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POSITIVE SOLUTIONS TO SCHRÖDINGER'S EQUATION AND THE EXPONENTIAL INTEGRABILITY OF THE BALAYAGE

by Michael W. FRAZIER & Igor E. VERBITSKY (*)

ABSTRACT. — Let $\Omega \subset \mathbb{R}^n$, for $n \geqslant 2$, be a bounded C^2 domain. Let $q \in L^1_{loc}(\Omega)$ with $q \geqslant 0$. We give necessary conditions and matching sufficient conditions, which differ only in the constants involved, for the existence of very weak solutions to the boundary value problem $(-\triangle - q)u = 0$, $u \geqslant 0$ on Ω, $u = 1$ on $\partial\Omega$, and the related nonlinear problem with quadratic growth in the gradient, $-\triangle u = |\nabla u|^2 + q$ on Ω, $u = 0$ on $\partial\Omega$. We also obtain precise pointwise estimates of solutions up to the boundary.

A crucial role is played by a new "boundary condition" on q which is expressed in terms of the exponential integrability on $\partial\Omega$ of the balayage of the measure $\delta q \, \mathrm{d}x$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. This condition is sharp, and appears in such a context for the first time. It holds, for example, if $\delta q \, \mathrm{d}x$ is a Carleson measure in Ω, or if its balayage is in $BMO(\partial\Omega)$, with sufficiently small norm. This solves an open problem posed in the literature.

RÉSUMÉ. — Soit $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ un domaine C^2 borné. Soit $q \in L^1_{\mathrm{loc}}(\Omega)$, avec $q \geqslant 0$. Nous obtenons des conditions nécessaires et des conditions suffisantes correspondantes — dont seules les constantes impliquées diffèrent — pour l'éxistence de solutions très faibles au problème aux limites $(-\Delta - q)u = 0$, $u \geqslant 0$ sur Ω et $u = 1$ sur $\partial\Omega$, et au problème non linéaire associé, avec une croissance quadratique par rapport au gradient, $-\Delta u = |\nabla u|^2 + q$ sur Ω et $u = 0$ sur $\partial\Omega$. Nous parvenons aussi à des estimations ponctuelles précises des solutions jusqu'à la frontière.

Un rôle crucial est joué par une nouvelle "condition aux limites" portant sur q, exprimée en terme d'intégrabilité exponentielle sur $\partial\Omega$ du balayage de la mesure $\delta q \, \mathrm{d}x$, où $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. Cette condition est optimale, et elle apparaît dans un tel contexte pour la première fois. Elle est notamment remplie si $\delta q \, \mathrm{d}x$ est une mesure de Carleson dans Ω, ou si son balayage, de norme suffisament petite, est dans $\mathrm{BMO}(\partial\Omega)$. Cela résout un problème qui était resté en suspens jusqu'à présent.

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1. Introduction

Let $n \geqslant 2$ and let Ω be a bounded C^2 domain in $\mathbb{R}^n$. Let q be a nonnegative, locally integrable function on Ω. Our main results give conditions for the existence of positive solutions of the following two problems fundamental to the mathematical theory of the Schrödinger operator $-\triangle - q$ (see e.g. [7] for q in Kato's class):

$$\begin{cases} -\triangle u = qu + 1, & u \geqslant 0 \text{ in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

$$\begin{cases} -\triangle u = qu, & u \geqslant 0 \text{ in } \Omega, \\ \quad u = 1 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

For (1.2), we also obtain results for $\Omega = \mathbb{R}^n$, $n \geqslant 3$. Analogous theorems for more general operators, including the fractional Laplacian $(-\triangle)^\alpha$, and domains Ω whose modified Green's function is quasi-metric (see [12]), are more technical and will be considered elsewhere.

Our results solve an open problem on the existence of solutions to (1.2), as well as the corresponding nonlinear problem (1.20) with quadratic growth in the gradient discussed below, which was posed in 1999 in [19].

Equations (1.1) and (1.2) have formal solutions as follows. Let $G(x,y)$ be the Green's function on Ω associated with the Laplacian $-\triangle$. Let G denote the corresponding Green's potential operator:

$$Gf(x) = \int_\Omega G(x,y) f(y)\,\mathrm{d}y, \quad x \in \Omega. \tag{1.3}$$

Let $G_1 = G$ and define G_j inductively for $j \geqslant 2$ by

$$G_j(x,y) = \int_\Omega G_{j-1}(x,z)\, G(z,y)\, q(z)\,\mathrm{d}z. \tag{1.4}$$

We define the minimal Green's function associated with the Schrödinger operator $-\triangle - q$ to be

$$\mathcal{G}(x,y) = \sum_{j=1}^{\infty} G_j(x,y). \tag{1.5}$$

The corresponding Green's operator is

$$\mathcal{G}f(x) = \int_\Omega \mathcal{G}(x,y) f(y)\,\mathrm{d}y.$$

We let

$$u_0(x) = \mathcal{G}1(x) = \int_\Omega \mathcal{G}(x,y)\,\mathrm{d}y$$

and

$$u_1(x) = 1 + \mathcal{G}q(x) = 1 + \int_\Omega \mathcal{G}(x,y)\, q(y)\, \mathrm{d}y. \tag{1.6}$$

Then u_0 is a formal solution of (1.1) and u_1, called the Feynman–Kac gauge in [7], is a formal solution of (1.2). The main issue is whether these formal solutions are finite a.e., and consequently solve the corresponding boundary value problems in a certain generalized sense. Problem (1.2) is more delicate than (1.1) because we must estimate $\mathcal{G}q$ for (1.2) instead of $\mathcal{G}1$ for (1.1).

We emphasize that our only *a priori* assumptions on the potential q are that $q \in L^1_{loc}(\Omega)$ and $q \geqslant 0$. For potentials in Kato's class, both u_0 and u_1 are finite a.e. if and only if the spectrum of the Schrödinger operator is positive on $L^2(\Omega)$, or equivalently (1.7) below holds for some $\beta \in (0,1)$. In that case, u_0 and u_1 are uniformly bounded by positive constants both from above and below. This is a consequence of the so-called Gauge Theorem (see e.g. [7]), which is no longer true for the general classes of potentials considered in this paper.

Let $\delta(x) = \mathrm{dist}(x, \partial\Omega)$, for $x \in \Omega$. Let $C_0^\infty(\Omega)$ be the class of C^∞ functions with compact support in Ω, and let $L_0^{1,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the Dirichlet norm

$$\|f\|_{L_0^{1,2}(\Omega)} = \|\nabla f\|_{L^2(\Omega,\, \mathrm{d}x)}.$$

Theorem 1.1. — *Suppose Ω is a bounded C^2 domain in $\mathbb{R}^n$, for $n \geqslant 2$, and $q \in L^1_{loc}(\Omega)$, $q \geqslant 0$.*

(1) *Suppose there exists $\beta \in (0,1)$ such that*

$$\int_\Omega h^2 q \,\mathrm{d}x \leqslant \beta^2 \int_\Omega |\nabla h|^2 \,\mathrm{d}x \quad \text{for all } h \in C_0^\infty(\Omega). \tag{1.7}$$

Then $u_0 = \mathcal{G}1 \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta q \,\mathrm{d}x) \cap L_0^{1,2}(\Omega)$, u_0 is a minimal positive weak solution of (1.1), and there exist constants $C > 0$ depending only on Ω and β, and $C_1 > 0$ depending only on Ω, such that

$$u_0(x) \leqslant C_1 \delta(x) e^{C \frac{G(\delta q)(x)}{\delta(x)}}, \quad \text{for all } x \in \Omega. \tag{1.8}$$

(2) *Conversely, if (1.1) has a positive very weak solution u, then (1.7) holds with $\beta = 1$ and there exist positive constants $c > 0$ and $c_1 > 0$ depending only on Ω such that*

$$u(x) \geqslant c_1 \delta(x) e^{c \frac{G(\delta q)(x)}{\delta(x)}} \quad \text{for a.e. } x \in \Omega. \tag{1.9}$$

The minimal positive weak solution $u_0 = \mathcal{G}1$ in Theorem 1.1 is clearly unique in the class $L^1(\Omega, \delta q\, \mathrm{d}x) \cap L_0^{1,2}(\Omega)$. It is easy to see, as a consequence of Theorem 1.1, that an analogue of part (1) of Theorem 1.1 with $u_0 = \mathcal{G}f$ holds for any bounded measurable function f in place of the function 1 on the right hand side of (1.1), and part (2) for any measurable function f bounded below by a positive constant.

Condition (1.7) was studied originally by V.G. Maz'ya for general open sets $\Omega \subset \mathbb{R}^n$ and Borel measures $\mathrm{d}\omega$ in place of $q\,\mathrm{d}x$, and characterized in terms of capacities associated with $L_0^{1,2}(\Omega)$ (see [25, §2.5.2]).

For equation (1.1), the existence of a weak solution under assumption (1.7) follows by well-known techniques (see e.g. [8]). Also the lower estimate (1.9) in Theorem 1.1 is known (see [15] and the literature cited there). What is new here is the upper estimate (1.8), whose proof relies on results in [12]. This upper estimate is, in turn, critical for our results regarding (1.2). The more difficult nature of (1.2) compared to (1.1) is exhibited in our results in two ways: we must consider solutions of (1.2) in the "very weak" sense (see Definitions 2.4 and 2.8), and, most importantly, a new condition (1.10) that controls the behavior of q near $\partial\Omega$ is needed for (1.2) but not for (1.1).

Let $P(x,y)$ be the Poisson kernel for Ω, and let P^* denote the balayage operator (formally adjoint to the Poisson integral) defined by

$$P^* f(y) = \int_\Omega P(x,y) f(x)\, \mathrm{d}x, \quad y \in \partial\Omega.$$

Let $\mathrm{d}\sigma$ be surface measure on $\partial\Omega$.

Theorem 1.2. — *Suppose Ω is a bounded C^2 domain in $\mathbb{R}^n$, for $n \geqslant 2$, and $q \in L^1_{loc}(\Omega)$, $q \geqslant 0$.*

(1) *Suppose there exists $\beta \in (0,1)$ such that (1.7) holds and*

$$\int_{\partial\Omega} e^{CP^*(\delta q)}\, \mathrm{d}\sigma < +\infty \tag{1.10}$$

where C is the constant in (1.8). Then $u_1 = 1 + \mathcal{G}q$ is a positive very weak solution of (1.2) with

$$\|u_1\|_{L^1(\Omega,\,\mathrm{d}x)} \leqslant |\Omega| + C_1 \int_{\partial\Omega} e^{CP^*(\delta q)}\, \mathrm{d}\sigma, \tag{1.11}$$

for some constant $C_1 > 0$ depending only on Ω. Also, there exist positive constants C_2, C_3 depending only on Ω and β such that

$$u_1(x) \leqslant C_2 \int_{\partial\Omega} e^{C_3 \int_\Omega G(x,y) \frac{P(y,z)}{P(x,z)} q(y)\, \mathrm{d}y} P(x,z)\, \mathrm{d}\sigma(z), \quad \text{for all } x \in \Omega. \tag{1.12}$$

(2) *Conversely, if* (1.2) *has a positive very weak solution* u*, then* (1.7) *holds with* $\beta = 1$, (1.10) *holds with the same constant* c *as in* (1.9)*, and*

$$\int_{\partial\Omega} e^{cP^*(\delta q)}\, \mathrm{d}\sigma \leqslant C_4 \left(\|u\|_{L^1(\Omega,\, \mathrm{d}x)} + |\partial\Omega|\right), \tag{1.13}$$

for some constant $C_4 > 0$ *depending only on* Ω*. Moreover, there exist positive constants* c_1, c_2 *depending only on* Ω *such that*

$$u(x) \geqslant c_1 \int_{\partial\Omega} e^{c_2 \int_\Omega G(x,y) \frac{P(y,z)}{P(x,z)} q(y)\, \mathrm{d}y} P(x,z)\, \mathrm{d}\sigma(z), \quad \textit{for all } x \in \Omega. \tag{1.14}$$

We observe that under the assumptions of Theorem 1.2(1), it follows that $u_1 \in L^{1,2}_{loc}(\Omega)$ (see [21, Theorem 6.2]). If we assume $q \in L^1(\Omega)$, in addition to (1.7) with $\beta \in (0,1)$, then $u_1 - 1 \in L^{1,2}_0(\Omega)$, and u_1 is a weak solution to (1.2), instead of just very weak (see e.g. [16]).

