{R}$ *and* $p \in (1, \infty)$.

(8) *If* $\mathcal{R}$ *be a positive Rockland operator of homogeneous degree* ν, *then, for any* $a, s \in \mathbb{R}$ *and* $p \in (1, \infty)$, *the operator* $(\mathrm{I} + \mathcal{R})^{\frac{a}{\nu}}$ *maps continuously* $L^p_s(G)$ *to* $L^p_{s-a}(G)$ *and the operator* $\mathcal{R}^{\frac{a}{\nu}}$ *maps continuously* $\dot{L}^p_s(G)$ *to* $\dot{L}^p_{s-a}(G)$.

(9) *For any* $s \in \mathbb{R}$ *and* $p \in (1, \infty)$, *the Banach spaces* $L^{p'}_{-s'}(G)$ *and* $\dot{L}^{p'}_{-s}(G)$ *are the duals of* $L^p_s(G)$ *and* $\dot{L}^p_s(G)$ *respectively.*

(10) *The inhomogeneous and homogeneous Sobolev spaces satisfy the properties of interpolation in the sense of Theorem 4.8 and Proposition 4.13 respectively.*

In Part (9) as in the rest of the paper, if $p \in (1, \infty)$ then $p' \in (1, \infty)$ is its conjugate exponent given via

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The inclusion in (1.1) may be viewed as a *Sobolev embedding*, $C(G)$ denoting the Banach space of continuous bounded functions on G. The inclusions in (1.2) yield the *Hardy–Littlewood–Sobolev inequality* in this context. The inequality in Part (4) may be called the *Gagliardo–Nirenberg inequality*.

Our analysis also yields other results. For instance, the case $p = 2$ yields Hilbert spaces, see Section 4.7. Also the limiting case $p = 1$ may be included in the homogeneous inclusion in (1.2) with $b = 0$ and in Part (4) above if one replaces $L^1(G)$ with the weak-L^1 space $L^{1,\infty}(G)$. However, for the sake of clarity, we present mainly the case L^p, $p \in (1, \infty)$ in order to present unified results.

This paper is organised as follows. After some preliminaries about graded Lie groups and their homogeneous structure in Section 2, we first define the fractional powers of a positive Rockland operator in Section 3, as well as its Riesz and Bessel potentials. This enables us to define our (inhomogeneous and homogeneous) Sobolev spaces in Section 4, where we also show that they satisfy the properties expected from Sobolev spaces.

2. Preliminaries

In this section, after defining graded Lie groups, we recall their homogeneous structure as well as the definition and some properties of their Rockland operators.

2.1. Graded and homogeneous Lie groups

Here we recall briefly the definition of graded nilpotent Lie groups and their natural homogeneous structure. A complete description of the notions of graded and homogeneous nilpotent Lie groups may be found in [11] or [14, Chapter 1], see also [12].

We will be concerned with graded Lie groups G which means that G is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits an $\mathbb{N}$-gradation $\mathfrak{g} = \oplus_{\ell=1}^{\infty} \mathfrak{g}_\ell$ where the $\mathfrak{g}_\ell$, $\ell = 1, 2, \ldots$, are vector subspaces of $\mathfrak{g}$, almost all equal to $\{0\}$, and satisfying $[\mathfrak{g}_\ell, \mathfrak{g}_{\ell'}] \subset \mathfrak{g}_{\ell+\ell'}$ for any $\ell, \ell' \in \mathbb{N}$. This implies that the group G is nilpotent. Examples of such groups are the Heisenberg group and, more generally, all stratified groups (which by definition correspond to the case $\mathfrak{g}_1$ generating the full Lie algebra $\mathfrak{g}$).

We construct a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ adapted to the gradation, by choosing a basis $\{X_1, \ldots X_{n_1}\}$ of $\mathfrak{g}_1$ (this basis is possibly reduced to $\emptyset$), then $\{X_{n_1+1}, \ldots, X_{n_1+n_2}\}$ a basis of $\mathfrak{g}_2$ (possibly $\{0\}$ as well as the others) and so on. Via the exponential mapping $\exp_G : \mathfrak{g} \to G$, we identify the points $(x_1, \dots, x_n) \in \mathbb{R}^n$ with the points $x = \exp_G(x_1 X_1 + \dots + x_n X_n)$ in G. Consequently we allow ourselves to denote by $C(G)$, $\mathcal{D}(G)$ and $\mathcal{S}(G)$ etc, the spaces of continuous functions, of smooth and compactly supported functions or of Schwartz functions on G identified with $\mathbb{R}^n$, and similarly for distributions with the duality notation $\langle \cdot, \cdot \rangle$.

This basis also leads to a corresponding Lebesgue measure on $\mathfrak{g}$ and the Haar measure dx on the group G, hence $L^p(G) \cong L^p(\mathbb{R}^n)$. The group convolution of two functions f and g, for instance integrable, is defined via

$$(f * g)(x) := \int_G f(y) g(y^{-1}x)\, \mathrm{d}y.$$

The convolution is not commutative: in general, $f * g \neq g * f$. However, apart from the lack of commutativity, group convolution and the usual convolution on $\mathbb{R}^n$ share many properties. For example, we have

$$\langle f * g, h \rangle = \langle f, h * \tilde{g} \rangle, \quad \text{with} \quad \tilde{g}(x) = g(x^{-1}). \tag{2.1}$$

And the Young convolutions inequalities hold: if $f_1 \in L^p(G)$ and $f_2 \in L^q(G)$ with $1 \leqslant p, q, r \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, then $f_1 * f_2 \in L^r(G)$ and

$$\|f_1 * f_2\|_r \leqslant \|f_1\|_p \|f_2\|_q . \tag{2.2}$$

The coordinate function $x = (x_1, \dots, x_n) \in G \mapsto x_j \in \mathbb{R}$ is denoted by x_j. More generally we define for every multi-index $\alpha \in \mathbb{N}_0^n$, $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, as a function on G. Similarly we set $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ in the universal enveloping Lie algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$.

For any $r > 0$, we define the linear mapping $D_r : \mathfrak{g} \to \mathfrak{g}$ by $D_r X = r^\ell X$ for every $X \in \mathfrak{g}_\ell$, $\ell \in \mathbb{N}$. Then the Lie algebra $\mathfrak{g}$ is endowed with the family of dilations $\{D_r, r > 0\}$ and becomes a homogeneous Lie algebra in the sense of [14]. We re-write the set of integers $\ell \in \mathbb{N}$ such that $\mathfrak{g}_\ell \neq \{0\}$ into the increasing sequence of positive integers $\upsilon_1, \dots, \upsilon_n$ counted with multiplicity, the multiplicity of $\mathfrak{g}_\ell$ being its dimension. In this way, the integers $\upsilon_1, \dots, \upsilon_n$ become the weights of the dilations and we have $D_r X_j = r^{\upsilon_j} X_j$, $j = 1, \dots, n$, on the chosen basis of $\mathfrak{g}$. The associated group dilations are defined by

$$D_r(x) = r \cdot x := (r^{\upsilon_1} x_1, r^{\upsilon_2} x_2, \dots, r^{\upsilon_n} x_n), \quad x = (x_1, \dots, x_n) \in G,\ r > 0.$$

In a canonical way this leads to the notions of homogeneity for functions and operators. For instance the degree of homogeneity of x^α and X^α, viewed respectively as a function and a differential operator on G, is

$$[\alpha] = \sum_j \upsilon_j \alpha_j.$$

Indeed, let us recall that a vector of $\mathfrak{g}$ defines a left-invariant vector field on G and, more generally, that the universal enveloping Lie algebra of $\mathfrak{g}$ is isomorphic with the left-invariant differential operators; we keep the same notation for the vectors and the corresponding operators.

Recall that a *homogeneous quasi-norm* on G is a continuous function $|\cdot| : G \to [0, +\infty)$ homogeneous of degree 1 on G which vanishes only at 0. This often replaces the Euclidean norm in the analysis on homogeneous Lie groups:

Proposition 2.1.

(1) *Any homogeneous quasi-norm $|\cdot|$ on G satisfies a triangle inequality up to a constant:*

$$\exists\, C \geqslant 1 \quad \forall\, x, y \in G \quad |xy| \leqslant C(|x| + |y|).$$

It partially satisfies the reverse triangle inequality:

$$\forall\, b \in (0,1) \quad \exists\, C = C_b \geqslant 1 \quad \forall\, x, y \in G \qquad |y| \leqslant b|x| \Longrightarrow \big||xy| - |x|\big| \leqslant C|y|. \tag{2.3}$$

(2) *Any two homogeneous quasi-norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent in the sense that*

$$\exists\, C > 0 \quad \forall\, x \in G \quad C^{-1}|x|_2 \leqslant |x|_1 \leqslant C|x|_2.$$

(3) *A concrete example of a homogeneous quasi-norm is given via*

$$|x|_{\nu_o} := \Big(\sum_{j=1}^{n} x_j^{2\nu_o/\upsilon_j} \Big)^{1/2\nu_o},$$

with ν_o a common multiple to the weights $\upsilon_1, \ldots, \upsilon_n$.

Various aspects of analysis on G can be developed in a comparable way with the Euclidean setting sometimes replacing the topological dimension $n = \sum_{\ell=1}^{\infty} \dim \mathfrak{g}_\ell$ of the group G by its *homogeneous dimension*

$$Q := \sum_{\ell=1}^{\infty} \ell \dim \mathfrak{g}_\ell = \upsilon_1 + \upsilon_2 + \ldots + \upsilon_n.$$

For example, there is an analogue of polar coordinates on homogeneous groups with Q replacing n:

Proposition 2.2. — *Let $|\cdot|$ be a fixed homogeneous quasi-norm on G. Then there is a (unique) positive Borel measure σ on the unit sphere $\mathfrak{S} := \{x \in G : |x| = 1\}$, such that for all $f \in L^1(G)$, we have*

$$\int_G f(x)\,\mathrm{d}x = \int_0^\infty \int_{\mathfrak{S}} f(ry) r^{Q-1}\,\mathrm{d}\sigma(y)\,\mathrm{d}r. \tag{2.4}$$

Another example is the following property regarding kernels or operators of type ν (see [12] and [14, Chapter 6A]):

Definition 2.3. — *A distribution $\kappa \in \mathcal{D}'(G)$ which is smooth away from the origin and homogeneous of degree $\nu - Q$ is called a* kernel of type *$\nu \in \mathbb{C}$ on G. The corresponding convolution operator $f \in \mathcal{D}(G) \mapsto f * \kappa$ is called an* operator of type *ν.*

The next statement summarise the properties of the operators of type ν used in the paper:

Proposition 2.4. — *Let G be a graded group.*

(1) *An operator of type ν with $\nu \in [0, Q)$ is $(-\nu)$-homogeneous and extends to a bounded operator from $L^p(G)$ to $L^q(G)$ whenever $p, q \in (1, \infty)$ satisfy $\frac{1}{p} - \frac{1}{q} = \frac{\Re\nu}{Q}$.*
(2) *Let κ be a smooth function away from the origin homogeneous of degree ν with $\Re\nu = -Q$. Then κ is a kernel of type ν, that is, κ coincides on $G\backslash\{0\}$ with the restriction of a distribution in $\mathcal{D}'(G)$ by definition, if and only if its mean value is zero, that is, when $\int_{\mathfrak{S}} \kappa \, \mathrm{d}\sigma = 0$ where σ is the measure on the unit sphere of a homogeneous quasi-norm given by the polar change of coordinates, see Proposition 2.2. (This condition is independent of the choice of a homogeneous quasi-norm.)*
(3) *Let κ be a kernel of type $s \in [0, Q)$. Let T be a homogeneous left differential operator of degree ν_T. If $s - \nu_T \in [0, Q)$, then $T\kappa$ is a kernel of type $s - \nu_T$.*
(4) *Suppose κ_1 is a kernel of type $\nu_1 \in \mathbb{C}$ with $\Re\nu_1 > 0$ and κ_2 is a kernel of type $\nu_2 \in \mathbb{C}$ with $\Re\nu_2 \geqslant 0$. We assume $\Re(\nu_1 + \nu_2) < Q$. Then $\kappa_1 * \kappa_2$ is well defined as a kernel of type $\nu_1 + \nu_2$. Moreover if $f \in L^p(G)$ where $1 < p < Q/(\Re(\nu_1 + \nu_2))$ then $(f * \kappa_1) * \kappa_2$ and $f * (\kappa_1 * \kappa_2)$ belong to $L^q(G)$, $\frac{1}{q} = \frac{1}{p} - \frac{\Re(\nu_1+\nu_2)}{Q}$, and they are equal.*

We will also need the following theorem which is a classical consequence of the theorem of singular integrals in the context of spaces of homogeneous type [6, Chapter III]:

Theorem 2.5. — *Let T be an operator bounded on $L^2(G)$ and invariant under left translation. Let $\kappa \in \mathcal{S}'(G)$ be its right convolution kernel, that is, $T\phi = \phi * \kappa$ for any $\phi \in \mathcal{S}(G)$.*

We assume that κ coincides with a continuously differentiable function on $G\backslash\{0\}$ and that for one (then any) choice of quasi-norm $|\cdot|$ on G, the *following quantities*

$$\int_{|x|\geqslant 1} |\kappa(x)|\,\mathrm{d}x, \qquad \sup_{0<|x|<1} |x|^{Q+\upsilon_j}|X_j\kappa(x)|, \quad j=1,\ldots,n,$$

and

$$\sup_{0<|x|<1} |x|^{Q}|\kappa(x)|,$$

are finite. Then T is bounded on $L^p(G)$ for any $p>2$.

The approximations of the identity may be constructed on G as on their Euclidean counterpart, replacing the topological dimension and the abelian convolution with the homogeneous dimension and the group convolution:

Lemma 2.6. — *Let $\phi \in L^1(G)$. Then the functions ϕ_t, $t>0$, defined via $\phi_t(x) = t^{-Q}\phi(t^{-1}x)$, are integrable and $\int \phi_t = \int \phi$ is independent of t. Furthermore, for any f in $L^p(G)$, $C_o(G)$, $\mathcal{S}(G)$ or $\mathcal{S}'(G)$, the sequence of functions $f*\phi_t$ and $\phi_t * f$, $t>0$, converges towards $(\int\phi)\,f$ as $t\to 0$ in $L^p(G)$, $C_o(G)$, $\mathcal{S}(G)$ and $\mathcal{S}'(G)$ respectively.*

In Lemma 2.6 and in the whole paper, $C_o(G)$ denotes the space of continuous functions on G which vanish at infinity. This means that $f \in C_o(G)$ when for every $\epsilon>0$ there exists a compact set K outside which we have $|f|<\epsilon$. Endowed with the supremum norm $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(G)}$, it is a Banach space.

Recall that $\mathcal{D}(G)$, the space of smooth and compactly supported functions, is dense in $L^p(G)$ for $p\in[1,\infty)$ and in $C_o(G)$ (in which case we set $p=\infty$).

In Theorem 2.8, we will see that the heat semi-group associated to a positive Rockland operator gives an approximation of the identity which is commutative.

2.2. Rockland operators

Here we recall the definition of Rockland operators and their main properties.

The definition of a Rockland operator uses the representations of the group. Here we consider only continuous unitary representations of G. We will often denote by π such a representation, by $\mathcal{H}_\pi$ its Hilbert space and by $\mathcal{H}_\pi^\infty$ the subspace of smooth vectors. The corresponding infinitesimal representation on the Lie algebra $\mathfrak{g}$ and its extension to the universal enveloping Lie algebra $\mathfrak{U}(\mathfrak{g})$ are also denoted by π. We recall that $\mathfrak{g}$ and $\mathfrak{U}(\mathfrak{g})$ are identified with the spaces of left-invariant vector fields and of left-invariant differential operators on G respectively.

Definition 2.7. — *A* Rockland operator *on G is a left-invariant differential operator $\mathcal{R}$ which is homogeneous of positive degree and satisfies the Rockland condition:*

(R) *for each unitary irreducible representation π on G, except for the trivial representation, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$, that is,*

$$\forall\, v \in \mathcal{H}_\pi^\infty \qquad \pi(\mathcal{R})v = 0 \implies v = 0.$$

Although the definition of a Rockland operator would make sense on a homogeneous Lie group (in the sense of [14]), it turns out (see [22], see also [10, Lemma 2.2]) that the existence of a (differential) Rockland operator on a homogeneous group implies that the homogeneous group may be assumed to be graded.

Some authors may have different conventions than ours regarding Rockland operators: for instance some choose to consider right-invariant operators and some definitions of a Rockland operator involves only the principal part. The analysis however would be exactly the same.

In 1979, Helffer and Nourrigat proved in [19] that the property in (R) is equivalent to the hypoellipticity of the operator. Rockland operators may be viewed as an analogue of elliptic operators (with a high degree of homogeneity) in a non-abelian subelliptic context. In the stratified case, one can check easily that any (left-invariant negative) *sub-Laplacian*, that is

$$\mathcal{L} = Z_1^2 + \ldots + Z_{n'}^2 \tag{2.5}$$

with $Z_1, \ldots, Z_{n'}$ forming any basis of the first stratum $\mathfrak{g}_1$, is a Rockland operator. More generally it is not difficult to see that the operator

$$\sum_{j=1}^{n} (-1)^{\frac{\nu_o}{\upsilon_j}} c_j X_j^{2\frac{\nu_o}{\upsilon_j}} \quad \text{with} \quad c_j > 0, \tag{2.6}$$

is a Rockland operator of homogeneous degree $2\nu_o$ if ν_o is any common multiple of $\upsilon_1, \ldots, \upsilon_n$. Hence Rockland operators do exist on any graded Lie group (not necessarily stratified). Furthermore, if $\mathcal{R}$ is a Rockland operator, then one can show easily that its powers $\mathcal{R}^k$, $k \in \mathbb{N}$, and its complex conjugate $\bar{\mathcal{R}}$ are also Rockland operators.

If a Rockland operator $\mathcal{R}$ which is formally self-adjoint, that is, $\mathcal{R}^* = \mathcal{R}$ as elements of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, is fixed, then it admits a self-adjoint extension on $L^2(G)$ [14, p. 131]. In this case we will denote by $\mathcal{R}_2$ the self-adjoint extension and by E its spectral measure:

$$\mathcal{R}_2 = \int_{\mathbb{R}} \lambda \, \mathrm{d}E(\lambda). \tag{2.7}$$

2.3. Positive Rockland operators and their heat kernels

In this section we summarise properties of positive Rockland operators that are important for our analysis.

A Rockland operator $\mathcal{R}$ is positive when $\mathcal{R}$ is formally self-adjoint, that is, $\mathcal{R}^* = \mathcal{R}$ in $\mathfrak{U}(\mathfrak{g})$, and satisfies

$$\forall\, f \in \mathcal{D}(G) \qquad \int_G \mathcal{R}f(x)\overline{f(x)} \, \mathrm{d}x \geqslant 0.$$

Note that if G is stratified and $\mathcal{L}$ is a (left-invariant negative) sub-Laplacian, then $-\mathcal{L}$ is a positive Rockland operator. The example in (2.6) is a positive Rockland operator. Hence positive Rockland operators always exist on any graded Lie group. Moreover if $\mathcal{R}$ is a positive Rockland operator, then its powers $\mathcal{R}^k$, $k \in \mathbb{N}$, and its complex conjugate $\bar{\mathcal{R}}$ are also positive Rockland operators.

