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Wenshuai JIANG, Feng WANG & Xiaohua ZHU

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BERGMAN KERNELS FOR A SEQUENCE OF ALMOST KÄHLER–RICCI SOLITONS

by Wenshuai JIANG, Feng WANG & Xiaohua ZHU (*)

Abstract. — In this paper, we prove the partial C^0-estimate conjecture of Tian for an almost Kähler–Einstein metrics sequence of Fano manifolds, or more general, an almost Kähler–Ricci solitons sequence. This generalizes Donaldson–Sun–Tian's result for a Kähler–Einstein metrics sequence of Fano manifolds. As an application, we prove that the Gromov–Hausdorff limit of sequence is homeomorphic to a log terminal Q-Fano variety which admits a Kähler–Ricci soliton on its smooth part.

Résumé. — Dans ce papier, nous montrons une conjecture due à Tian concernant une estimation C^0 partielle pour une suite de métriques de Kähler–Einstein tordues sur les variétés de Fano, ou plus généralement, pour une suite des solitons de Kähler–Ricci tordus. Ceci généralise les résultats de Donaldson–Sun–Tian pour une suite de métriques de Kähler–Einstein sur les variétés de Fano. Comme application, nous démontrons que la limite de Gromov–Hausdorff de la suite est homéomorphe à une variété de Q-Fano à singularités log terminales qui admet un soliton de Kähler–Ricci sur sa partie régulière.

1. Introduction

Let M^n be an n-dimensional Fano manifold and g a Kähler metric of M with its Kähler form ω_g in $2\pi c_1(M)$. Then g induces a hermitian metric h of the anti-canonical line bundle K_M^{-1} such that $\Theta(K_M^{-1}, h) = \omega_g$. Also h induces a hermitian metric (for simplicity, we still use the notation h) of l-multiple line bundle K_M^{-l}. As usual, the L^2-inner product on $H^0(M, K_M^{-l})$ is given by

$$(s_1, s_2) = \int_M \langle s_1, s_2\rangle_h \,\mathrm{dv}_g, \quad \forall\, s_1, s_2 \in H^0(M, K_M^{-l}). \tag{1.1}$$

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Choosing a unit orthogonal basis $\{s_i\}$ of $H^0(M, K_M^{-l})$ with respect to the inner product $(\,\cdot\,,\,\cdot\,)$ in (1.1), we define the Bergman kernel of (M, K_M^{-l}, h) by

$$\rho_l(x) = \Sigma_i |s_i|_h^2(x).$$

Clearly, $\rho_l(x)$ is independent of the choice of basis $\{s_i\}$. In [22], Tian proposed a conjecture for the existence of uniformly positive lower bound of $\rho_l(x)$:

Conjecture 1.1. — *Let $\{(M_i, g^i)\}$ be a Kähler–Einstein metrics sequence of Fano manifolds of n-dimension with constant scalar curvature n. Then there exists an integer number l_0 such that for any integer $l > 0$ there exists a uniform constant $c_l > 0$ with property:*

$$\rho_{ll_0}(M_i, g^i) \geqslant c_l. \tag{1.2}$$

Here c_l depends only on l, n.

The above conjecture was recently solved by Donaldson–Sun [9] and Tian [23], independently. The estimate (1.2) is usually called the partial C^0-estimate. Very recently, (1.2) is generalized to a sequence of conical Kähler–Einstein metrics by Tian [25]. This estimate plays a crucial role in his solution of the famous YTD conjecture for the existence problem of Kähler–Einstein metrics with positive scalar curvature. The YTD conjecture is also solved by Chen–Donaldson–Sun independently [6].

Theorem 1.2 (Tian, Chen–Donaldson–Sun). — *A Fano manifold admits a Kähler–Einstein metric if and only if it is K-stable.*

The notion of K-stability was first introduced by Tian [21] and it was reformulated by Donaldson in terms of test-configurations [8].

There are several generalization of (1.2) after the work by Donaldson–Sun and Tian. For examples, Phong–Song–Strum extended (1.2) to a sequence of Kähler–Ricci solitons [19] and Jiang extended (1.2) to a sequence of Kähler metrics on Fano manifolds of 3-dimension with uniformly lower bound of Ricci curvature and other uniformly geometric qauntities [13]. In present paper, we want to generalize the estimate (1.2) to an almost Kähler–Einstein metrics sequence on Fano manifolds, or more general, an almost Kähler–Ricci solitons sequence (see Definitions 3.5, 7.3). We prove

Theorem 1.3. — *Let $\{(M_i, g^i)\}$ be an almost Kähler–Einstein metrics (or an almost Kähler–Ricci solitons) sequence of Fano manifolds of dimension $n \geqslant 2$. Then there exists an integer number l_0 such that for any integer $l > 0$ there exists a uniformly constant $c_l > 0$ with property:*

$$\rho_{ll_0}(M_i, g^i) \geqslant c_l. \tag{1.3}$$

Here the constant c_l depends only on l, n, and some uniform geometric constants (cf. Section 9).

As an application of Theorem 1.3 together with the main results in [30] and [31], we prove

Corollary 1.4. — *Let $\{(M_i, g^i)\}$ be an almost Kähler–Einstein metrics (or an almost Kähler–Ricci solitons) sequence of Fano manifolds of dimension $n \geqslant 2$. Then $\{(M_i, g^i)\}$ converges subsequently to a metric space (M_∞, g_∞) in Gromov–Hausdorff topology with properties:*

- *(i) The real codimension of singularities of (M_∞, g_∞) is at least 4;*
- *(ii) g_∞ is a Kähler–Einstein metric (or a Kähler–Ricci soliton) on the regular part of M_∞;*
- *(iii) M_∞ is homeomorphic to a log terminal Q-Fano variety.*

In case of Kähler–Einstein metrics sequence with positive scalar curvature, (i) and (ii) in Corollary 1.4 follow from the Cheeger–Colding–Tian compactness theorem [5]. Donaldson–Sun proved the part (iii) [9] (also see [14]). Note that any Q-Fano variety, which admits a Kähler–Einstein metric, is automatically log terminal according to Proposition 3.8 in [1].(1)

There are important examples of almost Kähler–Einstein metrics sequence and almost Kähler–Ricci solitons sequence:

(1) Tian and Wang constructed a sequence of almost Kähler–Einstein metrics arising from solutions of certain complex Monge–Ampère equations on a Fano manifold with the Mabuchi's K-energy bounded below [26].

(2) Tian constructed a sequence of almost Kähler–Einstein metrics modified from conical Kähler–Einstein metrics on a Fano manifold with cone angles going to 2π [25].

(3) Wang and Zhu constructed a sequence of almost Kähler–Ricci solitons arising from solutions of certain complex Monge–Ampère equations on a Fano manifold with the modified K-energy bounded below [30], [31].

Thus Theorem 1.3 and Corollary 1.4 hold for these examples. In particular, we give an alternative proof for Tian's result of (1.3) for conical Kähler–Einstein metrics sequence with cone angles going to 2π [25].

(1) The result also holds for a Q-Fano variety, which admits a Kähler–Ricci soliton, according to the proof of Proposition 3.8.

Remarks 1.5.

(1) Li proved recently that the lower boundedness of K-energy is equivalent to the K-semistablity [14], althought his proof depended on the construction of test-configurations from the work of Tian, Chen–Donaldson–Sun to the proof of Theorem 1.2. It is reasonable to believe that there is an analogy of Li's result to describe the modified K-energy in sense of modified K-semistability (cf. [33], [2], [29]).

(2) If there is a new proof for Li's result, Theorem 1.3 for example 1) will give an alternative proof to Theorem 1.2 (cf. [25], [24], [17]).

At last we describe the proof of Theorem 1.3 briefly. As in [9] (or [25], [23]), the main idea is to construct locally nontrivial almost holomorphic sections over the sequence $\{(M_i, g^i)\}$ by using the rescaling method, then to get global holomorphic sections by solving $\overline{\partial}$-equation. Our difficulty is lack of locally strong convergence of $\{(M_i, g^i)\}$. To overcome it, we use Ricci flow to smooth $\{(M_i, g^i)\}$ locally to approximate them as done in [26], [31]. Although the approximation of $\{(M_i, g^i)\}$ is local and depends on the time t in Ricci flows, the approximated metrics are locally convergent as long as t is fixed. With difference to [9], we construct locally nontrivial almost holomorphic sections and solve $\overline{\partial}$-equation with respect to the approximated metrics, not to the original metrics $\{(M_i, g^i)\}$, see Proposition 5.1, Section 5. Since the proof of Proposition 5.1 will depend on the gradient estimate of holomorphic sections (cf. Lemma 3.1, Proposition 3.6, Proposition 7.5), we shall control scalar curvatures and gradients of Ricci potentials along Ricci flows by using the Moser iteration method (cf. Proposition 2.3, Lemma 7.4). Once Proposition 5.1 is available, we are able to estimate holomorphic sections with respect to $\{(M_i, g^i)\}$ (cf. Sections 6, 7). The technique used is to compare the hermitian metrics and L^2-norm of holomorphic sections between the approximated metrics and the original metrics (cf. Lemma 6.2, Lemma 7.7).

The organization of paper is as follows. In Section 2, we give some estimates for scalar curvatures and Ricci potentials along the Ricci flow, then, in Section 3, we use them to give the C^0-estimate and the gradient estimate for holomorphic sections on K_M^{-l}. Section 4 is devoted to construct nontrivial almost holomorphic sections by using the trivial bundle on the tangent cone. The nontrivial holomorphic sections, which depend on time t, will be constructed in Section 5. Theorem 1.3 will be proved in Sections 6, 7, according to almost Kähler–Einstein metrics and almost Kähler–Ricci solitons, respectively, while its proof is completed in Section 9. In Section 8, we prove Corollary 1.4.

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2. Estimates from Kähler Ricci flow

In this section, we give some necessary estimates for the scalar curvatures and Ricci potentials along the Kähler–Ricci flow. Let M be an n-dimensional Fano manifold and g a Kähler metric of M with its Kähler form ω_g in $2\pi c_1(M)$. Let $g_t = g(\cdot, t)$ be a solution of normalized Kähler Ricci flow,

$$\begin{cases} \frac{\partial}{\partial t} g = -\operatorname{Ric}(g) + g, \\ g_0 = g(\,\cdot\,, 0) = g. \end{cases} \tag{2.1}$$

Recall Zhang's estimate for Sobolev constants of g_t [35].

Lemma 2.1. — *Let g_t be the solution of* (2.1)*. Suppose that there exists a Sobolev constant C_s of g such that the following inequality holds,*

$$\left(\int_M f^{\frac{2n}{n-1}} \,\mathrm{dv}_g\right)^{\frac{n-1}{n}} \leqslant C_s \left(\int_M f^2 \,\mathrm{dv}_g + \int_M |\nabla f|^2 \,\mathrm{dv}_g\right), \ \ \forall f \in C^1(M). \tag{2.2}$$

Then there exist two uniform constants $A = A(C_s, -\inf_M R(g), V)$ and $C_0 = C_0(C_s, -\inf_M R(g), V)$ such that for any $f \in C^1(M)$ it holds

$$\left(\int_M f^{\frac{2n}{n-1}} \,\mathrm{dv}_{g_t}\right)^{\frac{n-1}{n}} \leqslant A \int_M (|\nabla f|^2 + (R_t + C_0) f^2) \,\mathrm{dv}_{g_t}, \tag{2.3}$$

where R_t are scalar curvatures of g_t.

By using the Moser iteration, we have

Lemma 2.2. — *Let $\Delta = \Delta_t$ be Lapalace operators associated to g_t. Suppose that $f \geqslant 0$ satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leqslant af, \quad \forall\, t \in (0,1), \tag{2.4}$$

where $a \geqslant 0$ is a constant. Then for any $t \in (0,1)$, it holds

$$\sup_{x \in M} f(x,t) \leqslant \frac{C}{t^{\frac{n+1}{p}}} \left(\int_{\frac{t}{2}}^t \int_M |f(x,\tau)|^p \,\mathrm{dv}_{g_\tau} dtau\right)^{\frac{1}{p}}, \tag{2.5}$$

where $C = C(a, p, C_s, -\inf R(g), V)$, $p \geqslant 1$ *and* C_s *is the Sobolev constant of* g *in* (2.2).

Proof. — Multiplying both sides of (2.4) by f^p, we have

$$\int_M f^p f'_\tau \,\mathrm{dv}_{g_\tau} - \int_M f^p \Delta f \,\mathrm{dv}_{g_\tau} \leqslant a \int_M f^{p+1} \,\mathrm{dv}_{g_\tau}.$$

Taking integration by parts, we get

$$\frac{1}{p+1}\int_M (f^{p+1})'_\tau \,\mathrm{dv}_{g_\tau} + \frac{4p}{(p+1)^2}\int_M |\nabla f^{\frac{p+1}{2}}|^2 \,\mathrm{dv}_{g_\tau} \leqslant a \int_M f^{p+1} \,\mathrm{dv}_{g_\tau}.$$

Using the relation

$$\frac{d}{\mathrm{d}\tau}\int_M f^{p+1} \,\mathrm{dv}_{g_\tau} = \int_M (f^{p+1})'_\tau \,\mathrm{dv}_{g_\tau} + \int_M f^{p+1}(n-R) \,\mathrm{dv}_{g_\tau},$$

It follows

$$\begin{aligned} \frac{1}{p+1}\frac{d}{\mathrm{d}\tau}\int_M f^{p+1} \,\mathrm{dv}_{g_\tau} + \frac{1}{p+1}\int_M R f^{p+1} \,\mathrm{dv}_{g_\tau} & \\ + \frac{4p}{(p+1)^2}\int_M |\nabla f^{\frac{p+1}{2}}|^2 \,\mathrm{dv}_{g_\tau} & \\ \leqslant (a + \frac{n}{p+1}) \int_M f^{p+1} \,\mathrm{dv}_{g_\tau}. & \end{aligned}$$

Thus

$$\begin{aligned} \frac{d}{\mathrm{d}\tau}\int_M f^{p+1} \,\mathrm{dv}_{g_\tau} + \int_M (R + C_0) f^{p+1} \,\mathrm{dv}_{g_\tau} + 2\int_M |\nabla f^{\frac{p+1}{2}}|^2 \,\mathrm{dv}_{g_\tau} & \\ \leqslant ((p+1)a + n + C_0) \int_M f^{p+1} \,\mathrm{dv}_{g_\tau} \,\mathrm{dv}_{g_\tau}. & \end{aligned} \tag{2.6}$$

For any $0 \leqslant \sigma' \leqslant \sigma \leqslant 1$, we define

$$\psi(\tau) = \begin{cases} 0, \tau \leqslant \sigma' t \\ \frac{\tau - \sigma' t}{(\sigma - \sigma')t}, \sigma' t \leqslant \tau \leqslant \sigma t \\ 1, \sigma t \leqslant \tau \leqslant t. \end{cases}$$