We can distinguish condition (1.10) for Theorem 1.2 from (1.7) in Theorem 1.1 via the example $q(x) = a\delta(x)^{-2}$, with $aC < 1$ where C is the constant in Hardy's inequality

$$\int_\Omega \frac{h^2}{\delta(x)^2}\, \mathrm{d}x \leqslant C \int_\Omega |\nabla h|^2\, \mathrm{d}x \ \text{ for all } \ h \in C_0^\infty(\Omega).$$

Then (1.7) holds, and hence the conclusions of Theorem 1.1(1) follow for equation (1.1). However, if (1.2) had a positive very weak solution u, then by (1.13), $P^*(\delta q)$ would be exponentially integrable on Ω. By Jensen's inequality, we would then have $\int_\Omega \delta(x) q(x)\, \mathrm{d}x < \infty$, which fails for $q(x) = a\delta(x)^{-2}$.

We remark that the additional condition $\int_\Omega \delta(x) q(x)\, \mathrm{d}x < \infty$, or equivalently $Gq < +\infty$ a.e., combined with (1.7) for any $\beta \in (0,1)$, is generally not enough (unless $n = 1$) to ensure that u_1 is a very weak solution to (1.2).

Theorem 1.2 leads to conditions for the existence of a very weak solution to (1.2) in terms of Carleson measures and BMO. For a measure μ on Ω, define the Carleson norm of μ by

$$\|\mu\|_C = \sup_{r>0, x\in\partial\Omega} r^{1-n} \mu(\{y \in \Omega : |y-x| < r\}).$$

For $f \in L^1(\partial\Omega, \mathrm{d}\sigma)$, define $U_r(x) = \{y \in \partial\Omega : |y - x| < r\}$ and

$$\|f\|_{BMO(\partial\Omega)} = \sup_{r>0, x\in\partial\Omega} |\sigma(U_r(x))|^{-1} \int_{U_r(x)} |f - f_{U_r(x)}|\, \mathrm{d}\sigma,$$

where $f_{U_r(x)} = |\sigma(U_r(x))|^{-1} \int_{U_r(x)} f\, \mathrm{d}\sigma$ is the average of f on $U_r(x)$.

Corollary 1.3. — *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, for $n \geqslant 2$, and let $q \in L^1_{loc}(\Omega)$, $q \geqslant 0$. Suppose* (1.7) *holds for some $\beta \in (0,1)$. Then there exist $\epsilon_1, \epsilon_2 > 0$, depending only on Ω and β such that if*

$$\text{(A)} \quad \|P^*(\delta q)\|_{BMO(\partial\Omega)} < \epsilon_1,$$

or

$$\text{(B)} \quad \|\delta q \, \mathrm{d}x\|_C < \epsilon_2,$$

then $u_1 \in L^1(\Omega, \mathrm{d}x)$ and u_1 is a positive very weak solution of (1.2).

For the case $\Omega = \mathbb{R}^n$, $n \geqslant 3$, we denote by $I_2 f = (-\triangle)^{-1} f$ the Newtonian potential of f:

$$I_2 f(x) = c_n \int_{\mathbb{R}^n} \frac{f(y)\,\mathrm{d}y}{|x-y|^{n-2}}, \quad x \in \mathbb{R}^n,$$

where c_n is a positive normalization constant. Let $G(x,y) = c_n |x-y|^{2-n}$ be the kernel of I_2.

Theorem 1.4. — *Let $n \geqslant 3$.*

(1) *Suppose there exists $\beta \in (0,1)$ such that*

$$\int_{\mathbb{R}^n} h^2 q \, \mathrm{d}x \leqslant \beta^2 \int_{\mathbb{R}^n} |\nabla h|^2 \, \mathrm{d}x \ \text{ for all } \ h \in C_0^\infty(\mathbb{R}^n), \tag{1.15}$$

and

$$\int_{\mathbb{R}^n} \frac{q(y)\,\mathrm{d}y}{(1+|y|)^{n-2}} < +\infty. \tag{1.16}$$

Then $u_1 = 1 + \mathcal{G}q$ is a positive minimal solution (in the distributional sense) to

$$\begin{cases} -\triangle u = qu \ \text{ on } \mathbb{R}^n, \\ \liminf\limits_{x\to\infty} u(x) = 1. \end{cases} \tag{1.17}$$

Also,

$$u_1(x) \leqslant e^{C\, I_2 q(x)}, \ \text{ for all } x \in \mathbb{R}^n, \tag{1.18}$$

where C depends only on β and n.

(2) *Conversely, if there is a positive (distributional) solution u of* (1.17), *then* (1.15) *holds with $\beta = 1$,* (1.16) *holds, and*

$$u(x) \geqslant e^{c\, I_2 q(x)}, \ \text{ for all } \ x \in \mathbb{R}^n, \tag{1.19}$$

where c depends only on n.

Earlier results related to Theorem 1.4 can be found in [19], [26], [27] and the references given there. Condition (1.15) is the so-called trace inequality which expresses the continuous imbedding of $L_0^{1,2}(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n, q\,\mathrm{d}x)$. The class of functions q (or more generally measures ω) such that (1.15) holds is well understood, and several characterizations are known (see [3], [25], and the literature cited there).

Theorems 1.1, 1.2, and 1.4 are the model cases of more general results for wider classes of operators, including fractional Laplacians, and domains Ω (Lipschitz and NTA domains), as well as more general right-hand sides and boundary data, that we plan to address in a forthcoming paper.

The Feynman–Kac gauge u_1 is closely related, via a formal substitution $v = \log u_1$, to a generalized solution of the nonlinear boundary value problem with quadratic growth in the gradient:

$$\begin{cases} -\triangle v = |\nabla v|^2 + q & \text{in } \Omega \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.20}$$

However, it is well known that the relation between (1.2) and (1.20) is not as simple as the formal substitution suggests (see [10]). Nevertheless, we obtain the following result.

Theorem 1.5. — *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded C^2 domain, where $n \geqslant 2$, and $q \in L^1_{loc}(\Omega)$, $q \geqslant 0$.*

(1) *Suppose there exists $\beta \in (0,1)$ such that (1.7) holds, and (1.10) holds. Then $v = \log u_1$ is a very weak solution of (1.20) with $v \in L^{1,2}_{loc}(\Omega)$.*
(2) *Conversely, if (1.20) has a very weak solution in $L^{1,2}_{loc}(\Omega)$, then (1.7) holds with $\beta = 1$, and (1.10) holds with some small constant $c = c(\Omega) > 0$.*

A similar problem for the superquadratic equation

$$-\triangle v = |\nabla v|^s + q,$$

with $s > 2$, was solved in [19], where a thorough discussion of such problems and more details can be found. We remark that no additional condition like (1.10) is required for $s > 2$. Theorem 1.5 resolves the case $s = 2$, which was stated as an open problem in [19].

Regarding solutions to (1.20), we refer also to Ferone and Murat [9] where the existence of finite energy solutions $v \in L_0^{1,2}(\Omega)$ is proved for $q \in L^{\frac{n}{2}}(\Omega)$ ($n \geqslant 3$), with sufficiently small norm; in that case $u_1 - 1 = e^v - 1 \in L_0^{1,2}(\Omega)$. In [11], these results are extended to $q \in L^{\frac{n}{2},\infty}(\Omega)$. (See also [1], [2], [16] where the existence of such solutions is obtained for $q \in L^1(\Omega)$

satisfying (1.7) with $\beta \in (0,1)$.) Clearly, for $q \in L^{\frac{n}{2},\infty}(\Omega)$, the assumptions of Corollary 1.3, and hence Theorem 1.5, are satisfied; that is, (1.7) holds, and δq is a Carleson measure, which yields (1.10).

In Section 2, we discuss very weak solutions for Schrödinger equations. The proofs of Theorems 1.1, 1.2, and 1.4 are given in Section 3. In Section 4, we discuss the nonlinear equation (1.20) and prove Theorem 1.5, using techniques from potential theory.

We would like to thank Fedor Nazarov for valuable conversations related to the content of this paper, which is a continuation and application of [12].

2. Very Weak Solutions

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain with Green's function $G(x,y)$, where $n \geqslant 2$. Recall that $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. We will use the following well-known estimates repeatedly:

$$G(x,y) \approx \frac{\delta(x)\,\delta(y)}{|x-y|^{n-2}(|x-y|+\delta(x)+\delta(y))^2}, \quad n \geqslant 3, \tag{2.1}$$

$$G(x,y) \approx \ln\left(1 + \frac{\delta(x)\,\delta(y)}{|x-y|^2}\right), \quad n = 2, \tag{2.2}$$

for all $x, y \in \Omega$, where "$\approx$" means that the ratio of the two sides is bounded above and below by positive constants depending only on Ω (see [29], [30] for $n \geqslant 3$; [7, Theorem 6.23] for $n = 2$).

Our main results hold with obvious modifications for more general domains Ω for which estimates (2.1), (2.2) hold, in particular for $C^{1,1}$ domains.

Estimates (2.1), (2.2) yield a cruder upper estimate

$$G(x,y) \leqslant C\frac{\delta(x)}{|x-y|^{n-1}}, \quad n \geqslant 2, \tag{2.3}$$

for all $x, y \in \Omega$. This is obvious if $n \geqslant 3$; for $n = 2$, notice that

$$\ln\left(1 + \frac{\delta(x)\,\delta(y)}{|x-y|^2}\right) \leqslant \frac{\delta(x)\,\delta(y)}{|x-y|^2}.$$

Hence, for $\delta(y) \leqslant 2|x-y|$, we have,

$$G(x,y) \leqslant C\frac{\delta(x)}{|x-y|}.$$

For $\delta(y) > 2|x-y|$, using the inequality $\delta(y) \leqslant |x-y| + \delta(x)$, we see that $|x-y| < \delta(x)$ and $\delta(y) < 2\delta(x)$. Hence, in this case,

$$G(x,y) \leqslant C\ln\left(1 + \frac{\delta(x)\,\delta(y)}{|x-y|^2}\right) \leqslant C\ln\left(1 + \frac{2\delta(x)^2}{|x-y|^2}\right) \leqslant C\frac{\delta(x)}{|x-y|},$$

which verifies (2.3) for $n = 2$.

The preceding estimates yield

$$G1(x) = \int_\Omega G(x,y)\,\mathrm{d}y \approx \delta(x), \quad n \geqslant 2, \tag{2.4}$$

for all $x \in \Omega$. Indeed, the lower bound $G1(x) \geqslant c\,\delta(x)$ follows from the well-known estimate $G(x,y) \geqslant c\,\delta(x)\delta(y)$, which is an obvious consequence of (2.1), (2.2). The upper bound in (2.4) follows by integrating both sides of (2.3) with respect to dy over a ball $B(x,R)$ with $R = \mathrm{diam}(\Omega)$ so that $\Omega \subset B(x,R)$:

$$G1(x) \leqslant C\,\delta(x) \int_{B(x,R)} \frac{\mathrm{d}y}{|x-y|^{n-1}} = C_1\,\delta(x).$$

Our first goal is to define a very weak solution for Schrödinger equations. We begin by defining very weak solutions for Poisson's equation with Dirichlet boundary conditions. We will use the class of test functions

$$C_0^2(\overline{\Omega}) = \{h \in C^2(\overline{\Omega}) : h = 0 \text{ on } \partial\Omega\}.$$

Definition 2.1. — *Suppose $f \in L^1(\Omega, \delta\,\mathrm{d}x)$. A function $u \in L^1(\Omega, \mathrm{d}x)$ is a very weak solution of the Dirichlet problem*

$$\begin{cases} -\triangle u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

if

$$-\int_\Omega u\,\triangle h\,\mathrm{d}x = \int_\Omega h\,f\,\mathrm{d}x, \tag{2.6}$$

for all $h \in C_0^2(\overline{\Omega})$.

The following lemma concerning the existence and uniqueness of very weak solutions is well known (see [6, Lemma 1]). For convenience we supply a simple proof which shows additionally that the very weak solution is given by the Green's potential Gf, defined by (1.3).

Lemma 2.2.

(1) *Let $f \in L^1(\Omega, \delta\,\mathrm{d}x)$. Then there exists a unique very weak solution $u \in L^1(\Omega, \mathrm{d}x)$ of (2.5) given by $u = Gf$.*
(2) *If $f \geqslant 0$ a.e. and $Gf(x_0) < +\infty$ for some $x_0 \in \Omega$, then $f \in L^1(\Omega, \delta\,\mathrm{d}x)$ and $u = Gf \in L^1(\Omega, \mathrm{d}x)$ is a very weak solution of (2.5).*

Proof. — Let us first prove (1). The proof of uniqueness follows [6]. Suppose both v and w are very weak solutions of (2.5). Let $\phi \in C_0^\infty(\Omega)$ and let $h = G\phi$. Then $h \in C_0^2(\overline{\Omega})$ and $-\triangle h = \phi$ on Ω. Consequently

$$\int_\Omega (v-w)\,\phi\,\mathrm{d}x = -\int_\Omega (v-w)\,\triangle h\,\mathrm{d}x = 0,$$

by (2.6). Since this equation holds for every $\phi \in C_0^\infty(\Omega)$, we obtain $v = w$.