Let us fix a positive Rockland operator $\mathcal{R}$ on G. By functional calculus (see (2.7)), we can define the spectral multipliers

$$e^{-t\mathcal{R}_2} := \int_0^\infty e^{-t\lambda} \, \mathrm{d}E(\lambda), \quad t > 0,$$

which form the *heat semigroup* of $\mathcal{R}$. The operators $e^{-t\mathcal{R}_2}$ are invariant under left-translations and are bounded on $L^2(G)$. Therefore the Schwartz kernel theorem implies that each operator $e^{-t\mathcal{R}_2}$ admits a unique distribution $h_t \in \mathcal{S}'(G)$ as its convolution kernel:

$$e^{-t\mathcal{R}_2} f = f * h_t, \quad t > 0, \ f \in \mathcal{S}(G).$$

The distributions h_t, $t > 0$, are called the *heat kernels* of $\mathcal{R}$. We summarise their main properties in the following theorem:

THEOREM 2.8. — *Let $\mathcal{R}$ be a positive Rockland operator on G which is homogeneous of degree $\nu \in \mathbb{N}$. Then each distribution h_t is Schwartz and* *we have:*

$$\forall\, s,t>0 \qquad h_t * h_s = h_{t+s}, \tag{2.8}$$

$$\forall\, x\in G,\ t,r>0 \qquad h_{r^\nu t}(rx) = r^{-Q} h_t(x), \tag{2.9}$$

$$\forall\, x\in G \qquad h_t(x) = \overline{h_t(x^{-1})}, \tag{2.10}$$

$$\int_G h_t(x)\,\mathrm{d}x = 1. \tag{2.11}$$

The function $h: G\times\mathbb{R}\to\mathbb{C}$ *defined by*

$$h(x,t) := \begin{cases} h_t(x) & \text{if } t>0 \text{ and } x\in G,\\ 0 & \text{if } t\leqslant 0 \text{ and } x\in G,\end{cases}$$

is smooth on $(G\times\mathbb{R})\backslash\{(0,0)\}$ *and satisfies* $(\mathcal{R}+\partial_t)h=\delta_{0,0}$ *where* $\delta_{0,0}$ *is the delta-distribution at* $(0,0)\in G\times\mathbb{R}$*. Having fixed a homogeneous quasi-norm* $|\cdot|$ *on* G*, we have for any* $N\in\mathbb{N}_0$, $\alpha\in\mathbb{N}_0^n$ *and* $\ell\in\mathbb{N}_0$*:*

$$\exists\, C=C_{\alpha,N,\ell}>0 \quad \forall\, t\in(0,1] \quad \sup_{|x|=1} |\partial_t^\ell X^\alpha h_t(x)| \leqslant C_{\alpha,N} t^N. \tag{2.12}$$

Consequently

$$\forall\, x\in G,\ t>0 \qquad h_t(x) = t^{-\frac{Q}{\nu}} h_1(t^{-\frac{1}{\nu}}x), \tag{2.13}$$

and for $x\in G\backslash\{0\}$ *fixed,*

$$X_x^\alpha h(x,t) = \begin{cases} O(t^{-\frac{Q+[\alpha]}{\nu}}) \text{ as } t\to\infty,\\ O(t^N) \text{ for all } N\in\mathbb{N}_0 \text{ as } t\to 0.\end{cases} \tag{2.14}$$

Inequalities (2.14) *are also valid for any* x *in a fixed compact subset of* $G\backslash\{0\}$.

Theorem 2.8 was proved by Folland and Stein in [14, Chapter 4.B].

Remark 2.9. — If the group is stratified and $\mathcal{R}=-\mathcal{L}$ where $\mathcal{L}$ is a (negative) sub-Laplacian, then $\mathcal{R}$ is of order two and the proof relies on Hunt's theorem [20]. The heat kernel is not only real-valued but also non-negative. Furthermore the heat semigroup is positivity preserving and is a contraction not only on L^2 but on each L^p-space. Cf. [12, Theorem 3.1] and [14, Chapter 1.G].

More precise estimates of heat kernels of general positive Rockland operators were obtained in [8, 9] and [2] but we will not use them in this paper.

In the proof by Folland and Stein in [14, Chapter 4.B], the following technical property, which we will use later on, is also shown:

LEMMA 2.10. — *Let $\mathcal{R}$ be a positive Rockland operator of a graded Lie group $G \sim \mathbb{R}^n$ with homogeneous degree ν. If m is a positive integer such that $m\nu \geqslant \frac{n}{2}$, then the functions in the domain of $\mathcal{R}^m$ are continuous on Ω and for any compact subset Ω of G, there exists a constant $C = C_{\Omega,\mathcal{R},G,m}$ such that*

$$\forall\, \phi \in \mathrm{Dom}(\mathcal{R}^m) \qquad \sup_{x\in\Omega} |\phi(y)| \leqslant C \left(\|\phi\|_{L^2} + \|\mathcal{R}^m \phi\|_{L^2} \right).$$

This is a weak form of Sobolev embeddings. We will later on obtain stronger results of this kind in Theorem 4.24.

We end this section with the following result of Liouville's type:

THEOREM 2.11. — *If $\mathcal{R}$ is a positive Rockland operator and $f \in \mathcal{S}'(G)$ a distribution satisfying $\mathcal{R}f = 0$ then f is a polynomial.*

Proof. — As $\mathcal{R}$ is a positive Rockland operator, $\bar{\mathcal{R}} = \mathcal{R}^t$ is also Rockland and they are both hypoelliptic, see [19]. The conclusion follows by applying the Liouville theorem for homogeneous Lie groups proved by Geller in [17]. □

3. Fractional powers of positive Rockland operators

This section is devoted to the fractional powers of positive Rockland operators. We will carry out the construction on the scale of L^p-spaces for $1 \leqslant p \leqslant \infty$, with $L^\infty(G)$ substituted by the space $C_o(G)$ of continuous functions vanishing at infinity. Then we discuss the essential properties of such an extension. Eventually we define its complex powers - in particular purely imaginary - and the associated Riesz and Bessel potentials.

3.1. Positive Rockland operators on L^p

Here we define and study the analogue $\mathcal{R}_p$ of the operator $\mathcal{R}$ on $L^p(G)$ or $C_o(G)$. This analogue will be defined as the infinitesimal generator of the heat convolution semigroup. Hence we start by proving the following properties:

PROPOSITION 3.1. — *The operators $f \mapsto f * h_t$, $t > 0$, form a strongly continuous semi-group on $L^p(G)$ for any $p \in [1,\infty)$ and on $C_o(G)$ if $p = \infty$. This semi-group is also equibounded:*

$$\forall\, t > 0,\ \forall\, f \in L^p(G) \text{ or } C_o(G) \qquad \|f * h_t\|_p \leqslant \|h_1\|_1 \|f\|_p .$$

Furthermore for any $p \in [1,\infty]$ (finite or infinite) and any $f \in \mathcal{D}(G)$,

$$\lim_{t\to 0}\left\|\frac{1}{t}(f*h_t - f) - \mathcal{R}f\right\|_p = 0. \tag{3.1}$$

Proof of Proposition 3.1. — If $f \in \mathcal{D}(G)$, then $f \in \mathrm{Dom}(\mathcal{R}) \subset \mathrm{Dom}(\mathcal{R}_2)$ and for any $s,t>0$, by functional calculus,

$$f*h_{t+s} = e^{-(t+s)\mathcal{R}_2}f = e^{-t\mathcal{R}_2}e^{-s\mathcal{R}_2}f = (f*h_s)*h_t$$

and, by the Young convolution inequalities for $p \in [1,\infty]$ (see (2.2)),

$$\|f*h_t\|_p \leqslant \|h_t\|_1\,\|f\|_p$$

with $\|h_t\|_1 = \|h_1\|_1 < \infty$ by Theorem 2.8. By density of $\mathcal{D}(G)$ in $L^p(G)$ for $p\in[1,\infty)$ and $C_o(G)$ for $p=\infty$, this implies that the operators $f\mapsto f*h_t$, $t>0$, form a strongly continuous equibounded semi-group on $L^p(G)$ for any $p\in[1,\infty)$ and on $C_o(G)$.

Let us prove the convergence in (3.1) for $p=\infty$ for a function $f\in\mathcal{D}(G)$. First let us prove that for any compact subset $\Omega\subset G$, we have

$$\sup_{\Omega}\left|\frac{1}{t}\left(f*h_t - f\right) - \mathcal{R}f\right| \longrightarrow_{t\to 0} 0. \tag{3.2}$$

Since $\mathcal{D}(G)\subset\mathrm{Dom}(\mathcal{R})$ and $e^{-t\mathcal{R}_2}f = f*h_t$, we have for any integer $m'\in\mathbb{N}_0$:

$$\begin{aligned}
\frac{1}{t}\mathcal{R}^{m'}\left(f*h_t - f\right) - \mathcal{R}^{m'+1}f &= \frac{1}{t}\mathcal{R}_2^{m'}\left(e^{-t\mathcal{R}_2}f - f\right) - \mathcal{R}_2^{m'+1}f\\
&= \frac{1}{t}\left(e^{-t\mathcal{R}_2}\mathcal{R}_2^{m'}f - \mathcal{R}_2^{m'}f\right) - \mathcal{R}_2^{m'+1}f\\
&= \frac{1}{t}\left((\mathcal{R}^{m'}f)*h_t - \mathcal{R}^{m'}f\right) - \mathcal{R}^{m'+1}f,
\end{aligned}$$

and this tends to 0 in $L^2(G)$ as $t\to 0$. Now (3.2) follows from this L^2-convergence and applying Lemma 2.10, to $\phi = \frac{1}{t}\left(f*h_t - f\right) - \mathcal{R}f$.

We fix a homogeneous quasi-norm $|\cdot|$ on G, for example the one in Part (3) of Proposition 2.1. Denoting $\bar{B}_R := \{x\in G,\ |x|\leqslant R\}$ the closed ball about 0 of radius R, we choose $R\geqslant 1$ such that $\bar{B}_R$ contains the support of f. Let $C_o = C_b$ be the constant in the reverse triangle inequality, see (2.3), for $b=\frac{1}{2}$. We choose $\Omega = \bar{B}_{2C_oR}$ the closed ball about 0 and with radius $2C_oR$. If $x\notin\Omega$, then since f is supported in $\bar{B}_R\subset\Omega$,

$$\left(\frac{1}{t}\left(f*h_t - f\right) - \mathcal{R}f\right)(x) = \frac{1}{t}f*h_t(x) = \frac{1}{t}\int_{|y|\leqslant R} f(y)h_t(y^{-1}x)\,\mathrm{d}y,$$

hence

$$\left|\frac{1}{t}f*h_t(x)\right| \leqslant \frac{\|f\|_\infty}{t}\int_{|y|\leqslant R}|h_t(y^{-1}x)|\,\mathrm{d}y = \frac{\|f\|_\infty}{t}\int_{|xt^{\frac{1}{\nu}}z^{-1}|\leqslant R}|h_1(z)|\,\mathrm{d}z,$$

as h_t satisfies (2.13). Simple manipulations, more precisely the reverse triangle inequality in (2.3) and the polar change of variable in Proposition 2.2, yield

$$\begin{aligned}\int_{|xt^{\frac{1}{\nu}}z^{-1}|\leqslant R}|h_1(z)|\,\mathrm{d}z &\leqslant \int_{|z|>t^{-\frac{1}{\nu}}R/2}|h_1(z)|\,\mathrm{d}z\\ &\leqslant \left\||\cdot|^{Q+2\nu}h_1\right\|_\infty \int_{|z|>t^{-\frac{1}{\nu}}R/2}|z|^{-Q-2\nu}\,\mathrm{d}z \leqslant Ct^2,\end{aligned}$$

as $h_1 \in \mathcal{S}(G)$. Consequently taking the supremum in the complementary on Ω

$$\sup_{\Omega^c}\left|\frac{1}{t}\left(f*h_t-f\right)-\mathcal{R}f\right| \leqslant C't \longrightarrow_{t\to 0} 0.$$

This shows the convergence in (3.1) for $p=\infty$.

We now proceed in a similar way to prove the convergence in (3.1) for p finite. As above we fix $f\in\mathcal{D}(G)$ supported in $\bar{B}_R$. We decompose

$$\begin{aligned}\left\|\frac{1}{t}\left(f*h_t-f\right)-\mathcal{R}f\right\|_p \leqslant \left\|\frac{1}{t}(f*h_t-f)-\mathcal{R}f\right\|_{L^p(\bar{B}_{2C_oR})}\\ +\left\|\frac{1}{t}(f*h_t-f)-\mathcal{R}f\right\|_{L^p(B^c_{2C_oR})}.\end{aligned}$$

For the first term,

$$\left\|\frac{1}{t}(f*h_t-f)-\mathcal{R}f\right\|_{L^p(\bar{B}_{2C_oR})} \leqslant |\bar{B}_{2C_oR}|^{\frac{1}{p}}\left\|\frac{1}{t}(f*h_t-f)-\mathcal{R}f\right\|_\infty \underset{t\to 0}{\longrightarrow} 0,$$

as we have already proved the convergence in (3.1) for $p=\infty$. For the second term, we adapt the argument given for $p=\infty$, that is, first we observe that

$$\begin{aligned}\left\|\frac{1}{t}(f*h_t-f)-\mathcal{R}f\right\|_{L^p(B^c_{2C_oR})} &= \frac{1}{t}\left\|f*h_t\right\|_{L^p(B^c_{2C_oR})}\\ &\leqslant \frac{\|f\|_\infty}{t}\left(\int_{|x|>2C_oR}\left(\int_{|y|<R}|h_t(y^{-1}x)|\,\mathrm{d}y\right)^p\mathrm{d}x\right)^{\frac{1}{p}},\end{aligned}$$

and then, using simple manipulations and the properties of the heat kernels (see Theorem 2.8), we have

$$\int_{|x|>2C_oR}\left(\int_{|y|<R}|h_t(y^{-1}x)|\,\mathrm{d}y\right)^p\mathrm{d}x \leqslant C_{R,p}\int_{|z|>R/2}|h_t(z)|^p\,\mathrm{d}z \leqslant C'_{R,p}t^2.$$

This yields the convergence in (3.1) for p finite. □

Definition 3.2. — *Let $\mathcal{R}$ be a positive Rockland operator on G.*

For $p \in [1,\infty)$, we denote by $\mathcal{R}_p$ the operator such that $-\mathcal{R}_p$ is the infinitesimal generator of the semi-group of operators $f \mapsto f * h_t$, $t>0$, on the Banach space $L^p(G)$.

We also denote by $\mathcal{R}_{\infty_o}$ the operator such that $-\mathcal{R}_{\infty_o}$ is the infinitesimal generator of the semi-group of operators $f \mapsto f * h_t$, $t>0$, on the Banach space $C_o(G)$.

For the moment it seems that $\mathcal{R}_2$ denotes the self-adjoint extension of $\mathcal{R}$ on $L^2(G)$ and minus the generator of $f \mapsto f * h_t$, $t>0$, on $L^2(G)$. In the sequel, in fact in Theorem 3.3 below, we show that the two operators coincide and there is no conflict of notation.

Theorem 3.3. — *Let $\mathcal{R}$ be a positive Rockland operator on G and $p \in [1,\infty) \cup \{\infty_o\}$.*

(1) *The operator $\mathcal{R}_p$ is closed. The domain of $\mathcal{R}_p$ contains $\mathcal{D}(G)$, and for $f \in \mathcal{D}(G)$ we have $\mathcal{R}_p f = \mathcal{R} f$.*
(2) *The positive Rockland operator $\bar{\mathcal{R}}_p$ is the infinitesimal generator of the strongly continuous semi-group $\{f \mapsto f * \bar{h}_t\}_{t>0}$ on $L^p(G)$ for $p \in [1,\infty)$ and on $C_o(G)$ for $p=\infty_o$.*
(3) *If $p \in (1,\infty)$ then the dual of $\mathcal{R}_p$ is $\bar{\mathcal{R}}_{p'}$. The dual of $\mathcal{R}_{\infty_o}$ restricted to $L^1(G)$ is $\bar{\mathcal{R}}_1$. The dual of $\mathcal{R}_1$ restricted to $C_o(G) \subset L^\infty(G)$ is $\bar{\mathcal{R}}_{\infty_o}$.*
(4) *If $p \in [1,\infty)$, the operator $\mathcal{R}_p$ is the maximal restriction of $\mathcal{R}$ to $L^p(G)$, that is, the domain of $\mathcal{R}_p$ consists of all the functions $f \in L^p(G)$ such that the distributional derivative $\mathcal{R} f$ is in $L^p(G)$ and $\mathcal{R}_p f = \mathcal{R} f$. In particular, the operator $\mathcal{R}_2$ obtained from Definition 3.2 coincides with the self-adjoint extension of $\mathcal{R}$ on $L^2(G)$.*

   The operator $\mathcal{R}_{\infty_o}$ is the maximal restriction of $\mathcal{R}$ to $C_o(G)$, that is, the domain of $\mathcal{R}_{\infty_o}$ consists of all the function $f \in C_o(G)$ such that the distributional derivative $\mathcal{R} f$ is in $C_o(G)$ and $\mathcal{R}_p f = \mathcal{R} f$.
(5) *If $p \in [1,\infty)$, the operator $\mathcal{R}_p$ is the smallest closed extension of $\mathcal{R}|_{\mathcal{D}(G)}$ on $L^p(G)$. For $p=2$, $\mathcal{R}_2$ is the self-adjoint extension of $\mathcal{R}$ on $L^2(G)$.*

Proof. — Part (1) is a consequence of Proposition 3.1. Intertwining with the complex conjugate, this implies that $\{f \mapsto f * \bar{h}_t\}_{t>0}$ is also a strongly continuous semi-group on $L^p(G)$ whose infinitesimal operator coincides with $\bar{\mathcal{R}} = \mathcal{R}^t$ on $\mathcal{D}(G)$. This shows Part (2).

For Part (3), we observe that using (2.1) and (2.10), we have

$$\forall\, f_1, f_2 \in \mathcal{D}(G) \qquad \langle f_1 * h_t, f_2 \rangle = \langle f_1, f_2 * \bar{h}_t \rangle. \tag{3.3}$$

Thus we have for any $f,g \in \mathcal{D}(G)$ and $p \in [1,\infty)\cup\{\infty_o\}$

$$\left\langle \frac{1}{t}(e^{-t\mathcal{R}_p}f-f),g\right\rangle = \frac{1}{t}\langle f*h_t-f,g\rangle = \frac{1}{t}\langle f,g*\bar{h}_t-g\rangle = \frac{1}{t}\langle f,e^{-t\bar{\mathcal{R}}_{p'}}g-g\rangle.$$

Here the brackets refer to the duality in the sense of distribution. Taking the limit as $t\to 0$ of the first and last expressions proves Part (3).

We now prove Part (4) for any $p\in[1,\infty)\cup\{\infty_o\}$. Let $f\in \mathrm{Dom}(\mathcal{R}_p)$ and $\phi\in\mathcal{D}(G)$. Since $\mathcal{R}$ is formally self-adjoint, we know that $\mathcal{R}^t=\bar{\mathcal{R}}$, and by Part (1), we have $\mathcal{R}_q\phi=\mathcal{R}\phi$ for any $q\in[1,\infty)\cup\{\infty_o\}$. Thus by Part (3) we have

$$\langle \mathcal{R}_p f,\phi\rangle = \langle f,\bar{\mathcal{R}}_{p'}\phi\rangle = \langle f,\mathcal{R}^t\phi\rangle = \langle \mathcal{R}f,\phi\rangle,$$

and $\mathcal{R}_p f=\mathcal{R}f$ in the sense of distributions. Thus

$$\mathrm{Dom}(\mathcal{R}_p)\subset\{f\in L^p(G)\ :\ \mathcal{R}f\in L^p(G)\}.$$

We now prove the reverse inclusion. Let $f\in L^p(G)$ such that $\mathcal{R}f\in L^p(G)$. Let also $\phi\in\mathcal{D}(G)$. The following computations are justified by the properties of $\mathcal{R}$ and h_t (see Theorem 2.8), Fubini's Theorem, and (3.3):

$$\begin{aligned}\langle f*h_t-f,\phi\rangle &= \langle f,\phi*\bar{h}_t-\phi\rangle = \langle f,\int_0^t \partial_s(\phi*\bar{h}_s)\,\mathrm{d}s\rangle\\ &= \langle f,\int_0^t -\bar{\mathcal{R}}(\phi*\bar{h}_s)\,\mathrm{d}s\rangle = -\langle f,\bar{\mathcal{R}}\int_0^t(\phi*\bar{h}_s)\,\mathrm{d}s\rangle\\ &= -\langle \mathcal{R}f,\int_0^t \phi*\bar{h}_s\,\mathrm{d}s\rangle = -\int_0^t\langle \mathcal{R}f,\phi*\bar{h}_s\rangle\,\mathrm{d}s\\ &= -\int_0^t\langle(\mathcal{R}f)*h_s,\phi\rangle\,\mathrm{d}s = -\langle\int_0^t(\mathcal{R}f)*h_s\,\mathrm{d}s,\phi\rangle.\end{aligned}$$

Therefore,

$$\frac{1}{t}(f*h_t-f) = -\frac{1}{t}\int_0^t(\mathcal{R}f)*h_s\,\mathrm{d}s.$$

This converges towards $-\mathcal{R}f$ in $L^p(G)$ as $t\to 0$ by the general properties of averages of strongly continuous semigroups on a Banach space. This shows $f\in\mathrm{Dom}(\mathcal{R}_p)$ and concludes the proof of (iv).

Part (5) follows from (4). This also shows that the self-adjoint extension of $\mathcal{R}$ coincides with $\mathcal{R}_2$ as defined in Definition 3.2 and concludes the proof of Theorem 3.3. □

Theorem 3.3 has the following consequences which will enable us to define the fractional powers of $\mathcal{R}_p$.