Then by (2.6), we have

$$\begin{aligned} \frac{d}{\mathrm{d}\tau}(\psi \int_M f^{p+1} \,\mathrm{dv}_{g_\tau}) + \psi \int_M \left[(R + C_0) f^{p+1} + 2|\nabla f^{\frac{p+1}{2}}|^2 \right] \mathrm{dv}_{g_\tau} & \\ \leqslant [\psi((p+1)a + n + C_0) + \psi'] \int_M f^{p+1} \,\mathrm{dv}_{g_\tau}. & \end{aligned}$$

It follows

$$\sup_{\sigma t\leqslant\tau\leqslant t}\int_M f^{p+1}\,\mathrm{dv}_{g_\tau}+\int_{\sigma t}^t\int_M\left[(R+C_0)f^{p+1}+2|\nabla f^{\frac{p+1}{2}}|^2\right]\mathrm{dv}_{g_\tau}$$
$$\leqslant((p+1)a+n+C_0+\frac{1}{(\sigma-\sigma')t})\int_{\sigma' t}^t\int_M f^{p+1}\,\mathrm{dv}_{g_\tau}.$$

Thus by Lemma 2.1, we get

$$\int_{\sigma t}^t\int_M f^{(p+1)(1+\frac{1}{n})}\,\mathrm{dv}_{g_\tau}$$
$$\leqslant\int_{\sigma t}^t\int_M(f^{p+1}\,\mathrm{dv}_{g_\tau})^{\frac{1}{n}}\left(\int_M f^{(p+1)\frac{n}{n-1}}\,\mathrm{dv}_{g_\tau}\right)^{\frac{n-1}{n}}$$
$$\leqslant\left(\sup_{\sigma t\leqslant\tau\leqslant t}\int_M f^{p+1}\,\mathrm{dv}_{g_\tau}\right)^{\frac{1}{n}}\int_{\sigma t}^t A\int_M\left[(R+C_0)f^{p+1}+2\left|\nabla f^{\frac{p+1}{2}}\right|^2\right]\mathrm{dv}_{g_\tau}$$
$$\leqslant A\left((p+1)a+n+C_0+\frac{1}{(\sigma-\sigma')t}\right)^{\frac{n+1}{n}}\left(\int_{\sigma' t}^t\int_M f^{p+1}\,\mathrm{dv}_{g_\tau}\right)^{\frac{n+1}{n}}.$$

By choosing $\sigma'=\frac{1}{2}+\frac{1}{4}\sigma_k,\sigma=\frac{1}{2}+\frac{1}{4}\sigma_{k+1}$, where $\sigma_k=\sum_{l=0}^k(\frac{1}{2})^l-1$, and replacing p by $p_{k+1}=(p_k+1)^{\frac{n+1}{n}}-1$ with $p_0=p\geqslant 0$ in the above inequalty, then iterating k we will get the desired estimate (2.5). □

By Lemma 2.2, we prove

Proposition 2.3. — *Let $u=u_t$ and $R=R_t$ be Ricci potentials and scalar curvatures of solutions g_t in* (2.1), *respectively. Suppose that (M,g) satisfies*

$$\mathrm{Ric}(g)\geqslant-\Lambda^2 g\quad\text{and}\quad\mathrm{diam}(M,g)\leqslant D.\tag{2.7}$$

Then there exists a constant $C(n,\Lambda,D)$ such that

$$|\nabla u|^2(x,t)\leqslant\frac{C}{t^{(n+1)(n+\frac{3}{2})}}\int_{\frac{t}{2}}^t\int_M|R-n|\,\mathrm{dv}_{g_\tau}\tag{2.8}$$

and

$$|R-n|(x,t)\leqslant\frac{C}{t^{(n+1)(n+\frac{7}{2})+n}}\int_{\frac{t}{2}}^t\int_M|R-n|\,\mathrm{dv}_{g_\tau},\quad\forall\ 0<t\leqslant 1.\tag{2.9}$$

Proof. — By a direct computation, we have the the following evolution formulas for $|\nabla u|$ and R, respectively,

$$\left(\frac{\partial}{\partial t}-\Delta\right)|\nabla u|^2=-|\nabla\nabla u|^2-|\nabla\bar{\nabla}u|^2+|\nabla u|^2\leqslant|\nabla u|^2\tag{2.10}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right) R = R - n + |\mathrm{Ric}(g) - g|^2. \tag{2.11}$$

It follows

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(R + n\Lambda + |\nabla u|^2) &= R - n - |\nabla\nabla u|^2 + |\nabla u|^2 \\ &\leqslant R + n\Lambda + |\nabla u|^2.\end{aligned} \tag{2.12}$$

Note that by (2.11) and the maximum principle, $R(g_t) + n\Lambda \geqslant 0$. Moreover, it was showed in [13] that there exists a uniform constant $C = C(\Lambda, D)$ such that

$$\int_0^1 \int_M (R + n\Lambda + |\nabla u|^2)\,\mathrm{dv}_g\,\mathrm{d}t \leqslant C.$$

Thus applying Lemma 2.2 to (2.12), we get

$$(R + n\Lambda + |\nabla u|^2)(x,t) \leqslant \frac{C}{t^{n+1}}. \tag{2.13}$$

In particular,

$$|\nabla u|^2(x,t) \leqslant \frac{C}{t^{n+1}} \quad \text{and} \quad R \leqslant \frac{C}{t^{n+1}}. \tag{2.14}$$

Next we estimate the C^0-norm of u_t. By Lemma 2.1 together with (2.14), we have the Sobolev inequality,

$$\begin{aligned}\left(\int_M f^{\frac{2n}{n-1}}\,\mathrm{dv}_{g_t}\right)^{\frac{n-1}{n}} &\leqslant A \int_M (|\nabla f|^2 + (R(x,t) + C_0) f^2)\,\mathrm{dv}_{g_t} \\ &\leqslant A \int_M \left(|\nabla f|^2 + \frac{C}{t^{n+1}} f^2\right)\mathrm{dv}_{g_t}.\end{aligned}$$

The inequality implies (cf. [12], [34]),

$$\mathrm{vol}(B(x,1)) \geqslant C t^{n(n+1)}, \quad \forall\, x \in M.$$

Since $\mathrm{vol}(M) = V$, it is easy to derive

$$\mathrm{diam}(M, g_t) \leqslant \frac{V}{C t^{n(n+1)}}.$$

Thus by (2.14), we obtain

$$\mathrm{osc}_M\, u(x,t) \leqslant \frac{C}{t^{(n+1)(n+\frac{1}{2})}}. \tag{2.15}$$

By (2.15), we can improve (2.14) to (2.8). In fact, by applying Lemma 2.2 to (2.10), we have

$$\begin{aligned}
|\nabla u|^2(x,t) &\leqslant \frac{C}{t^{n+1}}\int_{\frac{t}{2}}^{t}\int_M |\nabla u|^2 \,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau \\
&= \frac{C}{t^{n+1}}\int_{\frac{t}{2}}^{t}\int_M -u\Delta u \,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau \\
&\leqslant \frac{C}{t^{n+1}}\operatorname{osc}_{(x,\tau)\in M\times[\frac{t}{2},t]}|u|(x,\tau)\int_{\frac{t}{2}}^{t}\int_M |R-n| \,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau \\
&\leqslant \frac{C'}{t^{(n+1)(n+\frac{3}{2})}}\int_{\frac{t}{2}}^{t}\int_M |R-n| \,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau, \qquad (2.16)
\end{aligned}$$

where the constant C' depends only on $n,\ \Lambda,\ D$. This proves (2.8).

To get (2.9), we use the evolution equation as same as (2.12),

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)(|\nabla u|^2+R-n) &= R-n-|\nabla\nabla u|^2+|\nabla u|^2 \\
&\leqslant |\nabla u|^2+R-n.
\end{aligned}$$

Then applying Lemma 2.2, we see

$$\begin{aligned}
(|\nabla u|^2+R-n)_+ &\leqslant \frac{C}{t^{n+1}}\int_{\frac{t}{2}}^{t}\int_M \left||\nabla u|^2+R-n\right| \,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau \\
&\leqslant \frac{C}{t^{(n+1)(n+\frac{5}{2})}}\int_{\frac{t}{2}}^{t}\int_M |R-n| \,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau.
\end{aligned}$$

Here we have used (2.16) in the last inequality. Thus by (2.16) again, we have

$$(R-n)_+(t) \leqslant \frac{C}{t^{(n+1)(n+\frac{5}{2})}}\int_{\frac{t}{2}}^{t}\int_M |R-n| \,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau. \qquad (2.17)$$

In fact, from the proof of Lemma 2.2, this holds for all $\tau\in[2t/3,t]$, i.e.,

$$(R-n)_+(\tau) \leqslant \frac{C_0}{t^{(n+1)(n+\frac{5}{2})}}\int_{\frac{t}{2}}^{t}\int_M |R-n|\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau := A(t).$$

On the other hand, by the evolution equation (2.11) of R, we have

$$\left(\frac{\partial}{\partial \tau}-\Delta_\tau\right)(A(t)+n-R) \leqslant A(t)+n-R, \quad \tau\in[2t/3,t].$$

Note that $A(t)+n-R(\tau)\geqslant 0$ for all $\tau\in[2t/3,t]$. Hence applying Lemma 2.2 again, we get

$$\begin{aligned}(A(t)+n-R)(x,t) &\leqslant \frac{C''}{t^{n+1}}\int_{\frac{2t}{3}}^{t}\int_M (A(t)+n-R)\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau\\ &\leqslant \frac{C''}{t^{n+1}}\int_{\frac{2t}{3}}^{t}\int_M |n-R|\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau+\frac{A(t)VC}{t^n}.\end{aligned}$$

Therefore, combining the above inequality with (2.17), we obtain (2.9). ☐

3. Estimates for holomorphic sections

In this section, we use the estimates in Section 2 to give the C^0-estimate and the gradient estimate for holomorphic sections with respect to g_t. Let (M,g) be a Kähler metric as in Section 2 and $L=K_M^{-1}$ its anti-canonical line bundle with induced Hermitian metric h by g. In the rest of paper, we always use notations $\|\cdot\|_g$ and $\|\cdot\|_h$ to denote the L^∞-norm. We begin with the following lemma.

Lemma 3.1. — *Suppose that the Ricci potential u of g satisfies*

$$\|\nabla u\|_g\leqslant 1. \tag{3.1}$$

Then for $s\in H^0(M,L^l)$ we have

$$\|s\|_h+l^{-\frac12}\|\nabla s\|_h\leqslant C(C_s,n)l^{\frac n2}\left(\int_M |s|^2\,\mathrm{dv}_g\right)^{\frac12}, \tag{3.2}$$

where C_s is the Sobolev constant of (M,g).

Proof. — Note that

$$\Delta|s|_h^2=|\nabla s|_h^2-nl|s|_h^2.$$

It follows

$$-\Delta|s|_h^2\leqslant nl|s|_h^2. \tag{3.3}$$

Thus applying the standard Moser iteration method, we get

$$\|s\|_h\leqslant C(C_s,n)l^{\frac n2}\left(\int_M |s|^2\,\mathrm{dv}_g\right)^{\frac12}. \tag{3.4}$$

On the other hand, we have the following Bochner formula,

$$\Delta|\nabla s|_h^2=|\nabla\nabla s|^2+|\bar\nabla\nabla s|^2-(n+2)l|\nabla s|^2+\langle \mathrm{Ric}(\nabla s,\cdot),\nabla s\rangle.$$

Then we can also apply the Moser iteration to obtain a L^∞-estimate for $|\nabla s|_h^2$ as done for $|s|_h^2$. In fact, it suffices to deal with the extra integral terms like $\langle \mathrm{Ric}(\nabla s, .), \nabla s\rangle |\nabla s|^{2p}$. But those terms can be controlled by the integral of $(|\nabla\nabla s|^2 + |\bar\nabla\nabla s|^2)|\nabla s|_h^{2p}$ by taking integral by parts with the help of the condition (3.1). As a consequence, we obtain

$$\begin{aligned}\|\nabla s\|_h &\leqslant C(C_s, n) l^{\frac{n}{2}} \left(\int_M |\nabla s|^2 \,\mathrm{dv}_g\right)^{\frac{1}{2}} \\ &\leqslant C(C_s, n) l^{\frac{n+1}{2}} \left(\int_M |s|^2 \,\mathrm{dv}_g\right)^{\frac{1}{2}}.\end{aligned} \tag{3.5}$$

Therefore, combining (3.4) and (3.5), we derive (3.6). □

Remark 3.2. — Using the same argument in Lemma 3.1, we can prove: If (M, g) satisfies

$$\mathrm{Ric}(\omega_g) \geqslant -\Lambda^2\omega_g + \sqrt{-1}\partial\bar\partial u,$$

for some u with $|\nabla u|_g \leqslant A$, then

$$\|s\|_h + l^{-\frac{1}{2}}\|\nabla s\|_h \leqslant C(C_s, A, \Lambda) l^{\frac{n}{2}} \left(\int_M |s|^2 \,\mathrm{dv}_g\right)^{\frac{1}{2}}, \ \forall\ s \in H^0(M, L^l). \tag{3.6}$$

Lemma 3.3. — *Let (M, g) be a Kähler metric as in Lemma 3.1. Let $l \geqslant 4n$. Then for any $\sigma \in C^\infty(\Gamma(M, L^l))$, there exists a solution $v \in C^\infty(\Gamma(M, L^l))$ such that $\bar\partial v = \bar\partial\sigma$ with property:*

$$\int_M |v|_h^2 \leqslant 4l^{-1} \int_M |\bar\partial\sigma|_h^2. \tag{3.7}$$

Proof. — The existence part comes from the Hömander L^2-theory. It suffice to verify (3.7), which is equivalent to prove that the first eigenvalue $\lambda_1(\bar\partial, L^l)$ of $\Delta_{\bar\partial}$ is greater than $\frac{l}{4}$, where $\Delta_{\bar\partial}$ denotes the Lapalce operator defined on $L^2(T^*M \bigotimes L^l)$.