Next we prove that if $f \in L^1(\Omega, \delta\,\mathrm{d}x)$ then $u = Gf$ is a very weak solution. Without loss of generality we may assume that $f \geqslant 0$. By Fubini's theorem and the symmetry of G,

$$\begin{aligned}\|u\|_{L^1(\Omega,\,\mathrm{d}x)} &= \int_\Omega \int_\Omega G(x,y)\,f(y)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_\Omega G1(y)\,f(y)\,\mathrm{d}y \leqslant C\int_\Omega \delta(y)\,f(y)\,\mathrm{d}y < +\infty,\end{aligned}$$

by (2.4).

Let $f_k \in C_0^\infty(\Omega)$ be a sequence of nonnegative functions such that $\|f - f_k\|_{L^1(\Omega,\delta\,\mathrm{d}x)} \to 0$ as $k \to +\infty$. Denote by $u_k = Gf_k$ the solution to (2.5) with f_k in place of f. By Green's theorem,

$$-\int_\Omega u_k\,\triangle h\,\mathrm{d}x = \int_\Omega h\,f_k\,\mathrm{d}x, \tag{2.7}$$

for every $h \in C_0^2(\overline{\Omega})$. Note that

$$\|(u-u_k)\triangle h\|_{L^1(\Omega,\,\mathrm{d}x)} \leqslant \|\triangle h\|_{L^\infty(\Omega)}\|u-u_k\|_{L^1(\Omega,\,\mathrm{d}x)},$$

where by Fubini's theorem

$$\begin{aligned}\|u-u_k\|_{L^1(\Omega,\,\mathrm{d}x)} = \|G(f-f_k)\|_{L^1(\Omega,\,\mathrm{d}x)} &\leqslant C\|(f-f_k)\delta\|_{L^1(\Omega,\,\mathrm{d}x)}\\ &\leqslant C\,\|f-f_k\|_{L^1(\Omega,\delta\,\mathrm{d}x)} \to 0.\end{aligned}$$

Note that since $h \in C_0^2(\overline{\Omega})$, we have $|h(x)| \leqslant C\delta(x)$. Hence, passing to the limit as $k \to +\infty$ on both sides of (2.7) proves that $u = Gf$ is a very weak solution. This proves statement (1) of Lemma 2.2.

To prove statement (2), assume that $Gf(x_0) < +\infty$ for some $x_0 \in \Omega$, where $f \geqslant 0$ a.e. Since $u = Gf$ is superharmonic in Ω (see e.g. [5, Theorem 3.3.1]), it follows by the mean value inequality that

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} Gf(x)\,\mathrm{d}x \leqslant Gf(x_0) < +\infty,$$

for some ball $B(x_0,r)$ such that $0 < r < \frac{1}{2}\delta(x)$. By Fubini's theorem,

$$\int_{B(x_0,r)} Gf(x)\,\mathrm{d}x = \int_\Omega G\chi_{B(x_0,r)}(y)f(y)\,\mathrm{d}y.$$

Since $G\chi_{B(x_0,r)}(y) \geqslant C\,\delta(y)$ for all $y \in \Omega$, it follows that $f \in L^1(\Omega, \delta\,\mathrm{d}x)$. Thus by statement (1), $u = Gf \in L^1(\Omega, \mathrm{d}x)$ is a very weak solution of (2.5). □

Remark 2.3. — We can extend Definition 2.1 and Lemma 2.2 to the case where f is replaced with a signed Radon measure ω on Ω such that $\int_\Omega \delta\,\mathrm{d}|\omega| < \infty$. In this case, we say that $u \in L^1(\Omega, \mathrm{d}x)$ is a very weak solution of

$$\begin{cases} -\triangle u = \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.8}$$

if

$$-\int_\Omega u\,\triangle h\,\mathrm{d}x = \int_\Omega h\,\mathrm{d}\omega,$$

for all $h \in C_0^2(\overline{\Omega})$. Then by [24, Theorem 1.2.2], $u(x) = G\omega(x) = \int_\Omega G(x,y)\,\mathrm{d}\omega(y)$ is the unique very weak solution of (2.8). For future reference in §4, we note that the proof of [24, Theorem 1.2.2] shows that if $\int_\Omega \delta\,\mathrm{d}|\omega| < \infty$, then $G\omega \in W^{1,p}_{loc}(\Omega)$ for $1 \leqslant p < n/(n-1)$.

We now use the above definition of very weak solutions of the Poisson equation to define very weak solutions of the Schrödinger equation. For the following definition, and subsequent lemma, we do not require $q \geqslant 0$.

Definition 2.4. — *Let $q \in L^1_{loc}(\Omega, \mathrm{d}x)$ and let $f \in L^1(\Omega, \delta(x)\,\mathrm{d}x)$. A function $u \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta(x)|q(x)|\,\mathrm{d}x)$ is a very weak solution to the Schrödinger equation*

$$\begin{cases} -\triangle u = q\,u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

if

$$-\int_\Omega u\,\triangle h\,\mathrm{d}x = \int_\Omega h\,u\,q\,\mathrm{d}x + \int_\Omega h\,f\,\mathrm{d}x, \tag{2.10}$$

for all $h \in C_0^2(\overline{\Omega})$.

Formally, applying the Green's operator G to both sides of the equation $-\triangle u = qu + f$ yields the integral equation

$$u(x) = G(qu+f)(x) = \int_\Omega G(x,y)u(y)q(y)\,\mathrm{d}y + \int_\Omega G(x,y)f(y)\,\mathrm{d}y. \tag{2.11}$$

By a solution of (2.11) we mean a function u such that u and $G(qu+f)$ are finite and equal a.e. The relationship between very weak solutions of (2.9) and solutions of (2.11) is made clear by the following lemma.

Lemma 2.5. — *Suppose $q \in L^1_{loc}(\Omega, \mathrm{d}x), f \in L^1(\Omega, \delta(x)\,\mathrm{d}x)$, and $u \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta(x)|q(x)|\,\mathrm{d}x)$. Then u is a very weak solution to the Schrödinger equation* (2.9) *if and only if u is a solution to the integral equation $u = G(qu + f)$.*

Proof. — Suppose $h \in C^2_0(\overline{\Omega})$. By our assumptions, $qu \in L^1(\Omega, \delta\,\mathrm{d}x)$. Hence by Lemma 2.2,

$$-\int_\Omega G(qu)\triangle h\,\mathrm{d}x = \int_\Omega quh\,\mathrm{d}x, \qquad -\int_\Omega G(f)\triangle h\,\mathrm{d}x = \int_\Omega fh\,\mathrm{d}x.$$

If we assume $u = G(qu + f)$ a.e., then

$$-\int_\Omega u\triangle h\,\mathrm{d}x = -\int_\Omega G(qu+f)\triangle h\,\mathrm{d}x = \int_\Omega (qu+f)h\,\mathrm{d}x,$$

for all $h \in C^2_0(\overline{\Omega})$, so u is a very weak solution of (2.9). Conversely, suppose u is a very weak solution of (2.9). For any $\phi \in C^\infty_0(\Omega)$, then $h = G\phi$ satisfies $h \in C^2_0(\overline{\Omega})$ and $-\triangle h = \phi$. Hence

$$\begin{aligned}\int_\Omega u\phi\,\mathrm{d}x = -\int_\Omega u\triangle h\,\mathrm{d}x &= \int_\Omega h(qu+f)\,\mathrm{d}x \\ &= -\int_\Omega G(qu+f)\triangle h\,\mathrm{d}x = \int_\Omega G(qu+f)\phi\,\mathrm{d}x.\end{aligned}$$

Since $\phi \in C^\infty_0(\Omega)$ is arbitrary, $u = G(qu + f)$ a.e. □

We now return to our standing assumption that $q \geqslant 0$. The following Corollary will be useful.

Corollary 2.6. — *Suppose $f \in L^1(\Omega, \delta\,\mathrm{d}x)$ and $f \geqslant 0$. Suppose $u \geqslant 0$ satisfies $u = G(qu + f)$. Then $u \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta q\,\mathrm{d}x)$.*

Proof. — By assumption, $u < \infty$ a.e. In particular, $u(x_0) < \infty$ for some $x_0 \in \Omega$. Since $Gf \geqslant 0$, we have that $G(qu)(x_0) < \infty$. By Lemma 2.2(2), we have $qu \in L^1(\Omega, \delta\,\mathrm{d}x)$, or $u \in L^1(\Omega, \delta q\,\mathrm{d}x)$.

We integrate the equation $u = G(qu + f)$ over Ω. Since all terms are nonnegative, Fubini's theorem gives

$$\begin{aligned}\int_\Omega u(x)\,\mathrm{d}x = \int_\Omega u(x)\,G1(x)\,q(x)\,\mathrm{d}x &+ \int_\Omega G1(x)\,f(x)\,\mathrm{d}x \\ \approx \int_\Omega u(x)\,\delta(x)\,q(x)\,\mathrm{d}x &+ \int_\Omega \delta(x)\,f(x)\,\mathrm{d}x < \infty\end{aligned}$$

since $u \in L^1(\Omega, \delta q\,\mathrm{d}x)$. Hence $u \in L^1(\Omega, \mathrm{d}x)$. □

Lemma 2.7. — *Suppose $f \geqslant 0$ and $\mathcal{G}(f) < \infty$ a.e. Then $u = \mathcal{G}f$ satisfies $u \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta q\,\mathrm{d}x), u(x) = G(qu + f)(x)$ for all $x \in \Omega$, and u is a very weak solution of* (2.9).

Proof. — We first observe that $Gf \leqslant \mathcal{G}f < +\infty$ a.e., and hence $f \in L^1(\Omega, \delta\,\mathrm{d}x)$ and $Gf \in L^1(\Omega, \mathrm{d}x)$, by Lemma 2.2(2). Hence Gf is finite a.e. Note that for $j \geqslant 2$,

$$\begin{aligned}\int_\Omega G_j(x,y)f(y)\,\mathrm{d}y &= \int_\Omega\int_\Omega G(x,z)G_{j-1}(z,y)q(z)\,\mathrm{d}z\,f(y)\,\mathrm{d}y\\ &= \int_\Omega G(x,z)q(z)\int_\Omega G_{j-1}(z,y)f(y)\,\mathrm{d}y\,\mathrm{d}z\\ &= G\left(q\int_\Omega G_{j-1}(\cdot,y)f(y)\,\mathrm{d}y\right)(x)\end{aligned}$$

by Fubini's theorem. Hence

$$\begin{aligned}\mathcal{G}f(x) &= Gf(x) + \sum_{j=2}^\infty \int_\Omega G_j(x,y)f(y)\,\mathrm{d}y\\ &= Gf(x) + G\left(q\int_\Omega \sum_{j=2}^\infty G_{j-1}(\cdot,y)f(y)\,\mathrm{d}y\right)(x)\\ &= G(f + q\,\mathcal{G}f)(x),\end{aligned}$$

or $u(x) = G(qu+f)(x)$, for all $x \in \Omega$. Since u and Gf are finite a.e., so is $G(qu)$. Hence by Corollary 2.6, $u \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta q\,\mathrm{d}x)$. By Lemma 2.5, u is a very weak solution of (2.9). □

Positive very weak solutions of the Schrödinger equation are in general not unique (see [26]). However, if $f \geqslant 0$ and (2.11) has a nonnegative solution, then $\mathcal{G}f$ is the minimal solution, in the sense that if $u \geqslant 0$ satisfies (2.11) then $\mathcal{G}f(x) \leqslant u(x)$ for a.e. x. To see this fact, define $G_jf(x) = \int_\Omega G_j(x,y)f(y)\,\mathrm{d}y$ for $G_j(x,y)$ defined by (1.3) and (1.4), and define $Tg = G(gq)$. In the proof of the previous lemma, we showed that $G_jf = T(G_{j-1}f)$. Hence, substituting $G(uq+f) = Tu + Gf$ for u repeatedly,

$$\begin{aligned}u = Tu + Gf &= T(Tu+Gf) + Gf\\ &= T^2u + T(Gf) + Gf = T^2u + G_2f + Gf.\end{aligned}$$

Iterating, we obtain $u = T^ku + \sum_{j=1}^k G_jf$, and letting $k \to \infty$ shows that $u \geqslant \mathcal{G}(f)$. Hence $\mathcal{G}(f)$ is called the *minimal very weak solution* of (2.9). Thus, the only issue regarding the existence of a very weak solution of (2.9) is whether $\mathcal{G}(f) < \infty$ a.e.