Corollary 3.4. — *We keep the same setting and notation as in Theorem 3.3.*

(1) *The operator $\mathcal{R}_p$ is injective on $L^p(G)$ for $p \in [1,\infty)$ and $\mathcal{R}_{\infty_o}$ is injective on $C_o(G)$, namely,*

$$\text{for } p \in [1,\infty) \cup \{\infty_o\} \ : \qquad \forall\, f \in \mathrm{Dom}(\mathcal{R}_p) \qquad \mathcal{R}_p f = 0 \Longrightarrow f = 0.$$

(2) *If $p \in (1,\infty)$ then the operator $\mathcal{R}_p$ has dense range in $L^p(G)$. The operator $\mathcal{R}_{\infty_o}$ has dense range in $C_o(G)$. The closure of the range of $\mathcal{R}_1$ is the closed subspace $\{\phi \in L^1(G) : \int_G \phi = 0\}$ of $L^1(G)$.*

(3) *For $p \in [1,\infty) \cup \{\infty_o\}$, and any $\mu > 0$, the operator $\mu\mathrm{I} + \mathcal{R}_p$ is invertible on $L^p(G)$, $p \in [1,\infty)$, and on $C_o(G)$ for $p = \infty_o$, and the operator norm of $(\mu\mathrm{I} + \mathcal{R}_p)^{-1}$ is*

$$\left\|(\mu\mathrm{I} + \mathcal{R}_p)^{-1}\right\|_{\mathscr{L}(L^p(G))} \leqslant \|h_1\|\, \mu^{-1}, \tag{3.4}$$

or

$$\left\|(\mu\mathrm{I} + \mathcal{R}_{\infty_o})^{-1}\right\|_{\mathscr{L}(C_o(G))} \leqslant \|h_1\|\, \mu^{-1}.$$

Remark 3.5. — Note that the properties (1) and (2) above were proved in the stratified case using the existence of a unique homogeneous fundamental solution of the sub-Laplacian, cf. [12, Propositions 2.19 and 3.9]. In our proof below, we replace the use of this fundamental solution (which does not necessarily exist for Rockland operators) with the Liouville type property given in Theorem 2.11.

In the stratified case, property (3) follows from the Hille–Yosida theorem and the heat semigroup being a contraction on each L^p space, see the proof of [12, Theorem 3.15].

Proof of Corollary 3.4. — Let $f \in \mathrm{Dom}(\mathcal{R}_p)$ be such that $\mathcal{R}_p f = 0$ for $p \in [1,\infty) \cup \{\infty_o\}$. By Theorem 3.3(4), $f \in \mathcal{S}'(G)$ and $\mathcal{R} f = 0$. Consequently by Liouville's theorem, see Theorem 2.11, f is a polynomial. Since f is also in $L^p(G)$ for $p \in [1,\infty)$ or in $C_o(G)$ for $p = \infty_o$, f must be identically zero. This proves (1).

For (2), let Ψ be a bounded linear functional on $L^p(G)$ if $p \in [1,\infty)$ or on $C_o(G)$ if $p = \infty_o$ such that Ψ vanishes identically on $\mathrm{Range}(\mathcal{R}_p)$. Then Ψ can be realised as the integration against a function $f \in L^{p'}(G)$ if $p \in [1,\infty)$ or a measure also denoted by $f \in M(G)$ if $p = \infty_o$. Using the distributional notation, we have

$$\Psi(\phi) = \langle f, \phi \rangle \qquad \forall\, \phi \in L^p(G) \quad \text{or} \quad \forall\, \phi \in C_o(G).$$

Then for any $\phi \in \mathcal{D}(G)$, we know that $\phi \in \mathrm{Dom}(\mathcal{R}_p)$ and $\mathcal{R}_p\phi = \mathcal{R}\phi$ by Theorem 3.3 (1) thus

$$0 = \Psi(\mathcal{R}_p(\phi)) = \langle f, \mathcal{R}(\phi)\rangle = \langle \bar{\mathcal{R}} f, \phi\rangle,$$

since $\mathcal{R}^t = \bar{\mathcal{R}}$. Hence $\bar{\mathcal{R}}f = 0$. By Liouville's theorem, see Theorem 2.11, this time applied to the positive Rockland operator $\bar{\mathcal{R}}$, we see that f is a polynomial. This implies that $f \equiv 0$, since f is also a function in $L^{p'}(G)$ in the case $p \in (1,\infty)$, whereas for $p = \infty_o$, f is in $M(G)$ thus an integrable polynomial on G. For $p = 1$, f being a measurable bounded function and a polynomial, f must be constant, i.e. $f \equiv c$ for some $c \in \mathbb{C}$. This shows that if $p \in (1,\infty) \cup \{\infty_o\}$ then $\Psi = 0$ and $\mathrm{Range}(\mathcal{R}_p)$ is dense in $L^p(G)$ or $C_o(G)$, whereas if $p = 1$ then $\Psi : L^1(G) \ni \phi \mapsto c\int_G \phi$. This shows (2) for $p \in (1,\infty) \cup \{\infty_o\}$.

Let us study more precisely the case $p = 1$. It is easy to see that

$$\int_G X\phi(x)\,\mathrm{d}x = -\int_G \phi(x)\ (X1)(x)\,\mathrm{d}x = 0$$

holds for any $\phi \in L^1(G)$ such that $X\phi \in L^1(G)$. Consequently, for any $\phi \in \mathrm{Dom}(\mathcal{R}_1)$, we know that ϕ and $\mathcal{R}\phi$ are in $L^1(G)$ thus $\int_G \mathcal{R}_1\phi = 0$. So the range of $\mathcal{R}_1$ is included in

$$S := \{\phi \in L^1(G)\ :\ \int_G \phi = 0\} \supset \mathrm{Range}(\mathcal{R}_1).$$

Moreover, if Ψ_1 a bounded linear functional on S such that Ψ_1 is identically 0 on $\mathrm{Range}(\mathcal{R}_1)$, by the Hahn–Banach Theorem, it can be extended into a bounded linear function Ψ on $L^1(G)$. As Ψ vanishes identically on $\mathrm{Range}(\mathcal{R}_1) \subset S$, we have already proven that Ψ must be of the form $\Psi : L^1(G) \ni \phi \mapsto c\int_G \phi$ for some constant $c \in \mathbb{C}$ and its restriction to S is $\Psi_1 \equiv 0$. This concludes the proof of Part (2).

Let us prove Part (3). Integrating the formula

$$(\mu + \lambda)^{-1} = \int_0^\infty e^{-t(\mu+\lambda)}\,\mathrm{d}t$$

against the spectral measure $\mathrm{d}E(\lambda)$ of $\mathcal{R}_2$, we have formally

$$(\mu\mathrm{I} + \mathcal{R}_2)^{-1} = \int_0^\infty e^{-t(\mu\mathrm{I}+\mathcal{R}_2)}\,\mathrm{d}t, \tag{3.5}$$

and the convolution kernel of the operator on the right-hand side is (still formally) given by

$$\kappa_\mu(x) := \int_0^\infty e^{-t\mu} h_t(x)\,\mathrm{d}t.$$

From the properties of the heat kernel h_t (see Theorem 2.8), we see that κ_μ is continuous on G and that

$$\|\kappa_\mu\|_1 \leqslant \int_0^\infty e^{-t\mu} \|h_t\|_1 \, \mathrm{d}t = \|h_1\| \int_0^\infty e^{-t\mu} \, \mathrm{d}t = \frac{\|h_1\|}{\mu} < \infty.$$

Therefore $\kappa_\mu \in L^1(G)$, the operator $\int_0^\infty e^{-t(\mu \mathrm{I} + \mathcal{R}_2)} \, \mathrm{d}t$ is bounded on $L^2(G)$. Furthermore Formula (3.5) holds (it suffices to consider integration over $[0, N]$ with $N \to \infty$).

For any $\phi \in \mathcal{D}(G)$ and $p \in [1, \infty) \cup \{\infty_o\}$, Theorem 3.3 (4) implies

$$(\mu \mathrm{I} + \mathcal{R}_p)\phi = (\mu \mathrm{I} + \mathcal{R})\phi = (\mu \mathrm{I} + \mathcal{R}_2)\phi \in \mathcal{D}(G),$$

thus

$$\kappa_\mu * ((\mu \mathrm{I} + \mathcal{R}_p)\phi) = \kappa_\mu * ((\mu \mathrm{I} + \mathcal{R}_2)\phi) = \phi.$$

This yields that the operator $(\mu \mathrm{I} + \mathcal{R}_p)^{-1} : \phi \mapsto \phi * \kappa_\mu$ bounded on $L^p(G)$ if $p \in [1, \infty)$ and on $C_o(G)$ if $p = \infty_o$. Furthermore its operator norm is $\leqslant \|\kappa_\mu\|_1 \leqslant \|h_1\| \, \mu^{-1}$. □

3.2. Fractional powers of operators $\mathcal{R}_p$

In this section we study the fractional powers of the operators $\mathcal{R}_p$ and $\mathrm{I} + \mathcal{R}_p$. As explained in the proof of the following theorem, these fractional powers are defined as the powers of a Komatsu-non-negative operator. In the special case of $p = 2$, it coincides with the powers defined by functional calculus.

Theorem 3.6. — *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group G. We consider the operators $\mathcal{R}_p$ defined in Definition 3.2. Let $p \in [1, \infty) \cup \{\infty_o\}$.*

(1) *Let $\mathcal{A}$ denote either $\mathcal{R}$ or $\mathrm{I} + \mathcal{R}$.*

(a) *For every $a \in \mathbb{C}$, the operator $\mathcal{A}_p^a$ is closed and injective with $(\mathcal{A}_p^a)^{-1} = \mathcal{A}_p^{-a}$. We have $\mathcal{A}_p^0 = \mathrm{I}$, and for any $N \in \mathbb{N}$, $\mathcal{A}_p^N$ coincides with the usual powers of differential operators on $\mathcal{S}(G)$. Furthermore, the operator $\mathcal{A}_p^a$ is invariant under left translations.*

(b) *For any $a, b \in \mathbb{C}$, in the sense of operator graph, we have $\mathcal{A}_p^a \mathcal{A}_p^b \subset \mathcal{A}_p^{a+b}$. If $\mathrm{Range}(\mathcal{A}_p)$ is dense then the closure of $\mathcal{A}_p^a \mathcal{A}_p^b$ is $\mathcal{A}_p^{a+b}$. Moreover for any $N \in \mathbb{N}$, $\mathrm{Dom}(\mathcal{A}^N) \cap \mathrm{Range}(\mathcal{A}^N)$ is dense in $\mathrm{Range}(\mathcal{A}_p)$.*

(c) *Let* $a_o \in \mathbb{C}_+$.
- *If* $\phi \in \text{Range}(\mathcal{A}_p^{a_o})$ *then* $\phi \in \text{Dom}(\mathcal{A}_p^a)$ *for all* $a \in \mathbb{C}$ *with* $0 < -\Re a < \Re a_o$ *and the function* $a \mapsto \mathcal{A}_p^a \phi$ *is holomorphic in* $\{a \in \mathbb{C} :\ -\Re a_o < \Re a < 0\}$.
- *If* $\phi \in \text{Dom}(\mathcal{A}_p^{a_o})$ *then* $\phi \in \text{Dom}(\mathcal{A}_p^a)$ *for all* $a \in \mathbb{C}$ *with* $0 < \Re a < \Re a_o$ *and the function* $a \mapsto \mathcal{A}_p^a \phi$ *is holomorphic in* $\{a \in \mathbb{C} :\ 0 < \Re a < \Re a_o\}$.

(d) *If* $p \in (1, \infty)$ *then the dual of* $\mathcal{A}_p$ *is* $\bar{\mathcal{A}}_{p'}$. *The dual of* $\mathcal{A}_{\infty_o}$ *restricted to* $L^1(G)$ *is* $\bar{\mathcal{A}}_1$. *The dual of* $\mathcal{A}_1$ *restricted to* $C_o(G) \subset L^\infty(G)$ *is* $\bar{\mathcal{A}}_{\infty_o}$.

(e) *If* $a, b \in \mathbb{C}_+$ *with* $\Re b > \Re a$, *then*

$$\exists\, C = C_{a,b} > 0 \quad \forall\, \phi \in \text{Dom}(\mathcal{A}_p^b) \qquad \left\|\mathcal{A}_p^a \phi\right\| \leqslant C \left\|\phi\right\|^{1-\frac{\Re a}{\Re b}} \left\|\mathcal{A}_p^b \phi\right\|^{\frac{\Re a}{\Re b}}.$$

(f) *For any* $a \in \mathbb{C}_+$, $\text{Dom}(\mathcal{A}_p^a)$ *contains* $\mathcal{S}(G)$.

(g) *If* $f \in \text{Dom}(\mathcal{A}_p^a) \cap L^q(G)$ *for some* $q \in [1, \infty) \cup \{\infty_o\}$, *then* $f \in \text{Dom}(\mathcal{A}_q^a)$ *if and only if* $\mathcal{A}_p^a f \in L^q(G)$, *in which case* $\mathcal{A}_p^a f = \mathcal{A}_q^a f$.

(2) *For each* $a \in \mathbb{C}_+$, *the operators* $(\mathrm{I} + \mathcal{R}_p)^a$ *and* $\mathcal{R}_p^a$ *are unbounded and their domains satisfy* $\text{Dom}\left[(\mathrm{I} + \mathcal{R}_p)^a\right] = \text{Dom}(\mathcal{R}_p^a) = \text{Dom}\left[(\mathcal{R}_p + \epsilon \mathrm{I})^a\right]$ *for all* $\epsilon > 0$,

(3) *If* $0 < \Re a < 1$ *and* $\phi \in \text{Range}(\mathcal{R}_p)$ *then*

$$\mathcal{R}_p^{-a} \phi = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-t\mathcal{R}_p} \phi \, \mathrm{d}t,$$

in the sense that $\lim_{N\to\infty} \int_0^N$ *converges in the norm of* $L^p(G)$ *or* $C_o(G)$.

(4) *If* $a \in \mathbb{C}_+$, *then the operator* $(\mathrm{I} + \mathcal{R}_p)^{-a}$ *is bounded and for any* $\phi \in \mathcal{X}$ *with* $\mathcal{X} = L^p(G)$ *or* $C_o(G)$, *we have*

$$(\mathrm{I} + \mathcal{R}_p)^{-a} \phi = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-t(\mathrm{I}+\mathcal{R}_p)} \phi \, \mathrm{d}t,$$

in the sense of absolute convergence: $\int_0^\infty t^{a-1} \left\|e^{-t(\mathrm{I}+\mathcal{R}_p)} \phi\right\|_{\mathcal{X}} \mathrm{d}t < \infty$.

(5) *For any* $a, b \in \mathbb{C}$, *the two (possibly unbounded) operators* $\mathcal{R}_p^a$ *and* $(\mathrm{I} + \mathcal{R}_p)^b$ *commute.*

(6) *For any* $a \in \mathbb{C}$, *the operator* $\mathcal{R}_p^a$ *is homogeneous of degree* νa.

Here we say that two (possibly unbounded) operators A and B commute when

$$x \in \text{Dom}(AB) \cap \text{Dom}(BA) \Longrightarrow ABx = BAx.$$

Let us recall that the domain of the product AB of two (possibly unbounded) operators A and B on the same Banach space $\mathcal{X}$ is formed by the elements $x \in \mathcal{X}$ such that $x \in \mathrm{Dom}(B)$ and $Bx \in \mathrm{Dom}(A)$.

In Theorem 3.6, Γ denotes the usual Gamma function, that is, the meromorphic extension of the function defined via $z \mapsto \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t$ on $\{\Re z > 0\}$.

Proof of Theorem 3.6. — In the case $p = 2$, the statement follows by functional calculus and Part (4) of Theorem 3.3. We may therefore assume $p \neq 2$. By Theorem 3.3 (1), the operator $\mathcal{R}_p$ is closed and densely defined. By Corollary 3.4, it is injective and Komatsu-non-negative in the sense that $(-\infty, 0)$ is included in its resolvant set and it satisfies Property (3.4). Necessarily $\mathrm{I} + \mathcal{R}_p$ also satisfies these properties. Furthermore $-(\mathrm{I} + \mathcal{R}_p)$ generates an exponentially stable semigroup:

$$\left\| e^{-t(\mathrm{I}+\mathcal{R}_p)} \right\|_{\mathscr{L}(L^p(G))} \leqslant e^{-t} \left\| e^{-t\mathcal{R}_p} \right\|_{\mathscr{L}(L^p(G))} \leqslant e^{-t} \left\| h_1 \right\|_1,$$

and similarly for $C_o(G)$.

Most of the statements then follow from the general properties of fractional powers constructed via the Balakrishnan formulae, see [21]. More precisely, from the Balakrishnan formula, [21, Section 3.1], for any $N \in \mathbb{N}$, $\mathcal{A}_p^N$ coincides with the usual powers of differential operators on $\mathcal{S}(G)$. Then Part (1a) follows from the general properties of the construction, see [21, Section 7]. Parts (5) and (6) are true for the operator given via the Balakrishnan formula, therefore they hold for our fractional powers. Part (1d) is obtained in the same way together with the properties of duality.

For Part (1b), see [21, Proposition 1.1.3 (iii) and Section 7]. For Part (1c), see Corollary 5.1.13 together with Proposition 7.1.5 and its proof both in [21]. For Part (1e), see [21, Corollary 5.1.13]. Part (1f) follows from Parts (1a) and (1c). Part (1g) is certainly true for any $f \in \mathcal{S}(G)$ and $\Re a > 0$ via the Balakrishnan formulae. By analyticity (see Part (1c)) it is true for any $a \in \mathbb{C}$. The density of $\mathcal{D}(G)$ in $L^p(G)$ (or $C_o(G)$ if $p = \infty_o$) together with the maximality of $\mathcal{A}_p^a$ and the uniqueness of distributional convergence imply the result. For Part (2), see [21, Theorem 5.1.7]. For Parts (3) and (4), see [21, Lemma 6.1.5]. This concludes the proof of Theorem 3.6. □

3.3. Imaginary powers of $\mathcal{R}_p$ and $\mathrm{I} + \mathcal{R}_p$

In this section, we prove that imaginary powers of a positive Rockland operator $\mathcal{R}$ as well as $\mathrm{I} + \mathcal{R}$ are bounded operators on $L^p(G)$, $p \in (1, \infty)$

by applying the consequence of the theorem of singular integrals stated in Theorem 2.5. The operator bounds will follow as a general semi-group property.

Let us show that if $\mathcal{R}$ is a positive Rocklamd operator, then the imaginary powers of $\mathrm{I}+\mathcal{R}_p$ are bounded on $L^p(G)$:

Proposition 3.7. — *Let $\mathcal{R}$ be a positive Rocklamd operator. For any $\tau\in\mathbb{R}$ and $p\in(1,\infty)$, the operator $(\mathrm{I}+\mathcal{R}_p)^{i\tau}$ is bounded on $L^p(G)$. For any $p\in(1,\infty)$, there exists $C=C_{p,\mathcal{R}}>0$ and $\theta\in[0,\pi]$ such that*

$$\forall\,\tau\in\mathbb{R}\qquad \left\|(\mathrm{I}+\mathcal{R}_p)^{i\tau}\right\|_{\mathscr{L}(L^p(G))}\leqslant Ce^{\theta|\tau|}.$$

For any $p\in(1,\infty)$ and $a\in\mathbb{C}$, $\mathrm{Dom}((\mathrm{I}+\mathcal{R}_p)^a)=\mathrm{Dom}((\mathrm{I}+\mathcal{R}_p)^{\Re a})$.

In the proof of Proposition 3.7, we will need the following estimates for integral of the heat kernel h_t of a positive Rockland operator $\mathcal{R}$:

Lemma 3.8.