Note that the following two identities hold for any $\theta \in \Omega^{0,1}(L^l)$,

$$-\Delta_{\bar\partial}\theta = \bar\nabla^*\bar\nabla\theta + \mathrm{Ric}(\theta, \cdot) + l\theta$$

and

$$-\Delta_{\bar\partial}\theta = \nabla^*\nabla\theta - (n-1)l\theta.$$

It follows

$$-\Delta_{\bar\partial}\theta = \left(1 - \frac{1}{2n}\right)\bar\nabla^*\bar\nabla + \left(1 - \frac{1}{2n}\right)\mathrm{Ric}(\theta, \cdot) + \frac{1}{2n}\nabla^*\nabla\theta + \frac{l}{2}\theta. \tag{3.8}$$

Then taking integration by parts, we have

$$
\begin{aligned}
-\int_M \langle \Delta_{\bar\partial}\theta, \theta\rangle &= \left(1-\frac{1}{2n}\right)\int_M |\bar\nabla\theta|^2 + \frac{1}{2n}\int_M |\nabla\theta|^2 + \frac{l}{2}\int_M |\theta|^2 \\
&\quad + \left(1-\frac{1}{2n}\right)\int_M (|\theta|^2 + \langle \nabla\bar\nabla u(\theta,\,\cdot\,), \theta\rangle) \\
&= \left(1-\frac{1}{2n}\right)\int_M |\bar\nabla\theta|^2 + \frac{1}{2n}\int_M |\nabla\theta|^2 + \frac{l}{2}\int_M |\theta|^2 \\
&\quad + \left(1-\frac{1}{2n}\right)\int_M |\theta|^2 - \left(1-\frac{1}{2n}\right)\int_M \langle \bar\nabla u, (\langle\nabla\theta,\theta\rangle + \theta, \bar\nabla\theta)\rangle.
\end{aligned}
$$

Using the condition (3.1), we get

$$
\begin{aligned}
&-\int_M \langle \Delta_{\bar\partial}\theta, \theta\rangle \\
&\qquad \geqslant \left(1-\frac{1}{2n}\right)\int_M |\bar\nabla\theta|^2 + \frac{1}{2n}\int_M |\nabla\theta|^2 + \frac{l}{2}\int_M |\theta|^2 \\
&\qquad\quad + \left(1-\frac{1}{2n}\right)\int_M |\theta|^2 - \left(1-\frac{1}{2n}\right)\int_M \left[\frac{1}{2n}(|\bar\nabla\theta|^2 + |\nabla\theta|^2) + n|\theta|^2\right] \\
&\qquad \geqslant \left(\frac{l}{2} - n\right)\int_M |\theta|^2. \qquad (3.9)
\end{aligned}
$$

Now we can choose $l \geqslant 4n$ to see that $\lambda_1(\bar\partial, L) \geqslant \frac{l}{4}$ as required. □

Remark 3.4. — If the upper bound of $|\nabla u|$ is replaced by a constant C, the coefficient at the last inequality in (3.9) will be $\frac{l}{2} - nC^2$. Then by choosing $l \geqslant 4nC^2$, one can also get (3.7). This was proved in [27].

Let us recall the definition of an almost Kähler–Einstein metrics sequence of Fano manifolds [26].

DEFINITION 3.5. — *We say that Kähler metrics g^i $(i \to \infty)$ on Fano manifolds M_i is a sequence of an almost Kähler–Einstein metrics if they satisfy:*

$$
\begin{aligned}
&\text{(i)}\ \operatorname{Ric}(g^i) \geqslant -\Lambda^2 g^i \ \textit{and}\ \operatorname{diam}(M_i, g^i) \leqslant D; \\
&\text{(ii)}\ \int_{M_i} |\operatorname{Ric}(g^i) - g^i|\,\mathrm{dv}_{g^i} \to 0; \qquad (3.10)\\
&\text{(iii)}\ \int_0^1\int_{M_i} |R(g^i_t) - n|\,\mathrm{dv}_{g^i_t} dt \to 0,\ \textit{as}\ i \to \infty.
\end{aligned}
$$

Here $\Lambda > 0$, $D > 0$ are two uniform constants, g^i are normalized so that $\omega_{g^i} \in 2\pi c_1(M_i)$ and g^i_t are the solutions of (2.1) *with the initial metrics g^i.*

We note that $\operatorname{vol}(M_i, g^i) = (2\pi)^n c_1(M_i)^n \geqslant V$ *for some uniform constant* V *by the normalization.*

Applying Lemma 3.1 and Lemma 3.3 to almost Kähler–Einstein metrics with the help of gradient estimate (2.8) in Proposition 2.3, we have the following proposition.

Proposition 3.6. — *Let* $\{(M_i, g^i)\}$ *be an almost Kähler–Einstein metrics sequence of Fano manifolds which satisfy* (3.10). *Then for any* $t \in (0,1)$ *there exists an integer* $N = N(t)$ *such that for any* $i \geqslant N$ *and* $l \geqslant 4n$ *it holds,*

$$\|s\|_{h_t^i} + l^{-\frac{1}{2}} \|\nabla s\|_{h_t^i} \leqslant C l^{\frac{n}{2}} \left(\int_M |s|^2 \,\mathrm{dv}_{g_t^i} \right)^{\frac{1}{2}}, \quad s \in H^0(M_i, K_{M_i}^{-l}), \tag{3.11}$$

and

$$\int_{M_i} |v|_{h_t^i}^2 \leqslant 4 l^{-1} \int_{M_i} |\bar\partial \sigma|^2. \tag{3.12}$$

Here v *solves* $\bar\partial v = \bar\partial \sigma$ *as in Lemma 3.3, the norms of* $|\cdot|_{h_t^i}$ *are induced by* g_t^i*, and* C *is a uniform constant independent of* t*.*

Proof. — A well-known result shows that the Sobolev constants C_s of $(M_i,\, g^i)$ depend only on the constants Λ, D and V (cf. [15]). Then by (2.8) in Proposition 2.3, for any $t \in (0,1)$, there exists $N = N(t)$ such that

$$\|\nabla u^i\|_{g_t^i} \leqslant 1, \quad \forall\, i \geqslant N,$$

where u^i are Ricci potentials of g_t^i . Thus we can apply Lemma 3.1 to get (3.11). Similarly, we can get (3.12) from Lemma 3.3. □

4. Construction of locally almost holomorphic sections

Let $\{(M_i, g^i)\}$ be an almost Kähler–Einstein metrics sequence of Fano manifolds as in Section 3 and (M_∞, g_∞) its Gromov–Hausdorff limit. It was proved by Tian and Wang that the regular part $\mathcal{R}$ of M_∞ is an open Kähler manifold and the real codimension of singularities of M_∞ is at least 4 [26]. Moreover, according to Proposition 5.1 in that paper, we have

Lemma 4.1. — *Let* $x \in M_\infty$*. Then there exist constants* $\epsilon = \epsilon(n)$ *and* $r_0 = r_0(n, C)$ *such that if* $\operatorname{vol}(B_x(r)) \geqslant (1-\epsilon)\omega_{2n} r^{2n}$ *for some* $r \leqslant r_0$*, then* $B_x(\frac{r}{2}) \subseteq \mathcal{R}$*,* $\operatorname{Ric}(g_\infty) = g_\infty$ *in* $B_x(\frac{r}{2})$*), and*

$$\|\nabla^l \operatorname{Rm}\|_{C^0(B_x(\frac{r}{2}))} \leqslant \frac{C}{r^{l+2}},$$

where the constant C depends only on l, and the constants Λ and D in (3.10).

Recall that a tangent cone C_x at $x \in M_\infty$ is a Gromov–Hausdorff limit defined by

$$(C_x, g_x, x) = \lim_{j\to\infty} \left(M_\infty, \frac{g_\infty}{r_j^2}, x\right), \tag{4.1}$$

where $\{r_j\}$ is some sequence which goes to 0. Without the loss of generality, we may assume that $l_j = \frac{1}{r_j^2}$ are integers. Since (C_x, g_x, x) is a metric cone, $g_x = \text{hess}\,\frac{\rho_x^2}{2}$, where $\rho_x = \text{dist}(x,\cdot)$ is a distance function staring from x in C_x.

Denote the regular part of (C_x, g_x, x) by $\mathcal{CR}$, which consists of points in C_x with flat cones. By Lemma 4.1, we get

Lemma 4.2. — *$\mathcal{CR}$ is an open Kähler–Ricci flat manifold. Moreover, for any compact set $K \subset \mathcal{CR}$, there exist a sequence of $(K_j \subset \mathcal{R}, \frac{1}{r_j^2} g_\infty)$ which converges to K in C^∞-topology.*

Proof. — Let ϵ be a small number chosen as in Lemma 4.1. Then for any $y \in \mathcal{CR}$, there exists some small r such that $\hat{B}_y(r) \subset C_x$ and

$$\text{vol}(\hat{B}_y(r)) \geqslant \left(1 - \frac{\epsilon}{2}\right) \omega_{2n} r^{2n}.$$

Thus there exists a sequence of $y_\alpha \in M_\infty$ such that

$$\text{vol}(B_{y_\alpha}(r r_\alpha)) \geqslant (1-\epsilon)\, \omega_{2n} (r r_\alpha)^{2n},$$

where the sequence $\{r_\alpha\}$ is chosen as in (4.1). By Lemma 4.1, it follows

$$\| \operatorname{Rm}(\tilde{g}_\infty)\|_{C^l(\tilde{B}_{y_\alpha}(\frac{r}{2}))} \leqslant \frac{C_l}{r^{l+2}},$$

where $\tilde{g}_\infty = \frac{g_\infty}{r_\alpha^2}$ and $\tilde{B}_{y_\alpha}(\frac{r}{2}) \subset M_\infty$ is a $\frac{r}{2}$-geodesic ball with respect to $\tilde{g}_\infty$. Hence, by the Cheeger–Gromov convergence theorem [11], $(\tilde{B}_{y_\alpha}(\frac{r}{2}), \tilde{g}_\infty)$ converge to $(\hat{B}_y(\frac{r}{2}), g_x)$ in C^∞-topology. In particular, $B_{y_\alpha}(\frac{r_\alpha r}{2}) \subset \mathcal{R}$ and $\hat{B}_y(\frac{r}{2}) \subset \mathcal{CR}$. This implies that $\mathcal{CR}$ is an open manifolds. Moreover, $\mathcal{CR}$ is a Kähler–Ricci flat manifold since each $(B_{y_\alpha}(\frac{r_\alpha r}{2}), g_\infty)$ is an open Kähler–Einstein manifold. If K is a compact set of $\mathcal{CR}$, then by taking finite small geodesic covering balls, one can find a sequence $\{(K_j \subset \mathcal{R}, \frac{1}{r_j^2} g_\infty)\}$ which converges to (K, g_x) in C^∞-topology. □

Define an open set $V(x;\delta)$ of $\mathcal{CR}$ by

$$V(x;\delta) = \left\{ y \in C_x \,\middle|\, \text{dist}(y, S_x) \geqslant \delta, d(y,x) \leqslant \frac{1}{\delta} \right\}, \tag{4.2}$$

where $S_x = C_x \setminus \mathcal{CR}$. The following lemma shows that there exists a "nice" cut-off function on C_x which supported in $V(x;\delta)$.

Lemma 4.3. — *For any $\eta, \delta > 0$, there exist some $\delta_1 < \delta$ and a cut-off function β on C_x which supported in $V(x;\delta_1)$ with property: $\beta = 1$ in $V(x;\delta)$ and*

$$\int_{C_x} |\nabla \beta|^2 e^{-\frac{\rho_x^2}{2}} \,\mathrm{dv}_{g_x} \leqslant \eta.$$

Lemma 4.3 is in fact a corollary of following fundamental lemma.

Lemma 4.4. — *Let (X^m, d, μ) be a measured metric space such that*

$$\mu(B_y(r)) \leqslant C_0 r^m, \quad \forall\, r \leqslant 1,\ y \in X. \tag{4.3}$$

Let Z be a closed subset of X with $\mathcal{H}^{m-2}(Z) = 0$. Suppose that there exists a nonnegative function $f \leqslant 1$ on X such that

$$\int_X f d\mu \leqslant 1.$$

Then for any $x \in X$, $\eta > 0$ and $\delta > 0$, there exist a positive $\delta_1 \leqslant \delta$ and a cut-off function $\beta \geqslant 0$, which supported in $B_x(\frac{1}{\delta_1}) \setminus Z_{\delta_1}$ with property: $\beta = 1$ in $B_x(\frac{1}{\delta}) \setminus Z_\delta$ and

$$\int_X f |\operatorname{Lip}(\beta)|^2 d\mu \leqslant \eta. \tag{4.4}$$

Here $Z_{\delta_1} = \{x' \in X \mid \operatorname{dist}(x', Z) \leqslant \delta_1\}$ *and* $\operatorname{Lip}(\beta)(z) = \sup_{w \to z} \left| \frac{f(w)-f(z)}{d(w,z)} \right|$.

Proof. — Let $R \geqslant \sqrt{\frac{8}{\eta}} + \frac{2}{\delta}$. Since $\mathcal{H}^{m-2}(Z) = 0$, then for any $\kappa > 0$, we can take finite geodesic balls $B_{x_i}(r_i)$ ($r_i \leqslant \delta$) with $x_i \in Z$ to cover $B_x(R) \bigcap Z$ such that

$$\Sigma_i r_i^{m-2} \leqslant \kappa.$$

Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a cut-off function which satisfies:

$$\zeta(t) = 1,\ \text{for } t \leqslant \frac{1}{2}; \quad \zeta(t) = 0,\ \text{for } t \geqslant 1; \quad |\zeta'(t)| \leqslant 2.$$

Set

$$\chi(y) = \min_i \left\{ 1 - \zeta\left(\frac{d(y, x_i)}{r_i}\right) \right\}$$

and

$$\beta(y) = \zeta\left(\frac{\epsilon}{d(y,x)}\right) \zeta\left(\frac{d(y,x)}{R}\right) \chi(y),$$

where $\epsilon \leqslant \frac{\delta}{2}$. Then it is easy to see that β is supported in $B_x(R) \setminus \cup B_{x_i}(\frac{r_i}{2})$ with $\beta \equiv 1$ in $B_x(\frac{1}{\delta}) \setminus Z_\delta$. Moreover,

$$\begin{aligned}\int_X f|\operatorname{Lip}\beta|^2 d\mu &\leqslant 4C_0\Sigma_i r_i^{-2} r_i^m + 4C_0\epsilon^{m-2} + \frac{4}{R^2}\\ &\leqslant 4C_0\kappa + 4C_0\epsilon^{2n-2} + \frac{\eta}{2}.\end{aligned}$$

Thus, if we choose ϵ and κ such that $4C_0\kappa + 4C_0\epsilon^{2n-2} \leqslant \frac{\eta}{2}$, then we get (4.4). By choosing $\delta_1 \leqslant \min\{\frac{\epsilon}{2}, \frac{1}{2R}\}$ such that

$$Z_{\delta_1} \cap B_x(R) \subseteq \cup B_{x_i}(\frac{r_i}{2}),$$

we can also get $\operatorname{supp}(\beta) \subset B_x(\frac{1}{\delta_1}) \setminus Z_{\delta_1}$. Hence β satisfies all conditions required in the lemma. □

Proof of Lemma 4.3. — Applying Lemma 4.4 to $X = C_x$, $Z = S_x$, $f = e^{-\frac{\rho_x^2}{2}}$, we get the lemma. □

By Lemma 4.2, we see that for any $\delta > 0$ there exists a sequence of $K_j \subset (M_\infty, r_j^{-2}g_\infty)$ which converges to $V(x;\delta)$. Let $L_0 = (C_x, \mathbb{C})$ be the trivial holomorphic bundle over C_x with a hermitian metric $h_0 = e^{-\frac{\rho_x^2}{2}}$. Then h_0 induces the Chern connection ∇_0 with its curvature

$$\Theta(L_0, \nabla_0) = \Theta^{\nabla_0} = g_x.$$

In the following we show that a sufficiently large multiple line bundles of $K_{\mathcal{R}}^{-1}|_{K_j}$ will approximate to L_0 over $V(x;\delta)$. This is in fact an application of the following fundamental lemma.

