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CORRIGENDUM TO "MATHER DISCREPANCY AND THE ARC SPACES"

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by Shihoko ISHII

Abstract. — This paper gives a correction of a theorem in "Mather discrepancy and the arc spaces".

Résumé. — Dans cette note nous corrigeons un théorème de « Discrépance de Mather et les espaces d'arcs ».

In this paper, we make a correction of the statement of Theorem 4.7 in [2], where (v) was misstated as:

"The tangent cone of (X, x) has a reduced irreducible component."

This statement should be corrected as:

"Let $\overline{b} : \overline{Y} \to X$ be the composite of the blow up $b : Y \to X$ at the point $x \in X$ and the normalization $\nu : \overline{Y} \to Y$. Then, the fiber scheme $\overline{b}^{-1}(x) = \overline{E}$ has a reduced irreducible component."

The whole statement of the corrected theorem is as follows:

Theorem 4.7. — *For a singularity (X, x) of dimension n the following are equivalent:*

(i) $\widehat{\mathrm{mld}}(x; X, \mathcal{O}_X) = n$*;*
(ii) $\lambda_m = 0$ *for every* $m \in \mathbb{N}$*;*
(iii) $\lambda_m^0 = 0$ *for every* $m \in \mathbb{N}$*;*
(iv) $\lambda_1^0 = 0$*;*

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(v) *Let $\overline{b} : \overline{Y} \to X$ be the composite of the blow up $b : Y \to X$ at the point $x \in X$ and the normalization $\nu : \overline{Y} \to Y$. Then, the fiber scheme $\overline{b}^{-1}(x) = \overline{E}$ has a reduced irreducible component, where a reduced irreducible component means an irreducible component which is reduced at the generic point.*

Here, we give a proof for the relevant parts under this alteration.

Proof. — The proof in [2] of equivalence among (i), (ii) and (iii) is not affected by the change in (v). The implication (iii) $\Rightarrow$ (iv) is obvious. The implication (iv) $\Rightarrow$ (v) is proved as follows:

Let $E \subset Y$ be the scheme theoretic fiber of x by the blow up $b : Y \to X$ Let $g : \widetilde{Y} \to Y$ be a log resolution of (Y, E) and let g be factored as

$$\widetilde{Y} \xrightarrow{h} \overline{Y} \xrightarrow{\nu} Y.$$

Then, by the same argument in the corresponding part of the proof in the paper [2], we obtain

$$\dim g(E'_{reg}) = n - 1,$$

where E' is the scheme theoretic fiber of x by the morphism $b \circ g : \widetilde{Y} \to X$ and E'_{reg} is the locus of non-singular points of E'. Therefore we obtain

$$\dim h(E'_{reg}) = n - 1.$$

As h is isomorphic at the generic point of each irreducible component of $h(E'_{reg})$, this shows that $\overline{E} = \overline{b}^{-1}(x)$ is reduced at an irreducible component, which implies (v).

For the proof of (v)$\Rightarrow$(iii), we show that we can reduce the discussion into the case that E has a reduced component and Y is non-singular at the generic point of the component. Then the discussion in the proof of the corresponding part in [2] would work.

Let $\overline{E_0} \subset \overline{Y}$ be an irreducible component of $\overline{E}$ with the coefficient 1 in $\overline{E}$ and let $E_0 \subset Y$ be the irreducible components of E corresponding to $\overline{E_0}$. Let e and $\overline{e}$ be the generic points of E_0 and $\overline{E_0}$, respectively. The normalization $\nu : \overline{Y} \to Y$ induces a homomorphism

$$\widehat{\nu}^* : \widehat{\mathcal{O}_{Y,e}} \to \widehat{\mathcal{O}_{\overline{Y},\overline{e}}}.$$

of k-algebras. Let $\mathcal{O}_0$ be the image of $\widehat{\nu}^*$:

$$\widehat{\mathcal{O}_{Y,e}} \twoheadrightarrow \mathcal{O}_0 \subset \widehat{\mathcal{O}_{\overline{Y},\overline{e}}}.$$

Then $\operatorname{Spec} \mathcal{O}_0$ is an analytic branch of Y at e that is dominated by $\operatorname{Spec} \widehat{\mathcal{O}_{\overline{Y},\overline{e}}}$. Here, as $\overline{Y}$ is non-singular at $\overline{e}$ we have

$$\widehat{\mathcal{O}_{\overline{Y},\overline{e}}} = K[\![s]\!],$$

for some extension field K of k. For $f = \sum_i a_i s^i \in K[\![s]\!]$, we denote the lowest degree i with $a_i \neq 0$ by $\operatorname{ord}_s f$ and call it the order of f with respect to the variable s.

We will show that $\mathcal{O}_0$ is a regular local ring. For that, we first prove that $\mathcal{O}_0$ contains an element of order 1 with respect to s. Assume contrary, then every element of $\mathcal{O}_0$ is either a unit or an element of order greater than 1. Let $\ell \in \widehat{\mathcal{O}_{Y,e}}$ be the defining equation of E in Y around e. We also denote by ℓ the images of ℓ in $\mathcal{O}_0$ and in $\widehat{\mathcal{O}_{\overline{Y},\overline{e}}}$ by abuse of notation. Then, in particular, $\operatorname{ord}_s \ell \geqslant 2$. As ℓ is also the defining equation of $\overline{E}$ in $\overline{Y}$ around $\overline{e}$ by the assumption on $\overline{E}$. Then, the above inequality shows that $\overline{E}$ is not reduced at $\overline{e}$, which yields a contradiction.

Now we may assume there is an element $s' \in \mathcal{O}_0$ with order 1 with respect to s. As $\widehat{\mathcal{O}_{\overline{Y},\overline{e}}} = K[\![s]\!] = K[\![s']\!]$, we may assume that $s \in \mathcal{O}_0$, by replacing s by s'. By Cohen's structure theorem, the residue field K' of the complete local ring $\mathcal{O}_0$ is contained in $\mathcal{O}_0$ and therefore we obtain

$$K'[\![s]\!] \subset \mathcal{O}_0\,.$$

Note that the base field k is of characteristic 0. Then the extension $K' \hookrightarrow K$ of fields is separable, therefore it is étale. Now as $K'[\![s]\!] \to K[\![s]\!]$ is étale and $\mathcal{O}_0 \to K[\![s]\!]$ is flat, it follows that

$$K'[\![s]\!] \to \mathcal{O}_0$$

is étale by [1, IV, 17.7.7]. Therefore, $\mathcal{O}_0$ is also regular and $\operatorname{ord}_s(\ell) = 1$.

Now one branch of Y at e is non-singular and E is reduced at the the generic point. We restrict the discussion onto this branch. So, we may assume that Y is non-singular at e and E is reduced at e. Then, the proof of (v) $\Rightarrow$ (iii) in [2] completes the proof. $\square$

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Weichen Gu kindly provides us the following example which shows a contradiction to the previous statement in the Theorem 4.7 in [2]. The author would like to thank him.

Example. — Let $X \subset \mathbb{A}^5$ be a hypersurface defined by

$$y^2 - x_1x_2x_3x_4 = 0\,.$$

Then, the tangent cone has no reduced component, but $(X, 0)$ satisfies (iv). We should also note that X satisfies the condition (v).

BIBLIOGRAPHY

[1] A. Grothendieck, "Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas", *Publ. Math., Inst. Hautes Étud. Sci.* **32** (1967), p. 1-361.

[2] S. Ishii, "Mather discrepancy and the arc spaces", *Ann. Inst. Fourier* **63** (2013), no. 1, p. 89-111.

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