(1) *For any homogeneous quasi-norm $|\cdot|$, any multi-index $\alpha\in\mathbb{N}_0^n$, and any real number a with $0<a<\frac{Q+[\alpha]}{\nu}$, there exists a constant $C>0$ such that*

$$\int_0^\infty t^{a-1}|X^\alpha h_t(x)|\,\mathrm{d}t\leqslant C|x|^{-Q-[\alpha]+\nu a}.$$

For any homogeneous quasi-norm $|\cdot|$, any multi-index $\alpha\in\mathbb{N}_0^n$, there exists a constant $C>0$ such that

$$\int_0^\infty |X^\alpha h_t(x)|e^{-t}\,\mathrm{d}t\leqslant C|x|^{-Q-[\alpha]}.$$

(2) *For any homogeneous quasi-norm $|\cdot|$, any multi-index $\alpha\in\mathbb{N}_0^n$, and any $t>0$, we have*

$$\int_{|x|\geqslant 1/2}|X^\alpha h_t(x)|\,\mathrm{d}x\leqslant t^{-\frac{[\alpha]}{\nu}}\left\|X^\alpha h_1\right\|_{L^1}.$$

(3) *For any homogeneous quasi-norm $|\cdot|$, any multi-index $\alpha\in\mathbb{N}_0^n$, any $N\in\mathbb{N}$ and any $t\in(0,1)$, there exists a constant $C>0$ such that*

$$\int_{|x|\geqslant 1/2}|X^\alpha h_t(x)|\,\mathrm{d}x\leqslant Ct^N.$$

Proof of Lemma 3.8. — Let us prove Part (1). We write

$$\int_0^\infty t^{a-1}|X^\alpha h_t(x)|\,\mathrm{d}t=\int_0^{|x|^\nu}+\int_{|x|^\nu}^\infty.$$

For the second integral, we use the property of homogeneity of h_t (see (2.9) or (2.13))

$$\int_{|x|^\nu}^{\infty} \leqslant \left(\frac{Q+[\alpha]}{\nu} - a\right)^{-1} \|X^\alpha h_1\|_\infty \, |x|^{\nu\left(a-\frac{Q+[\alpha]}{\nu}\right)}.$$

As $h_1 \in \mathcal{S}(G)$, $\|X^\alpha h_1\|_\infty$ is finite. For the first integral, we use again (2.9) to obtain

$$\int_0^{|x|^\nu} \leqslant C_1 a^{-1} |x|^{\nu\left(a-\frac{Q+[\alpha]}{\nu}\right)},$$

where $C_1 := \sup_{|y|=1, 0\leqslant t_1\leqslant 1} |X^\alpha h_{t_1}(y)|$ is finite by (2.12). Combining the two estimates above shows the estimates for the first integral in Part (1). We proceed in the same way for the second one:

$$\int_0^\infty |X^\alpha h_t(x)| e^{-t}\, \mathrm{d}t = \int_0^{|x|^\nu} + \int_{|x|^\nu}^\infty.$$

We have (with C_1 as above)

$$\int_0^{|x|^\nu} \leqslant C_1 |x|^{\nu\left(a-\frac{Q+[\alpha]}{\nu}\right)}, \quad \text{whereas} \quad \int_{|x|^\nu}^\infty \leqslant \|X^\alpha h_1\|_\infty \, |x|^{-(Q+[\alpha])}.$$

We conclude in the same way as above and Part (1) is proved.

Let us prove Part (2). The property of homogeneity of h_t (see (2.13)) together with $h_1 \in \mathcal{S}(G)$ imply

$$\int_{|x|\geqslant 1/2} |X^\alpha h_t(x)|\, \mathrm{d}x = t^{-\frac{[\alpha]}{\nu}} \int_{t^{\frac{1}{\nu}}|x'|\geqslant 1/2} |X^\alpha h_1(x')|\, \mathrm{d}x' \leqslant t^{-\frac{[\alpha]}{\nu}} \int_G |X^\alpha h_1|,$$

having used the change of variable $x' = t^{-\frac{1}{\nu}} x$. This shows Part (2).

Let us prove Part (3). The properties of the heat kernel, especially (2.9) and (2.12), imply

$$\int_{|x|\geqslant 1/2} |X^\alpha h_t(x)|\, \mathrm{d}x \leqslant C t^N \int_{|x|\geqslant 1/2} |x|^{-[\alpha]-Q-\nu N}\, \mathrm{d}x,$$

where $C = \sup_{|x'|=1, 0<t'<1} t'^{-N} |X^\alpha h_{t'}(x')| < \infty$. This shows Part (3) and concludes the proof of Lemma 3.8. ☐

Proof of Proposition 3.7. — Let us fix $\tau \in \mathbb{R}\backslash\{0\}$. By functional calculus, $(\mathrm{I}+\mathcal{R}_2)^{i\tau}$ is bounded on $L^2(G)$. The formula

$$\forall\, \lambda > 0 \qquad \lambda^{i\tau} = \frac{\lambda}{\Gamma(1-i\tau)} \int_0^\infty t^{-i\tau} e^{-\lambda t}\, \mathrm{d}t.$$

and the functional calculus of $\mathcal{R}_2$ implies that the right convolution kernel of $(\mathrm{I}+\mathcal{R}_2)^{i\tau}$ is the tempered distribution κ which coincides with the smooth function away from 0 given via

$$\kappa(x) = \frac{1}{\Gamma(1-i\tau)} \int_0^\infty t^{-i\tau}(\mathrm{I}+\mathcal{R})h_t(x)e^{-t}\,\mathrm{d}t, \qquad x \neq 0,$$

because of Part (1) of Lemma 3.8. The rest of Lemma 3.8 imply easily that the hypotheses of Theorem 2.5 are satisfied. This together with Theorem 3.6 (1f) and (1g) show that for each $\tau \in \mathbb{R}$, $(\mathrm{I}+\mathcal{R}_p)^{i\tau}$ is bounded for $p \geqslant 2$. By duality, it is bounded for any $p \in (1,\infty)$. The properties of the semi-group [21, Corollary 7.1.2 and Proposition 8.1.1] imply the rest of the statement in Proposition 3.7. □

Let us now prove the homogeneous case, that is, that the imaginary powers of a positive Rockland operator are bounded on $L^p(G)$:

Proposition 3.9. — *For any $\tau \in \mathbb{R}$ and $p \in (1,\infty)$, the operator $\mathcal{R}_p^{i\tau}$ is bounded on $L^p(G)$. For any $p \in (1,\infty)$, there exists $C = C_{p,\mathcal{R}} > 0$ and $\theta \in [0,\pi]$ such that*

$$\forall\, \tau \in \mathbb{R} \qquad \left\|\mathcal{R}_p^{i\tau}\right\|_{\mathscr{L}(L^p(G))} \leqslant Ce^{\theta|\tau|}.$$

For any $p \in (1,\infty)$ and $a \in \mathbb{C}$, $\mathrm{Dom}(\mathcal{R}_p^a) = \mathrm{Dom}(\mathcal{R}_p^{\Re a})$.

Proof of Proposition 3.9. — Let $p \in (1,\infty)$ and $\tau \in \mathbb{R}$. Let us denote by $\mathcal{R}_{p,i\tau}$ the (possibly unbounded) operator given as the strong limit in $L^p(G)$ of $(\epsilon+\mathcal{R}_p)^{i\tau}\phi$ as $\epsilon \to 0$, for $\phi \in \mathrm{Dom}((\epsilon+\mathcal{R}_p)^{i\tau})$ for any $\epsilon \in (0,\epsilon_0)$ for some small $\epsilon_0 > 0$ and such that this strong limit exists. The domain of $\mathcal{R}_{p,i\tau}$ is naturally the subspace of all those functions ϕ. Note that the homogeneity of $\mathcal{R}$ implies

$$(\epsilon+\mathcal{R}_p)^{i\tau}\phi = (\mathrm{I}+\epsilon^{-1}\mathcal{R}_p)^{i\tau}\phi = (\mathrm{I}+\mathcal{R}_p)^{i\tau}\{\phi(\epsilon^{-1/\nu}\cdot\,)\}(\epsilon^{1/\nu}\cdot\,),$$

for any $\epsilon > 0$ and any $\phi \in L^p(G)$ such that $\phi(\epsilon^{-1/\nu}\cdot) \in \mathrm{Dom}((\mathrm{I}+\mathcal{R}_p)^{i\tau})$. By Proposition 3.7, $\mathrm{Dom}((\mathrm{I}+\mathcal{R}_p)^{i\tau}) = L^p(G)$ and the operator $(\mathrm{I}+\mathcal{R}_p)^{i\tau}$ is bounded. Therefore we have for all $\phi \in L^p(G)$ and $\epsilon > 0$,

$$\begin{aligned} &\phi \in \mathrm{Dom}((\epsilon+\mathcal{R}_p)^{i\tau}) \\ &\qquad\qquad \text{and} \quad \left\|(\epsilon+\mathcal{R}_p)^{i\tau}\phi\right\|_{L^p(G)} \leqslant \left\|(\mathrm{I}+\mathcal{R}_p)^{i\tau}\right\|_{\mathscr{L}(L^p(G))} \|\phi\|\,. \end{aligned}$$

Consequently, $\mathcal{R}_{p,i\tau}$ extends to a bounded operator on $L^p(G)$. This implies [21, Theorem 7.4.6] that $\mathcal{R}_p^{i\tau}$ is also a bounded operator on $L^p(G)$. As in the inhomogeneous case, the properties of the semi-group [21, Corollary 7.1.2 and Proposition 8.1.1] imply the rest of the statement in Proposition 3.9. □

Remark 3.10. — In the stratified case, the L^p-boundedness of imaginary powers of the sub-Laplacian $-\mathcal{L}$ and $\mathrm{I}+(-\mathcal{L})$ follows from general properties of semigroups preserving positivity together with the Laplace transform, see [12, Proposition 3.14 and Lemma 3.13]. More precisely, the boundedness follows from Littlewood–Paley theory and the study of square functions associated with the semi-group.

Note that in the case of a sub-Laplacian, the proof in [12] yields a bound of the operator norm by $|\Gamma(1-i\tau)|^{-1}$, up to a constant of p.

In our case, we applied a consequence of the theorem of Singular Integrals via Theorem 2.5 to obtain the L^p-boundedness. We can follow the constants in the proof of this theorem as well as in our application: $\left\|(\mathrm{I}+\mathcal{R}_2)^{i\tau}\right\|_{\mathscr{L}(L^2(G))} \leqslant 1$ and the constants $C_0, C_1, \dots C_n$ in the statement of Theorem 2.5 in our context are bounded by $(1+c)$ where $c := |\Gamma(1-i\tau)|^{-1}$. Thus one can show that $\left\|(\mathrm{I}+\mathcal{R}_p)^{i\tau}\right\|_{\mathscr{L}(L^p(G))}$ is bounded up to a constant of p, by $(1+c)^{2|\frac{1}{p}-\frac{1}{2}|}$.

However, we do not need these precise bounds as the bounds obtained from the general theory of semigroups as stated in Proposition 3.7 will be sufficient for our purpose in the proofs of interpolation properties in Theorem 4.8 and Proposition 4.13.

3.4. Riesz and Bessel potentials

Mimicking the usual terminology in the Euclidean setting, we call the operators $\mathcal{R}^{-a/\nu}$ for $\{a \in \mathbb{C},\ 0 < \Re a < Q\}$ and $(\mathrm{I}+\mathcal{R})^{-a/\nu}$ for $a \in \mathbb{C}_+$, the *Riesz potential* and the *Bessel potential*, respectively. In the sequel we will denote their kernels by $\mathcal{I}_a$ and $\mathcal{B}_a$, respectively, as defined in the following:

Corollary 3.11. — *We keep the setting and notation of Theorem 3.3.*

(1) *Let $a \in \mathbb{C}$ with $0 < \Re a < Q$. The integral*

$$\mathcal{I}_a(x) := \frac{1}{\Gamma(a/\nu)} \int_0^\infty t^{\frac{a}{\nu}-1} h_t(x)\, \mathrm{d}t,$$

converges absolutely for every $x \neq 0$. This defines a distribution $\mathcal{I}_a$ which is a kernel of type a, that is, smooth away from the origin and $(a-Q)$-homogeneous.

For any $p \in (1,\infty)$, if $\phi \in \mathcal{S}(G)$ or, more generally, if $\phi \in L^q(G) \cap L^p(G)$ where $q \in [1,\infty)$ is given by $\frac{1}{q} - \frac{1}{p} = \frac{\Re a}{Q}$, then

$$\phi \in \mathrm{Dom}(\mathcal{R}_p^{-\frac{a}{\nu}}) \quad \textit{and} \quad \mathcal{R}_p^{-\frac{a}{\nu}} \phi = \phi * \mathcal{I}_a \in L^p(G).$$

(2) *Let $a \in \mathbb{C}_+$. The integral*

$$\mathcal{B}_a(x) := \frac{1}{\Gamma(\frac{a}{\nu})} \int_0^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x)\, \mathrm{d}t,$$

converges absolutely for every $x \neq 0$. The function $\mathcal{B}_a$ is always smooth away from 0 and integrable on G. If $\Re a > Q/2$, then $\mathcal{B}_a \in L^2(G)$. For each $a \in \mathbb{C}_+$, the operator $(\mathrm{I} + \mathcal{R}_p)^{-a/\nu}$ is a bounded convolution operator on $L^p(G)$ for $p \in [1, \infty)$ or $C_o(G)$ for $p = \infty$, with the same (right convolution) kernel $\mathcal{B}_a$. If $a, b \in \mathbb{C}_+$, then as integrable functions, we have $\mathcal{B}_a * \mathcal{B}_b = \mathcal{B}_{a+b}$.

Remark 3.12. — If the homogeneous degree of the Rockland operator $\mathcal{R}$ is $\nu < Q$ then $\mathcal{R}$ admits a unique homogeneous fundamental solution (see [12, Theorem 2.1]). Part (1) above shows that $\mathcal{I}_\nu$ is this fundamental solution. This is the case for instance for sub-Laplacians and the existence of such a unique homogeneous fundamental solution is used extensively in [12]. However one can find Rockland operators of degree $\nu \geqslant Q$ (see (2.6)). Moreover one can construct a graded group such that all its Rockland operators will have this property: it suffices to consider the abelian group $(\mathbb{R}^n, +)$ with well chosen anisotropic dilations or the three dimensional Heisenberg group $\tilde{\mathbb{H}}_1$ with a graded non-stratified structure defined in Section 4.8.

Proof of Corollary 3.11. — The absolute convergence and the smoothness of $\mathcal{I}_a$ and $\mathcal{B}_a$ follow from the estimates in (2.14) while the homogeneity of $\mathcal{I}_a$ follows from the properties of the heat kernel (especially (2.9)) and a the change of variable. By Proposition 2.4, the operator $\mathcal{S}(G) \ni \phi \mapsto \phi * \mathcal{I}_a$ is homogeneous of degree $-a$, and admits a bounded extension $L^q(G) \to L^p(G)$ when $\frac{1}{p} - \frac{1}{q} = \frac{\Re(a)}{Q}$. The rest of Part (1) follows from Theorem 3.6.

By Theorem 2.8, we have $\int_G |h_t| = \|h_1\|_{L^1} < \infty$ for all $t > 0$, so

(3.6)
$$\int_G |\mathcal{B}_a(x)|\, \mathrm{d}x \leqslant \frac{1}{|\Gamma(\frac{a}{\nu})|} \int_0^\infty t^{\frac{\Re a}{\nu}-1} e^{-t} \int_G |h_t(x)|\, \mathrm{d}x\, \mathrm{d}t = \frac{\Gamma(\frac{\Re a}{\nu})}{|\Gamma(\frac{a}{\nu})|} \|h_1\|_{L^1(G)},$$

and $\mathcal{B}_a$ is integrable. By Theorem 3.6 Part (4), the integrable function $\mathcal{B}_a$ is the convolution kernel of $(\mathrm{I} + \mathcal{R}_p)^{-a/\nu}$.

Let us show the square integrability of $\mathcal{B}_a$. We assume $\Re a > 0$. We compute for any $R > 0$:

$$\begin{aligned}\int_{|x|<R} |\mathcal{B}_a(x)|^2\, \mathrm{d}x &= \int_{|x|<R} \mathcal{B}_a(x) \overline{\mathcal{B}_a(x)}\, \mathrm{d}x \\ &= |\Gamma\left(\frac{a}{\nu}\right)|^{-2} \int_{|x|<R} \int_0^\infty \int_0^\infty t^{\frac{a}{\nu}-1} s^{\frac{\bar{a}}{\nu}-1} e^{-(t+s)} h_t(x) \bar{h}_s(x)\, \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}s.\end{aligned}$$

From the properties of the heat kernel (see (2.10) and (2.8)) we see that

$$\int_{|x|<R} h_t(x)\bar{h}_s(x)\,\mathrm{d}x = \int_{|x|<R} h_t(x)h_s(x^{-1})\,\mathrm{d}x \underset{R\to\infty}{\longrightarrow} h_t * h_s(0) = h_{t+s}(0) = (t+s)^{-\frac{Q}{\nu}} h_1(0).$$

Therefore,

$$\int_G |\mathcal{B}_a(x)|^2\,\mathrm{d}x \leqslant \frac{h_1(0)}{|\Gamma(a/\nu)|^2}\int_0^\infty\int_0^\infty (st)^{\frac{\Re a}{\nu}-1} e^{-(t+s)}(t+s)^{-\frac{Q}{\nu}}\,\mathrm{d}t\,\mathrm{d}s \tag{3.7}$$

$$= \frac{h_1(0)}{|\Gamma(a/\nu)|^2}\int_{s'=0}^1 (s'(1-s'))^{\frac{\Re a}{\nu}-1}\,\mathrm{d}s' \int_{u=0}^\infty e^{-u}u^{2(\frac{\Re a}{\nu}-1)-\frac{Q}{\nu}+1}\,\mathrm{d}u, \tag{3.8}$$

after the change of variables $u = s+t$ and $s' = s/u$. The integrals over s' and u converge when $\Re a > Q/2$. Thus $\mathcal{B}_a$ is square integrable under this condition. The rest of the proof of Corollary 3.11 follows easily from the properties of the fractional powers of $\mathrm{I}+\mathcal{R}$. □

Corollary 3.13. — *We keep the notation of Corollary 3.11. Then for any $p \in [1,\infty)$, if $a > Q/p'$ then $\mathcal{B}_a \in L^p$.*

Proof of Corollary 3.13. — For $p \in [1,2]$, thanks to the estimates in (3.6) (together with Stirling's estimates) and (3.7), the map $f \mapsto (f, \mathcal{B}_z)$ satisfies the hypotheses of the complex interpolation (as stated in [24, Chapter V §4], the image space being a singleton). This yields the result for $p \in [1,2]$. For $p > 2$, we write $a = a_1 + a_2$ with $a_1 > Q/2$ and apply Young's inequalities (see (2.2)) to $\mathcal{B}_a = \mathcal{B}_{a_1} * \mathcal{B}_{a_2}$ with space L^p, L^2, L^q. This shows that $\mathcal{B}_a \in L^p$ when $a > Q/2 + Q/q' = Q/p'$. □

We now state the following technical lemma which will be useful in the sequel.

Lemma 3.14. — *We keep the notation of Corollary 3.11.*

(1) *For any $\phi \in \mathcal{S}(G)$ and $a \in \mathbb{C}_+$, the function $\phi * \mathcal{B}_a$ is Schwartz.*
(2) *Let $a \in \mathbb{C}$ and $\phi \in \mathcal{S}(G)$. Then $(\mathrm{I}+\mathcal{R}_p)^a\phi$ does not depend on $p \in [1,\infty)\cup\{\infty_o\}$. If $a \in \mathbb{N}$, $(\mathrm{I}+\mathcal{R}_p)^a\phi$ coincides with $(\mathrm{I}+\mathcal{R})^a\phi$. If $a \in \mathbb{C}_+$, we have*

$$(\mathrm{I}+\mathcal{R}_p)^a(\phi * \mathcal{B}_{a\nu}) = ((\mathrm{I}+\mathcal{R}_p)^a\phi) * \mathcal{B}_{a\nu} = \phi \quad (p \in [1,\infty)\cup\{\infty_o\}). \tag{3.9}$$

(3) *For any $N \in \mathbb{N}$, $(\mathrm{I}+\mathcal{R})^N(\mathcal{S}(G)) = \mathcal{S}(G)$.*

Proof. — Let $|\cdot|$ be a homogeneous quasi-norm on G and $N \in \mathbb{N}$. We see that

$$\int_G |x|^N|\mathcal{B}_a(x)|\,\mathrm{d}x \leqslant \frac{1}{|\Gamma(\frac{a}{\nu})|}\int_0^\infty t^{\frac{\Re a}{\nu}-1}e^{-t}\int_G |x|^N|h_t(x)|\,\mathrm{d}x\,\mathrm{d}t,$$

and using the homogeneity of the heat kernel (see (2.13)) and the change of variables $y = t^{-\frac{1}{\nu}}x$, we get

$$\int_G |x|^N |h_t(x)| \,\mathrm{d}x = \int_G |t^{\frac{1}{\nu}} y|^N |h_1(y)| \,\mathrm{d}y = c_N t^{\frac{N}{\nu}},$$

where $c_N = \left\| |y|^N h_1(y) \right\|_{L^1(\mathrm{d}y)}$ is a finite constant since $h_1 \in \mathcal{S}(G)$. Thus,

$$\int_G |x|^N |\mathcal{B}_a(x)| \,\mathrm{d}x \leqslant \frac{c_N}{|\Gamma(\frac{a}{\nu})|} \int_0^\infty t^{\frac{\Re a}{\nu} - 1 + \frac{N}{\nu}} e^{-t} \,\mathrm{d}t < \infty,$$

and $x \mapsto |x|^N \mathcal{B}_a(x)$ is integrable.