Lemma 4.5. — *Let (V, g) be a C^2 open Riemannian manifold and U, $U' \subset\subset V$ be two pre-compact open subsets of V with $U \subset\subset U'$. Then for any positive number ϵ, there exist a small number $\delta = \delta(U', g, \epsilon)$ and a positive integer $N = N(U, g, \epsilon)$, which depends on the fundamental group of U, the metric g on U, and the small ϵ such that the following is true: if a hermitian complex line bundle (L, h) over V with associated connection ∇ satisfies*

$$|\,\Theta^\nabla\,|_g \leqslant \delta, \text{ in } U', \tag{4.5}$$

where Θ^∇ is the curvature with respect to connection ∇, then there exist a positive integer $l \leqslant N$ and a section ψ of $L^{\otimes l}$ over U with $|\psi|_h \equiv 1$ which satisfies

$$|D^{\nabla^{\otimes l}}\psi|_{h,g} \leqslant \epsilon, \text{ in } U. \tag{4.6}$$

Proof. — The proof seems standard.

First we show that (L, U) is a flat bundle with respect to some connection. Let $B_{x_i}(r_i)$ ($r_i \leqslant 1$) be finite convex geodesic balls in V such that $\bar{U} \subset \cup B_{x_i}(r_i) \subset U'$. Then for $y \in B_{x_i}(r_i)$ there exists a minimal geodesic curve γ_y in $B_{x_i}(r_i)$, which connects x_i and y. Picking any vector $s_i \in L_{x_i}$ with $|s_i| = 1$ and using the parallel transportation, we define a parallel vector field by

$$e_i(y) = \mathrm{Para}_{\gamma_y}(s_i), \ \forall\, y \in B_{x_i}(r_i).$$

In particular, $De_i(x_i) = 0$. Let T be a vector field, which is tangent to γ_y, and X another vector field with $[T, X] = 0$. Then

$$D_T[D_X e_i] = D_X[D_T e_i] + \Theta^\nabla(T, X)\, e_i = \Theta^\nabla(T, X)\, e_i. \tag{4.7}$$

By the condition (4.5), it follows

$$|De_i|_{h,g} \leqslant C(U', g)|\Theta^\nabla|_{(U',g)} \leqslant C(U', g)\,\delta, \text{ in } B_{x_i}(r_i). \tag{4.8}$$

This implies that the transformation function g_{ij} of L is nearly constant in $B_{x_i} \cap B_{x_j}$. Since the first Chern class lies in the secondary integral cohomology group, L is topologically trivial as long as δ is small, i.e., $c_1(L) = 0$. Hence, there exist some complex functions f_i over $B_{x_i}(r_i)$ such that

$$|Df_i| \leqslant C(U', g)\,\delta \ll 1, \tag{4.9}$$

and the transition functions for $\tilde{e_i} = f_i e_i$ are constant. As a consequence, we can define an associated connection ∇' on L to h such that

$$|\tilde{e_i}|_h = 1 \quad \text{and} \quad D^{\nabla'}\tilde{e_i} = 0.$$

In fact, if we set $\nabla' = \nabla + \alpha \otimes e_i$, then locally,

$$D^{\nabla'}\tilde{e_i} = D^\nabla(f_i e_i) + \alpha(\tilde{e_i}) = f_i D^\nabla e_i + df_i \otimes e_i + f_i \alpha \otimes e_i.$$

Thus

$$\alpha = -\frac{1}{f_i}(df_i + \langle f_i D^\nabla e_i, e_i\rangle_h)$$

which is uniquely determined by requiring $D^{\nabla'}(\tilde{e_i}) = 0$. Therefore, (L, ∇') is a flat bundle over U with respect to ∇'. Moreover, by (4.8) and (4.9), we have

$$\|\nabla - \nabla'\|_{(U,g)} = \|\alpha\|_{(U,g)} \leqslant C(U', g)\|\Theta^\nabla\|_{(U',g)} \leqslant C(U', g)\delta. \tag{4.10}$$

Next we note that the holonomy group of a flat bundle over U is an element of $Hom(\pi_1(U), \mathbb{S}^1) \cong G \times \mathbb{T}^k$ for some finite group G with order m_1, where k is the Betti number of $\pi_1(U)$. By the pigeon-hole principle, we see that for any γ-neighborhood $W \subseteq \mathbb{T}^k$ of the identity there exists a positive integer $m_2 = m_2(\gamma)$ such that for any element $\rho \in \mathbb{T}^k$, $\rho^a \in W$

for some number a ($1 \leqslant a \leqslant m_2$). As a consequence, for any element $t \in G \times \mathbb{T}^k$, there exists l ($1 \leqslant l \leqslant N = m_1 m_2$) such that $t^l \in W$. Hence, there exist l and a smooth section ψ of $L^{\otimes l}$ over U by perturbing a parallel vector field in $L^{\otimes l}$ such that

$$\||\psi|_h - 1|, |\psi^{\nabla'^{\otimes l}}|_{h,g} \leqslant C(U', g)\gamma(\delta), \text{ in } U.$$

Moreover, By (4.10), we can normalize ψ by $|\psi|_h \equiv 1$ so that (4.6) is true. The lemma is proved. ☐

Proposition 4.6. — *Let $x \in M_\infty$ and $\delta_1 > 0$. Then for any $\epsilon > 0$, there exist a positive integer $N = N(V(x;\delta_1), \epsilon)$ and a large integer j_0 such that for $j \geqslant j_0$ there exist $l = l(j) \leqslant N$, and a sequence of $K_j \subseteq M_\infty$ and a sequence of pairs of isomorphisms (ϕ_j, ψ_j) with property:*

$$\begin{array}{ccc} L_0 & \xrightarrow{\psi_j} & K_{\mathcal{R}}^{-ll_j}|_{K_j} \\ \Big\downarrow & & \Big\downarrow \\ V(x;\delta_1) & \xrightarrow{\phi_j} & K_j, \end{array} \tag{4.11}$$

which satisfy

$$\phi_j^*(ll_j g_\infty) \to g_x, \text{ as } j \to \infty,$$

and

$$|D\psi_j|_{g_x} \leqslant \epsilon, \text{ in } V(x;\delta_1).$$

Proof. — Define an open set U of $\mathcal{CR}$ by

$$U = U(x;\epsilon_1,\epsilon_2,R) = \{y \in C_x \,|\, \mathrm{dist}(\bar{y}, S_x) \geqslant \epsilon_1, \epsilon_2 \leqslant d(y,x) \leqslant R\},$$

where $\bar{y}$ is the projection to the section Y of $C_x = C(Y)$. Then there exist some ϵ_1, ϵ_2 and R such that

$$V(x;\delta_1) \subseteq U(x;\epsilon_1,\epsilon_2,R).$$

Moreover, we can choose a sequence of integers $l_j = \frac{1}{r_j^2}$ such that

$$(M_\infty, l_j g_\infty, x) \to (C_x, g_x, x), \text{ as } j \to \infty.$$

Hence by Lemma 4.2, there exist a sequence of $\tilde{K}_j \subseteq M_\infty$ and a sequence of diffeomorphisms $\tilde{\phi}_j$ from $U(x;\epsilon_1, \frac{\epsilon_2}{\sqrt{N}}, R)$ to $\tilde{K}_j$ such that $\tilde{\phi}_j^*(l_j g_\infty) \to g_x$, where $N = N(U, g_x, \epsilon)$ is a large integer as determined in Lemma 4.5.

Let h_∞ be the induced hermitian metric on $K_{\mathcal{R}}^{-1}$ by g_∞ on the regular part $\mathcal{R}$ of M_∞. Let

$$(L_j, h) = \tilde{\phi}_j^*(K_{\mathcal{R}}^{-l_j}, h_\infty^{\otimes l_j}) \otimes (L_0, h_0)^*$$

be a product complex line bundle on U, where h is an induced hermitian metric by h_∞ and h_0 with associated connection ∇_j on L_j for each j. Clearly,

$$\|\Theta^{\nabla_j}\|_{(U',g_x)} \leqslant \delta \ll 1,$$

as long as j is large enough, where $U' \subset\subset \mathcal{C}R$ is an open set such that $\bar{U} \subset\subset U'$. Applying Lemma 4.5 to L_j over U', we see that there exist some positive integer $l = l(j) \leqslant N$ and a section ψ' on $L_j^{\otimes l}$ such that

$$\|D^{\nabla_j^{\otimes l}}\psi'\|_{(U,g_x)} \leqslant \epsilon.$$

Let $Y_{\epsilon_1} = U(x;\epsilon_1,\epsilon_2,R)\bigcap Y$ and $\tilde{\psi}$ an extension section over $U(x;\epsilon_1, \frac{\epsilon_2}{\sqrt{l}}, R)$ of the restriction of ψ' on Y_{ϵ_1} by the parallel transportation along rays from x. Clearly,

$$\|\tilde{\psi}\|_{\otimes^l h} \equiv 1.$$

Moreover, by the formula (4.7), it is easy to see

$$\|D^{\nabla_j^{\otimes l}}\tilde{\psi}\|_{(U(x;\epsilon_1,\frac{\epsilon_2}{\sqrt{l}},R),g_x)} \leqslant \frac{\sqrt{l}}{\epsilon_2}(\epsilon + C_0R^2\delta), \tag{4.12}$$

where the constant C_0 depends only on (Y, g_x). Thus we have pairs of isomorphisms $(\tilde{\phi}_j, \tilde{\psi}_j)$ with property:

$$\begin{array}{ccc} L_0^l & \xrightarrow{\tilde{\psi}_j} & K_{M_\infty}^{-l_j l}|_{K_j} \\ \downarrow & & \downarrow \\ (U(x;\epsilon_1,\frac{\epsilon_2}{\sqrt{l}},R), g_x) & \xrightarrow{\tilde{\phi}_j} & (K_j, l_j g_\infty), \end{array} \tag{4.13}$$

which satisfy

$$|D\tilde{\psi}_j|_{g_x} \leqslant 2\frac{\sqrt{l}}{\epsilon_2}\epsilon, \tag{4.14}$$

as long as j is large enough.

Rescaling $U(x;\epsilon_1,\epsilon_2,R)$ into $U(x;\epsilon_1,\frac{\epsilon_2}{\sqrt{l}},R)$ by

$$\mu_l:\ y \to \frac{y}{\sqrt{l}}, \quad y \in U(x;\epsilon_1,\epsilon_2,R).$$

We have isometrics

$$\mu_l^* L_0^l \cong L_0, \quad \mu_l^* g_x = \frac{g_x}{l}.$$

By (4.13), it follows

$$\begin{array}{ccc} L_0 & \xrightarrow{\tilde{\psi}_j\circ(\mu_l^*)^{-1}} & K_{M_\infty}^{-l_j l}|_{K_j} \\ \downarrow & & \downarrow \\ (U(x;\epsilon_1,\epsilon_2,R),\frac{g_x}{l}) & \xrightarrow{\tilde{\phi}_j\circ\mu_l} & (K_j, l_j g_\infty). \end{array} \tag{4.15}$$

Let

$$\phi_j = \tilde{\phi}_j \circ \mu_l, \text{ and } \psi_j = \tilde{\psi}_j \circ (\mu_l^*)^{-1}.$$

Note that $V(x;\delta_1) \subseteq U(x;\epsilon_1,\epsilon_2,R)$. Then $K_j = \phi_j(V(x;\delta_1))$ is well-defined. Hence, rescaling the metric $\frac{g_x}{l}$ back to g_x, we get from (4.14),

$$|D\psi_j|_{g_x} \leqslant 2\frac{\epsilon}{\epsilon_2}, \text{ in } V(x;\delta_1). \tag{4.16}$$

Replacing $2\frac{\epsilon}{\epsilon_2}$ by ϵ, we prove the proposition. □

Proposition 4.6 will be used to construct locally nontrivial holomorphic sections of holomorphic line bundles over a sequence of Kähler manifolds in next section.

5. $\bar{\partial}$-equation and construction of holomorphic sections

In this section, we apply Proposition 4.6 to construct global holomorphic sections by solving the $\bar{\partial}$-equation on Fano manifolds with almost Kähler–Einstein metrics. We will use the rescaling method as done on Kähler–Einstein manifolds in [9], [23].

Proposition 5.1. — *Let $\{(M_i, g^i)\}$ be an almost Kähler–Einstein sequence of Fano manifolds and (M_∞, g_∞) be its Gromov–Hausdorff limit. Then for any sequence of $p_i \in M_i$ which converges to $x \in M_\infty$, there exist two large number l_x and i_0, and a small time t_x such that for any $i \geqslant i_0$ there exists a holomorphic section $s_i \in \Gamma(K_{M_i}^{-l_x}, h^i_{t_x})$ which satisfies*

$$\int_{M_i} |s_i|^2_{h^i_{t_x}} \,\mathrm{dv}_{g^i_{t_x}} \leqslant 1 \quad \text{and} \quad |s_i|_{h^i_{t_x}}(p_i) \geqslant \frac{1}{8}. \tag{5.1}$$

Here g^i_t is solution of (2.1) with the initial g^i and $h^i_{t_x}$ is the hermitian metric of $K_{M_i}^{-l_x}$ induced by $g^i_{t_x}$.

Proof. — As in Section 4, we let

$$(C_x, \omega_x, x) = \lim_{j\to\infty}\left(M_\infty, \frac{g_\infty}{r_j^2}, x\right).$$

Choose a δ so that $\delta \leqslant (2\pi)^{-\frac{n}{2}}\frac{C_1}{64}$, where C_1 is a constant chosen as in (3.11). We consider the $\overline{\partial}$-equation for sections on the trivial line bundle $L_0 = (V(x;\delta), \mathbb{C})$,

$$\bar{\partial}\sigma = f, \quad \forall\, f \in \Gamma^\infty((TV^*)^{(0,1)} \otimes L_0).$$

Then the standard C^0-estimate for the elliptic equation shows

$$\|\sigma\|_{C^0(V(x;2\delta))} \leqslant C_2\left(\|f\|_{C^0(V(x;\delta))} + \delta^{-n}\left[\int_{V(x;\delta)} |\sigma|^2 \,\mathrm{dv}_{g_x}\right]^{\frac{1}{2}}\right), \tag{5.2}$$

where the constant C_2 depends on the metric g_x.