We adapt Definition 2.4 of a very weak solution to the case of non-zero boundary conditions. If $g \in L^1(\partial\Omega, \mathrm{d}\sigma)$, then $P(g)$, the Poisson integral of g, defined $P(g)(x) = \int_{\partial\Omega} P(x,y)g(y)\,\mathrm{d}\sigma(y)$, is harmonic on Ω and has boundary values $g(y)$ σ-a.e. The following definition does not require $q \geqslant 0$.

Definition 2.8. — *Let $q \in L^1_{loc}(\Omega, \mathrm{d}x)$, $f \in L^1(\Omega, \delta(x)\,\mathrm{d}x)$, $g \in L^1(\partial\Omega, \mathrm{d}\sigma)$, and $qP(g) \in L^1(\delta\,\mathrm{d}x)$. A function $u \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta(x)|q(x)|\,\mathrm{d}x)$ is a very weak solution of the Schrödinger equation*

$$\begin{cases} -\triangle u = q\,u + f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2.12}$$

if $u = v + P(g)$, where v is a very weak solution of

$$\begin{cases} -\triangle v = q\,v + f + qP(g) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Our definition is not entirely standard, but it is equivalent to the standard definition (see e.g. [24, Definition 1.1.2]) since both of them result in the integral representation $u = G(qu) + G(f) + P(g)$.

In the case of (1.2), we have $f = 0$ and $g = 1$ in (2.12). Then $P(1) = 1$, so any very weak solution of (1.2) has the form $u = v + 1$, where v is a very weak solution of $-\triangle v = qv + q$ on Ω, $v = 0$ on $\partial\Omega$. If we assume $q \in L^1(\Omega, \delta\,\mathrm{d}x)$ then Lemma 2.5 gives that $v \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta|q|\,\mathrm{d}x)$ is a very weak solution of $-\triangle v = qv + q$ on Ω, $v = 0$ on $\partial\Omega$ if and only if v is a solution of the integral equation $v = G(qv + q)$. Hence $u \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta|q|\,\mathrm{d}x)$ is a very weak solution of (1.2) if and only u is a solution of the integral equation $u = 1 + G(qu)$.

We now return to the assumption that $q \geqslant 0$. As for Lemma 2.7, the only issue regarding whether the formal solution u_1 in (1.6) yields a very weak solution to (1.2) is whether the expression in (1.6) is finite a.e.

Lemma 2.9. — *Suppose $u_1 = 1 + \mathcal{G}(q) < \infty$ a.e. Then u_1 is a very weak solution of* (1.2).

Proof. — By Lemma 2.7, $v = \mathcal{G}(q)$ is a very weak solution of $-\triangle v = qv + q$ on Ω, $v = 0$ on $\partial\Omega$. □

In fact, u_1 is the minimal positive weak solution of (1.2). To see this fact, suppose $u \geqslant 0$ is a positive weak solution of (1.2). The equation $u = 1 + G(qu)$ shows that $u \geqslant 1$. Hence $v = u - 1$ is a positive solution to the integral equation $v = G(qv + q)$, hence a positive very weak solution to (2.9) with $f = q$. By the minimality of the positive solution $\mathcal{G}(q)$ of (2.9) with $f = q$, we have $\mathcal{G}(q) \leqslant v$, and hence $u_1 = 1 + \mathcal{G}(q) \leqslant 1 + v = u$.

3. Positive Solutions to Schrödinger Equations

In Lemma 2.5, we reduced the solution (in the very weak sense) of (2.9) to the associated integral equation $u = G(qu) + Gf$. We define the integral operator T by

$$Tf(x) = G(fq)(x) = \int_\Omega G(x,y)\, f(y) q(y)\, \mathrm{d}y, \quad x \in \Omega. \tag{3.1}$$

Our first step in proving Theorem 1.1 is to relate condition (1.7) to the norm of T on $L^2(\Omega, q\,\mathrm{d}x)$.

Lemma 3.1. — *Suppose $\Omega \subset \mathbb{R}^n$, for $n \geqslant 1$, is a bounded C^2 domain. Then T maps $L^2(q\,\mathrm{d}x)$ to itself boundedly if and only if* (1.7) *holds for some β, and $\|T\|_{L^2(\Omega,q\,\mathrm{d}x)\to L^2(\Omega,q\,\mathrm{d}x)} = \beta^2$, where β is the least constant in* (1.7).

Proof. — Recall that $L_0^{1,2}(\Omega)$ is the homogeneous Sobolev space of order 1, that is, the closure of $C_0^\infty(\Omega)$ with respect to the Dirichlet norm $\|\nabla f\|_{L^2(\Omega,\,\mathrm{d}x)}$. The dual of $L_0^{1,2}$ is isometrically isomorphic to the space $L^{-1,2}(\Omega)$ (and vice versa). For $f \in L^1(\Omega,\, \mathrm{d}x)$ (or more generally a finite signed measure), we have

$$\|f\|^2_{L^{-1,2}(\Omega)} = \int_\Omega |\nabla G(f)|^2 \,\mathrm{d}x = \int_\Omega f G f \,\mathrm{d}x \tag{3.2}$$

(see [22, §I.4, Theorem 1.20]). Also note that by duality, the inequality

$$\|fq\|_{L^{-1,2}(\Omega)} \leqslant \alpha \|f\|_{L^2(\Omega,q\,\mathrm{d}x)}, \text{ for all } f \in L^2(\Omega, q\,\mathrm{d}x) \tag{3.3}$$

is equivalent to the inequality

$$\|h\|_{L^2(\Omega,q\,\mathrm{d}x)} \leqslant \alpha \|h\|_{L_0^{1,2}(\Omega)} \text{ for all } h \in C_0^\infty(\Omega). \tag{3.4}$$

For example, if (3.3) holds and $h \in C_0^\infty(\Omega)$, then

$$\begin{aligned}\|h\|_{L^2(\Omega,q\,\mathrm{d}x)} &= \alpha \sup_{\{\phi: \|\phi\|_{L^2(\Omega,q\,\mathrm{d}x)} \leqslant 1/\alpha\}} \left| \int_\Omega h\phi q \,\mathrm{d}x \right| \\ &\leqslant \alpha \sup_{\{\phi: \|\phi q\|_{L^{-1,2}(\Omega)} \leqslant 1\}} \left| \int_\Omega h\phi q\,\mathrm{d}x \right| \\ &\leqslant \alpha \sup_{\{\psi: \|\psi\|_{L^{-1,2}(\Omega)} \leqslant 1\}} \left| \int_\Omega h\psi\,\mathrm{d}x \right| = \alpha \|h\|_{L_0^{1,2}(\Omega)}.\end{aligned}$$

Since $G(x,y)$ is symmetric, T is a self-adjoint operator on $L^2(\Omega, q\,\mathrm{d}x)$, and hence

$$\|T\|_{L^2(\Omega,q\,\mathrm{d}x)\to L^2(\Omega,q\,\mathrm{d}x)} = \sup_{\{f: \|f\|_{L^2(\Omega,q\,\mathrm{d}x)} \leqslant 1\}} \left| \langle Tf, f\rangle_{L^2(\Omega,q\,\mathrm{d}x)} \right|.$$

In computing this supremum, we can assume $f \in \mathcal{B} = \{f \in C_0^\infty(\Omega) : \|f\|_{L^2(q\,\mathrm{d}x)} \leqslant 1\}$. For $f \in \mathcal{B}$, we have that $fq \in L^1(\Omega)$. Hence we obtain

$$\begin{aligned}\|T\|_{L^2(\Omega, q\,\mathrm{d}x)\to L^2(\Omega, q\,\mathrm{d}x)} &= \sup_{f\in\mathcal{B}} \left|\langle G(fq), fq\rangle_{L^2(\Omega,\,\mathrm{d}x)}\right| \\ &= \sup_{f\in\mathcal{B}} \|fq\|^2_{L^{-1,2}(\Omega)},\end{aligned} \tag{3.5}$$

by (3.2).

Suppose (1.7) holds for all $h \in C_0^\infty(\Omega)$. Since (3.4) implies (3.3), we have

$$\|T\|_{L^2(\Omega, q\,\mathrm{d}x)\to L^2(\Omega, q\,\mathrm{d}x)} = \sup_{f\in\mathcal{B}} \|fq\|^2_{L^{-1,2}(\Omega)} \leqslant \beta^2.$$

Conversely, if T is bounded on $L^2(q\,\mathrm{d}x)$, then by (3.5), we have

$$\|fq\|^2_{L^{-1,2}(\Omega)} \leqslant \|T\| \|f\|^2_{L^2(\Omega, q\,\mathrm{d}x)},$$

first for all $f \in C_0^\infty(\Omega)$, but then as a consequence of density, for all $f \in L^2(\Omega, q\,\mathrm{d}x)$. Since (3.3) implies (3.4), we obtain $\|h\|^2_{L^2(\Omega, q\,\mathrm{d}x)} \leqslant \|T\| \|h\|^2_{L^{1,2}(\Omega)}$ for all $h \in C_0^\infty(\Omega)$. Hence $\beta^2 \leqslant \|T\|_{L^2(\Omega, q\,\mathrm{d}x)\to L^2(\Omega, q\,\mathrm{d}x)}$. ☐

The next step utilizes estimates from [12]. In that paper, a general σ-finite measure space $(X, \mathrm{d}\omega)$ and integral operator T defined by $Tf(x) = \int_X K(x, y) f(y)\,\mathrm{d}\omega(y)$ are considered. Here $K : X \times X \to (0, \infty]$ is symmetric and quasi-metrically modifiable, which means that there exists a measurable function $m : X \to (0, \infty)$ (the "modifier"), such that for $\tilde{K}(x, y) = \frac{K(x,y)}{m(x)m(y)}$ we have that $d(x, y) = 1/\tilde{K}(x, y)$ satisfies the quasi-metric condition

$$d(x, y) \leqslant \kappa(d(x, z) + d(z, y))$$

for some constant $\kappa > 0$ and all $x, y, z \in X$. For $j \geqslant 2$, we define $K_j(x, y) = \int_X K_{j-1}(x, z)K(z, y)\,\mathrm{d}\omega(z)$. Then the j^{th} iterate T^j of T has the form $T^j f(x) = \int_X K_j(x, y)f(y)\,\mathrm{d}\omega(y)$. The formal solution to the equation $v = Tv + m$ is

$$v_0 = m + \sum_{j=1}^{\infty} T^j m,$$

for a modifier m. Then [12, Corollary 3.5] states that there exists $c > 0$ depending only on κ such that

$$me^{c(Tm)/m} \leqslant v_0, \tag{3.6}$$

and, if in addition $\|T\|_{L^2(\omega)\to L^2(\omega)} < 1$, then there exists a constant $C > 0$ depending only on κ and $\|T\|$ such that

$$v_0 \leqslant me^{C(Tm)/m}. \tag{3.7}$$

To apply this result to our case, we let $X = \Omega$, $\mathrm{d}\omega = q(y)\,\mathrm{d}y$ and $K(x,y) = G(x,y)$. Note that (3.1) holds for T defined on X as above. As noted in [13, p. 118] or [12, p. 905], the equivalence (2.1) in the case $n \geqslant 3$ combined with (2.4) shows that K is quasi-metrically modifiable with modifier $m(x) = \delta(x) = \operatorname{dist}(x, \partial\Omega)$. (We take this opportunity to note a misprint in [12]: the power of $|x-y| + \delta(x) + \delta(y)$ in equation (1.6) should be α, not $\alpha/2$; this error has no bearing on the validity of the results in that paper.) For $n = 2$, it remains true that $m = \delta$ is a modifier for K; this fact follows from estimates (2.2) and (2.4) (see [17, Propositions 8.6 and 9.6]). Then by (1.3) and (1.4), we have $K_j(x,y) = G_j(x,y)$ for all $j \geqslant 1$. Hence

$$\begin{aligned} T^j(G1)(x) &= \int_\Omega G_j(x,y) q(y) \int_\Omega G(y,z)\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_\Omega \int_\Omega G_j(x,y) G(y,z) q(y)\,\mathrm{d}y\,\mathrm{d}z = G_{j+1}1(x), \end{aligned}$$

where G_j is the integral operator defined by $G_j f(x) = \int_\Omega G_j(y) f(y)\,\mathrm{d}y$. Hence

$$u_0 = \mathcal{G}1 = \sum_{j=1}^{\infty} G_j 1 = G1 + \sum_{j=1}^{\infty} G_{j+1} 1 = G1 + \sum_{j=1}^{\infty} T^j(G1). \tag{3.8}$$

However, we noted in (2.4), $G1$ and δ are pointwise equivalent. Hence $u_0 \approx \delta + \sum_{j=1}^{\infty} T^j \delta = v_0$. Therefore by (3.6), there exist constants $c_1 > 0$ and $c > 0$ such that

$$u_0 \geqslant c_1 \delta e^{cG(q\delta)/\delta}, \tag{3.9}$$

and, if we assume $\|T\|_{L^2(\Omega, q\,\mathrm{d}x) \to L^2(\Omega, q\,\mathrm{d}x)} < 1$, then (3.7) gives the estimate

$$u_0 \leqslant C_1 e^{CG(q\delta)/\delta}, \tag{3.10}$$

for some constants $C_1 > 0$ and $C > 0$.