Let $C_o \geqslant 1$ denote the constant in the triangle inequality for $|\cdot|$ (see Proposition 2.1). Let also $\phi \in \mathcal{S}(G)$. We have for any $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$:

$$\begin{aligned}
(1+|x|)^N \left| \tilde{X} \left[\phi * \mathcal{B}_a\right](x) \right| &\leqslant (1+|x|)^N \left| \tilde{X}\phi \right| * |\mathcal{B}_a|(x) \\
&\leqslant C_o^N \left| (1+|\cdot|)^N \tilde{X}\phi \right| * \left| (1+|\cdot|)^N \mathcal{B}_a \right|(x) \\
&\leqslant C_o^N \left\| (1+|\cdot|)^N \tilde{X}\phi \right\|_\infty \left\| (1+|\cdot|)^N \mathcal{B}_a \right\|_{L^1(G)}.
\end{aligned}$$

This shows that that $\phi * \mathcal{B}_a \in \mathcal{S}(G)$ (for a description of the Schwartz class, see [14, Chapter 1D] and Part (1) is proved.

Part (2) follows easily from Theorem 3.6 and Corollary 3.11.

Let us prove Part (3). By Theorem 3.3 (4), we have the inclusion $(\mathrm{I} + \mathcal{R})^N(\mathcal{S}(G)) \subset \mathcal{S}(G)$. The reverse inclusion $\mathcal{S}(G) \subset (\mathrm{I} + \mathcal{R})^N(\mathcal{S}(G))$ follows from (3.9) and Theorem 3.3 (4). So for any $N \in \mathbb{N}$, $\mathcal{S}(G)$ is included in $\mathrm{Dom}\left[(\mathrm{I} + \mathcal{R}_p)^N\right] \cap \mathrm{Range}\left[(\mathrm{I} + \mathcal{R}_p)^N\right]$ and we can apply the analyticity results of Theorem 3.6: the function $a \mapsto (\mathrm{I} + \mathcal{R}_p)^a\phi$ is holomorphic in $\{a \in \mathbb{C} : -N < \Re a < N\}$. We observe that by Corollary 3.11 (2), if $-N < \Re a < 0$, all the functions $(\mathrm{I} + \mathcal{R}_p)^a\phi$ coincide with $\phi * \mathcal{B}_{a\nu}$ for any $p \in [1, \infty) \cup \{\infty_o\}$. This shows that for each $a \in \mathbb{C}$ fixed, $(\mathrm{I} + \mathcal{R}_p)^a\phi$ is independent of p. This concludes the proof of Lemma 3.14. □

4. Sobolev spaces on graded Lie groups

In this section we define the Sobolev spaces associated to a positive Rockland operator $\mathcal{R}$ and show that they satisfy similar properties to the Euclidean Sobolev spaces. We will show that the constructed spaces are actually independent of the choice of a positive Rockland operator $\mathcal{R}$ on a graded Lie group with which we start our construction.

4.1. Definition and first properties of Sobolev spaces

We first need the following lemma:

LEMMA 4.1. — *We keep the notation of Theorem 3.6. For any $s \in \mathbb{R}$ and $p \in [1,\infty) \cup \{\infty_o\}$, the domain of the operator $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ contains $\mathcal{S}(G)$, and the map*

$$f \longmapsto \left\|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\right\|_{L^p(G)}$$

defines a norm on $\mathcal{S}(G)$. We denote it by

$$\|f\|_{L^p_s(G)} := \left\|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f\right\|_{L^p(G)}.$$

Moreover, any sequence in $\mathcal{S}(G)$ which is Cauchy for $\|\cdot\|_{L^p_s(G)}$ is convergent in $\mathcal{S}'(G)$.

We have allowed ourselves to write $\|\cdot\|_{L^\infty(G)} = \|\cdot\|_{L^{\infty_o}(G)}$ for the supremum norm. We may also write $\|\cdot\|_\infty$ or $\|\cdot\|_{\infty_o}$.

Proof. — The domain of $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ contains $\mathcal{S}(G)$: by Theorem 3.6 Part (2) for $s>0$, by Corollary 3.11 (2) for $s<0$ and, trivially for $s=0$ since $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} = \mathrm{I}$. Since the operator $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ is linear and injective (by Theorem 3.6 (1)), the map $\left\|(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}\cdot\right\|_{L^p(G)}$ defines a norm on $\mathcal{S}(G)$. Clearly $\|\cdot\|_{L^p_0(G)} = \|\cdot\|_p$, so the case of $s=0$ is trivial. Let us assume $s>0$. By Corollary 3.11 (2), the operator $(\mathrm{I}+\mathcal{R}_p)^{-\frac{s}{\nu}}$ is bounded on $L^p(G)$. Hence we have $\|\cdot\|_{L^p(G)} \leqslant C\,\|\cdot\|_{L^p_s(G)}$ on $\mathcal{S}(G)$. Consequently a $\|\cdot\|_{L^p_s(G)}$-Cauchy sequence of Schwartz functions converge in L^p-norm thus in $\mathcal{S}'(G)$. Now let us assume $s<0$. Let $\{f_\ell\}_{\ell\in\mathbb{N}}$ be a sequence of Schwartz functions which is Cauchy for the norm $\|\cdot\|_{L^p_s(G)}$. By (3.9) we have $f_\ell = \left((\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f_\ell\right) * \mathcal{B}_s$. Furthermore, if $\phi\in\mathcal{S}(G)$ then using (2.1) and (2.10), we have

$$\int_G f_\ell(x)\phi(x)\,\mathrm{d}x = \int_G \left((\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f_\ell\right)(x)\ \left(\phi * \mathcal{B}_s\right)(x)\,\mathrm{d}x. \tag{4.1}$$

By assumption the sequence $\{(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}} f_\ell\}_{\ell\in\mathbb{N}}$ is $\|\cdot\|_{L^p(G)}$-Cauchy thus convergent in $L^p(G)$. By Lemma 3.14, $\phi * \mathcal{B}_s \in \mathcal{S}(G)$. Therefore, the right hand-side of (4.1) is convergent as $\ell\to\infty$. Hence the scalar sequence $\langle f_\ell,\phi\rangle$ converges for any $\phi\in\mathcal{S}(G)$. This shows that the sequence $\{f_\ell\}$ converges in $\mathcal{S}'(G)$. ☐

Lemma 4.1 allows us to define the Sobolev spaces:

DEFINITION 4.2. — *Let $\mathcal{R}$ be a positive Rockland operator on G. We consider its L^p-analogue $\mathcal{R}_p$ and the powers of $(\mathrm{I}+\mathcal{R}_p)^a$ as defined in Theorems 3.3 and 3.6. Let $s\in\mathbb{R}$.*

If $p \in [1, \infty)$, the Sobolev space $L^p_{s,\mathcal{R}}(G)$ *is the subspace of $\mathcal{S}'(G)$ obtained by completion of $\mathcal{S}(G)$ with respect to the* Sobolev norm

$$\|f\|_{L^p_{s,\mathcal{R}}(G)} := \left\|(\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}} f\right\|_{L^p(G)}, \quad f \in \mathcal{S}(G).$$

If $p = \infty_o$, the Sobolev space $L^{\infty_o}_{s,\mathcal{R}}(G)$ *is the subspace of $\mathcal{S}'(G)$ obtained by completion of $\mathcal{S}(G)$ with respect to the* Sobolev norm

$$\|f\|_{L^{\infty_o}_{s,\mathcal{R}}(G)} := \left\|(\mathrm{I} + \mathcal{R}_{\infty_o})^{\frac{s}{\nu}} f\right\|_{L^\infty(G)}, \quad f \in \mathcal{S}(G).$$

When the Rockland operator $\mathcal{R}$ is fixed, we may allow ourselves to drop the index $\mathcal{R}$ in $L^p_{s,\mathcal{R}}(G) = L^p_s(G)$ to simplify the notation.

We will see later that the Sobolev spaces do not depend on the Rockland operator $\mathcal{R}$, see Theorem 4.20.

By construction the Sobolev space $L^p_s(G)$ endowed with the Sobolev norm is a Banach space which contains $\mathcal{S}(G)$ as a dense subspace and is included in $\mathcal{S}'(G)$. The Sobolev spaces share many properties with their Euclidean counterparts.

Theorem 4.3. — *Let $\mathcal{R}$ be a positive Rockland operator on G. We consider the associated Sobolev spaces $L^p_s(G)$ for $p \in [1, \infty) \cup \{\infty_o\}$ and $s \in \mathbb{R}$.*

(1) *If $s = 0$, then $L^p_0(G) = L^p(G)$ for $p \in [1, \infty)$ with $\|\cdot\|_{L^p_0(G)} = \|\cdot\|_{L^p(G)}$, and $L^{\infty_o}_0(G) = C_o(G)$ with $\|\cdot\|_{L^{\infty_o}_0(G)} = \|\cdot\|_{L^\infty(G)}$.*

(2) *If $s > 0$, then for any $a \in \mathbb{C}$ with $\Re a = s$, we have*

$$L^p_s(G) = \mathrm{Dom}\left[(\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}}\right] = \mathrm{Dom}(\mathcal{R}_p^{\frac{a}{\nu}}) \subsetneq L^p(G),$$

and the following norms are equivalent to $\|\cdot\|_{L^p_s(G)}$:

$$f \longmapsto \|f\|_{L^p(G)} + \left\|(\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}} f\right\|_{L^p(G)}, \; f \longmapsto \|f\|_{L^p(G)} + \left\|\mathcal{R}_p^{\frac{s}{\nu}} f\right\|_{L^p(G)}.$$

(3) *Let $s \in \mathbb{R}$ and $f \in \mathcal{S}'(G)$.*
   - *If $p \in (1, \infty)$ and $f \in L^p_s(G)$ then $(\mathrm{I} + \mathcal{R}_p)^{s/\nu} f \in L^p(G)$ in the sense that the linear mapping $\mathcal{S}(G) \ni \phi \mapsto \langle f, (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{s/\nu}\phi\rangle$ extends to a bounded functional on $L^{p'}(G)$. The converse is also true.*
   - *If $f \in L^1_s(G)$ then $(\mathrm{I} + \mathcal{R}_1)^{s/\nu} f \in L^1(G)$ in the sense that the linear mapping $\mathcal{S}(G) \ni \phi \mapsto \langle f, (\mathrm{I} + \bar{\mathcal{R}}_{\infty_o})^{s/\nu}\phi\rangle$ extends to a bounded functional on $C_o(G)$ which is realised as a measure given by an integrable function. The converse is also true.*
   - *Let $p = \infty_o$. If $f \in L^{\infty_o}_s(G)$ then $(\mathrm{I} + \mathcal{R}_{\infty_o})^{s/\nu} f \in C_o(G)$ and the linear mapping $\mathcal{S}(G) \ni \phi \mapsto \langle f, (\mathrm{I} + \bar{\mathcal{R}}_1)^{s/\nu}\phi\rangle$ extends to a*

bounded functional on $L^1(G)$. If $s \geqslant 0$ this linear mapping is realised as integration against a function in $C_o(G)$. Conversely, for any s, if the linear mapping $\mathcal{S}(G) \ni \phi \mapsto \langle f, (\mathrm{I}+\bar{\mathcal{R}}_1)^{s/\nu}\phi\rangle$ extends to a bounded functional on $L^1(G)$ and if this bounded extension is realised as integration against function in $C_o(G)$ then $f \in L^{\infty_o}_s(G)$.

(4) *If $a, b \in \mathbb{R}$ with $a < b$ and $p \in [1,\infty)$, then the following continuous strict inclusions hold*

$$\mathcal{S}(G) \subsetneq L^p_b(G) \subsetneq L^p_a(G) \subsetneq \mathcal{S}'(G),$$

and an equivalent norm for $L^p_b(G)$ is

$$L^p_b(G) \ni f \longmapsto \|f\|_{L^p_a(G)} + \left\|\mathcal{R}^{\frac{b-a}{\nu}} f\right\|_{L^p_a(G)}.$$

If $p = \infty_o$, the same properties hold for $a, b \geqslant 0$.

Furthermore if $c \in (a, b)$, then there exists a positive constant $C = C_{a,b,c}$ such that we have for any $f \in L^p_b$

$$\|f\|_{L^p_c} \leqslant C \, \|f\|^{1-\theta}_{L^p_a} \, \|f\|^{\theta}_{L^p_b} \quad \textit{where } \theta := (c-a)/(b-a).$$

In Part (3) above, for the case $p = \infty_o$, the bounded functional can be identified with a function in $L^\infty(G)$ and not (in general) with a function in $L^{\infty_o}_0(G) = C_o(G)$ if $s < 0$. The converse does not hold in this case.

From now on, we will often use the notation $L^p_0(G)$ since this allows us not to distinguish between the cases $L^p_0(G) = L^p(G)$ when $p \in [1,\infty)$ and $L^p_0(G) = C_o(G)$ when $p = \infty_o$.

We can now prove Theorem 4.3.

Proof of Theorem 4.3, Part (1). — This is true since $(\mathrm{I}+\mathcal{R}_p)^{\frac{0}{\nu}} = \mathrm{I}$. ☐

Proof of Theorem 4.3, Part (2). — Let $s > 0$. Clearly $L^p_s(G)$ coincides with the domain of the unbounded operator $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ (see Theorem 3.6 (2)) hence it is a proper subspace of $L^p(G)$. As the operator $(\mathrm{I}+\mathcal{R}_p)^{-\frac{s}{\nu}}$ is bounded on $L^p(G)$, we have $\|\cdot\|_{L^p(G)} \leqslant C\|\cdot\|_{L^p_s(G)}$ on $L^p_s(G)$. So $\|\cdot\|_{L^p(G)} + \|\cdot\|_{L^p_s(G)}$ is a norm on $L^p_s(G)$ which is equivalent to the Sobolev norm. By Theorem 3.6, the operators $\mathcal{R}_p^{\frac{s}{\nu}}$ and $(\mathrm{I}+\mathcal{R}_p)^{\frac{s}{\nu}}$ share the same domain. Hence Part (2) follows from general functional analysis, especially the closed graph theorem. ☐

Proof of Theorem 4.3,Part (3). — This follows from Part (2) in the case $s \geqslant 0$. We now consider the case $s < 0$. By Lemma 3.14 and Corollary 3.11, the mapping

$$T_{s,p',f} : \mathcal{S}(G) \ni \phi \longmapsto \langle f, (\mathrm{I}+\bar{\mathcal{R}}_{p'})^{s/\nu}\phi\rangle = \langle f, \phi * \bar{\mathcal{B}}_{-s}\rangle$$

is well defined for any $f \in \mathcal{S}'(G)$. If $T_{s,p',f}$ admits a bounded extension to a functional on $L_0^{p'}(G)$, then we denote this extension $\tilde{T}_{s,p',f}$ and we have $\|\tilde{T}_{s,p',f}\|_{\mathscr{L}(L^{p'})} = \|f\|_{L^p_s(G)}$. This is certainly so if $f \in \mathcal{S}(G)$. The proof of Part (3). follows from the following observation: a sequence $\{f_\ell\}_{\ell\in\mathbb{N}}$ of Schwartz functions is convergent for the Sobolev norm $\|\cdot\|_{L^p_s(G)}$ if and only if $\{\tilde{T}_{s,p',f_\ell}\}$ is convergent in $L_0^{p'}(G)$. □

Note that the proof of Part (3) of Theorem 4.3 implies easily

Lemma 4.4. — *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group G. We consider the associated Sobolev spaces $L^p_{s,\mathcal{R}}(G)$ for $p \in [1,\infty) \cup \{\infty_o\}$ and $s \in \mathbb{R}$.*

For any $s \in \mathbb{R}$ and $p \in [1,\infty)$, the dual space of $L^p_{s,\mathcal{R}}(G)$ is isomorphic to $L^{p'}_{-s,\bar{\mathcal{R}}}(G)$ via the distributional duality, where p' is the conjugate exponent of p if $p \in (1,\infty)$, and $p' = \infty_o$ if $p = 1$.

For any $s \leqslant 0$ and $p = \infty_o$, the dual space of $L^{\infty_o}_{s,\mathcal{R}}(G)$ is isomorphic to $L^1_{-s,\bar{\mathcal{R}}}(G)$ via the distributional duality.

Lemma 4.4 will be improved in Proposition 4.22 once we show (see Theorem 4.20) that Sobolev spaces are indeed independent of the considered Rockland operator.

Proof of Theorem 4.3, Part (4). — If $0 < a < b$, then by Parts (1) and (2), which are already proven, and Theorem 3.6 on fractional powers, there is a constant $C = C_{a,b} > 0$ such that

$$\forall\, f \in L^p_b(G) \quad \|f\|_{L^p_a(G)} \leqslant C\, \|f\|_{L^p(G)}^{1-\frac{a}{b}}\, \|f\|_{L^p_b(G)}^{\frac{a}{b}} < \infty,$$

and we have the inclusions $L^p_b(G) \subset L^p_a(G) \subset L^p(G)$ for any $p \in [1,\infty) \cup \{\infty_o\}$. These inclusions are strict since the operator $(\mathrm{I} + \mathcal{R}_p)^{\frac{a-b}{\nu}}$ is unbounded, see Theorem 3.6 (2).

We can now use the duality results given in Lemma 4.4 since its proof relies on Part (3) shown above. Together with the duality result in Theorem 3.6, this yields the reverse inclusions for $b < a < 0$ if $p \neq \infty_o$. Since $L^p(G) = L^p_0(G)$, we have obtained the inclusion of Part (4) for any $a, b \in \mathbb{R}$ with $a \leqslant b$.

Let $f \in L^p_b(G)$ with $a \leqslant b$ and $p \in [1,\infty) \cup \{\infty_o\}$ with the additional property that $a, b \geqslant 0$ if $p = \infty_o$. So $f \in L^p_a(G)$ and one obtains easily by

Theorem 3.6 (1) and Part (2) above

$$\|f\|_{L^p_b(G)} = \left\|(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\right\|_{L^p_{b-a}(G)}$$
$$\asymp \left\|(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\right\|_{L^p(G)} + \left\|\mathcal{R}_p^{\frac{b-a}{\nu}} (\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\right\|_{L^p(G)}.$$

By Theorem 3.6 (5), we can commute the operators $\mathcal{R}_p^{\frac{b-a}{\nu}}$ and $(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}}$ in this last expression. Consequently, for any $f \in L^p_b(G)$, we obtain

$$\|f\|_{L^p_b(G)} \asymp \|f\|_{L^p_a(G)} + \left\|\mathcal{R}_p^{\frac{b-a}{\nu}} f\right\|_{L^p_a(G)}.$$

The end of Part (4) follows from Theorem 3.6 (1). □

This concludes the proof of Theorem 4.3 which yields the two following corollaries:

Corollary 4.5. — *We keep the setting and notation of Theorem 4.3. Let $s < 0$ and $p \in [1, \infty) \cup \{\infty_o\}$. Let $f \in \mathcal{S}'(G)$.*

The tempered distribution f is in $L^p_s(G)$ if and only if the mapping $\phi \in \mathcal{S}(G) \mapsto \langle f, \phi * \bar{\mathcal{B}}_{-s}\rangle$ extends to a bounded linear functional on $L^{p'}_0(G)$ with the additional property that for $p = 1$ this functional on $C_o(G)$ is realised as a measure given by an integrable function, and if $p = \infty_o$, this functional on $L^1(G)$ is realised by integration against a function in $C_o(G)$.