Let $0 < \eta \leqslant \frac{\delta^{2n}}{1000C_2^2}$ and β a cut-off function supported in $V(x;\delta_1)$ constructed in Lemma 4.3. Let K_j be the sequence of open sets with rescaled metrics $\frac{1}{r_j^2}g_\infty$ in M_∞ which converge to $V(x;\delta_1)$ and ψ_j be the sequence of isomorphisms from L_0 to $K_{\mathcal{R}}^{-ll_j}|_{K_j}$ constructed in Proposition 4.6, where $l = l(l_j) \leqslant N = N(V(x;\delta_1), \epsilon)$ and $l_j = \frac{1}{r_j^2}$. Set $\tau_j = \psi_j(\beta e)$, where e is a unit basis of L_0. Then $\{\tau_j\}$ is a sequence of smooth sections of $K_{\mathcal{R}}^{-l_j l}$ supported in $\psi_j(V(x;\delta_1))$. Moreover, τ_j satisfies the following property as long as j is large enough:

$$\begin{aligned}
&\text{(i) } \|\tau_j\|^2_{C^0(\phi_i(V(x;\delta)\bigcap B_x(3\delta)))} \geqslant \frac{3}{4}e^{-3\delta^2} \geqslant \frac{1}{2};\\
&\text{(ii) } \int_{M_\infty} |\tau_j|^2 \,\mathrm{dv}_{g_\infty} \leqslant \frac{3}{2}\frac{r_j^{2n}}{l^n}(2\pi)^n;\\
&\text{(iii) } |\bar{\partial}_{J\infty}\tau_j| \leqslant \frac{\eta}{8}, \text{ in } V(x;\delta);\\
&\text{(iv) } \int_{M_\infty} |\bar{\partial}_{J_\infty}\tau_j|^2 \,\mathrm{dv}_{g_\infty} \leqslant \frac{3}{2}r_j^{2n-2}\frac{\eta}{l^{n-1}}.
\end{aligned} \tag{5.3}$$

On the other hand, from the proof of Lemma 4.2, we see that there exists t_0, which depends on $V(x;\delta_1)$ such that for any sufficiently large j it holds

$$\mathrm{vol}\left(B_y\left(\sqrt{t_0}\frac{r_j}{\sqrt{l}}\right)\right) \geqslant (1-\epsilon)\,\mathrm{vol}\left(B_0\left(\sqrt{t_0}\frac{r_j}{\sqrt{l}}\right)\right), \quad \forall\, y \in K_j,$$

where ϵ is a small constant chosen as in Lemma 4.1. Note that there is a sequence of sets $B_i \subseteq M_i$ for fixed K_j such that (B_i, g_i) converge to (K_j, g_∞) in Gromov–Hausdorff topology. By Colding's volume convergence theorem [7], it follows

$$\mathrm{vol}\left(B_{y'}\left(\sqrt{t_0}\frac{r_j}{\sqrt{l}}\right)\right) \geqslant (1-2\epsilon)\,\mathrm{vol}\left(B_0\left(\sqrt{t_0}\frac{r_j}{\sqrt{l}}\right)\right), \quad \forall\, y' \in B_i,\ i \ll 1.$$

Applying Theorem A.3 with $X = 0$ in Appendix to each ball $(B_{y'}(\sqrt{t_0}\frac{r_j}{\sqrt{l}}) \subset M_i, g^i)$, there exists $t_0' \leqslant t_0$ independent of i such that the sectional curvature of $g^i(t_0'\frac{r_j^2}{l})$ on $B_{y'}(\sqrt{t_0'}\frac{r_j}{\sqrt{l}})$ is uniformly bounded, where $g^i(t) = g^i_t$. Thus by Cheeger–Gromov's convergence theorem [11], there exists a sequence of diffeomorphisms $\varphi_i : K_j \to B_i$ such that

$$\begin{aligned} \varphi_i^* g^i(t_0'\frac{r_j^2}{l}) &\to g_\infty, \\ \varphi_i^* J_i &\to J_\infty, \\ \varphi_i^* K_{M_i}^{-1} &\to K_{\mathcal{CR}}^{-1}, \end{aligned}$$

in C^∞-topology. Hence, if we let $v_i = (\varphi_i)_*\tau_{j_0} \in \Gamma(M_i, K_{M_i}^{-ll_{j_0}})$ for some large integer $l_{j_0} = \frac{1}{r_{j_0}^2}$ and $l = l(l_{j_0}) \leqslant N$, then there exists a large integer i_0 such that for any $i \geqslant i_0$ it holds:

$$(5.4)\qquad \begin{aligned} &(\mathrm{i}')\ \ |v_i|_{h^i_{t_x}} \geqslant \frac{3}{8}, \quad \text{in } (\varphi_i \circ \psi_{j_0})(V(x;2\delta) \bigcap B_x(3\delta)); \\ &(\mathrm{ii}')\ \int_{M_i} |v_i|^2_{h^i_{t_x}} \,\mathrm{dv}_{g^i_{t_x}} \leqslant \frac{5}{2} r_{j_0}^{2n-2}\frac{\eta}{l^{n-1}}; \\ &(\mathrm{iii}')\ |\bar\partial_{J_i} v_i|_{h^i_{t_x}} \leqslant \frac{1}{4}\eta, \quad \text{in } (\varphi_i \circ \psi_{j_0})(V(x;\delta)); \\ &(\mathrm{iv}')\ \int_{M_i} |\bar\partial_{J_i} v_i|^2_{h^i_{t_x}} \,\mathrm{dv}_{g^i_{t_x}} \leqslant \frac{5}{4} r_{j_0}^{2n-2}\frac{\eta}{l^{n-1}}. \end{aligned}$$

Here $t_x = t_0' r_{j_0}^2/l$ and $h^i_{t_x}$ are hermitian metrics of $K_{M_i}^{-ll_{j_0}}$ induced by $g^i_{t_x}$.

By solving $\bar\partial$-equations for $K_{M_i}^{-ll_{j_0}}$-valued (0,1)-form σ_i,

$$\bar\partial \sigma_i = \bar\partial v_i, \text{ in } M_i,$$

we get the L^2-estimates from (3.7) and (iv$'$) in (5.4),

$$(5.5)\qquad \|\sigma_i\|^2_{L^2(M_i, g^i_{t_x})} \leqslant \frac{4}{ll_{j_0}} \int_{M_i} |\bar\partial_{J_i} v_i|^2 \,\mathrm{dv}_{g^i_{t_x}} \leqslant \frac{5\eta}{l^n l_{j_0}^n}.$$

Hence, by (5.2) and $iii')$ in (5.4), we derive

$$
\begin{aligned}
|\sigma_i|_{h^i_{t_x}}(q) \leqslant 2C_2 \Bigg(& \sup_{(\varphi_i\circ\psi_{j_0})(V(x;\delta))} |\bar\partial v_i|_{h^i_{t_x}} \\
& + \delta^{-n}\left[(ll_{j_0})^n \int_{(\varphi_i\circ\psi_{j_0})(V(x;\delta))} |\sigma_i|^2_{h^i_{t_x}} \,\mathrm{dv}_{g^i_{t_x}}\right]^{\frac12}\Bigg) \\
\leqslant\ & 2C_2\left(\frac14\eta + \delta^{-n}\left[(ll_{j_0})^n\int_{M_i}|\sigma_i|^2\,\mathrm{dv}_{g^i_{t_x}}\right]^{\frac12}\right) \\
\leqslant\ & 2C_2\left(\frac14\eta + \delta^{-n}\left[(ll_{j_0})^n\frac{5\eta}{l^n l_{j_0}} r_{j_0}^{2n-2}\right]^{\frac12}\right) \\
\leqslant\ & 5C_2\left(\frac14\eta+\delta^{-n}\sqrt{\eta}\right) \leqslant \frac18, \quad \forall\, q\in(\varphi_i\circ\psi_{j_0})(V(x;2\delta)).
\end{aligned}
\tag{5.6}
$$

Let $s_i = v_i - \sigma_i$. Then s_i is a holomorphic section of $K_{M_i}^{-ll_{j_0}}$. By $i')$ in (5.4) and (5.6), we have

$$
|s_i|_{h^i_{t_x}}(q_1) \geqslant \frac38 - \frac18 = \frac14, \quad \forall\, q_1\in(\varphi_i\circ\psi_{j_0})(V(x;2\delta)\bigcap B_x(3\delta)).
$$

Moreover, by (ii′) in (5.4), it is easy to see that

$$
\begin{aligned}
\int_{M_i}|s_i|^2_{h^i_{t_x}}\,\mathrm{dv}_{g^i_{t_x}} &\leqslant 2\left(\int_{M_i}|v_i|^2_{h^i_{t_x}}\,\mathrm{dv}_{g^i_{t_x}} + \int_{M_i}|\sigma_i|^2_{h^i_{t_x}}\,\mathrm{dv}_{g^i_{t_x}}\right) \\
&\leqslant 4(2\pi)^n\frac{r_{j_0}^{2n}}{l^n}.
\end{aligned}
\tag{5.7}
$$

Thus by the estimate (3.11), we get

$$
\|\nabla s_i\|_{h^i_{t_x}} \leqslant \sqrt{4(2\pi)^n}\,C_1\sqrt{l}\,r_{j_0}^{-1}.
\tag{5.8}
$$

Since $d(p_i,q_1)\leqslant 4\frac{r_{j_0}}{\sqrt l}\delta$, we deduce

$$
\begin{aligned}
|s_i|_{h^i_{t_x}}(p_i) &\geqslant |s_i|_{h^i_{t_x}}(q_1) - 4\frac{r_{j_0}}{\sqrt l}\delta\|\nabla s_i\|_{h^i_{t_x}} \\
&\geqslant |s_i|_{h^i_{t_x}}(q_1) - 8\sqrt{(2\pi)^n}C_1\delta \geqslant \frac18.
\end{aligned}
$$

This proves the theorem when l_x is chosen by ll_{j_0}. □

6. Proof of Theorem 1.3–I

In this section, we use Proposition 5.1 to give a weak version proof of Theorem 1.3 for an almost Kähler–Einstein metrics sequence. Namely, we prove

Theorem 6.1. — *Let (M_i, g^i) be an almost Kähler–Einstein metrics sequence of Fano manifolds and (M_∞, g_∞) its Gromov–Hausdorff limit. Then there exists an integer $l_0 > 0$, which depends only on (M_∞, g_∞) such that for any integer $l > 0$ there exists a uniform constant $c_l > 0$ with property:*

$$\rho_{ll_0}(M_i, g^i) \geqslant c_l. \tag{6.1}$$

The proof of Theorem 6.1 depends on the following lemma.

Lemma 6.2. — *Let (M, g) be a Fano manifold with $\omega_g \in 2\pi c_1(M)$ which satisfies*

$$\mathrm{Ric}(g) \geqslant -\Lambda^2 g \quad \text{and} \quad \mathrm{diam}(M, g) \leqslant D. \tag{6.2}$$

Let g_t be a solution of (2.1) with the initial metric g. Then there exists a small $t_0 = t_0(l, \Lambda, D)$ such that the following is true: if $s \in \Gamma(M, K_M^{-l})$ is a holomorphic section with $\int_M |s|^2_{h_t} \,\mathrm{dv}_{g_t} = 1$ for some $t \leqslant t_0$ which satisfies

$$|s|^2_{h_t}(p) \geqslant c > 0,$$

then

$$|s|^2_h(p) \geqslant c' > 0 \quad \text{and} \quad \int_M |s|^2_h \,\mathrm{dv}_g \leqslant c''. \tag{6.3}$$

Here h_t and h are hermitian metrics of $K_{M_i}^{-l}$ induced by g_t and g, respectively, and $c', c'' > 0$ are uniform constants depending only on c, l, Λ and D.

Proof. — Let $\omega_{g_t} = \omega_g + \sqrt{-1}\partial\bar\partial\phi$. Namely, ϕ is a Kähler potential of g_t. Then $\phi = \phi(x,t)$ satisfies

$$\frac{\partial}{\partial t}\phi = \log\frac{(\omega_g + \sqrt{-1}\partial\bar\partial\phi)^n}{\omega_g^n} + \phi - f_g, \tag{6.4}$$

where f_g is the Ricci potential of g normalized by

$$\int_M f_g \,\mathrm{dv}_g^n = 0.$$

Since

$$\Delta f_g = \mathrm{R}(g) - n \geqslant -(n-1)\Lambda^2 - n,$$

by using the Green formula, we see

$$f_g(x) \leqslant -\int_M G(x,\cdot)\Delta f_g \leqslant C(\Lambda, D).$$

Here we used the fact that the Green function G is uniformly bounded below since the metric g satisfies the condition (6.2) (cf. [13]). Thus applying the maximum principle to (6.4), it follows

$$\phi \geqslant -C(\Lambda, D).$$

On the other hand, integrating both sides of (6.4), we have

$$\begin{aligned}\frac{d}{dt}\int_M \phi\,\mathrm{dv}_g &= \int_M \log\frac{(\omega_g+\sqrt{-1}\partial\bar\partial\phi)^n}{\omega_g^n}\,\mathrm{dv}_g + \int_M \phi\,\mathrm{dv}_g - \int_M f_g\,\mathrm{dv}_g\\ &\leqslant \int_M \phi\,\mathrm{dv}_g + C.\end{aligned}$$

Here we use an element inequality $\log ff \geqslant -C$ to the log term. It follows

$$\int_M \phi\,\mathrm{dv}_g \leqslant Ce^t \leqslant eC.$$

Hence by using the Green formula to ϕ, we can also get

$$\phi \leqslant C'(\lambda, D).$$

As a consequence, we derive

$$e^{-C'l}|\cdot|_h \leqslant |\cdot|_{h_t} = e^{-l\phi}|\cdot|_h \leqslant e^{Cl}|\cdot|_h. \tag{6.5}$$

Therefore to prove Lemma 6.2, it suffice to prove

Claim 6.3. — *Let $s \in \Gamma(M, K_M^{-l})$ be a holomorphic section. Suppose that*

$$\int_M |s|_h^2\,\mathrm{dv}_g = 1.$$

Then

$$\int_M |s|_{h_t}^2\,\mathrm{dv}_{g_t} \geqslant c(l, \Lambda, D) > 0. \tag{6.6}$$

Since

$$\begin{aligned}\frac{\partial}{\partial t}\frac{(\omega_g+\sqrt{-1}\partial\bar\partial\phi)^n}{\omega_g^n} &= \Delta'\frac{\partial\phi}{\partial t}\\ &= -R(g_t)+n \leqslant \lambda = \lambda(\Lambda),\end{aligned}$$

$$\mathrm{vol}_{g_t}(\Omega) \leqslant e^{\lambda t}\,\mathrm{vol}_g(\Omega), \quad \forall\ \Omega\subset M. \tag{6.7}$$

It follows

$$\begin{aligned}\mathrm{vol}_{g_t}(\Omega) = V - \mathrm{vol}_{g_t}(M\setminus\Omega) &\geqslant V - e^{\lambda t}\,\mathrm{vol}_g(M\setminus\Omega)\\ &\geqslant \mathrm{vol}_g(\Omega) - 2V\lambda t.\end{aligned} \tag{6.8}$$

By the estimate (3.4), we see

$$|s(x)|_h^2 \leqslant H = H(\Lambda, D).$$

Then

$$\int_0^H \mathrm{vol}_g\{x\in M \mid |s(x)|_h^2 \geqslant s\}\,\mathrm{d}s = \int_M |s|_h^2\,\mathrm{dv}_g.$$

Hence, by using (6.5) and (6.8), we get

$$\begin{aligned}\int_M |s|^2_{h_t}\,\mathrm{dv}_{g_t} &\geqslant \int_0^H \mathrm{vol}_{g_t}\{x\in M\,|\,|s(x)|^2_{h_t}\geqslant s\}\,\mathrm{d}s\\ &\geqslant \int_0^H \mathrm{vol}_{g_t}\{x\in M\,|\,|s(x)|^2_{h}\geqslant e^{C'l}s\}\,\mathrm{d}s\\ &\geqslant e^{-C'l}\int_0^{e^{C'l}H}\left[\mathrm{vol}_{g}\{x\in M\,|\,|s(x)|^2_{h}\geqslant s\}-2V\lambda t\right]\mathrm{d}s\\ &\geqslant e^{-C'l}\left(1-2\lambda VHe^{C'l}t\right).\end{aligned}$$

Therefore, by choosing $t_0\leqslant (4\lambda VHe^{C'l})^{-1}$, we derive (6.6). The claim is proved. □

Proof of the Theorem 6.1. — By Proposition 5.1, we see that for any $x\in M_\infty$ and a sequence $\{p_i\subset M_i\}$ which converges to x, there exist two large number l_x and i_0, a small time t_x such that there exists a holomorphic section $s_i\in\Gamma(K_{M_i}^{-l_x},h^i_{t_x})$ for any $i\geqslant i_0$ with $\int_{M_i}|s_i|^2_{h^i_{t_x}}\,\mathrm{dv}_{g^i}\leqslant 1$ which satisfies

$$|s_i|_{h^i_{t_x}}(p_i)\geqslant \frac{1}{8},$$

where $h^i_{t_x}$ is the hermitian metric of $K_{M_i}^{-l_x}$ induced by $g^i_{t_x}$. By Lemma 6.2, it follows that there exists a constant $c(l_x,\Lambda,D)$ and a holomorphic section $\hat{s}_i\in\Gamma(K_{M_i}^{-l_x},h_i)$ for any $i\geqslant i_0$ with $\int_{M_i}|\hat{s}_i|^2_{h_i}\,\mathrm{dv}_{g^i}=1$ which satisfies

$$|\hat{s}_i|_{h_i}(p_i)\geqslant c_x=c(l_x,\Lambda,D),$$

where h_i is the hermitian metric of $K_{M_i}^{-l_x}$ induced by g^i.