Proof of Theorem 1.1. — First suppose (1.7) holds for some $\beta \in (0,1)$. We note that we then have the inequality

$$\int_\Omega h^2 q\,\mathrm{d}x \leqslant \beta^2 \|h\|^2_{L_0^{1,2}(\Omega)} \tag{3.11}$$

for all $h \in L_0^{1,2}(\Omega)$, by an approximation argument, as follows. Let $h_n \in C_0^\infty(\Omega)$ be a sequence of functions converging to h in $L_0^{1,2}(\Omega)$. Then by the Sobolev imbedding theorem, h_n converges to h in L^{p^*} for some $p^* \geqslant 1$, so by passing to a subsequence we can assume h_n converges to h a.e. Because of (1.7), h_n is Cauchy in $L^2(\Omega, q\,\mathrm{d}x)$ and hence converges in $L^2(\Omega, q\,\mathrm{d}x)$ to some function $\tilde{h}$. Since there is a subsequence of h_n converging $q\,\mathrm{d}x$-a.e. to

$\tilde{h}$, we must have $\tilde{h} = h$ a.e. with respect to $q\,\mathrm{d}x$. Hence we can let $n \to \infty$ in $\int_\Omega h_n^2 q\,\mathrm{d}x \leqslant \beta^2 \int_\Omega |\nabla h_n|^2\,\mathrm{d}x$ to obtain (3.11).

Observe that $G1 \in C(\overline{\Omega})$, $G1 = 0$ on $\partial\Omega$, and, by (3.2) and (2.4),

$$\int_\Omega |\nabla G1|^2\,\mathrm{d}x = \int_\Omega G1\,\mathrm{d}x \leqslant C \int_\Omega \delta(x)\,\mathrm{d}x < \infty.$$

Hence $G1 \in L_0^{1,2}(\Omega)$. By the remark in the last paragraph, $G1 \in L^2(\Omega, q\,\mathrm{d}x)$. Since $G1 \approx \delta$, this means that $\delta q \in L^1(\Omega, \delta\,\mathrm{d}x)$. By Lemma 2.2(1), $G(\delta q) \in L^1(\Omega,\,\mathrm{d}x)$. In particular, $G(\delta q) < \infty$ a.e.

By our assumption (1.7) and Lemma 3.1, the operator T defined by (3.1) has $\|T\|_{L^2(\Omega, q\,\mathrm{d}x) \to L^2(\Omega, q\,\mathrm{d}x)} \leqslant \beta^2 < 1$. Hence by (3.10), $u_0 = \mathcal{G}1$ satisfies (1.8) and $u_0 < \infty$ a.e. By Lemma 2.7, $u_0 \in L^1(\Omega,\,\mathrm{d}x) \cap L^1(\Omega,\,\delta q\,\mathrm{d}x)$, and u_0 is a positive very weak solution of (2.9).

Since $\|T\|_{L^2(\Omega, q\,\mathrm{d}x) \to L^2(\Omega, q\,\mathrm{d}x)} < 1$, the operator $(I - T)^{-1} = \sum_{j=0}^\infty T^j$ is bounded on $L^2(q\,\mathrm{d}x)$. Hence $u_0 \in L^2(q,\,\mathrm{d}x)$, by (3.8). Since $u_0 = G(u_0 q + 1)$, we have

$$\begin{aligned}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x &= \int_\Omega |\nabla G(u_0 q + 1)|^2\,\mathrm{d}x \\ &= \int_\Omega G(u_0 q + 1)(u_0 q + 1)\,\mathrm{d}x = \int_\Omega (u_0^2 q + u_0)\,\mathrm{d}x.\end{aligned}$$

by (3.2). Since $u_0 = G(u_0 q + 1)$ is 0 on $\partial\Omega$, we obtain $u_0 \in L_0^{1,2}(\Omega)$.

We now show that u_0 is a weak solution of (1.1). Since $u_0 \in L_0^{1,2}(\Omega)$, we must show that

$$\int_\Omega \nabla u_0 \cdot \nabla h\,\mathrm{d}x = \int_\Omega h u_0 q + h\,\mathrm{d}x,$$

for all $h \in L_0^{1,2}(\Omega)$. Let h_n be a sequence in $C_0^\infty(\Omega)$ converging to h in the norm on $L_0^{1,2}(\Omega)$. Then

$$\int_\Omega \nabla u_0 \cdot \nabla h_n\,\mathrm{d}x = -\int_\Omega u_0 \triangle h_n\,\mathrm{d}x = \int_\Omega h_n u_0 q + h_n\,\mathrm{d}x$$

because u_0 is a very weak solution of (1.1). The left side converges as $n \to \infty$ to $\int_\Omega \nabla u_0 \cdot \nabla h\,\mathrm{d}x$, because h_n converges to h in $L_0^{1,2}(\Omega)$. By (3.11), which we now know is valid for all h in $L_0^{1,2}(\Omega)$, we have that h_n converges to h in $L^2(\Omega, q\,\mathrm{d}x)$. We also know that $u_0 \in L^2(\Omega, q\,\mathrm{d}x)$. Hence using the Cauchy–Schwarz inequality in $L^2(\Omega, q\,\mathrm{d}x)$ we see that $\int_\Omega h_n u_0 q\,\mathrm{d}x$ converges to $\int_\Omega h u_0 q\,\mathrm{d}x$. The imbedding of $L_0^{1,2}(\Omega,\,\mathrm{d}x)$ in $L^1(\Omega,\,\mathrm{d}x)$ shows that $\int_\Omega h_n\,\mathrm{d}x$ converges to $\int_\Omega h\,\mathrm{d}x$. Therefore u_0 is a weak solution of (1.1).

Now suppose $u \in L^1(\Omega,\, \mathrm{d}x) \cap L^1(\Omega,\, \delta q\, \mathrm{d}x)$ and u is a positive very weak solution of (1.1). By Definition 2.1 and Lemma 2.5, u satisfies the integral equation $u = G(qu) + G1 = Tu + G1$ a.e., for T defined by (3.1). Since $G1 \geqslant 0$, we have $T(u) \leqslant u$ a.e., with $u \geqslant G1 > 0$ and $u < \infty$ a.e. Hence by Schur's test for integral operators, we have $\|T\|_{L^2(\Omega, q\,\mathrm{d}x) \to L^2(\Omega, q\,\mathrm{d}x)} \leqslant 1$. By Lemma 3.1, it follows that (1.7) holds with $\beta = 1$. Since $u_0 = \mathcal{G}1$ is the minimal positive very weak solution of (1.1), we have $u_0 \leqslant u$, hence (1.9) holds because of (3.9). □

We turn now to equation (1.2). By Lemma 2.9, the essential issue is whether $u_1 = 1 + \mathcal{G}(q)$ is finite a.e., or equivalently $u_1 \in L^1(\Omega)$. We will use the relation between u_0 and u_1 exhibited by the following simple lemma.

Lemma 3.2. — *Let Ω, q, u_0, and u_1 be as in Theorems 1.1 and 1.2. Then $u_1 \in L^1(\Omega,\, \mathrm{d}x)$ if and only if $u_0 \in L^1(\Omega, q\, \mathrm{d}x)$.*

Proof. — Since $u_0 = \mathcal{G}1$, Fubini's theorem and the symmetry of $\mathcal{G}(x, y)$ yield

$$\int_\Omega u_1\, \mathrm{d}x = \int_\Omega 1\, \mathrm{d}x + \int_\Omega \int_\Omega \mathcal{G}(x, y) q(y)\, \mathrm{d}y\, \mathrm{d}x = |\Omega| + \int_\Omega u_0 q\, \mathrm{d}y. \tag{3.12}$$ □

The following convergence lemma will be useful.

Lemma 3.3. — *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded C^2 domain, for $n \geqslant 2$. Suppose $q \in L^1(\Omega,\, \mathrm{d}x)$ and q has compact support in Ω. Suppose $\phi \in L^1(\Omega, q\, \mathrm{d}x)$. Let $z \in \partial\Omega$, and let $\{x_j\}_{j=1}^\infty$ be a sequence in Ω converging to z in the normal direction. Then*

$$\lim_{j\to\infty} \frac{G(\phi q)(x_j)}{\delta(x_j)} = \int_\Omega P(y, z)\phi(y) q(y)\, \mathrm{d}y.$$

Proof. — Recall that $P(y, z)$ is the normal derivative of $G(y, x)$ as $x \to z$, $x \in \Omega$. Hence $\lim_{j\to\infty} G(y, x_j)/\delta(x_j) = P(y, z)$. Since G is symmetric,

$$\lim_{j\to\infty} \frac{G(\phi q)(x_j)}{\delta(x_j)} = \lim_{j\to\infty} \int_\Omega \frac{G(y, x_j)}{\delta(x_j)} \phi(y) q(y)\, \mathrm{d}y.$$

There is some constant $c_1 > 0$ such that $|y - x_j| \geqslant c_1$ for all y belonging to the support of q and all sufficiently large j. Hence (2.1) shows that $G(y, x_j)/\delta(x_j)$ is bounded for all large enough j. The result follows by the dominated convergence theorem. □

We will need an elementary lemma on quasi-metric spaces due to Hansen and Netuka [18, Proposition 8.1 and Corollary 8.2]; in the context of normed spaces it was proved earlier by Pinchover [27, Lemma A.1].

Lemma 3.4. — *Suppose d is a quasi-metric on a set Ω with quasi-metric constant κ. Suppose $z \in X$. Then*

$$\tilde{\mathrm{d}}(x,y) = \frac{\mathrm{d}(x,y)}{\mathrm{d}(x,z)\cdot \mathrm{d}(y,z)}, \qquad x,y \in \Omega \setminus \{z\}, \tag{3.13}$$

is a quasi-metric on $\Omega \setminus \{z\}$ with quasi-metric constant $4\kappa^2$.

Proof of Theorem 1.2. — First suppose (1.7) holds for some $\beta \in (0,1)$ and (1.10) holds for the constant C in (1.8). By Theorem 1.1, $u_0 \in L^1(\Omega, \mathrm{d}x) \cap L^1(\Omega, \delta q \,\mathrm{d}x)$ and u_0 satisfies (1.8). By Corollary 2.6 and Lemma 2.7, $u_0 = G(u_0 q + 1)$ at every point of Ω.

Let $z \in \partial\Omega$. We claim that

$$\int_\Omega P(y,z) u_0(y) q(y)\,\mathrm{d}y \leqslant C_1 e^{CP^*(\delta q)(z)}, \tag{3.14}$$

where C_1 and C are the constants from (1.8).