Corollary 4.6. — *We keep the setting and notation of Theorem 4.3. Let $s \in \mathbb{R}$ and $p \in [1, \infty) \cup \{\infty_o\}$. Then $\mathcal{D}(G)$ is dense in $L^p_s(G)$.*

Proof of Corollary 4.6. — This is certainly true for $s \geqslant 0$ (see the proof of Parts (1) and (2) of Theorem 4.3). For $s < 0$, it suffices to proceed as in the last part of the proof of Part (3) with a sequence of functions $f_\ell \in \mathcal{D}(G)$. □

In the next statement, we show how to produce functions and converging sequences in Sobolev spaces using the convolution:

Proposition 4.7. — *We keep the setting and notation of Theorem 4.3. Here $a \in \mathbb{R}$ and $p \in [1, \infty) \cup \{\infty_o\}$.*

(1) *If $f \in L^p_0(G)$ and $\phi \in \mathcal{S}(G)$, then $f * \phi \in L^p_a$ for any a and p.*

(2) *If $f \in L^p_a(G)$ and $\psi \in \mathcal{S}(G)$, then*

$$(\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} (\psi * f) = \psi * \left((\mathrm{I}+\mathcal{R}_p)^{\frac{a}{\nu}} f\right), \tag{4.2}$$

and $\psi * f \in L^p_a(G)$ with $\|\psi * f\|_{L^p_a(G)} \leqslant \|\psi\|_{L^1(G)} \|f\|_{L^p_a(G)}$. Furthermore, assuming $\int_G \psi = 1$ and writing $\psi_\epsilon(x) := \epsilon^{-Q}\psi(\epsilon^{-1}x)$ for each $\epsilon > 0$, then $\{\psi_\epsilon * f\}$ converges to f in $L^p_a(G)$ as $\epsilon \to 0$.

Proof of Proposition 4.7. — Let us prove Part (1). Here $f \in L^p_0(G)$. By density of $\mathcal{S}(G)$ in $L^p_0(G)$, we can find a sequence of Schwartz functions $\{f_\ell\}$ converging to f in L^p_0-norm. Then $f_\ell * \phi \in \mathcal{S}(G)$ and for any $N \in \mathbb{N}$,

$$\mathcal{R}^N(f_\ell * \phi) = f_\ell * \mathcal{R}^N \phi \underset{\ell\to\infty}{\longrightarrow} f * \mathcal{R}^N \phi \quad \text{in } L^p_0(G),$$

thus $\mathcal{R}_p^N(f * \phi) = f * \mathcal{R}^N \phi \in L^p(G)$ and

$$\|f * \phi\|_{L^p_0(G)} + \left\|\mathcal{R}_p^N(f * \phi)\right\|_{L^p_0(G)} < \infty.$$

By Theorem 4.3 (4), this shows that $f * \phi$ is in $L^p_{\nu N}$ for any $N \in \mathbb{N}$, hence in any p-Sobolev spaces. This proves (1).

Let us prove Part (2). We observe that both sides of Formula (4.2) always make sense as convolutions of a Schwartz function with a tempered distribution. Formula (4.2) is clearly true if $a < 0$ by Corollary 3.11 (2) since then the $(\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}}$ is a convolution operator. Consequently (4.2) is true also for any $f, \psi \in \mathcal{S}(G)$ and $a \in \mathbb{R}$ by the analyticity result of Theorem 3.6 and Lemma 3.14. Using this result for Schwartz functions yields that Equality (4.2) holds as distributions for any $f \in L^p_a(G)$, $\phi \in \mathcal{S}(G)$, and $a \in \mathbb{R}$, since we have

$$\begin{aligned}\langle (\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}}(\psi * f), \phi\rangle &= \langle \psi * f, (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{\frac{a}{\nu}}\phi\rangle = \langle f, \tilde{\psi} * (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{\frac{a}{\nu}}\phi\rangle \\ &= \langle f, (\mathrm{I} + \bar{\mathcal{R}}_{p'})^{\frac{a}{\nu}}(\tilde{\psi} * \phi)\rangle = \langle (\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}} f, \tilde{\psi} * \phi\rangle.\end{aligned}$$

Taking the L^p-norm on both sides of Equality (4.2) yields

$$\left\|(\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}}(\psi * f)\right\|_p = \left\|\psi * \left((\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}} f\right)\right\|_p \leqslant \|\psi\|_1 \left\|(\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}} f\right\|_p.$$

Hence $\psi * f \in L^p_a(G)$ with L^p_a-norm $\leqslant \|\psi\|_1 \|f\|_{L^p_a(G)}$. Moreover, by Lemma 2.6,

$$\begin{aligned}\|\psi_\epsilon * f - f\|_{L^p_a(G)} &= \left\|(\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}}(\psi_\epsilon * f - f)\right\|_p \\ &= \left\|\psi_\epsilon * \left((\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}} f\right) - (\mathrm{I} + \mathcal{R}_p)^{\frac{a}{\nu}} f\right\|_p \longrightarrow_{\epsilon\to 0} 0,\end{aligned}$$

that is, $\{\psi_\epsilon * f\}$ converges to f in $L^p_a(G)$ as $\epsilon \to 0$. This proves (2). □

4.2. Interpolation between Sobolev spaces

In this section, we prove that interpolation between Sobolev spaces $L^p_a(G)$ works in the same way as its Euclidean counterpart.

THEOREM 4.8. — *Let $\mathcal{R}$ and $\mathcal{Q}$ be two positive Rockland operators on two graded Lie groups G and F. We consider their associated Sobolev spaces $L^p_a(G)$ and $L^q_b(F)$. Let $p_0, p_1, q_0, q_1 \in [1, \infty) \cup \{\infty_o\}$ and $a_0, a_1, b_0, b_1 \in$*

$\mathbb{R}$. We also consider a linear mapping T from $L^{p_0}_{a_0}(G) + L^{p_1}_{a_1}(G)$ to locally integrable functions on F. We assume that T maps $L^{p_0}_{a_0}(G)$ and $L^{p_1}_{a_1}(G)$ boundedly into $L^{q_0}_{b_0}(F)$ and $L^{q_1}_{b_1}(F)$, respectively.

Then T extends uniquely to a bounded mapping from $L^{p}_{a_t}(G)$ to $L^{q}_{b_t}(F)$ for $t \in [0,1]$ where a_t, b_t, p_t, q_t are defined by

$$\left(a_t, b_t, \frac{1}{p_t}, \frac{1}{q_t}\right) = t\left(a_0, b_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + (1-t)\left(a_1, b_1, \frac{1}{p_1}, \frac{1}{q_1}\right).$$

Proof of Theorem 4.8. — By duality (see Lemma 4.4 and the duality property in Theorem 3.6), it suffices to prove Theorem 4.8 in the case $a_1, b_1, a_0, b_0 \geqslant 0$. Furthermore, up to a change of notation, we may assume $b_1 \geqslant b_0$.

The idea of the proof is similar to the one of the Euclidean or stratified cases (see [12, Theorem 4.7]): we interpolate between the operators formally given by $T_z = (\mathrm{I}+\mathcal{Q})^{b_z/\nu_{\mathcal{Q}}} T (\mathrm{I}+\mathcal{R})^{-a_z/\nu_{\mathcal{R}}}$, where $\nu_{\mathcal{R}}$ and $\nu_{\mathcal{Q}}$ denote the degrees of homogeneity of $\mathcal{R}$ and $\mathcal{Q}$ respectively and the complex numbers a_z and b_z are defined by $(a_z, b_z) := z\,(a_0, b_0) + (1-z)\,(a_1, b_1)$ for z in $S := \{z \in \mathbb{C} : \Re z \in [0,1]\}$. More precisely, by Theorem 3.6 (1), we may assume that the fractional powers may not depend on the varying L^p-spaces which they are acting upon and we set for any $\phi \in \mathcal{S}(G)$ and $z \in S$,

$$U_z \phi := (\mathrm{I}+\mathcal{Q}_{q_1})^{\frac{b_1}{\nu_{\mathcal{Q}}}} T (\mathrm{I}+\mathcal{R})^{-\frac{a_z}{\nu_{\mathcal{R}}}} \phi.$$

Necessarily $U_z\phi \in L^{q_1}_0(F)$ and we can define its convolution with the integrable Bessel potential $\mathcal{B}_a$ corresponding to $\mathcal{Q}$, $\Re a > 0$, or apply the bounded operator $(\mathrm{I}+\mathcal{Q}_{q_1})^a$, $\Re a = 0$:

$$T_z\phi := \begin{cases} (U_z\phi) * \mathcal{B}_{b_1 - b_z} & \text{if } \Re(b_1 - b_z) > 0, \\ (\mathrm{I}+\mathcal{Q}_{q_1})^{\frac{b_z - b_1}{\nu_{\mathcal{Q}}}} (U_z\phi) & \text{if } \Re(b_1 - b_z) = 0. \end{cases}$$

Clearly for any $\psi \in \mathcal{S}(F)$, we have

$$\langle T_z\phi, \psi\rangle = \langle U_z\phi, (\mathrm{I}+\mathcal{Q})^{\frac{z(b_0 - b_1)}{\nu_{\mathcal{Q}}}} \psi\rangle, \tag{4.3}$$

and this expression is analytic on S by Lemma 3.14 and its proof.

Inside the strip S, Corollary 3.11 (2) and its proof, together with the bound of $\|\mathcal{B}_a\|_1$ in (3.6) with Sterling's estimates easily yield:

$$\forall\, z = x+iy \in S \qquad \ln|\langle T_z\phi, \psi\rangle| \leqslant \ln|y|(2|y| + O(\ln|y|))$$

with the constant from the notation O depending on $\phi, \psi, a_1, a_0, b_1, b_0$. Similarly (using also Proposition 3.7 for the case $\Re a = 0$) one shows that for $j = 0, 1$ we have

$$\forall\, y \in \mathbb{R} \qquad e^{-3|y|} \ln\|T_{j+iy}\|_{\mathscr{L}(L^{p_j}, L^{q_j})} \leqslant M_j,$$

for some finite positive constants M_0 and M_1 independent of y. The end of the proof is now similar to the stratified case, see the proof of Theorem 4.7 in [12]. $\square$

4.3. Homogeneous Sobolev spaces

Here we define and study the homogeneous version of our Sobolev spaces.

Definition 4.9. — *Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν on a graded Lie group G, and let $p \in (1,\infty)$. We denote by $\dot{L}^p_{s,\mathcal{R}}(G)$ the completion of the Schwartz space for the norm*

$$\|f\|_{\dot{L}^p_s(G)} := \left\|\mathcal{R}_p^{\frac{s}{\nu}} f\right\|_p, \quad f \in \mathrm{Dom}(\mathcal{R}_p^{s/\nu}).$$

As in the inhomogeneous case, we may write $\dot{L}^p_s(G)$ and omit the reference to the Rockland operator $\mathcal{R}$ when it is not useful.

Note that if $s \geqslant 0$ then $\mathcal{S}(G) \subset \mathrm{Dom}(\mathcal{R}_p^{s/\nu})$ by Theorem 3.6 (1f). However, even in the Euclidean case, we have $\mathcal{S}(\mathbb{R}^n) \not\subset \dot{L}^2_{-s}$ for $s > n/2$. Indeed one can construct a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\Delta^{-s/2}\phi \notin L^2$ by choosing ϕ such that its Fourier transform is identically one on a neighbourhood of 0 and compactly supported. And one can construct elements of $\dot{L}^2_s$, again for $s > n/2$, which are not tempered distributions.

We realise the elements of $L^p_{s,\mathcal{R}}(G)$ as follows:

Proposition 4.10. — *We continue with the notation of Definition 4.9. We realise an element f of $\dot{L}^p_{s,\mathcal{R}}(G)$ as a linear functional on* $\mathrm{Dom}(\mathcal{R}_{p'}^{-\frac{s}{\nu}})$ *satisfying*

$$\exists\, C > 0 \qquad \forall\, \phi \in \mathrm{Dom}(\bar{\mathcal{R}}_{p'}^{-\frac{s}{\nu}}) \qquad |f(\phi)| \leqslant C \left\|\bar{\mathcal{R}}_{p'}^{-\frac{s}{\nu}}\phi\right\|_{p'}. \tag{4.4}$$

In the case of $s \leqslant 0$, f can also be viewed as a tempered distribition in $L^p_s(G)$.

Proof. — If $(f_\ell)_{\ell\in\mathbb{N}_0}$ is a $\|\cdot\|_{\dot{L}^p_s(G)}$-Cauchy sequence in $\mathrm{Dom}(\mathcal{R}_p^{s/\nu})$, then one checks easily that $\phi \mapsto \lim_{\ell\to\infty}\langle f_\ell,\phi\rangle$ define a linear functional f satisfying (4.4) with $C = \liminf \|f_\ell\|_{\dot{L}^p_s(G)}$. This shows that two equivalent Cauchy sequences yields the same functional.

Let f be a linear functional satisfying (4.4). Then the linear functional $\phi \mapsto f(\bar{\mathcal{R}}_{p'}^{\frac{s}{\nu}}\phi)$ is defined on $\mathrm{Dom}(\bar{\mathcal{R}}_{p'}^{\frac{s}{\nu}})$ which is a dense subset of $L^{p'}$ by Theorem 3.6 (1) and Corollary 3.4. Hence it extends continuously to $L^{p'}$ and can be identified with an element of $L^p(G)$, which we denote by $\mathcal{R}^{\frac{s}{\nu}} f$. We fix

$\psi \in \mathcal{S}(G)$ with $\int_G \psi = 1$, and set $\psi_\epsilon(x) = \epsilon^{-Q}\psi(\epsilon^{-1}x)$ for $\epsilon > 0$. We have $(\mathcal{R}^{\frac{s}{\nu}} f) * \psi_\epsilon \in \mathrm{Dom}(\mathcal{R}^N)$ for any $N \in \mathbb{N}$, see the proof of Proposition 4.7 (1). Let us assume $s \leqslant 0$. By Theorem 3.6 (1), $(\mathcal{R}^{\frac{s}{\nu}} f) * \psi_\epsilon \in \mathrm{Dom}(\mathcal{R}_p^{-s/\nu})$ and we set

$$f_\epsilon := \mathcal{R}_p^{-s/\nu} \left((\mathcal{R}^{\frac{s}{\nu}} f) * \psi_\epsilon \right) \in \mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}}).$$

By Lemma 2.6, we have

$$\left\| \mathcal{R}_p^{\frac{s}{\nu}} f_\epsilon - \mathcal{R}_p^{\frac{s}{\nu}} f \right\|_{L^p} = \left\| (\mathcal{R}^{\frac{s}{\nu}} f) * \psi_\epsilon - \mathcal{R}_p^{\frac{s}{\nu}} f \right\|_{L^p} \longrightarrow_{\epsilon \to 0} 0.$$

From this, we deduce that (f_ϵ) is a Cauchy sequence in $\mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$, converging to f for $\|\cdot\|_{\dot{L}^p_s}$ in the case $s \leqslant 0$. Note that still in the case $s \leqslant 0$, by adding $\|\phi\|_p$ to the right-hand side of the estimate (4.4) and using Theorem 4.3 (2), f extends into a continuous linear mapping on $L^{p'}_{-s,\bar{\mathcal{R}}}$. Therefore f may be viewed as an element of $L^p_{s,\mathcal{R}}$ by Lemma 4.4.

In the case $s > 0$, we need to choose ψ in $\mathcal{R}^N(\mathcal{S}(G))$ for some integer $N > s$, which is always possible: we fix N to be the smallest integer strictly greater than s and choosing first $\tilde{\psi} \in \mathcal{S}(G)$ with $\tilde{c} := \int_G \mathcal{R}^N \tilde{\psi}(x)\, \mathrm{d}x \neq 0$, we set $\psi := \tilde{c}^{-1}\mathcal{R}^N\tilde{\psi}$. Proceeding as above, this ensures that $(\mathcal{R}^{\frac{s}{\nu}} f) * \psi_\epsilon$ is in the range of $\mathcal{R}^N$, hence in the domain of $\mathcal{R}_p^{-s/\nu}$. As above, we deduce that (f_ϵ) is a Cauchy sequence in $\mathrm{Dom}(\mathcal{R}_p^{\frac{s}{\nu}})$, converging to f for $\|\cdot\|_{\dot{L}^p_s}$ and this proves the case $s > 0$. □

We can now obtain the following properties of homogeneous Sobolev spaces:

Proposition 4.11. — *Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν on G. Let $s \in \mathbb{R}$ and $p \in (1, \infty)$. We denote by p' its conjugate exponent: $\frac{1}{p} + \frac{1}{p'} = 1$.*

(1) *If $s = 0$, then $\dot{L}^p_0(G) = L^p(G)$ for $p \in (1, \infty)$ with $\|\cdot\|_{\dot{L}^p_0(G)} = \|\cdot\|_{L^p(G)}$.*

(2) *If $f \in \dot{L}^p_s(G)$ then $\mathcal{R}_p^{s/\nu} f \in L^p(G)$ in the sense that the linear functional $\phi \mapsto f(\bar{\mathcal{R}}_{p'}^{s/\nu}\phi)$ extends continuously from the dense subspace $\mathrm{Dom}(\bar{\mathcal{R}}_{p'}^{s/\nu})$ to $L^{p'}(G)$.*

Conversely, if $g \in L^p(G)$ then $\mathcal{R}_p^{-s/\nu} g \in \dot{L}^p_s(G)$ in the sense that the linear functional $\phi \mapsto \langle g, \bar{\mathcal{R}}_{p'}^{-\frac{s}{\nu}}\phi \rangle$ satisfies (4.4).

More generally, for any $s_1 \in \mathbb{R}$, $\mathcal{R}_p^{s_1/\nu}$ maps $\dot{L}^p_s$ bijectively onto $\dot{L}^p_{s+s_1}$.

(3) *If* $s>0$, *then* $\mathcal{S}(G)\subset \mathrm{Dom}(\mathcal{R}_p^{s/\nu})\subset \dot{L}^p_s(G)$ *and we have* $L^p_s(G)=\dot{L}^p_s(G)\cap L^p(G)$*; the Sobolev norm is equivalent to*

$$\|\cdot\|_{L^p_s(G)}\asymp \|\cdot\|_{L^p(G)}+\|\cdot\|_{\dot{L}^p_s(G)}\,.$$

(4) *For* $p\in(0,1)$ *and any* $a,b,c\in\mathbb{R}$ *with* $a<c<b$, *there exists a positive constant* $C=C_{a,b,c}$ *such that we have for any* $f\in\dot{L}^p_b$

$$\|f\|_{\dot{L}^p_c}\leqslant C\,\|f\|^{1-\theta}_{\dot{L}^p_a}\,\|f\|^{\theta}_{\dot{L}^p_b}\quad \text{where } \theta:=(c-a)/(b-a).$$

Proof. — Part (1) is trivial. Part (2) follows from the proof of Proposition 4.10 and easy verifications. Part (3) follows from Part (2) of Theorem 4.3 and the density of $\mathcal{S}(G)$. Part (4) follows from Theorem 3.6 (1e) and Part (2) proved above. □

From Proposition 4.10 and Part (2) of Proposition 4.11, we also obtain the following duality result, which will be improved in Proposition 4.22:

LEMMA 4.12. — *Let* $\mathcal{R}$ *be a positive Rockland operator of homogeneous degree* ν *on* G. *Let* $s\in\mathbb{R}$ *and* $p\in(1,\infty)$. *We denote by* p' *its conjugate exponent:* $\frac{1}{p}+\frac{1}{p'}=1$.

The dual space of $\dot{L}^p_{s,\mathcal{R}}(G)$ *is isomorphic to* $\dot{L}^{p'}_{-s,\bar{\mathcal{R}}}(G)$ *via the duality:*

$$\langle f,g\rangle_{\dot{L}^p_{s,\mathcal{R}}(G)\times\dot{L}^{p'}_{-s,\bar{\mathcal{R}}}(G)}=\langle \mathcal{R}_p^{\frac{s}{\nu}}f,\bar{\mathcal{R}}_{p'}^{-\frac{s}{\nu}}g\rangle_{L^p(G)\times L^{p'}(G)}. \tag{4.5}$$

The proof of Lemma 4.12 is left to the reader.

The following interpolation property can be proved after a careful modification of the inhomogeneous proof:

PROPOSITION 4.13. — *Let* $\mathcal{R}$ *and* $\mathcal{Q}$ *be two positive Rockland operators on two graded Lie groups* G *and* F *respectively. We consider their associated homogeneous Sobolev spaces* $\dot{L}^p_a(G)$ *and* $\dot{L}^q_b(F)$. *Let* $p_0,p_1,q_0,q_1\in(1,\infty)$ *and* $a_0,a_1,b_0,b_1\in\mathbb{R}$.