Let $C=C(C_S,n)$ be the constant as in (3.6), which depending only on Λ and D. For each x, we choose $r_x=\frac{c_x}{2}l_x^{-\frac{n+1}{2}}C$. Then by the estimate (3.6), we get

$$|\hat{s}_i|_{h_i}(q)\geqslant \frac{c_x}{2},\quad \forall\, q\in B_{p^i}(r_x).$$

Take N balls $B_{x_\alpha}(\frac{r_{x_\alpha}}{2})$ to cover M_∞. Then it is easy to see that there exists $i_1\geqslant i_0$ such that $\cup_\alpha B_{p^i_\alpha}(r_{x_\alpha})=M_i$ for any $i\geqslant i_1$, where $\{p^i_\alpha\}$ is a set of N points in M_i. This shows that for any $q\in M_i$ $(i\geqslant i_1)$ there exist a ball $B_{p^i_\alpha}(r_{x_\alpha})$ and a holomorphic section $s^i_\alpha\in\Gamma(K_{M_i}^{-l_{x_\alpha}},h_i)$ such that $q\in B_{p_\alpha}(r^i_{x_\alpha})$, and $\int_{M_i}|s^i_\alpha|^2_{h_i}\,\mathrm{dv}_{g^i}=1$ and

$$|s^i_\alpha|_{h_i}(q)\geqslant c=\min_\alpha\{c_{x_\alpha}\}>0. \tag{6.9}$$

Set $l_0=\prod_\alpha l_{x_\alpha}$. Then by using a standard method (cf. [9], [25]), for any $q\in M_i$ $(i\geqslant i_1)$, one can construct another holomorphic section $s\in\Gamma(K_{M_i}^{-l_0},h_i)$

based on holomorphic sections s^i_α such that $\int_{M^i} |s|^2_{h_i}\,\mathrm{dv}_{g^i} = 1$ and

$$|s|_{h_i}(q) \geqslant c' > 0,$$

where $c' = c'(l_0, c)$. This proves the theorem for $l = 1$. One can also prove the theorem for general multiple $l \geqslant 1$ as above. □

7. Proof of Theorem 1.3–II

In this section, we prove Theorem 1.3 for the case of almost Kähler–Ricci solitons sequence. The proof can be finished step by step as for Theorem 6.1 while some necessary variant estimates should be done. We now assume that a Fano manifold M admits a non-trivial holomorphic vector field X, where X lies in an reductive Lie subalgebra η_r of space of holomorphic vector fields [28]. Consider a K_X-invariant g with $\omega_g \in 2\pi c_1(M)$ which satisfies the following geometric conditions:

$$\begin{aligned} &\text{(i)}\ \operatorname{Ric}(g) + L_X g \geqslant -\Lambda^2 g,\ |X|_g \leqslant A \text{ and } \operatorname{diam}(M, g) \leqslant D; \\ &\text{(ii)}\ R(g) \geqslant -C_0. \end{aligned} \tag{7.1}$$

Here $L_X g$ denotes the Lie derivative along X, and by the Hodge theorem, there exists a potential θ of X such that $L_X g = \sqrt{-1}\partial\bar\partial\theta$. In particular, under the condition (i), g has a uniform L^2-Sobolev constant $C_s = C_s(\Lambda, A, D)$ [31]. We note that the volume of (M, g) is uniformly bounded below by the normalized condition $\omega_g \in 2\pi c_1(M)$ and it is uniformly bounded above by the volume comparison theorem [32].

Let $g(\cdot, t) = g_t$ ($t \in (0, \infty)$) be a solution of following modified Kähler–Ricci flow with the above initial Kähler metric g,

$$\begin{cases} \frac{\partial}{\partial t} g = -\operatorname{Ric}(g) + g + L_X g, \\ g_0 = g(\,\cdot\,, 0) = g. \end{cases} \tag{7.2}$$

Clearly, all Kähler metrics g_t are K_X-invariant. Since the solution $g(\cdot, t)$ is just different to one of (2.1) by a family of holomorphic transformations generated by X, by Lemma 2.1, we have

Lemma 7.1. — *All g_t of (7.2) have Sobolev constants $C_s = C_s(\Lambda, A, D)$ uniformly bounded below. Namely, the following inequalities hold,*

$$\left(\int_M f^{\frac{2n}{n-1}}\,\mathrm{dv}_{g_t}\right)^{\frac{n-1}{n}} \leqslant C_s\left(\int_M f^2(R_t + \hat C_0)\,\mathrm{dv}_{g_t} + \int_M |\nabla f|^2\,\mathrm{dv}_{g_t}\right),$$

where $f \in C^1(M)$ and $\hat C_0$ is a uniform constant depending only on the lower bound C_0 of scalar curvature R of g.

Lemma 7.2. — *Let $\Delta = \Delta_t$ be the Lapalace operator associated to g_t. Suppose that $f \geqslant 0$ satisfies*

$$\left(\frac{\partial}{\partial t} - (\Delta + X)\right) f \leqslant af, \tag{7.3}$$

where a is a constant. Then for any $t \in (0,1)$, we have

$$\sup_{x\in M} f(x,t) \leqslant \frac{C_1(\Lambda, A, D, C_0)}{t^{\frac{n+1}{p}}} \left(\int_{\frac{t}{2}}^{t} \int_M |f(x,\tau)|^p \,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau\right)^{\frac{1}{p}}. \tag{7.4}$$

Proof. — As in the proof of Lemma 2.2 , multiplying both sides of (7.3) by f^p, we have

$$\int_M f^p f'_\tau \,\mathrm{dv}_{g_\tau} + p\int_M |\partial f|^2 f^{p-1}\,\mathrm{dv}_{g_\tau} - \int_M \langle \partial\theta, \partial f\rangle f^p\,\mathrm{dv}_{g_\tau} \leqslant a\int_M f^{p+1}\,\mathrm{dv}_{g_\tau}.$$

On the other hand, by (7.2), it is easy to see

$$\int_M f^p f'_\tau\,\mathrm{dv}_{g_\tau} = \frac{1}{p+1}\frac{d}{\mathrm{d}\tau}\left(\int_M f^{p+1}\,\mathrm{dv}_{g_\tau}\right) + \frac{1}{p+1}\int_M (R - n - \Delta\theta) f^{p+1}\,\mathrm{dv}_{g_\tau}.$$

Thus we get

$$\frac{1}{p+1}\frac{d}{\mathrm{d}\tau}\left(\int_M f^{p+1}\,\mathrm{dv}_{g_\tau}\right) + \frac{1}{p+1}\int_M (R-n) f^{p+1} dv_{g_\tau} + p\int_M |\partial f|^2 f^{p-1}\,\mathrm{dv}_{g_\tau} \leqslant a\int_M f^p\,\mathrm{dv}_{g_\tau}.$$

It follows

$$\frac{d}{\mathrm{d}\tau}\int_M f^{p+1}\,\mathrm{dv}_{g_\tau} + \int_M (R + \hat{C}_0) f^{p+1}\,\mathrm{dv}_{g_\tau} + 2\int_M |\nabla f^{\frac{p+1}{2}}|^2\,\mathrm{dv}_{g_\tau} \leqslant ((p+1)\,a + n + C_0)\int_M f^{p+1}\,\mathrm{dv}_{g_\tau}. \tag{7.5}$$

Note that (7.5) is similar to (2.6). Therefore, we can follow the argument in the proof of Lemma 2.2 to obtain (7.4). $\square$

Recall that according to [30] an almost (weak) Kähler–Ricci solitons sequence of Fano manifolds (M_i, g^i, X_i) $(i \to \infty)$ satisfy the condition (i) in (7.1) and

$$\text{(iii)} \quad \int_{M_i} |\operatorname{Ric}(g^i) - g^i - L_{X_i} g^i|\,\mathrm{dv}^n_{g^i} \to 0, \quad \text{as } i \to \infty. \tag{7.6}$$

As in [26], [31], we shall further assume that the solutions g^i_t of (7.1) with the initial metrics g^i satisfy

$$\begin{aligned}&\text{(iv) } |X^i|_{g^i_t} \leqslant \frac{B}{\sqrt{t}};\\ &\text{(v) } \int_0^1 dt \int_{M_i} |R(g^i_t) - \Delta\theta_{g^i_t} - n|\, \mathrm{dv}^n_{g^i_t} \to 0, \quad \text{as } i \to \infty,\end{aligned} \tag{7.7}$$

where B is a uniform constant. It was proved that under the condition (i) of (7.1), and (7.6) and (7.7) there exists a subsequence of $\{(M_i, g^i, X_i)\}$ which converges to a Kähler–Ricci soliton away from singularities of Gromov–Hausdorff limit with real codimension 4.

Definition 7.3. — *$\{(M_i, g^i, X_i)\}$ are called an almost Kähler–Ricci solitons sequence of Fano manifolds if* (7.1), (7.6) *and* (7.7) *are satisfied.*

The following is a key lemma in this section.

Lemma 7.4. — *Let $\{(M_i, g^i, X_i)\}$ be an almost Kähler–Ricci soliton sequence of Fano manifolds. Then there exists a uniform constant $C = C(\Lambda, D, B, C_0)$ such that for any $t \in (0,1)$ there exists $N = N(t)$ such that for any $i \geqslant N$ it holds*

$$|\nabla h^i_t| \leqslant C \text{ and } |R^i_t| \leqslant C.$$

Proof. — By

$$\begin{aligned}\left(\frac{\partial}{\partial t} - (\Delta + X)\right)&|\nabla(h-\theta)|^2\\ &= -|\nabla\bar{\nabla}(h-\theta)|^2 - |\nabla\nabla(h-\theta)|^2 + |\nabla(h-\theta)|^2\\ &\leqslant |\nabla(h-\theta)|^2,\end{aligned} \tag{7.8}$$

we apply Lemma 7.2 to get

$$\begin{aligned}|\nabla(h-\theta)|^2 &\leqslant \frac{C}{t^{n+1}} \int_{\frac{t}{2}}^t \int_M |\nabla(h-\theta)|^2 \,\mathrm{dv}_{g_\tau}\, \mathrm{d}\tau\\ &= \frac{C}{t^{n+1}} \int_{\frac{t}{2}}^t \int_M (\theta - h)\Delta(h-\theta) \,\mathrm{dv}_{g_\tau}\, \mathrm{d}\tau\\ &\leqslant \frac{C}{t^{n+1}} \int_{\frac{t}{2}}^t \int_M \mathrm{osc}_M(h-\theta)|R - n - \Delta\theta| \,\mathrm{dv}_{g_\tau}\, \mathrm{d}\tau.\end{aligned}$$

By (2.15), it follows

$$|\nabla(h-\theta)|^2 \leqslant \frac{C}{t^{(n+1)(n+\frac{3}{2})}} \int_{\frac{t}{2}}^t \int_M |R - n - \Delta\theta| \,\mathrm{dv}_{g_\tau}\, \mathrm{d}\tau. \tag{7.9}$$

On the other hand, by the evolution equation of $(\Delta+X)(h-\theta)$ [3],

$$\left(\frac{\partial}{\partial t}-(\Delta+X)\right)[(\Delta+X)(h-\theta)] = (\Delta+X)(h-\theta)+|\nabla\bar{\nabla}(h-\theta)|^2, \tag{7.10}$$

we have

$$\left(\frac{\partial}{\partial t}-(\Delta+X)\right)[(\Delta+X)(h-\theta)+|\nabla(h-\theta)|^2] \leqslant (\Delta+X)(h-\theta)+|\nabla(h-\theta)|^2.$$

Then applying Lemma 7.2, we get

$$(\Delta+X)(h-\theta)+|\nabla(h-\theta)|^2 \leqslant \frac{C}{t^{n+1}}\int_{\frac{t}{2}}^{t}\int_M |(\Delta+X)(h-\theta)+|\nabla(h-\theta)|^2|\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau. \tag{7.11}$$

Note that by (iii) in (7.6) we have

$$\int_{\frac{t}{2}}^{t}\int_M |X(h-\theta)|\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau \leqslant B\,\mathrm{vol}(M)[\int_{\frac{t}{2}}^{t}\int_M |\nabla(h-\theta)|^2\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau]^{\frac{1}{2}}.$$

It follows from (7.9),

$$\begin{aligned}\int_{\frac{t}{2}}^{t}\int_M &|(\Delta+X)(h-\theta)+|\nabla(h-\theta)|^2|\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau \\ &\leqslant \int_{\frac{t}{2}}^{t}\int_M |R-n-\Delta\theta|\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau \\ &\quad+\frac{CB\,\mathrm{vol}(M)}{t^{\frac{1}{2}(n+1)(n+\frac{1}{2})}}\left[\int_{\frac{t}{2}}^{t}\int_M |R-n-\Delta\theta|\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau\right]^{\frac{1}{2}} \\ &\quad+\frac{C}{t^{(n+1)(n+\frac{3}{2})}}\int_{\frac{t}{2}}^{t}\int_M |R-n-\Delta\theta|\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau.\end{aligned}$$

Thus inserting the above inequality into (7.11), we derive

$$\begin{aligned}(\Delta+X)(h-\theta)+|\nabla(h-\theta)|^2 &\leqslant \frac{C}{t^{(n+1)(n+\frac{5}{2})}}\left(\int_{\frac{t}{2}}^{t}\int_M |R-n-\Delta\theta|\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau\right. \\ &\quad\left.+\left[\int_{\frac{t}{2}}^{t}\int_M |R-n-\Delta\theta|\,\mathrm{dv}_{g_\tau}\,\mathrm{d}\tau\right]^{\frac{1}{2}}\right).\end{aligned} \tag{7.12}$$

Combining (7.9) and (7.12), we see that for any $t \in (0,1)$ there exists $N = N(t)$ such that

$$\left|\frac{1}{\sqrt{t}}\nabla(h-\theta)\right| \leqslant 1 \text{ and } R - n - \Delta\theta \leqslant 1, \quad \forall\, i \geqslant N(t). \tag{7.13}$$

It follows

$$\Delta\theta = -|\nabla\theta|^2 - X(h-\theta) - \theta \leqslant C.$$

As a consequence, we get $R \leqslant C$, and so $|R| \leqslant C$.