We first prove this claim under the additional assumption that q is compactly supported in Ω, so $q \in L^1(\Omega, \mathrm{d}x)$. Since δ is bounded above and below, away from 0, on the support of q, the condition $u_0 \in L^1(\Omega, \delta q)$ is equivalent to the condition $u_0 \in L^1(\Omega, q\,\mathrm{d}x)$. Let $\{x_j\}_{j=1}^\infty$ be a sequence in Ω converging to z in the normal direction. Applying Lemma 3.3 with $\phi = u_0$, we obtain

$$\int_\Omega P(y,z) u_0(y) q(y)\,\mathrm{d}y = \lim_{j\to\infty} \frac{G(u_0 q)(x_j)}{\delta(x_j)}.$$

Applying the equation $u_0 = G(u_0 q + 1)$ and (1.8),

$$\frac{G(u_0 q)(x_j)}{\delta(x_j)} \leqslant \frac{u_0(x_j)}{\delta(x_j)} \leqslant C_1 e^{CG(\delta q)(x_j)/\delta(x_j)}.$$

Taking the limit and applying Lemma 3.3 with $\phi = \delta \in L^1(\Omega, q\,\mathrm{d}x)$ gives

$$\int_\Omega P(y,z) u_0(y) q(y)\,\mathrm{d}y \leqslant C_1 e^{C\int_\Omega P(y,z)\delta(y) q(y)\,\mathrm{d}y} = C_1 e^{CP^*(\delta q)(z)}.$$

We now remove the assumption that q is compactly supported in Ω. Let $q \in L^1_{loc}(\Omega)$. Let $\{\Omega_k\}_{k=1}^\infty$ be an exhaustion of Ω by smooth subdomains with compact closure such that $\overline{\Omega}_{k+1} \subset \Omega_k$, $k = 1, 2, \ldots$. Define $q_k = q\chi_{\Omega_k}$. Then each q_k has compact support in Ω. Define the iterated Green's kernels $G_j^{(k)}(x,y)$ for $j = 1, 2, 3, \ldots$, and $\mathcal{G}^{(k)}$, as in (1.3), (1.4), and (1.5), except with q replaced by q_k. Let $u_k = \mathcal{G}^{(k)}1$. By repeated use of the monotone convergence theorem, $G_j^{(k)}(x,y)$ increases monotonically as $k \to$

∞ to $G_j(x,y)$ for each j, $\mathcal{G}^{(k)}(x,y)$ increases monotonically to $\mathcal{G}(x,y)$, and u_k increases monotonically to u_0. Applying the compact support case gives

$$\int_\Omega P(y,z)u_k(y)q_k(y)\,\mathrm{d}y \leqslant C_1 e^{CP^*(\delta q_k)(z)} \leqslant C_1 e^{CP^*(\delta q)(z)}.$$

Then the monotone convergence theorem yields (3.14).

We integrate (3.14) over $\partial\Omega$:

$$\int_{\partial\Omega}\int_\Omega P(y,z)u_0(y)q(y)\,\mathrm{d}y\,\mathrm{d}\sigma(z) \leqslant C_1\int_{\partial\Omega} e^{CP^*(\delta q)(z)}\,\mathrm{d}\sigma(z).$$

By Fubini's theorem and the fact that $P(1)=1$, the left side is just $\int_\Omega u_0 q\,\mathrm{d}x$. Hence (1.8) implies $u_0 \in L^1(\Omega, q\,\mathrm{d}x)$. By Lemma 3.2 and (3.12), we obtain $u_1 \in L^1(\Omega,\mathrm{d}x)$ and (1.11) holds. Hence $u_1 < \infty$ a.e. Then Lemma 2.9 shows that u_1 is a very weak solution of (1.2).

Next we prove the pointwise estimate (1.12). Since for all $x\in\Omega$ we have $\int_{\partial\Omega} P(x,z)\,\mathrm{d}\sigma(z) = 1$, it follows

$$\begin{aligned} u_1(x) = 1 + \mathcal{G}q(x) &= 1 + \sum_{j=1}^\infty \int_\Omega G_j(x,y)q(y)\,\mathrm{d}y \\ &= \int_{\partial\Omega}\left(P(x,z) + \sum_{j=1}^\infty \int_\Omega G_j(x,y)P(y,z)q(y)\,\mathrm{d}y\right)\mathrm{d}\sigma(z) \\ &= \int_{\partial\Omega}\sum_{j=0}^\infty T^j(P(\cdot,z))(x)\,\mathrm{d}\sigma(z). \end{aligned}$$

The following estimates of the Poisson kernel are well known (see [7]): there exist constants $c=c(\Omega)$, $C=C(\Omega)$ so that, for $x\in\Omega$ and $z\in\partial\Omega$:

$$\frac{c\,\delta(x)}{|x-z|^n} \leqslant P(x,z) \leqslant \frac{C\,\delta(x)}{|x-z|^n}. \tag{3.15}$$

Fix $z\in\partial\Omega$ for the moment. We claim that $m(x)=P(x,z)$ is a modifier for $K(x,y)=G(x,y)$. To see this fact, define a quasi-metric d on $\overline{\Omega}$, for $n\geqslant 3$, by:

$$d(x,y) = |x-y|^{n-2}\left[|x-y|^2 + \delta(x)^2 + \delta(y)^2\right], \quad x,y\in\overline{\Omega}.$$

Notice that, for $z\in\partial\Omega$, we have $d(x,z)\approx |x-z|^n$ since $|x-z|\geqslant\delta(x)$ and $\delta(z)=0$. Hence by (3.15),

$$m(x) \approx \delta(x)/\,\mathrm{d}(x,z), \qquad x\in\Omega.$$

Using (2.1) together with the preceding inequalities, we estimate the modified kernel $\tilde{K}$:

$$\tilde{K}(x,y) = \frac{G(x,y)}{m(x)\cdot m(y)} \approx \frac{\mathrm{d}(x,z)\cdot\mathrm{d}(y,z)}{\mathrm{d}(x,y)}. \tag{3.16}$$

By Lemma 3.4, $\tilde{K}$ is a quasi-metric kernel on $\overline{\Omega} \setminus \{z\}$, and hence on Ω. Notice that all the constants of equivalence depend only on Ω, but not on z.

Similarly, for $n = 2$, we invoke (2.2) to define a quasi-metric on $\overline{\Omega}$ using an extension by continuity of the quasi-metric originally defined on Ω by

$$\mathrm{d}(x,y) = \delta(x)\delta(y)\left[\ln\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right)\right]^{-1}, \quad x, y \in \Omega.$$

In other words, for $x \in \Omega$ and $z \in \partial\Omega$, we set

$$\begin{aligned}\mathrm{d}(x,z) &= \lim_{y\to z,\, y\in\Omega} \mathrm{d}(x,y)\\ &= \lim_{y\to z,\, \delta(y)\to 0} \delta(x)\delta(y)\left[\ln\left(1+\frac{\delta(x)\delta(y)}{|x-y|^2}\right)\right]^{-1} = |x-z|^2.\end{aligned}$$

The same formula will be used if both $x, z \in \partial\Omega$, so that

$$\mathrm{d}(x,z) = |x-z|^2 \quad \text{for all } x \in \overline{\Omega},\ z \in \partial\Omega.$$

Clearly, the extended function d satisfies the quasi-triangle inequality on $\overline{\Omega}$. Moreover, for $z \in \partial\Omega$, we have by (3.15),

$$m(x) = P(x,z) \approx \delta(x)/\,\mathrm{d}(x,z) \quad \text{for all } x \in \Omega.$$

By Lemma 3.4 the modified kernel $\tilde{K}(x,y) = \frac{G(x,y)}{m(x)m(y)}$ is a quasi-metric kernel on Ω, since it satisfies (3.16) as in the case $n \geqslant 3$.

Applying (3.6) and (3.7) to estimate $\sum_{j=0}^{\infty} T^j m$, we obtain:

$$\begin{aligned}c_1\, P(x,z)\, e^{c_2 \int_\Omega G(x,y)\frac{P(y,z)}{P(x,z)} q(y)\,\mathrm{d}y} &\leqslant \sum_{j=0}^{\infty} T^j(P(\cdot,z))(x)\\ &\leqslant C_2\, P(x,z)\, e^{C_3 \int_\Omega G(x,y)\frac{P(y,z)}{P(x,z)} q(y)\,\mathrm{d}y},\end{aligned}$$

where the constants do not depend on $x \in \Omega$ and $z \in \partial\Omega$. Substituting into the expression for u_1 above, we obtain (1.12) as well as the lower estimate

$$u_1(x) \geqslant c_1 \int_{\partial\Omega} e^{c_2 \int_\Omega G(x,y)\frac{P(y,z)}{P(x,z)} q(y)\,\mathrm{d}y} P(x,z)\,\mathrm{d}\sigma(z). \tag{3.17}$$

For the converse, suppose u is a positive very weak solution of (1.2). By the remarks after Definition 2.8, u satisfies the integral equation $u = 1 + G(qu) = 1 + Tu$, for T defined by (3.1). Hence $0 < u < \infty$ a.e., and $Tu \leqslant u$. By Schur's test, $\|T\|_{L^2(\Omega, q\,\mathrm{d}x)\to L^2(\Omega, q\,\mathrm{d}x)} \leqslant 1$. By Lemma 3.1, inequality (1.7) holds with $\beta = 1$.

Since $u_1 = 1 + \mathcal{G}q$ is the minimal positive very weak solution of (1.2), we have $\mathcal{G}q < u_1 \leqslant u$, hence $\mathcal{G}q < \infty$ a.e. By Lemma 2.2, $q \in L^1(\Omega, \delta\,\mathrm{d}x)$

and by Lemma 2.7, $\mathcal{G}q \in L^1(\Omega, \mathrm{d}x)$. Hence $u_1 = 1 + \mathcal{G}q \in L^1(\Omega, \mathrm{d}x)$. By Lemma 3.2, $u_0 \in L^1(\Omega, q\,\mathrm{d}x)$.

Let $z \in \partial\Omega$. We claim that there exists $c_2 > 0$ depending only on Ω such that

$$c_1 e^{cP^*(\delta q)(z)} \leqslant \int_\Omega P(y,z)u_0(y)q(y)\,\mathrm{d}y + c_2, \tag{3.18}$$

where c_1 and c are the constants from (1.9). By the same exhaustion process that was used in the forward direction, it is sufficient to prove (3.18) under the assumption that q has compact support in Ω. Under that assumption, we have that δ is bounded above, and below away from 0, on the support of q, so $\delta q \in L^1(\Omega, \mathrm{d}x)$. Let $\{x_j\}_{j=1}^\infty$ be a sequence of points in Ω which converge to z in the normal direction. By Lemma 3.3 with $\phi = \delta$,

$$c_1 e^{cP^*(\delta q)(z)} = e^{c\int_\Omega P(y,z)\delta(y)q(y)\,\mathrm{d}y} = \lim_{j\to\infty} c_1 e^{cG(\delta q)(x_j)/\delta(x_j)}.$$

By Theorem 1.1, u_0 satisfies the estimate in (1.9). Hence

$$c_1 e^{cG(\delta q)(x_j)/\delta(x_j)} \leqslant \frac{u_0(x_j)}{\delta(x_j)} = \frac{G(u_0 q)(x_j)}{\delta(x_j)} + \frac{G1(x_j)}{\delta(x_j)} \leqslant \frac{G(u_0 q)(x_j)}{\delta(x_j)} + c_2,$$

by (2.4). Because $u_0 \in L^1(\Omega, q\,\mathrm{d}x)$, taking the limit and applying Lemma 3.3 with $\phi = u_0$ gives (3.18).

Integrating (3.18) over $\partial\Omega$, applying Fubini's theorem, and using the fact that $P1 = 1$, we obtain

$$\begin{aligned} c_1 \int_{\partial\Omega} e^{cP^*(\delta q)(z)}\,\mathrm{d}\sigma(z) &\leqslant \int_{\partial\Omega}\left(\int_\Omega P(y,z)u_0(y)q(y)\,\mathrm{d}y + c_2\right)\mathrm{d}\sigma(z) \\ &= \int_\Omega u_0 q\,\mathrm{d}x + c_2|\partial\Omega| < \infty, \end{aligned}$$

since $u_0 \in L^1(\Omega, q\,\mathrm{d}x)$. By Lemma 3.2 and the minimality of u_1, we have $\int_\Omega u_0 q\,\mathrm{d}x \leqslant \int_\Omega u_1\,\mathrm{d}x \leqslant \int_\Omega u\,\mathrm{d}x$, which establishes (1.13).

Now (1.14) follows from (3.17), since $u \geqslant u_1$. □

Proof of Corollary 1.3. — By the John–Nirenberg theorem, $e^{\beta P^*(\delta q)}$ is integrable on $\partial\Omega$, for β less than a multiple of the reciprocal of the BMO norm of $P^*(\delta q)$. Hence if (A) holds for ϵ_1 small enough, then $\int_{\partial\Omega} e^{CP^*(\delta q)}\,\mathrm{d}\sigma < \infty$, and the conclusions follow from Theorem 1.2.

By a standard theorem (see e.g. [28], [14], p. 229), $P^*(\delta q) \in \mathrm{BMO}(\partial\Omega)$ with BMO norm bounded by a multiple of the Carleson norm of $\delta q\,\mathrm{d}x$. Therefore (B) for ϵ_2 sufficiently small implies (A). □

Condition (B) above actually yields (A) with every $\chi_E\,\delta q$ in place of δq, for any measurable $E \subset \Omega$, and the converse is also true (see [28]).