We also consider a linear mapping T *from* $\dot{L}^{p_0}_{a_0}(G)+\dot{L}^{p_1}_{a_1}(G)$ *to locally integrable functions on* F. *We assume that* T *maps* $\dot{L}^{p_0}_{a_0}(G)$ *and* $\dot{L}^{p_1}_{a_1}(G)$ *boundedly into* $\dot{L}^{q_0}_{b_0}(F)$ *and* $\dot{L}^{q_1}_{b_1}(F)$, *respectively.*

Then T *extends uniquely to a bounded mapping from* $\dot{L}^p_{a_t}(G)$ *to* $\dot{L}^q_{b_t}(F)$ *for* $t\in[0,1]$, *where* a_t,b_t,p_t,q_t *are defined by*

$$\left(a_t,b_t,\frac{1}{p_t},\frac{1}{q_t}\right)=(1-t)\left(a_0,b_0,\frac{1}{p_0},\frac{1}{q_0}\right)+t\left(a_1,b_1,\frac{1}{p_1},\frac{1}{q_1}\right).$$

Sketch of the proof of Proposition 4.13. — By duality (see Lemma 4.12) and up to a change of notation, it suffices to prove the case $a_1\geqslant a_0$ and

$b_1 \leqslant b_0$. The idea is to interpolate between the operators formally on $\mathcal{S}(G)$ given by

$$T_z = \mathcal{Q}^{z(b_1-b_0)/\nu_{\mathcal{Q}}} \mathcal{Q}^{b_0} T \mathcal{R}^{-a_0} \mathcal{R}^{z(a_0-a_1)/\nu_{\mathcal{R}}}, \quad z \in S, \tag{4.6}$$

with the same notation for $\nu_{\mathcal{R}}$, $\nu_{\mathcal{Q}}$, a_z, b_z and S as in the proof of Theorem 4.8. In (4.6), we have abused the notation regarding the fractional powers of $\mathcal{R}_p$ and $\mathcal{Q}_q$ and removed p and q thanks to by Theorem 3.6 (1). Moreover, Theorem 3.6 implies that on $\mathcal{S}(G)$, each operator T_z, $z \in S$, coincides with

$$T_z = \mathcal{Q}^{(1-z)(b_1-b_0)/\nu_{\mathcal{Q}}} \mathcal{Q}^{b_1} T \mathcal{R}^{-a_1} \mathcal{R}^{(1-z)(a_0-a_1)/\nu_{\mathcal{R}}},$$

and that for any $\phi \in \mathcal{S}(G)$ and $\psi \in \mathcal{S}(F)$, $z \mapsto \langle T_z\phi, \psi\rangle$ is analytic on S. We also have

$$|\langle T_z\phi, \psi\rangle| \leqslant \|T\|_{\mathscr{L}(\dot{L}^{p_1}_{a_1}, \dot{L}^{q_1}_{b_1})} \left\| \mathcal{R}^{\frac{-a_z+a_1}{\nu_{\mathcal{R}}}} \phi \right\|_{L^{p_1}} \left\| \bar{\mathcal{Q}}^{\frac{b_z-b_1}{\nu_{\mathcal{Q}}}} \psi \right\|_{L^{q_1'}}.$$

We have $\Re(-a_z + a_1) = (1 - \Re z)(a_1 - a_0) \in [0, a_1 - a_0]$ and $\Re(b_z - b_1) = (1 - \Re z)(b_1 - b_0) \in [b_1 - b_0, 0]$. If $\Re(-a_z + a_1) \in (0, a_1 - a_0)$, then Theorem 3.6 (1) implies

$$\left\| \mathcal{R}^{\frac{-a_z+a_1}{\nu_{\mathcal{R}}}} \phi \right\|_{p_1} \leqslant C \left(\|\phi\|_{p_1} + \left\| \mathcal{R}^{\frac{a_1-a_0}{\nu_{\mathcal{R}}}} \phi \right\|_{p_1} \right),$$

whereas if $\Re(-a_z + a_1) = 0$ or $a_1 - a_0$, that is, $-a_z + a_1 = iy$ or $-a_z + a_1 = a_1 - a_0 + iy$, then by Proposition 3.9

$$\left\| \mathcal{R}^{\frac{-a_z+a_1}{\nu_{\mathcal{R}}}} \phi \right\|_{p_1} \leqslant Ce^{\theta|y|} \|\phi\|_{p_1} \quad \text{or} \quad Ce^{\theta|y|} \left\| \mathcal{R}^{\frac{a_1-a_0}{\nu_{\mathcal{R}}}} \phi \right\|_{p_1} \quad \text{respectively.}$$

We have similar bounds for $\left\| \bar{\mathcal{Q}}^{\frac{b_z-b_1}{\nu_{\mathcal{Q}}}} \psi \right\|_{q_1'}$ and all these estimates imply that

$$e^{-|\Im z|} \ln |\langle T_z\phi, \psi\rangle|$$

is bounded uniformly with respect to $z \in S$ by a constant depending on $\phi, \psi, a_1, a_0, b_1, b_0$. We proceed in a similar way to show

$$\sup_{y\in\mathbb{R}} e^{-|y|} \ln \|T_{j+iy}\|_{\mathscr{L}(L^{p_j}, L^{q_j})} < \infty, \quad j = 0, 1.$$

We conclude the proof in the same way as for Theorem 4.8. $\square$

4.4. Differential operators acting on Sobolev spaces

In this section we show that left-invariant differential operators map Sobolev spaces to Sobolev spaces in the following way:

Theorem 4.14. — *If T is a left-invariant differential operator of homogeneous degree $\nu_T > 0$ (at most), then, for every $s \in \mathbb{R}$ and $p \in (1, \infty)$, T maps linearly boundedly $L^p_{s+\nu_T}(G)$ to $L^p_s(G)$, and $\dot{L}^p_{s+\nu_T}(G)$ to $\dot{L}^p_s(G)$.*

Remark 4.15. — The proof of the stratified case (see [12, p. 190]) uses at several stages the fact that the sub-Laplacian is of degree 2, and thus is not adaptable here.

The proof of Theorem 4.14 relies on the following property:

Lemma 4.16. — *Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν.*

If T is an operator of type 0, then, for each $N \in \mathbb{N}_0$, the operator $\mathcal{R}^N T \mathcal{R}_2^{-N}$ is of type 0.

For each $j = 1, \ldots, n$ and $N \in \mathbb{N}_0$, the operator $\mathcal{R}^N X_j \mathcal{R}_2^{-N-\frac{\upsilon_j}{\nu}}$ is of type 0.

Proof of Lemma 4.16. — Let $\mathcal{I}_s$ be the kernel of the Riesz operator of $\mathcal{R}$ as in Corollary 3.11. For each $j = 1, \ldots, n$, by Corollary 3.11 (1), $\mathcal{I}_{\upsilon_j}$ is a kernel of type $\upsilon_j \in (0, Q)$, so $X_j \mathcal{I}_{\upsilon_j}$ is a kernel of type 0 by Proposition 2.4 (3). More generally if κ is a kernel of type 0, then $\kappa * \mathcal{I}_{\nu_j}$ is a kernel of type ν_j by Proposition 2.4 (4) and $X_j(\kappa * \mathcal{I}_{\nu_j})$ is a kernel of type 0 by Proposition 2.4 (3). Proceeding recursively shows that

$$X^\alpha \left(\kappa * \mathcal{I}_{\upsilon_n}^{*\alpha_n} * \ldots * \mathcal{I}_{\upsilon_1}^{*\alpha_1}\right) \tag{4.7}$$

is a kernel of type 0 for any $\alpha \in \mathbb{N}_0^n$.

Let T be an operator of type 0. Denoting by $\kappa \in \mathcal{S}'(G)$ its kernel, the kernel of the operator $\mathcal{R}^N T \mathcal{R}_2^{-N}$ can be written as a linear combination of kernels of type 0 as in (4.7) with $[\alpha] = \nu N$. Thus $\mathcal{R}^N T \mathcal{R}_2^{-N}$ is an operator of type 0.

We can apply this to the operator $X_j \mathcal{R}_2^{-\frac{\upsilon_j}{\nu}}$ which is of type 0 since its kernel is $X_j \mathcal{I}_{\upsilon_j}$. □

Proof of Theorem 4.14. — It suffices to show Theorem 4.14 for any left-invariant differential operator T which is homogeneous of degree ν_T.

By Lemma 4.16, for any $N \in \mathbb{N}_0$ and $p \in (1, \infty)$, the operator $\mathcal{R}^N T \mathcal{R}_p^{-\frac{\nu_T}{\nu}-N}$ extends to an $L^p(G)$-bounded operator. Since $\mathcal{R}_p^{N+\frac{\nu_T}{\nu}}$ is injective, we obtain

$$\forall\, \psi \in \mathcal{S}(G) \qquad \left\|\mathcal{R}^N T\psi\right\|_{L^p(G)} \leqslant C_N \left\|\mathcal{R}_p^{\frac{\nu_T}{\nu}+N}\psi\right\|_{L^p(G)}.$$

This shows that T maps $\dot{L}^p_{N+\nu_T/\nu}$ to L^p_N boundedly. By duality and interpolation, this implies the boundedness of every ν_T-homogeneous left-invariant differential operator T from $\dot{L}^p_{s+\nu_T}(G)$ to $\dot{L}^p_s(G)$ for every $s \in \mathbb{R}$.

We also have

$$\|T\psi\|_p + \left\|\mathcal{R}^N T\psi\right\|_p \leqslant C_0 \left\|\mathcal{R}_p^{\frac{\nu_T}{\nu}}\psi\right\|_p + C_N \left\|\mathcal{R}_p^N \mathcal{R}_p^{\frac{\nu_T}{\nu}}\psi\right\|_p.$$

By Theorem 4.3, the left-hand side is equivalent to the Sobolev norm of $T\psi$ in $L^p_{\nu N}(G)$ whereas the right-hand side is equivalent to the Sobolev norm of ψ in $L^p_{\nu_T+\nu N}(G)$. Therefore T is continuous from $L^p_{\nu N+\nu_T}(G)$ to $L^p_{\nu N}(G)$. By duality and interpolation, this implies the boundedness of every ν_T-homogeneous left-invariant differential operator T from $L^p_{s+\nu_T}(G)$ to $L^p_s(G)$ for every $s \in \mathbb{R}$. □

We observe that we can modify the proof above to show:

Proposition 4.17. — *If T is an operator of type ν_T with $\Re\nu_T = 0$, then, for every $s \in \mathbb{R}$ and $p \in (1,\infty)$, T is bounded on $L^p_s(G)$ and on $\dot{L}^p_s(G)$.*

Proof of Proposition 4.17. — Let $\kappa \in \mathcal{S}'(G)$ be the kernel of T. Proceeding as in Lemma 4.16, we see that $X_j(\kappa * \mathcal{I}_{\nu_j})$ is a kernel of type ν_T by Proposition 2.4. Proceeding recursively shows that $X^\alpha\left(\kappa * \mathcal{I}_{\upsilon_n}^{*\alpha_n} * \ldots * \mathcal{I}_{\upsilon_1}^{*\alpha_1}\right)$ is a kernel of type ν_T for any $\alpha \in \mathbb{N}_0^n$. Consequently, the kernel of the operator $\mathcal{R}^N T \mathcal{R}_2^{-N}$ can be written as a linear combination of kernels of type ν_T and $\mathcal{R}^N T \mathcal{R}_2^{-N}$ is an operator of type ν_T. This implies that T is bounded on $\dot{L}^p_{\nu N}$ for any $N \in \mathbb{N}_0$, and subsequently using duality and interpolation and Theorem 4.3 that T is bounded on every homogeneous and inhomogeneous Sobolev space. □

4.5. Independence with respect to Rockland operators

In this Section, we show that the Sobolev spaces do not depend on a particular choice of a Rockland operator. Consequently Theorems 4.3 and 4.8, Corollary 4.6, and Proposition 4.7 hold independently of any chosen Rockland operator $\mathcal{R}$.

We will need the following property:

Lemma 4.18. — *Let $\mathcal{R}$ be a Rockland operator on G of homogeneous degree ν and let $\ell \in \mathbb{N}_0$.*

(1) *The map $\phi \mapsto \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$, is a norm on $\dot{L}^p_{\nu\ell}(G)$ equivalent to the Sobolev norm. The space $\dot{L}^p_{\nu\ell}(G)$ is the completion of the Schwartz space $\mathcal{S}(G)$ for this norm.*

(2) *The space $L^p_{\nu\ell}(G)$ is the collection of functions $f \in L^p(G)$ such that $X^\alpha f \in L^p(G)$ for any $\alpha \in \mathbb{N}_0^n$ with $[\alpha] = \nu\ell$. Moreover the map $\phi \mapsto \|\phi\|_p + \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$ is a norm on $L^p_{\nu\ell}(G)$ which is equivalent to the Sobolev norm.*

Remark 4.19. — Note that in the stratified case, it is proved that the Sobolev $L^p_1(G)$ space consists of the function $f \in L^p(G)$ such that $X_j f \in L^p(G)$ for X_j a basis of the first stratum. And this property is proved using the existence of a unique homogeneous fundamental solution of the sub-Laplacian, cf. Theorem 4.10 and the lemmata below entering its proof in [12].

Our lemma requires that the integers have to be multiple of the homogeneous degree of a positive Rockland operator. In our proof, we replace the use of this fundamental solution (which does not necessarily exist for Rockland operators) with Theorem 4.14.

Proof of Lemma 4.18. — Writing $\mathcal{R}^\ell = \sum_{[\alpha]=\ell\nu} c_{\alpha,\ell} X^\alpha$ we have on one hand,

$$\exists\, C > 0 \quad \forall\, \phi \in \mathcal{S}(G) \qquad \left\|\mathcal{R}^\ell \phi\right\|_p \leqslant \max |c_\alpha| \sum_{[\alpha]=\ell\nu} \|X^\alpha \phi\|_p .$$

On the other hand, by Theorem 4.14, X^α maps continuously $\dot{L}^p_{[\alpha]}(G)$ to $L^p(G)$. This implies that the maps

$$\phi \mapsto \left\|\mathcal{R}^\ell \phi\right\| = \|\phi\|_{\dot{L}^p_{\nu\ell}(G)} \qquad \text{and} \qquad \phi \mapsto \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$$

are two equivalent norms on $\mathcal{S}(G)$. This proves Part (1).

Adding $\|\phi\|_{L^p}$ on both sides of the inequality above implies by Theorem 4.3, part (2), that the maps

$$\phi \mapsto \|\phi\|_{L^p_{\nu\ell}(G)} \qquad \text{and} \qquad \phi \mapsto \|\phi\|_{L^p} + \sum_{[\alpha]=\nu\ell} \|X^\alpha \phi\|_p$$

are two equivalent norms on $\mathcal{S}(G)$. This proves Part (2). □

One may wonder whether Lemma 4.18 would be true not only for integer exponents of the form $s = \nu\ell$ but for any integer s. The answer is no, and this will be explained in Section 4.8 together with the construction of a different type of Sobolev spaces on a graded Lie group due to Goodman.

We can now show the main result of this section, that is, that the Sobolev spaces on graded Lie groups are independent of the chosen positive Rockland operators.

THEOREM 4.20. — *For each $p \in (1,\infty)$, the homogeneous (resp. inhomogeneous) L^p-Sobolev spaces on G associated with any positive Rockland operators coincide. Moreover the homogeneous (resp. inhomogeneous) Sobolev norms associated to two positive Rockland operators are equivalent.*

Proof of Theorem 4.20. — Let $\mathcal{R}_{(1)}$ and $\mathcal{R}_{(2)}$ be two positive Rockland operators on G of homogeneous degree ν_1 and ν_2, respectively. Then $\mathcal{R}_{(1)}^{\nu_2}$ and $\mathcal{R}_{(2)}^{\nu_1}$ are two positive Rockland operators with the same homogeneous degree $\nu = \nu_1\nu_2$. Their associated Sobolev spaces of exponent $\nu\ell = \nu_1\nu_2\ell$ for any $\ell \in \mathbb{N}_0$ coincide and have equivalent homogeneous (resp. inhomogeneous) Sobolev norms by Lemma 4.18. By duality and interpolation, this is true for any Sobolev spaces. $\square$

COROLLARY 4.21. — *Let $\mathcal{R}_{(1)}$ and $\mathcal{R}_{(2)}$ be two positive Rockland operators on G with degrees of homogeneity ν_1 and ν_2. Then for any $s \in \mathbb{R}$ and $p \in (1,\infty)$, the operators*

$$\mathcal{R}_{(1)}{}^{\frac{s}{\nu_1}}\mathcal{R}_{(2)}{}^{-\frac{s}{\nu_2}} \qquad \text{and} \qquad (\mathrm{I}+\mathcal{R}_{(1)})^{\frac{s}{\nu_1}}(\mathrm{I}+\mathcal{R}_{(2)})^{-\frac{s}{\nu_2}}$$

extends boundedly on $L^p(G)$.

Proof of Corollary 4.21. — For the inhomogeneous case, we view the operator $(\mathrm{I}+(\mathcal{R}_{(2)})_p)^{-\frac{a}{\nu_2}}$ as a bounded operator from $L^p(G)$ to $L^p_a(G)$ and use the norm $f \mapsto \left\|(\mathrm{I}+(\mathcal{R}_{(1)})_p)^{\frac{a}{\nu_1}}f\right\|_p$ on $L^p_a(G)$. We proceed similarly in the homogeneous case. $\square$

Thanks to Theorem 4.20, we can now improve our duality result given in Lemmata 4.4 and 4.12:

PROPOSITION 4.22.

(1) *For any $s \in \mathbb{R}$ and $p \in (1,\infty)$, the dual space of $\dot{L}^p_s(G)$ is isomorphic to $L^{p'}_{-s}(G)$ via the duality in (4.5) where p' is the conjugate exponent of p if $p \in (1,\infty)$, i.e. $\frac{1}{p}+\frac{1}{p'}=1$.*

(2) *For any $s \in \mathbb{R}$ and $p \in [1,\infty)$, the dual space of $L^p_s(G)$ is isomorphic to $L^{p'}_{-s}(G)$ via the distributional duality, where p' is the conjugate exponent of p if $p \in (1,\infty)$, and $p' = \infty_o$ if $p = 1$.*

For any $s \leqslant 0$ and $p = \infty_o$, the dual space of $L^{\infty_o}_s(G)$ is isomorphic to $L^1_{-s}(G)$ via the distributional duality.

If $p \in (1,\infty)$ then the Banach space $L^p_s(G)$ is reflexive. It is also the case for $s \leqslant 0$ and $p = \infty_o$, and for $s \geqslant 0$ and $p = 1$.

We can also show that multiplication by a bump function is continuous on Sobolev spaces:

Proposition 4.23. — *For any $\phi \in \mathcal{D}(G)$, $p \in (1,\infty)$ and $s \in \mathbb{R}$, the operator $f \mapsto f\phi$ defined for $f \in \mathcal{S}(G)$ extends continuously into a bounded map from $L^p_s(G)$ to itself.*

Proof. — The Leibniz' rule for the X_j's and the continuous inclusions in Theorem 4.3 (4) imply easily that for any fixed $\alpha \in \mathbb{N}_0^n$ there exist a constant $C = C_{\alpha,\phi} > 0$ and a constant $C' = C'_{\alpha,\phi} > 0$ such that

$$\forall\, f \in \mathcal{D}(G) \quad \|X^\alpha(f\phi)\|_p \leqslant C \sum_{[\beta]\leqslant[\alpha]} \left\|X^\beta f\right\|_p \leqslant C' \|f\|_{L^p_{[\alpha]}(G)} .$$

Lemma 4.18 yields the existence of a constant $C'' = C''_{\alpha,\phi} > 0$ such that

$$\forall\, f \in \mathcal{D}(G) \quad \|(f\phi)\|_{L^p_{\ell\nu}(G)} \leqslant C'' \|f\|_{L^p_{\ell\nu}(G)}$$

for any integer $\ell \in \mathbb{N}_0$ and any degree of homogeneity ν of a Rockland operator.

This shows the statement for the case $s = \nu\ell$. The case $s > 0$ follows by interpolation (see Theorem 4.8), and the case $s < 0$ by duality (see Proposition 4.22). □

4.6. Sobolev embeddings

In this section, we show the analogue of the classical fractional integration theorems of Hardy–Littlewood and Sobolev. The main difference is that the topological dimension n of $G \sim \mathbb{R}^n$ is replaced by the homogeneous dimension Q. The stratified case was proved by Folland in [12] (mainly Theorem 4.17 therein).

Theorem 4.24.