By (7.13), we have

$$\Delta\theta \geqslant R - n - 1 \geqslant -C.$$

Thus

$$|\nabla\theta|^2 = -X(h-\theta) - \theta - \Delta\theta \leqslant C. \tag{7.14}$$

Again by (7.13), we prove that $|\nabla h| \leqslant C$. □

By Lemma 7.1 and the scalar curvature estimate in Lemma 7.4, we see that for any $t \in (0,1)$ there exists an integer $N = N(t)$ such that the Sobolev constant C_s of g^i_t is uniformly bounded for any $i \geqslant N$. Then by the gradient estimate of Ricci potentials in Lemma 7.4, we can follow the arguments in Lemma 3.1 and Lemma 3.3 (also see Remark 3.2 and Remark 3.4) to get an analogy of Proposition 3.6.

Proposition 7.5. — *Let (M_i, g^i) be an almost Kähler–Ricci solitons sequence of Fano manifolds which satisfy (7.1), (7.6) and (7.7). Then for any $t \in (0,1)$ there exist integers $N = N(t)$ such that for any $i \geqslant N$ and $l \geqslant l_0$ it holds,*

$$\|s\|_{h^i_t} + l^{-\frac{1}{2}}\|\nabla s\|_{h^i_t} \leqslant C l^{\frac{n}{2}} \left(\int_{M_i} |s|^2 \,\mathrm{dv}_{g^i_t}\right)^{\frac{1}{2}}, \quad \forall\, s \in H^0(M_i, K^{-l}_{M_i}), \tag{7.15}$$

and

$$\int_{M_i} |v|^2_{h^i_t} \leqslant 4l^{-1} \int_{M_i} |\bar\partial\sigma|^2_{h^i_t}. \tag{7.16}$$

Here v is a solution of (3.7), the norms of $|\cdot|_{h^i_t}$ are induced by g^i_t, and the integer l_0 and the uniform constant C are both independent of t.

By Proposition 7.5, we can follow the arguments in Proposition 5.1 and Theorem 6.1 to prove

Theorem 7.6. — *Let (M_i, g^i) be an almost Kähler–Ricci solitons sequence of Fano manifolds and (M_∞, g_∞) its Gromov–Hausdorff limit. Then* *there exists an integer $l_0 > 0$ which depending only on (M_∞, g_∞) such that for any integer $l > 0$ there exists a uniform constant $c_l > 0$ with property:*

$$\rho_{ll_0}(M_i, g^i) \geqslant c_l. \tag{7.17}$$

Proof. — We give a sketch of proof of Theorem 7.6.

Step 1. — By the rescaling method as in proof of Proposition 5.1 with help of Proposition 7.5 and Theorem A.3, we have an analogy of Proposition 5.1: For any sequence of $p_i \in M_i$ which converge to $x \in M_\infty$, there exist two large number l_x and i_0, and a small time t_x such that for any $i \geqslant i_0$ there exists a holomorphic section $s_i \in \Gamma(K_{M_i}^{-l_x}, h_{t_x}^i)$ which satisfies

$$\int_{M_i} |s_i|^2_{h^i_{t_x}} \,\mathrm{dv}_{g^i_{t_x}} \leqslant 1 \quad \text{and} \quad |s_i|_{h^i_{t_x}}(p_i) \geqslant \frac{1}{8}, \tag{7.18}$$

where g_t^i is a solution of (7.2) with the initial metric g^i and $h^i_{t_x}$ is the hermitian metric of $K_{M_i}^{-l_x}$ induced by $g^i_{t_x}$ of (7.2) at $t = t_x$.

Step 2. — We can compare the C^0-norm of holomorphic sections with respect to the varying metrics g_t evolved in the flow (7.2). In fact, we have

Lemma 7.7. — *Let (M, g) be a Fano manifold with $\omega_g \in 2\pi c_1(M)$ which satisfies* (7.1)*, and g_t a solution of* (7.2) *with the initial g. Then there exists a small $t_0 = t_0(l, \Lambda, D)$ such that the following is true: if $s \in \Gamma(M, K_M^{-l})$ is a holomorphic section with*

$$\int_M |s|^2_{h_t} \,\mathrm{dv}_{g_t} = 1 \tag{7.19}$$

for some $t \leqslant t_0$ which satisfies

$$|s|_{h_t}(p) \geqslant c > 0, \tag{7.20}$$

then there is a holomorphic section s' of K_M^{-l} which satisfies

$$|s'|_h(p) \geqslant c' > 0 \quad \text{and} \quad \int_M |s'|^2_h \,\mathrm{dv}_g \leqslant c''.$$

Here h_t and h are the hermitian metrics of K_M^{-l} induced by g_t and g, respectively, and the constants c' and c'' depend only on c, l, Λ, A, C_0 and D.

Proof of Lemma 7.7. — Let Φ_t be a one-parameter subgroup generated by $-X$. Then $\Phi_t^* g_t$ is a solution of (2.1). It is clear that (7.19) also holds for $\Phi_t^* s, \Phi_t^* g_t, \Phi_t^* h_t$ and the condition (7.20) is equivalent to $|\Phi_t^* s|_{\Phi_t^* h_t}(\Phi_{-t}(p)) \geqslant c$. Since the Green functions associated to the metric

g is bounded below under the condition (i) of (7.1) (cf. [3], [16]), we can follow the argument in Lemma 6.2 for the metrics $\Phi_t^* g_t$ to obtain

$$|\Phi_t^* s|_h(\Phi_{-t}(p)) \geqslant \tilde{c} \text{ and } \int_M |\Phi_t^* s|_h^2 \,\mathrm{dv}_g \leqslant c'',$$

where the constant $\tilde{c}$ depends only on c, l, Λ, A and D. Let $s' = \Phi_t^* s$. Then by the gradient estimate of $|\nabla s'| \leqslant C(l, \Lambda, D, C_0, A)$, we have

$$|s'|_h(p) \geqslant |s'|_h(\Phi_{-t}(p)) - C(\Lambda, D, C_0, A)At \geqslant c'.$$

This proves Lemma 7.7. □

Step 3. — By using the covering argument as in Theorem 6.1 together with the results in Step 1 and Step 2, we can finish the proof of Theorem 7.6. □

8. Proof of Corollary 1.4

In this section, for simplicity, we just give a proof of Corollary 1.4 in case of almost Kähler–Einstein metrics sequences (M_i, g^i). We have known that the partial C^0-estimate holds for (M_i, g^i),

$$\rho_l(M_i, g^i) \geqslant c_l > 0, \tag{8.1}$$

for some integer l. Then, as an application of (8.1), we have

$$H^0(M_i, K_{M_i}^{-m}) \subseteq H^0(M_i, K_{M_i}^{-(m-l)}) \otimes H^0(M, K_{M_i}^{-l}), \tag{8.2}$$

where $m \geqslant l(n+2+[\Lambda^2])$ is any integer and the constant $-\Lambda^2$ is a uniform lower bound of Ricci curvature of (M_i, g^i) (cf. [14, Proposition 7])[2]

We need a strong version of (8.1) as follows.

Lemma 8.1. — *Let $x, y \in M_\infty$ be two different points and $p_i \to x, q_i \to y$ two sequenecs, where $p_i, q_i \in M_i$. Then there exist $\ell = \ell(n, \Lambda, D, x, y)$, which is a multiple of l, and two sections $s_x, s_y \in H^0(M_i, K_{M_i}^{-\ell})$ such that*

$$|s_x(p_i)|_{h_i} = |s_y(q_i)|_{h_i} = 1 \quad \textit{and} \quad s_x(q_i) = s_y(p_i) = 0, \quad \forall\ i \ll 1. \tag{8.3}$$

Proof. — As in the proof of Proposition 5.1, we can choose two compact sets $V(x; \delta_1^x), V(y; \delta_1^y)$ in C_x and C_y, respectively, such that $\phi_i \circ \psi_j(V(x; \delta_1^x))$ and $\phi_i \circ \psi_j(V(y; \delta_1^y))$ are disjoint as long as j and i are large enough. Let $v_i^x, \sigma_i^x, s_x^i \in \Gamma(M_i, K_{M_i}^{-l_x})$ and $v_i^y, \sigma_i^y, s_y^i \in \Gamma(M_i, K_{M_i}^{-l_y})$ be sections associated x and y, constructed there respectively. We may assume that

[2] There is a generalization of (8.2) under the Bakry–Emèry Ricci curvature condition in Appendix.

$l_x = l_y = \ell$ for a multiple of l. Moreover, by the C^0-estimate of σ_i^x in $V(x;\delta^x)$ in (5.6), we see that $|s_x^i(q_i)|$ is small. Similarly, $|s_y^i(p_i)|$ is also small. Now we define holomorphic sections

$$\tilde{s}_x^i = s_x^i - \frac{s_x^i(q_i)}{s_y^i(q_i)} s_y^i \text{ and } \tilde{s}_y^i = s_y^i - \frac{s_y^i(p_i)}{s_x^i(p_i)} s_x^i. \tag{8.4}$$

Clearly, $\tilde{s}_x(q_i) = \tilde{s}_y(p_i) = 0$. Then $s_x = \frac{\tilde{s}_x^i}{|\tilde{s}_x^i(p_i)|_{h_i}}$ and $s_y = \frac{\tilde{s}_y^i}{|\tilde{s}_y^i(q_i)|_{h_i}}$ will satisfy (8.3). □

By Lemma 8.1, we prove

Proposition 8.2. — *Let $\{(M_i, g^i)\}$ be a sequence of Fano manifolds with Ricci bounded from below and diameter bounded from above, and (M_∞, g_∞) its limit in Gromov–Hausdorff topology. Suppose that* (8.1) *and* (8.3) *in Lemma 8.1 hold. Then M_∞ is homeomorphic to an algebraic variety.*

Proof. — By (8.1), for any k, we can define holomorphisms

$$T_{kl,i} : M_i \to \mathbb{C}P^N,$$

where $N + 1 = \dim H^0(M_i, K_{M_i}^{-kl})$ is constant if i is large enough. Since $T_{kl,i}$ is uniformly Lipschitz by (3.6), we get a limit map

$$T_{kl,\infty} : M_\infty \to \mathbb{C}P^N.$$

On the other hand, the images W_i^{kl} of $T_{kl,i}$ have a chow limit W^{kl}, which coincides with the image of the map $T_{kl,\infty}$. Thus $T_{kl,\infty}$ maps M_∞ onto $W^{kl} = T_{kl,\infty}(M_\infty)$. We claim that $T_{(n+2+[\Lambda^2])l,\infty}$ is injective, so the proposition is proved.

By Lemma 8.3, for any $x, y \in M_\infty$, there are $p_i \to x$ and $q_i \to y$, and $s_x, s_y \in H^0(M_\infty, K_{M_i}^{-k_1 l})$ for some k_1 such that

$$|s_x|_{h_i}(p_i) = |s_y|_{h_i}(q_i)| = 1 \text{ and } s_x(q_i) = s_y(p_i) = 0. \tag{8.5}$$

This means $T_{k_1 l,\infty}(x) \neq T_{k_1 l,\infty}(y)$. We further show that

$$T_{(n+2+[\Lambda^2])l,\infty}(x) \neq T_{(n+2+[\Lambda^2])l,\infty}(y). \tag{8.6}$$

In fact, if (8.6) is not true, it is easy to see $T_{il,\infty}(x) = T_{il,\infty}(y)$ for any $i \leqslant n + 2 + [\Lambda^2]$. Then by (8.2), it follows

$$T_{kl,\infty}(x) = T_{kl,\infty}(y), \ \forall\ k,$$

which is contradiction with (8.5). Thus (8.6) is true. Hence $T_{(n+2+[\Lambda^2])l,\infty}$ must be injective. □

Proof of Corollary 1.4. — By the Gromov compactness theorem, there exists a subsequence $\{(M_{i_k}, g^{i_k})\}$ of $\{(M_i, g^i)\}$, which converges to (M_∞, g_∞). Then (i) and (ii) in Corollary 1.4 follow from a generalized Cheeger–Colding–Tian compactness theorem for almost Kähler–Einstein metrics sequence [26] (or almost Kähler–Ricci solitons sequence [31]). Thus it suffices to prove the part (iii). By Proposition 8.2, we know that M_∞ is homomorphic to an algebraic variety $W^{k_0 l}$, where $k_0 = n+2+[\Lambda^2]$. We further show that $W^{k_0 l}$ is a log terminal Q-Fano variety.

Let $H^0(M_\infty, K_{M_\infty}^{-k_0 l})$ be a space of bounded holomorphic sections of $K_{\mathcal{R}}^{-k_0 l}$ with an induced hermitian metric by g_∞. Then for any compact set $K \subseteq \mathcal{R} \subseteq M_\infty$, we know that there are $t_K > 0$ and $K_i \subseteq M_i$ such that $(K_i, g_i(t_K))$ converge to (K, g_∞) smoothly. Thus by the argument in Proposition 5.1 and Lemma 6.2, we can identify $H^0(M_\infty, K_{M_\infty}^{-k_0 l})$ with the limit of $H^0(M_i, K_{M_i}^{-k_0 l})$. But, from the proof in Proposition 8.2, the later is the same as $H^0(W^{k_0 l}, \mathcal{O}_{\mathbb{C}P^N}(1))$. This implies that M_∞ is homeomorphic to the normalization of $W^{k_0 l}$ since the codimension of singularities of $W^{k_0 l}$ is at least 2 (cf. [25] and [9]). Hence W^{kl_0} is normal. By [1], it remains to prove that $W^{k_0 l}$ is a Q-Fano variety.