We now turn to the case of $\Omega = \mathbb{R}^n$. For $0 < \alpha < n$, let $I_\alpha = (-\triangle)^{-\alpha/2}$ denote the Riesz potential defined by

$$I_\alpha f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)\, \mathrm{d}y,$$

for some constant $c_{n,\alpha} > 0$. If $f \geqslant 0$, then the Riesz potential $I_\alpha f(x)$ is finite a.e. in $\mathbb{R}^n$ if and only if

$$\int_{\mathbb{R}^n} \frac{f(y)\, \mathrm{d}y}{(|y|+1)^{n-\alpha}} < +\infty. \tag{3.19}$$

Otherwise $I_\alpha f \equiv +\infty$ on $\mathbb{R}^n$ ([22, §I.3]). If (3.19) holds, then

$$\liminf_{x\to\infty} I_\alpha f(x) = 0.$$

The kernel of I_2 is the Green's function $G(x,y) = c_n |x-y|^{2-n}$ and the Green's operator G coincides with I_2. Define the iterates G_j and $\mathcal{G}$ by (1.4) and (1.5).

We consider positive solutions u to the Schrödinger equation

$$-\triangle u = qu + f \quad \text{in } \Omega, \tag{3.20}$$

where $q \geqslant 0$ is a given non-negative potential and $f \geqslant 0$ is a function such that

$$\int_{\mathbb{R}^n} \frac{f(y)\, \mathrm{d}y}{(1+|y|)^{n-2}} < +\infty. \tag{3.21}$$

Equation (3.20) is understood in the distributional sense. Equivalently (see [22], Sec. I.5), $u \in L^1_{loc}(\mathbb{R}^n)$, $u \geqslant 0$, is a solution to (3.20) if

$$\int_{\mathbb{R}^n} \frac{u(y)\, q(y)\, \mathrm{d}y}{(1+|y|)^{n-2}} < +\infty,$$

and

$$u = I_2(qu) + I_2 f + c \quad \text{a.e.}, \tag{3.22}$$

where c is a non-negative constant and $\liminf_{x\to\infty} u(x) = c$.

Since $f = 1$ does not satisfy (3.21), we do not obtain conditions for the solvability of (1.1) on $\mathbb{R}^n$. On a bounded domain, the results for (1.1) in Theorem 1.1 were used to obtain our results in Theorem 1.2 for (1.2). Nevertheless we obtain results for (1.2) on $\mathbb{R}^n$.

We first note that Lemma 3.1 holds for $\Omega = \mathbb{R}^n$. Define the operator T by (3.1) with $\Omega = \mathbb{R}^n$. Define the homogeneous Sobolev space $L^{1,2}_0(\mathbb{R}^n)$ to be the closure of $C^\infty_0(\mathbb{R}^n)$ with respect to the norm $\|f\|_{L^{1,2}(\mathbb{R}^n)} = \|(-\triangle)^{1/2} f\|_{L^2(\mathbb{R}^n)}$. The dual of $L^{1,2}_0(\mathbb{R}^n)$ is isometrically isomorphic to

$L^{-1,2}(\mathbb{R}^n)$ defined via the norm $\|f\|_{L^{-1,2}(\mathbb{R}^n)} = \|(-\triangle)^{-1/2}f\|_{L^2(\mathbb{R}^n)}$. Because of the semi-group property $I_{\alpha/2} * I_{\alpha/2} = I_\alpha$ of the Riesz kernels, we have, for $f \geqslant 0$ (or if f is a finite signed measure),

$$\|f\|^2_{L^{-1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (I_1 f)^2 \, \mathrm{d}x = \int_{\mathbb{R}^n} f I_2 f \, \mathrm{d}x = \int_{\mathbb{R}^n} fGf \, \mathrm{d}x,$$

which is the analogue of (3.2). With this result, the proof of Lemma 3.1 carries over to $\mathbb{R}^n$ and we obtain that $\|T\|_{L^2(\mathbb{R}^n, q\,\mathrm{d}x) \to L^2(\mathbb{R}^n, q\,\mathrm{d}x)} = \beta^2$, where β is the least constant in (1.15).

Proof of Theorem 1.4. — First suppose (1.15) holds for some $\beta \in (0,1)$, and (1.16) holds. Then the equation $-\triangle u = qu$ with $\liminf_{x\to\infty} u(x) = 1$ is equivalent to $u = I_2(qu) + 1 = T(u) + 1$, by (3.22) with $f = 0$ and $c = 1$. By the analogue of Lemma 3.1 for $\mathbb{R}^n$ just noted, the operator T has norm less than 1 on $L^2(\mathbb{R}^n, q\,\mathrm{d}x)$. Since the Riesz kernel $G(x,y)$ is quasi-metric, [12, Theorem 3.1] (i.e. [12, Corollary 3.5], or (3.7) with $m = 1$) states that

$$1 + \sum_{j=1}^{\infty} T^j 1 \leqslant e^{CT1},$$

where C depends only on n and β. Note that

$$T^j 1(x) = \int_{\mathbb{R}^n} K_j(x,y) q(y) \, \mathrm{d}y = \int_{\mathbb{R}^n} G_j(x,y) q(y) \, \mathrm{d}y = G_j q(x),$$

since $K_j(x,y) = G_j(x,y)$ by (1.3) and (1.4). Hence

$$u_1 = 1 + \mathcal{G}(q) = 1 + \sum_{j=1}^{\infty} T^j 1 \leqslant e^{CT1} = e^{CI_2 q},$$

so (1.18) holds. By (1.16), $I_2 q < \infty$ a.e., so u_1 defines a positive solution to $-\triangle u = qu$ with $\liminf_{x\to\infty} u(x) = 1$.

Conversely, suppose u is a nonnegative solution of (1.17), or equivalently, $u = I_2(qu) + 1 = Tu + 1$. Then $1 \leqslant u < \infty$ a.e., so by Schur's test we have $\|T\|_{L^2(\mathbb{R}^n, q\,\mathrm{d}x) \to L^2(\mathbb{R}^n, q\,\mathrm{d}x)} \leqslant 1$, which, we have seen, implies (1.15) with $\beta = 1$. By iteration of the identity $u = Tu + 1$, we see that $u \geqslant u_1$, so u_1 is minimal among positive solutions. Applying the lower estimate from [12, Theorem 3.1] (i.e. (3.6) with $m = 1$), we have

$$u \geqslant u_1 \geqslant e^{cT1} = e^{cI_2 q},$$

where c depends only on n, so (1.19) holds. Since $u < \infty$ a.e., we conclude that $I_2 q < \infty$ a.e., so (1.16) holds. □

4. Nonlinear Equations with Quadratic Growth in the Gradient

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain and let $q \in L^1_{loc}(\Omega)$ with $q \geqslant 0$. The definition of very weak solutions of (1.20) is consistent with Definitions 2.1, 2.4, and Remark 2.3. A good reference to very weak solutions of elliptic equations is [24].

Definition 4.1. — *Let $q \in L^1(\Omega, \delta\, \mathrm{d}x)$. A function $v \in L^1(\Omega, \mathrm{d}x)$ such that $\int_\Omega |\nabla v|^2 \delta\, \mathrm{d}x < \infty$ is a very weak solution of* (1.20) *if*

$$-\int_\Omega v \triangle h \, \mathrm{d}x = \int_\Omega |\nabla v|^2 h \, \mathrm{d}x + \int_\Omega h\, q \, \mathrm{d}x, \quad \textit{for all } h \in C^2_0(\overline{\Omega}).$$

Lemma 2.5 and Corollary 2.6 show that if v is a very weak solution of (1.20) then v satisfies the integral equation

$$v = G(|\nabla v|^2 + q) \text{ a.e.,} \tag{4.1}$$

and if $q \geqslant 0, v < \infty$ a.e. and v satisfies (4.1), then $v \in L^1(\Omega, \mathrm{d}x)$, $\int_\Omega |\nabla v|^2 \delta\, \mathrm{d}x < \infty$, and v is a very weak solution of (1.20).

Corresponding formally to (1.20) under the substitution $v = \log u$ is equation (1.2). However the precise relation between very weak solutions to (1.2) and (1.20) is not as simple as it might appear, as shown by the next example which was first noted by Ferone and Murat in [10].

Remark 4.2. — Even for the case $q = 0$, there is a very weak solution v of (1.20) such that $u = e^v$ is not a very weak solution of (1.2). Let $v(x) = \log(1 + G(x, x_0))$, where $x_0 \in \Omega$ is a fixed pole. Then standard arguments show that v is a very weak solution of $-\triangle v = |\nabla v|^2$ on Ω with $v = 0$ on $\partial\Omega$. However, $u = 1 + G(x, x_0)$ satisfies $-\Delta u = \delta_{x_0}$ in Ω, so that u is not a very weak solution of (1.2).

We will see that if u_1 is the minimal positive very weak solution of (1.2), then $v = \log u_1$ is a very weak solution of (1.20). However, in general, if v is a very weak solution to (1.20) then $u = e^v$ is only a supersolution to (1.2), which is enough to prove Theorem 1.5.

Proof of Theorem 1.5. — First suppose that (1.7) holds for some $\beta \in (0, 1)$, and (1.10) holds. By Theorem 1.2, the Schrödinger equation (1.2) has a positive very weak solution $u(x) = 1 + \mathcal{G}q$. (This solution u was called u_1 in the statement of the theorem; we call it u in the proof to avoid ambiguity with a sequence $\{u_k\}_{k=1}^\infty$ which will be defined later.) Then $u \in L^1(\Omega, \mathrm{d}x)$ and u satisfies the integral equation $u = 1 + G(qu)$. Therefore $u : \Omega \to [1, +\infty]$ is defined everywhere as a positive superharmonic function in Ω and hence is quasi-continuous; moreover, $\mathrm{cap}(\{u = +\infty\}) = 0$ (see [5]). By Remark 2.3, $u \in W^{1,p}_{loc}(\Omega)$ when $p < \frac{n}{n-1}$. We remark that actually, as shown in [21], $u \in W^{1,2}_{loc}(\Omega)$, but the proof of this stronger property is somewhat involved, and it will not be used below.

Define $d\mu = -\triangle u = qu\,\mathrm{d}x$, where $qu \in L^1_{loc}(\Omega)$. Let $v = \log u$. Then $0 \leqslant v < +\infty$-a.e., v is superharmonic in Ω by Jensen's inequality, and $v \in W^{1,2}_{loc}(\Omega)$ (see [20, Theorem 7.48] and [23, §2.2]). We claim that

$$-\triangle v = |\nabla v|^2 + q \quad \text{in} \quad D'(\Omega). \tag{4.2}$$

To prove (4.2), we will apply the integration by parts formula

$$\int_\Omega g\,d\rho = -\langle g, \Delta r\rangle = \int_\Omega \nabla g \cdot \nabla r\,\mathrm{d}x, \tag{4.3}$$

where $g \in W^{1,2}(\Omega)$ is compactly supported and quasi-continuous in Ω, and $\rho = -\Delta r$ where $r \in W^{1,2}_{loc}(\Omega)$ is superharmonic (see e.g. [23, Theorem 2.39 and Lemma 2.33]). This proof would simplify if we could apply (4.3) with $g = h/u, \rho = \mu$, and $r = u$, for $h \in C_0^\infty(\Omega)$. However, we do not have $r \in W^{1,2}_{loc}(\Omega)$, so we require an approximation argument. For $k \in \mathbb{N}$, let

$$u_k = \min(u,\, e^k), \quad v_k = \min(v,\, k), \quad \text{and} \quad \mu_k = -\triangle u_k.$$

Clearly u_k and v_k are superharmonic, hence μ_k is a positive measure. Moreover, u_k and v_k belong to $W^{1,2}_{loc}(\Omega) \bigcap L^\infty(\Omega)$ (see [20, Corollary 7.20]).