(1) *If $1 < p < q < \infty$ and $a, b \in \mathbb{R}$ with $b - a = Q(\frac{1}{p} - \frac{1}{q})$ then we have the following continuous inclusion $\dot{L}^p_b \subset \dot{L}^q_a$, that is, for every $f \in \dot{L}^p_b$, we have $f \in \dot{L}^q_a$ and there exists a constant $C = C_{a,b,p,q,G} > 0$ independent of f such that*

$$\|f\|_{\dot{L}^q_a} \leqslant C \|f\|_{\dot{L}^p_b} .$$

(2) *If* $1 < p < q < \infty$ *and* $a, b \in \mathbb{R}$ *with* $b - a = Q(\frac{1}{p} - \frac{1}{q})$ *then we have the following continuous inclusion* $L^p_b \subset L^q_a$*, that is, for every* $f \in L^p_b$*, we have* $f \in L^q_a$ *and there exists a constant* $C = C_{a,b,p,q,G} > 0$ *independent of* f *such that*

$$\|f\|_{L^q_a} \leqslant C \, \|f\|_{L^p_b} .$$

(3) *If* $p \in (1, \infty)$ *and* $s > Q/p$ *then we have the following continuous inclusion* $L^p_s \subset C(G)$ *in the sense that any function* $f \in L^p_s(G)$ *admits a bounded continuous representative (still denoted by* f*). Furthermore there exists a constant* $C = C_{s,p,G} > 0$ *independent of* f *such that*

$$\|f\|_\infty \leqslant C \, \|f\|_{L^p_s(G)} .$$

Proof. — Let us prove Part (1). We fix a positive Rockland operator $\mathcal{R}$ of homogeneous degree ν and we assume that $b, a > 0$ and $p, q \in (1, \infty)$ satisfy $b - a = Q(\frac{1}{p} - \frac{1}{q})$. By Corollary 3.11 (1), $\mathcal{I}_{b-a}$ is a kernel of type $b - a$ and for any $p_1 \in (1, \infty)$ and $\phi \in \mathcal{S}(G)$, $\mathcal{R}_{p_1}^{\frac{b-a}{\nu}} \phi \in L^{p_1}_b$ and $\phi = (\mathcal{R}_{p_1}^{\frac{b-a}{\nu}} \phi) * \mathcal{I}_{b-a}$. By Proposition 2.4 (1), this implies with $p_1 = p$,

$$\|\phi\|_{L^q} \leqslant C \left\| \mathcal{R}_p^{\frac{b-a}{\nu}} \phi \right\|_{L^p} .$$

For the same reason we also have $\mathcal{R}_q^{\frac{a}{\nu}} \phi = \mathcal{R}_p^{\frac{b}{\nu}} * \mathcal{I}_{b-a} \phi$ and

$$\left\| \mathcal{R}_q^{\frac{a}{\nu}} \phi \right\|_{L^q} \leqslant C \left\| \mathcal{R}_p^{\frac{b}{\nu}} \phi \right\|_{L^p} .$$

This shows Part (1).

For Part (2), we add the two estimates above to obtain:

$$\|\phi\|_q + \left\| \mathcal{R}_q^{\frac{a}{\nu}} \phi \right\|_q \leqslant C \left(\left\| \mathcal{R}_p^{\frac{b-a}{\nu}} \phi \right\|_{L^p} + \left\| \mathcal{R}_p^{\frac{b}{\nu}} \phi \right\|_{L^p} \right).$$

By Theorem 4.3 (4), and density of $\mathcal{S}(G)$ in the Sobolev spaces, this shows Part (2) for $b > a > 0$. The result for any a, b follows by duality and interpolation (see Proposition 4.22 and Theorem 4.8). The proof of Part (2) is now complete.

Let us prove Part (3). Let $p \in (1, \infty)$ and $s > Q/p$. For any $f \in L^p_s(G)$, setting $f_s := (\mathrm{I} + \mathcal{R}_p)^{\frac{s}{\nu}} f$, we have $f = f_s * \mathcal{B}_s$ by Corollary 3.11 (2). The conclusion now follows from $\|\mathcal{B}_s\|_{p'} < \infty$ (see Corollary 3.13). □

From the Sobolev embedding theorem (Theorem 4.24 (2)) and the description of Sobolev spaces with integer exponent (Lemma 4.18) follows easily the following property:

Corollary 4.25. — *Let G be a graded Lie group, $p \in (1,\infty)$ and $s \in \mathbb{N}$. We assume that s is proportional to the homogeneous degree ν of a positive Rockland operator, that is, $\frac{s}{\nu} \in \mathbb{N}$, and that $s > Q/p$.*

Then if f is a distribution on G such that $f \in L^p(G)$ and $X^\alpha f \in L^p(G)$ when $\alpha \in \mathbb{N}_0^n$ satisfies $[\alpha] = s$, then f admits a bounded continuous representative (still denoted by f). Furthermore there exists a constant $C = C_{s,p,G} > 0$ independent of f such that

$$\|f\|_\infty \leqslant C \left(\|f\|_p + \sum_{[\alpha]=s} \|X^\alpha f\|_p \right).$$

4.7. Properties of $L^2_s(G)$

The case $L^2(G)$ has some special features, such as being a Hilbert space, that we will discuss here.

Many of the proofs in this paper could be simplified if we had just considered the case L^p with $p = 2$. For instance, let us consider a positive Rockland operator $\mathcal{R}$ and its self-adjoint extension $\mathcal{R}_2$ on $L^2(G)$. One can define the fractional powers of $\mathcal{R}_2$ and $\mathrm{I} + \mathcal{R}_2$ by functional calculus. Then one can obtain the properties of the kernels of the Riesz and Bessel potentials with similar methods as in Corollary 3.11.

The proof of the properties of the associated Sobolev spaces $L^2_s(G)$ would be the same in this particular case, maybe slightly helped occasionally by the Hölder inequality being replaced by the Cauchy–Schwartz inequality. A noticeable exception is that Lemma 4.18 can be obtained directly in the case L^p, $p = 2$, from the estimates due to Helffer and Nourrigat in [19].

The main difference between L^2 and L^p Sobolev spaces is the structure of Hilbert spaces of $L^2_s(G)$ whereas the other Sobolev spaces $L^p_s(G)$ are "only" Banach spaces:

Proposition 4.26 (Hilbert space L^2_s). — *Let G be a graded Lie group.*

(1) *For any $s \in \mathbb{R}$, the homogeneous Sobolev space $\dot{L}^2_s(G)$ is a Hilbert space with inner product given by*

$$(f, g)_{\dot{L}^2_s(G)} := \int_G \mathcal{R}_2^{\frac{s}{\nu}} f(x) \, \overline{\mathcal{R}_2^{\frac{s}{\nu}} g(x)} \, \mathrm{d}x,$$

where $\mathcal{R}$ is a positive Rockland operator of homogeneous degree ν.

If $s > 0$, an equivalent inner product is

$$(f, g)_{\dot{L}^2_s(G)} := \int_G f(x) \, \overline{g(x)} \, \mathrm{d}x \; + \; \int_G \mathcal{R}_2^{\frac{s}{\nu}} f(x) \, \overline{\mathcal{R}_2^{\frac{s}{\nu}} g(x)} \, \mathrm{d}x.$$

If $s=\nu\ell$ with $\ell\in\mathbb{N}_0$, an equivalent inner product is

$$(f,g)=\sum_{[\alpha]=\nu\ell}(X^\alpha f, X^\alpha g)_{L^2(G)}.$$

(2) *For any $s\in\mathbb{R}$, the inhomogeneous Sobolev space $L^2_s(G)$ is a Hilbert space with inner product given by*

$$(f,g)_{L^2_s(G)}:=\int_G(\mathrm{I}+\mathcal{R}_2)^{\frac{s}{\nu}}f(x)\,\overline{(\mathrm{I}+\mathcal{R}_2)^{\frac{s}{\nu}}g(x)}\,\mathrm{d}x,$$

where $\mathcal{R}$ is a positive Rockland operator of homogeneous degree ν.

If $s>0$, an equivalent inner product is

$$(f,g)_{L^2_s(G)}:=\int_G f(x)\,\overline{g(x)}\,\mathrm{d}x\;+\;\int_G\mathcal{R}_2^{\frac{s}{\nu}}f(x)\,\overline{\mathcal{R}_2^{\frac{s}{\nu}}g(x)}\,\mathrm{d}x.$$

If $s=\nu\ell$ with $\ell\in\mathbb{N}_0$, an equivalent inner product is

$$(f,g)=(f,g)_{L^2(G)}+\sum_{[\alpha]=\nu\ell}(X^\alpha f, X^\alpha g)_{L^2(G)}.$$

Proposition 4.26 is easily checked, using the structure of Hilbert space of $L^2(G)$.

4.8. Comparison with other definitions of Sobolev spaces

If the group G is stratified, then we can choose as positive Rockland operator $\mathcal{R}=-\mathcal{L}$ with $\mathcal{L}$ a (negative) sub-Laplacian. The corresponding Sobolev spaces have been developed by Folland in [12] for stratified groups, see also [23]. Folland showed that his Sobolev spaces do not depend on a particular choice of a sub-Laplacian [12, Corollary 4.14], and we have shown the same for our Sobolev spaces and Rockland operators in Theorem 4.20. Therefore, our Sobolev spaces coincide with Folland's in the stratified case, and gives new descriptions of Folland's Sobolev spaces.

For instance, let us consider the "simplest" case after the abelian case, that is, the three dimensional Heisenberg group $\mathbb{H}_1$, with Lie algebra $\mathfrak{h}_1=\mathbb{R}X\oplus\mathbb{R}Y\oplus\mathbb{R}T$ and canonical commutation relations $[X,Y]=T$. This is naturally a stratified group, with canonical (negative) sub-Laplacian $\mathcal{L}_{\mathbb{H}_1}:=X^2+Y^2$. We have obtained that the Sobolev spaces (in our sense or equivalently Folland's) may be defined using any of the positive Rockland operators

$$-\mathcal{L}_{\mathbb{H}_1},\quad\text{or }\mathcal{L}^2_{\mathbb{H}_1},\quad\text{or }\mathcal{L}^2_{\mathbb{H}_1}-T^2.$$

To compare our Sobolev spaces $L^p_s(G)$ with their Euclidean counterparts $L^p_s(\mathbb{R}^n)$, that is, for the abelian group $(\mathbb{R}^n,+)$, we can proceed as in [12], especially Theorem 4.16 therein. First there can be only local relations between our Sobolev Spaces and the Euclidean Sobolev spaces, since the coefficients of X_j's with respect to the abelian derivatives ∂_{x_k} are polynomials in the coordinate functions x_ℓ's, and conversely, the coefficients of ∂_{x_j}'s with respect to the abelian derivatives X_k are polynomials in the coordinate functions x_ℓ's. Hence we are led to define the following local Sobolev spaces for $s \in \mathbb{R}$ and $p \in (1, \infty)$:

$$L^p_{s,loc}(G) := \left\{ f \in \mathcal{D}'(G) : \phi f \in L^p_s(G) \quad \text{for all } \phi \in \mathcal{D}(G) \right\}. \tag{4.8}$$

By Proposition 4.23, $L^p_{s,loc}(G)$ contains $L^p_s(G)$. We can compare locally the Sobolev spaces on graded Lie groups and on their abelian counterpart:

Theorem 4.27. — *For any $p \in (1, \infty)$ and $s \in \mathbb{R}$,*

$$L^p_{s/\upsilon_1,loc}(\mathbb{R}^n) \subset L^p_{s,loc}(G) \subset L^p_{s/\upsilon_n,loc}(\mathbb{R}^n).$$

Above, $L^p_{s,loc}(\mathbb{R}^n)$ denotes the usual local Sobolev spaces, or equivalently the spaces defined by (4.8) in the case of the abelian (graded) Lie group $(\mathbb{R}^n, +)$. Recall that υ_1 and υ_n are respectively the smallest and the largest weights of the dilations. In particular, in the stratified case, $\upsilon_1 = 1$ and υ_n coincides with the number of steps in the stratification, and with the step of the nilpotent Lie group G. Hence in the stratified case we recover Theorem 4.16 in [12].

Proof of Theorem 4.27. — It suffices to show that the mapping $f \mapsto f\phi$ defined on $\mathcal{D}(G)$ extends boundedly from $L^p_{s/\upsilon_1}(\mathbb{R}^n)$ to $L^p_s(G)$ and from $L^p_s(G)$ to $L^p_{s/\upsilon_n,loc}(\mathbb{R}^n)$. By duality and interpolation (see Theorem 4.8 and Proposition 4.22), it suffices to show this for a sequence of increasing positive integers s.

For the $L^p_{s/\upsilon_1}(\mathbb{R}^n) \to L^p_s(G)$ case, we assume that s is divisible by the homogeneous degree of a positive Rockland operator. Then we use Lemma 4.18, the fact that the X^α may be written as a combination of the ∂_x^β with polynomial coefficients in the x_ℓ's and that $\max_{[\beta] \leqslant s} |\beta| = s/\upsilon_1$.

For the case of $L^p_s(G) \to L^p_{s/\upsilon_n,loc}(\mathbb{R}^n)$, we use the fact that the abelian derivative ∂_x^α, $|\alpha| \leqslant s$, may be written as a combination over the X^β, $|\beta| \leqslant s$, with polynomial coefficients in the x_ℓ's, that X^β maps $L^p \to L^p_{[\beta]}$ boundedly together with $\max_{|\beta| \leqslant s} [\beta] = s\upsilon_n$. $\square$

Proceeding as in [12, p. 192], one can convince oneself that Theorem 4.27 can not be improved.

In another direction, Sobolev spaces, and more generally Besov spaces, have been defined on any group of polynomial growth in [15] using left-invariant sub-Laplacians and an associated Littlewood–Paley decomposition. Considering stratified groups and homogeneous left-invariant sub-Laplacians (as in (2.5)), this gives another description of the Sobolev spaces in the stratified case which is equivalent to Folland's and to ours. However, for a general graded non-stratified Lie group, our Sobolev spaces may differ from the ones in [15] on any Lie group of polynomial growth. For instance, if we consider the three dimensional Heisenberg group endowed with the dilations

$$r \cdot (x, y, t) = (r^3 x, r^5 y, r^8 t). \tag{4.9}$$

We denote this group $\tilde{\mathbb{H}}_1$, it is graded but not stratified. The sub-Laplacian $\mathcal{L}_{\mathbb{H}_1}$ is not homogeneous and is of degree 10. One can check that that $\mathcal{L}_{\mathbb{H}_1}$ maps $L^2_{10}(\tilde{\mathbb{H}}_1) \to L^2(\tilde{\mathbb{H}}_1)$ and $L^2_2(\mathbb{H}_1) \to L^2(\mathbb{H}_1)$ boundedly and this can not be improved. Hence our Sobolev spaces on $\tilde{\mathbb{H}}_1$ differ from the Sobolev spaces based on the sub-Laplacian in [12] or equivalently in [15].

Sobolev spaces of integer exponents on graded Lie groups have already been defined by Goodman in [18, Section III.5.4]: the L^p Goodman–Sobolev spaces of order $s \in \mathbb{N}_0$ is the space of function $\phi \in L^p$ such that $X^\alpha \phi \in L^p$ for any $[\alpha] \leqslant s$. Goodman's definition does not use Rockland operators but makes sense only for integer exponents. Adapting the proof of Lemma 4.18, one could show easily that the L^p Goodman–Sobolev space of order $s \in \mathbb{N}_0$ always contains our Sobolev space $L^p_s(G)$, and in fact coincides with it if s is proportional to the homogeneous degree ν of a positive Rockland operator or for any s if the group is stratified.

However, this equality between Goodman–Sobolev spaces and our Sobolev spaces is not true on any general graded Lie group. For instance this does not hold on graded Lie groups whose weights are all strictly greater than 1. Indeed, the L^p Goodman–Sobolev space of order $s = 1$ is $L^p(G)$ which contains $L^p_1(G)$ stricly (see Theorem 4.3 (4)). An example of such a graded Lie group is the three dimensional Heisenberg group $\tilde{\mathbb{H}}_1$ with weights given by (4.9).

Again, one consequence of these strict inclusions together with our results is that the Goodman–Sobolev spaces do not satisfy interpolation properties in general.

BIBLIOGRAPHY

[1] G. Alexopoulos, "Spectral multipliers on Lie groups of polynomial growth", *Proc. Am. Math. Soc.* **120** (1994), no. 3, p. 973-979.

[2] P. Auscher, A. F. M. ter Elst & D. W. Robinson, "On positive Rockland operators", *Colloq. Math.* **67** (1994), no. 2, p. 197-216.

[3] H. Bahouri, C. Fermanian-Kammerer & I. Gallagher, "Refined inequalities on graded Lie groups", *C. R. Math. Acad. Sci. Paris* **350** (2012), no. 7-8, p. 393-397.

[4] H. Bahouri & I. Gallagher, "Paraproduit sur le groupe de Heisenberg et applications", *Rev. Mat. Iberoamericana* **17** (2001), no. 1, p. 69-105.

[5] M. Christ, D. Geller, P. Głowacki & L. Polin, "Pseudodifferential operators on groups with dilations", *Duke Math. J.* **68** (1992), no. 1, p. 31-65.

[6] R. R. Coifman & G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics, vol. 242, Springer, 1971, Étude de certaines intégrales singulières, v+160 pages.

[7] T. Coulhon, E. Russ & V. Tardivel-Nachef, "Sobolev algebras on Lie groups and Riemannian manifolds", *Am. J. Math.* **123** (2001), no. 2, p. 283-342.

[8] J. Dziubański, "On semigroups generated by subelliptic operators on homogeneous groups", *Colloq. Math.* **64** (1993), no. 2, p. 215-231.

[9] J. Dziubański, W. Hebisch & J. Zienkiewicz, "Note on semigroups generated by positive Rockland operators on graded homogeneous groups", *Studia Math.* **110** (1994), no. 2, p. 115-126.

[10] A. F. M. ter Elst & D. W. Robinson, "Spectral estimates for positive Rockland operators", in *Algebraic groups and Lie groups*, Austral. Math. Soc. Lect. Ser., vol. 9, Cambridge University Press, 1997, p. 195-213.

[11] V. Fischer & M. Ruzhansky, *Quantization on nilpotent Lie groups*, Progress in Mathematics, vol. 314, Birkhäuser/Springer, 2016, xiii+557 pages.

[12] G. B. Folland, "Subelliptic estimates and function spaces on nilpotent Lie groups", *Ark. Mat.* **13** (1975), no. 2, p. 161-207.

[13] G. B. Folland & E. M. Stein, "Estimates for the $\bar\partial_b$ complex and analysis on the Heisenberg group", *Commun. Pure Appl. Math.* **27** (1974), p. 429-522.

[14] ———, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press; University of Tokyo Press, 1982, xii+285 pages.

[15] G. Furioli, C. Melzi & A. Veneruso, "Littlewood-Paley decompositions and Besov spaces on Lie groups of polynomial growth", *Math. Nachr.* **279** (2006), no. 9-10, p. 1028-1040.

[16] I. Gallagher & Y. Sire, "Besov algebras on Lie groups of polynomial growth", *Studia Math.* **212** (2012), no. 2, p. 119-139.

[17] D. Geller, "Liouville's theorem for homogeneous groups", *Comm. Partial Differ. Equations* **8** (1983), no. 15, p. 1665-1677.

[18] R. W. Goodman, *Nilpotent Lie groups: structure and applications to analysis*, Lecture Notes in Mathematics, vol. 562, Springer, 1976, x+210 pages.

[19] B. Helffer & J. F. Nourrigat, "Caracterisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué", *Comm. Partial Differ. Equations* **4** (1979), no. 8, p. 899-958.

[20] G. A. Hunt, "Semi-groups of measures on Lie groups", *Trans. Am. Math. Soc.* **81** (1956), p. 264-293.

[21] C. Martínez Carracedo & M. Sanz Alix, *The theory of fractional powers of operators*, North-Holland Mathematics Studies, vol. 187, North-Holland Publishing Co., 2001, xii+365 pages.

[22] K. G. Miller, "Parametrices for hypoelliptic operators on step two nilpotent Lie groups", *Comm. Partial Differ. Equations* **5** (1980), no. 11, p. 1153-1184.

[23] K. Saka, "Besov spaces and Sobolev spaces on a nilpotent Lie group", *Tôhoku Math. J.* **31** (1979), no. 4, p. 383-437.

[24] E. M. Stein & G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, vol. 32, Princeton University Press, 1971, x+297 pages.

[25] N. T. Varopoulos, L. Saloff-Coste & T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, 1992, xii+156 pages.

Manuscrit reçu le 12 mai 2016,
révisé le 3 octobre 2016,
accepté le 27 octobre 2016.

Veronique FISCHER
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY,United Kingdom
v.c.m.fischer@bath.ac.uk

Michael RUZHANSKY
Imperial College London
180 Queen's Gate,
London SW7 2AZ, United Kingdom
m.ruzhansky@imperial.ac.uk