Let $h \in C_0^\infty(\Omega)$. We apply (4.3) with $g = h/u_k, \rho = \mu_k$, and $r = u_k$. Note that $u_k \geqslant 1$, g is compactly supported since h is, and $g \in W^{1,2}(\Omega)$ since $u_k \in W^{1,2}_{loc}(\Omega)$. Then by (4.3),

$$\begin{aligned}\int_\Omega \frac{h}{u_k}\,d\mu_k &= \int_\Omega \nabla\left(\frac{h}{u_k}\right)\cdot\nabla u_k\,\mathrm{d}x = \int_\Omega \frac{\nabla h}{u_k}\cdot\nabla u_k\,\mathrm{d}x - \int_\Omega \frac{|\nabla u_k|^2}{u_k^2} h\,\mathrm{d}x \\ &= \int_\Omega \nabla h \cdot \nabla v_k\,\mathrm{d}x - \int_\Omega |\nabla v_k|^2\,h\,\mathrm{d}x.\end{aligned}$$

Since u is superharmonic, u is lower semi-continuous, so the set $\{x \in \Omega : u(x) > e^k\} \equiv \{u > e^k\}$ is open, hence the measure $\mu_k = -\Delta u_k$ is supported on the set $\{u \leqslant e^k\}$ where $u = u_k$. Hence $u = u_k$ $d\mu_k$-a.e., and

$$\begin{aligned}\left|\int_\Omega \frac{h}{u}\,d\mu - \int_\Omega \frac{h}{u_k}\,d\mu_k\right| &\leqslant e^{-k}\int_{\{u\geqslant e^k\}} |h|d\mu + e^{-k}\int_{\{u=e^k\}} |h|\,d\mu_k \\ &\leqslant e^{-k}\int_\Omega |h|d\mu + e^{-k}\int_\Omega |h|\,d\mu_k \to 0\end{aligned}$$

as $k \to \infty$. Hence

$$\int_\Omega hq\,\mathrm{d}x = \int_\Omega \frac{h}{u}\,qu\,\mathrm{d}x = -\int_\Omega \frac{h}{u}\,\triangle u\,\mathrm{d}x = \int_\Omega \frac{h}{u}\,d\mu = \lim_{k\to\infty}\int_\Omega \frac{h}{u_k}\,d\mu_k.$$

Notice that $\nabla v_k = \nabla v$ a.e. on $\{v < k\}$, and $\nabla v_k = 0$ a.e. on $\{v \geqslant k\}$ (see [23, Corollary 1.43]). Hence,

$$\lim_{k\to\infty} \int_\Omega \nabla h \cdot \nabla v_k \,\mathrm{d}x = \int_\Omega \nabla h \cdot \nabla v \,\mathrm{d}x,$$
$$\lim_{k\to\infty} \int_\Omega |\nabla v_k|^2\, h \,\mathrm{d}x = \int_\Omega |\nabla v|^2\, h \,\mathrm{d}x$$

by the dominated convergence theorem. Passing to the limit as $k \to \infty$ in the equation above, we obtain

$$\int_\Omega hq \,\mathrm{d}x = \int_\Omega \nabla h \cdot \nabla v \,\mathrm{d}x - \int_\Omega |\nabla v|^2\, h \,\mathrm{d}x = -\int_\Omega v \triangle h \,\mathrm{d}x - \int_\Omega |\nabla v|^2\, h \,\mathrm{d}x,$$

which justifies equation (4.2).

The Riesz decomposition theorem states that a superharmonic function w can be written uniquely as $G(-\triangle w) + g$, where $-\triangle w$, understood in the distributional sense, is called the Riesz measure associated with w and g is the greatest harmonic minorant of w (see [5, §4.4]). Hence

$$v = G(-\triangle v) + g = G(|\nabla v|^2 + q) + g, \tag{4.4}$$

where g is the greatest harmonic minorant of v. Since $v \geqslant 0$, a harmonic minorant of v is 0, so $g \geqslant 0$. It follows from (4.4) and $u = G(uq) + 1$ that

$$g \leqslant v = \log u = \log\left(G(uq) + 1\right) \leqslant G(uq).$$

Since $G(uq)$ is a Green potential, the greatest harmonic minorant of $G(uq)$ is 0, therefore $g = 0$.

Hence we have $v = G(|\nabla v|^2 + q)$, which we have noted (see (4.1)) is equivalent to v being a very weak solution of (1.20).

Conversely, suppose $v \in W^{1,2}_{loc}(\Omega)$ is a very weak solution of equation (1.20), that is, $v = G(|\nabla v|^2 + q)$. Then $v \geqslant 0$. Let $v_k = \min(v, k)$ and $\nu_k = -\Delta v_k$, for $k = 1, 2, \dots$. Then $v_k \in W^{1,2}_{loc}(\Omega) \bigcap L^\infty(\Omega)$ is superharmonic, and

$$-\Delta v_k = |\nabla v_k|^2 + q\, \chi_{\{v<k\}} + \tilde{\nu}_k, \tag{4.5}$$

where $\tilde{\nu}_k$ is a nonnegative measure in Ω supported on $\{v = k\}$.

Let $u = e^v \geqslant 1$. Let $u_k = e^{v_k}$ and $\mu_k = -\Delta u_k$. Since $u_k \in W^{1,2}_{loc}(\Omega) \bigcap L^\infty(\Omega)$, it is easy to see that

$$\mu_k = -\Delta u_k = -\Delta v_k\, e^{v_k} - |\nabla v_k|^2\, e^{v_k} \geqslant 0. \tag{4.6}$$

Equation (4.6) is justified by using integration by parts (4.3) with $g = he^{v_k}$ where $h \in C_0^\infty(\Omega)$, and v_k in place of r:

$$\begin{aligned}\int_\Omega h\, e^{v_k}\, d\nu_k &= \int_\Omega \nabla(h\, e^{v_k}) \cdot \nabla v_k \,\mathrm{d}x \\ &= \int_\Omega e^{v_k}\, \nabla h \cdot \nabla v_k \,\mathrm{d}x + \int_\Omega h\, |\nabla v_k|^2\, e^{v_k} \,\mathrm{d}x \\ &= \int_\Omega \nabla h \cdot \nabla u_k \,\mathrm{d}x + \int_\Omega h\, |\nabla v_k|^2\, e^{v_k} \,\mathrm{d}x \\ &= \int_\Omega h\, d\mu_k + \int_\Omega h\, |\nabla v_k|^2\, e^{v_k} \,\mathrm{d}x.\end{aligned}$$

Hence,

$$\begin{aligned}\langle h, \mu_k\rangle = \int_\Omega \nabla h \cdot \nabla(e^{v_k}) \,\mathrm{d}x &= \int_\Omega e^{v_k} \nabla h \cdot \nabla v_k \,\mathrm{d}x \\ &= \int_\Omega h\, e^{v_k} \,\mathrm{d}\nu_k - \int_\Omega h\, |\nabla v_k|^2\, e^{v_k} \,\mathrm{d}x \\ &= \int_\Omega h\, e^{v_k}\ \chi_{\{v<k\}} q \,\mathrm{d}x + \int_\Omega h\, e^{v_k} \,\mathrm{d}\tilde{\nu}_k,\end{aligned}$$

where in the last expression we used (4.5). From the preceding estimates it follows that $\langle h, \mu_k\rangle \geqslant 0$ if $h \geqslant 0$, and consequently u_k is superharmonic, and

$$-\Delta u_k \geqslant q u_k\, \chi_{\{u_k < e^k\}}. \tag{4.7}$$

Clearly, $u = e^v < +\infty$-a.e., and $u = \lim_{k\to+\infty} u_k$ is superharmonic in Ω as the limit of the increasing sequence of superharmonic functions u_k. Since $\mu_k \to \mu$ in the sense of measures, where $\mu = -\triangle u$, (4.7) yields

$$-\Delta u \geqslant qu \quad \text{in } \Omega \tag{4.8}$$

in the sense of measures, where $qu \in L^1_{loc}(\Omega)$.

It follows from (4.8) that $\omega = -\Delta u - qu$ is a non-negative measure in Ω, so by the Riesz decomposition theorem

$$u = G(-\Delta u) + g = G(qu) + G\omega + g \geqslant G(qu) + g,$$

where g is the greatest harmonic minorant of u. Since $u \geqslant 1$, i.e., 1 is a harmonic minorant of u, it follows that $g \geqslant 1$, and consequently,

$$u \geqslant G(qu) + 1 = Tu + 1, \tag{4.9}$$

for T defined by (3.1). Since $u \geqslant Tu$, it follows by Schur's test that $\|T\|_{L^2(\Omega, q\,\mathrm{d}x)\to L^2(\Omega, q\,\mathrm{d}x)} \leqslant 1$, and hence (1.7) holds with $\beta = 1$ by Lemma 3.1.

Iterating (4.9) and taking the limit, we see that

$$\phi \equiv 1 + \mathcal{G}q = 1 + \sum_{j=1}^{\infty} G_j q = 1 + \sum_{j=1}^{\infty} T^j 1 \leqslant u < +\infty \quad \text{a.e.},$$

and

$$\phi = G(q\phi) + 1.$$

Hence ϕ is a positive very weak solution of (1.2). Thus (1.10) holds, by Theorem 1.2(2). □

Remarks 4.3.

1. — We remark in conclusion that the main results of this paper remain valid for any elliptic operator $\mathcal{L}$ whose Green's function $G^{\mathcal{L}}$ is equivalent to the Green's function G of the Laplacian (see [4]).

2. — Our main results also hold for general locally finite Borel measures ω in Ω in place of $q \in L^1_{loc}(\Omega)$, with minor adjustments in the proofs. Notice that condition (1.7) for ω in place of $q\,\mathrm{d}x$ implies that ω is absolutely continuous with respect to capacity (see [25]), and all solutions considered in this paper are superharmonic, i.e., finite quasi-everywhere in Ω; moreover, they actually lie in $W^{1,2}_{loc}(\Omega)$ (see [21, Theorem 6.2]).

3. — Concerning Theorem 1.2 and Theorem 1.5, suppose (1.7) holds with $\beta < 1$. Then a necessary and sufficient condition in order that $w = u_1 - 1 \in L^{1,2}_0(\Omega)$ is $\int_\Omega Gq\, q\,\mathrm{d}x < \infty$, i.e., $q \in L^{-1,2}(\Omega)$.

The sufficiency part of the last statement follows from the Lax–Milgram Lemma since $-\triangle w = qw + q$ where $q \in L^{-1,2}(\Omega)$; necessity is a consequence of the fact that $w = G(wq+q)$, so that

$$\begin{aligned}\int_\Omega |\nabla w|^2\,\mathrm{d}x &= \int_\Omega |\nabla G(wq+q)|^2\,\mathrm{d}x \\ &= \int_\Omega G(wq+q)\,(wq+q)\,\mathrm{d}x \geqslant \int_\Omega Gq\,q\,\mathrm{d}x.\end{aligned}$$

In particular, if (1.7) holds, and $q \in L^1(\Omega)$, then for all $h \in C^\infty_0(\Omega)$,

$$\begin{aligned}\left|\int_\Omega h\,q\,\mathrm{d}x\right| &\leqslant \Big(\int_\Omega h^2\,q\,\mathrm{d}x\Big)^{1/2} \|q\|^{1/2}_{L^1(\Omega)} \\ &\leqslant \beta^{1/2}\|\nabla h\|_{L^2(\Omega)}\|q\|^{1/2}_{L^1(\Omega)} < \infty.\end{aligned}$$

Hence, by duality $q \in L^{-1,2}(\Omega)$, and consequently $w = u_1 - 1 \in L^{1,2}_0(\Omega)$, for all $n \geqslant 2$ (see also [16], [1], [2]).

This also gives a weak solution $v \in L^{1,2}_0(\Omega)$ to (1.20) such that $e^v - 1 \in L^{1,2}_0(\Omega)$, as in [9], [11], if $q \geqslant 0$. For arbitrary distributions $q \in L^{-1,2}(\Omega)$, $w = u_1 - 1 \in L^{1,2}_0(\Omega)$ and $v = \log u_1 \in L^{1,2}_0(\Omega)$ is a weak solution to (1.20), provided q is form bounded with the upper form bound strictly less than 1 (see [21]).

Remarks 4.4.

1. — Concerning q which may change sign, our sufficiency results obviously hold if q is replaced with $|q|$ both in the spectral conditions (1.7), (1.15) and the balayage condition (1.10), and (1.16). The same is true as well for the upper estimates of solutions u_0 and u_1 given by (1.8), (1.12), (1.18).

2. — The lower estimates of solutions (1.9), (1.14), (1.18) are still true for q which may change sign, under some additional assumptions (see [15]). However, the upper pointwise estimates (1.8), (1.12), (1.18) are no longer true in general, unless we replace q with $|q|$.

3. — It is still unclear under which (precise) additional assumptions on the quadratic form of q the main existence results and upper estimates of solutions remain valid. Some results of this type are discussed in [21], but the prescribed boundary conditions considered in the present paper make the situation more complicated.

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