Let $\mathcal{S} = \mathrm{Sing}(M_\infty)$, $\hat{\mathcal{S}} = T_{k_0 l, \infty}(\mathcal{S})$, and let $W_s \subset \hat{\mathcal{S}}$ be the singular set of $W^{k_0 l}$. Then both W_s and $\hat{\mathcal{S}}$ lie in a subvariety of $W^{k_0 l}$ with codimension at least 2. Thus it suffices to prove that $W_s = \hat{\mathcal{S}}$ since $(W^{k_0 l}, \mathcal{O}_{\mathbb{C}P^N}(1)) = K^{-k_0 l}_{W^{k_0 l} \setminus \hat{\mathcal{S}}}$. In the following, we give a proof for the general limit Kähler–Ricci soliton (M_∞, g_∞) in Section 7 by using PDE method as in [9]. Namely, g_∞ satisfies an equation,

$$\mathrm{Ric}(g_\infty) - g_\infty - L_{X_\infty} g_\infty = 0, \text{ in } M_\infty \setminus \mathcal{S}, \tag{8.7}$$

where X_∞ is the limit holomorphic vector field of (M_i, X_i) on $M_\infty \setminus \mathcal{S}$ [31].

On contrary, we suppose that $W_s \neq \hat{\mathcal{S}}$. Then there exists some $x \in \mathcal{S}$ such that $p = T_{k_0 l, \infty}(x) \in W^{k_0 l} \setminus W_s$, a smooth point in $W^{k_0 l}$. Thus there exists a small ball B around p in $W^{k_0 l}$ with the standard holomorphic coordinates such that the induced Kähler form $\omega_0 = \frac{1}{k_0 l}\omega_{g_{FS}}$ by the Fubini–Study metric g_{KS} of the projective space is smooth on B. We may assume that $\omega_0 = \sqrt{-1}\partial\bar{\partial} v$ for some Kähler potential v on B.

Let ρ_∞ be the limit of $\rho_{k_0 l}(M_i, g^i)$ (perhaps replaced by a subsequence of $\rho_{k_0 l}(M_i, g^i)$) on $(M_\infty \setminus \mathcal{S}, g_\infty)$. Then ρ_∞ and $|\nabla \rho_\infty|_{g_\infty}$ are both uniformly bounded since $\rho_{k_0 l}(M_i, g^i)$ and $|\nabla \rho_{k_0 l}(M_i, g^i)|_{g^i}$ are all uniformly bounded by (3.6). Clearly, ρ_∞ satisfies

$$\omega_{g_\infty} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho_\infty, \text{ in } W^{k_0 l} \setminus \hat{\mathcal{S}}.$$

Let $u = v + \rho_\infty$. Then by (8.7), we see that u satisfies

$$\sqrt{-1}\partial\bar{\partial}(\log\det(u_{i\bar{j}}) + X_\infty(u) + u) = 0, \text{ in } B \setminus \hat{\mathcal{S}}.$$

It follows

$$\log\det(u_{i\bar{j}}) + X_\infty(u) + u = const., \text{ in } B \setminus \hat{\mathcal{S}}. \tag{8.8}$$

We claim that there exists a uniform C such that

$$C^{-1}\delta_{i\bar{j}} \leqslant u_{i\bar{j}} \leqslant C\delta_{i\bar{j}}, \text{ in } B \setminus \hat{\mathcal{S}}. \tag{8.9}$$

Since the basis in $H^0(M_\infty, K_{M_\infty}^{-k_0 l})$, which gives the embedding $T_{k_0 l,\infty}$, is uniformly C^1-bounded, we have

$$\omega_0 \leqslant C\omega_{g_\infty}, \text{ in } M_\infty.$$

On the other hand, by (7.14),

$$|X_\infty(\rho_\infty)| \leqslant |X_\infty|_{g_\infty}|\nabla\rho_\infty|_{g_\infty} \leqslant C, \text{ in } M_\infty.$$

Then $X_\infty(u)$ is uniformly bounded. Thus by (8.8), we see that $\det(u_{i\bar{j}})$ is uniformly positive and bounded. This implies (8.9).

By the above claim, we can apply the following lemma to show that u is a smooth function in a small neighborhood of p. But this is impossible by $x \in \mathcal{S}$. Hence $W^{k_0 l}$ must be a Q-Fano variety. □

Lemma 8.3. — *Let u be a smooth solution of* (8.8) *in $B \setminus \hat{\mathcal{S}}$, where B is a ball in the euclidean space in $\mathbb{C}^n$ and $\hat{\mathcal{S}}$ is a closed subset in $\mathbb{C}^n$ with real Hausdroff dimension less than $2n-1$. Suppose that u satisfies* (8.9). *Then u can be extended to a smooth function on $\frac{1}{4}B$.*

Proof. — By the Schaulder estimate for the equation (8.8), it suffices to get a $C^{2,\alpha}$-regularity of u in $\frac{1}{4}B$. We first do the $C^{1,1}$-estimate.

For any $0 < \epsilon < \frac{1}{8}$ and any unit vector v, we let the difference quotient

$$w = w_\epsilon = \frac{u(x+\epsilon v) + u(x-\epsilon v) - 2u(x)}{\epsilon^2}.$$

Then by the convexity of $\log\det$, we get from (8.8),

$$u^{i\bar{j}}w_{i\bar{j}} \geqslant e^g\frac{g(x+\epsilon v) + g(x-\epsilon v) - 2g(x)}{\epsilon^2}, \tag{8.10}$$

where $g = -u - X_\infty(u)$. Denote $(a_{\alpha\beta})$ to be the $2n \times 2n$ matrix of Riemannian metric of g_∞ and $(a^{\alpha\beta}) = \det(a_{\delta\gamma})(a_{\alpha\beta})^{-1}$. It is clear that (8.10) is equivalent to

$$(a_{\alpha\beta}w_\beta)_\alpha \geqslant l(x) + \frac{h(x+\epsilon v) - h(x)}{\epsilon}, \text{ in } \frac{3}{4}B \setminus \hat{\mathcal{S}}, \tag{8.11}$$

where $l = f(x+\epsilon v)\frac{e^g(x+\epsilon v)-e^g(x)}{\epsilon}$, $h = e^g f$ and $f = \frac{g(x)-g(x-\epsilon v)}{\epsilon}$. Note that X_∞ can be extended to a holomorphic vector field on B. Then by (8.9), w can be regarded as a weak sub-solution in (8.11) in whole $\frac{3}{4}B$. Thus by the L^∞-estimate arising from the Moser iteration, we have,

$$\sup_{\frac{1}{2}B}(w_\epsilon) \leqslant C(|w_\epsilon|_{L^p(\frac{3}{4}B)} + |l|_{L^{\frac{q}{2}}(B)} + |h|_{L^q(B)}), \tag{8.12}$$

where C depends only on $(a_{\alpha\beta})$, $p \geqslant 1$ and $q > 2n$. In fact, by Theorem 8.17 in [10], the estimate (8.12) holds for sub-solution w as follows,

$$(a_{\alpha\beta}w_\beta)_\alpha \geqslant l + <v, Dh>.$$

But Theorem 8.17 is also true when the term $< v, Dh >$ is replaced by the difference quotient $\frac{h(x+\epsilon v)-h(x)}{\epsilon}$.

Since g is uniformly Lipschitz in $B \setminus \hat{\mathcal{S}}$, l, h are L^∞-functions in B. On the other hand, by (8.9), $u \in W^{2,p}(\frac{3}{4}B)$ for any $p \geqslant 1$, and so $|w_\epsilon|_{L^p(\frac{3}{4}B)}$ is uniformly bounded. Thus the (8.12) implies that w_ϵ is uniformly bounded above. As a consequence, $C^{1,1}$-derivative u_{vv} is uniformly bounded above. By (8.9), we can also get a uniform lower bound of u_{vv}. Hence $C^{1,1}$-norm of u is uniformly bounded in $\frac{1}{2}B$.

Next to get $C^{2,\alpha}$-estimate of u in (8.8), we can apply Evans–Krylov theorem, Theorem 17.14 in [10] to $C^{1,1}$-solution of (8.8) in $\frac{1}{2}B$ directly. This is because (8.8) is strictly elliptic in B and $-u - X_\infty(u)$ is Lipschitz. Thus the lemma is proved. □

9. Conclusion

In the proofs of Theorem 6.1 and Theorem 7.6, the constants c_l in the estimates (6.1) and (7.6) may depend on the limit (M_∞, g_∞). In this section, we show that c_l just depends on n, l_0 and l, and the geometric uniform constants Λ and D in (i) of (3.10), or the constants Λ, D, C_0 and B in (7.1) and (iii) of (7.6). Thus we complete the proof of Theorem 1.3. For simplicity, we just consider the case of almost Kähler–Einstein Fano manifolds below.

Set a class of Kähler metrics on Fano manifolds by

$$\mathcal{K}_{\Lambda,D} = \left\{(M^n, g) \;\middle|\; \omega_g \in 2\pi c_1(M),\ \mathrm{Ric}(g) \geqslant -(n-1)\Lambda^2,\ \mathrm{diam}(M, g) \leqslant D\right\}.$$

It is known that $\mathcal{K}_{\Lambda,D}$ is precompact in Gromov–Hausdorff topology. Moreover, by Cheeger–Colding theory in [4], any Gromov–Hausdorff limit M_∞ in $\mathcal{K}_{\Lambda,D}$ contains singularities with codimension at least 2 and each tangent cone at $x \in M_\infty$ is a metric cone C_x, which also contains singularities with codimension at least 2.

Let $\mathcal{K}^0_{\Lambda,D}$ be a subset of $\mathcal{K}_{\Lambda,D}$ such that $\mathcal{H}^{2n-2}(\mathrm{Sing}(C_x)) = 0$ for any $x \in M_\infty$, where M_∞ is any Gromov–Hausdorff limit in $\mathcal{K}^0_{\Lambda,D}$. Then according to the proofs in Proposition 5.1 and Theorem 6.1, we have

Theorem 9.1. — *Let $(M, g) \in \mathcal{K}^0_{\Lambda,D}$ and g_t a solution of (2.1) with the initial metric g. Then there exist a small number $\delta = \delta(\Lambda, D, n)$ and a large integer $l_0 = l_0(n, \Lambda, D)$ such that the following is true: if g satisfies*

$$\int_0^1 \int_M |R_t - n| \,\mathrm{dv}_{g_t}\,\mathrm{d}t \leqslant \delta, \tag{9.1}$$

then for any integer l there exists a uniform constant $c = c(n, l, \Lambda, D) > 0$ such that

$$\rho_{ll_0}(M, g) \geqslant c. \tag{9.2}$$

Proof. — By Theorem 6.1, we see that for any $Y \in \bar{\mathcal{K}}^0_{\Lambda,D}$, there exist a small number $\delta_Y > 0$, a large integer l_Y and a uniform constant $c_Y > 0$ such that if $M \in \mathcal{K}_{\Lambda,D}$ satisfies

$$\mathrm{d}_{GH}((M, g), (Y, g_Y)) \leqslant \delta_Y, \quad \int_0^1 \int_M |R_t - n| \,\mathrm{dv}_{g_t}\,\mathrm{d}t \leqslant \delta_Y,$$

then

$$\rho_{l_Y}(M, g) \geqslant c_Y.$$

Since $\bar{\mathcal{K}}_{\Lambda,D}$ is compact, we can cover it by finite balls $B_{Y_i}(\delta_{Y_i})(1 \leqslant i \leqslant N)$ in Gromov–Hausdorff topology. Putting $l_0 = \Pi l_{Y_i}$, $\delta = \min\{\delta_{Y_i}\}$ and $c = \min\{c_{Y_i}\}$. Then we get (9.2) for $l = 1$, if (M, g) satisfies (9.1). (9.2) is also true for general l as in the proof of Theorem 6.1. □

Theorem 1.3 follows from Theorem 9.1.

Appendix A.

In this appendix, we first use Siu's lemma to generalize the finite generation formula (8.2) under the Bakry-Émery Ricci curvature condition (i) in (7.1), then we recall a version of Perelman's pseudolocality theorem with the condition (ii) in (7.1).

The following lemma can be found in [20].

Lemma A.1. — *Let (M^n, g) be a compact complex manifold, G a holomorphic line bundle, E a holomorphic line bundle with a hermitian metric $e^{-\psi}$ whose Ricci curvature is positive. Let $\{s_i\}_{1 \leqslant i \leqslant p}$ be a basis of $H^0(M, G)$*

and $|s|^2 = \Sigma_{i=1}^p |s_i|^2$. *Then for any* $f \in H^0(M,(n+k+1)G+E+K_M)$ *which satisfies*

$$\int_M \frac{|f|^2 e^{-\psi}}{|s|^{2(n+k+1)}}\,\mathrm{dv}_g < +\infty,$$

there are some $h_i \in H^0(M,(n+k)G+E+K_M)$ $(k \geqslant 1)$ *such that* $f = \Sigma_{i=1}^p h_j \otimes s_j$ *and each* h_i *satisfies*

$$\int_M \frac{|h_j|^2 e^{-\psi}}{|s|^{2(n+k)}}\,\mathrm{dv}_g \leqslant \frac{n+k}{k}\int_M \frac{|f|^2 e^{-\psi}}{|s|^{2(n+k+1)}}\,\mathrm{dv}_g.$$

Proposition A.2. — *Let* (M,g) *be a Kähler manifold with*

$$\mathrm{Ric}(g) + \mathrm{Hess}\,u \geqslant -Cg,$$

where $X = \nabla_{\bar\partial} u$ *is a holomorphic vector field and* $|u| \leqslant A$. *Assume that*

$$c' \geqslant \rho_l(M,g) \geqslant c > 0 \tag{A.1}$$

for some $l \in \mathbb{N}$. *Then for any* $s \in H^0(M,K_M^{-m})$ *with* $m \geqslant (n+2)l + C + 1$, *there are* $u_i \in H^0(M,K_M^{-(m-l)})$ *such that* $s = \Sigma_{i=0}^N u_i \otimes s_i$, *where* $\{s_i\}$ *is an orthonormal basis of* $H^0(M,K_M^{-l})$. *Moreover, each* u_i *satisfies*

$$\int_M |u_i|^2_{h^{\otimes m-l}}\,\mathrm{dv}_g \leqslant (n+1)e^{2A}\left(\frac{c'}{c}\right)^{\frac{m}{l}} \int_M |s|^2_{h^{\otimes m}}\,\mathrm{dv}_g. \tag{A.2}$$

Proof. — Putting $L = K_M^{-1}$ and $m - [C] - 1 = (n+k+1)l + r$ $(0 \leqslant r < l)$, we decompose mL as

$$mL = (n+k+1)(lL) + ((m-(n+k+1)l)L - K_M) + K_M.$$

Let h and ω_g^n be two hermitian metrics on L such that

$$\Theta(L,h) = g,\ \Theta(L,\omega_g^n) = \mathrm{Ric}(g).$$

Denote the line bundle $(m-(n+k+1)l)L - K_M$ by E. Then $h_1 = h^{\otimes m-(n+k+1)l} \otimes e^{-u} \otimes \omega_g^n$ is a hermitian metric on E. It is easy to see

$$\Theta(E,h_1) = (m-(n+k+1)l)\omega_g + \mathrm{Ric}(g) + \sqrt{-1}\partial\bar\partial u \geqslant \omega_g.$$

Now applying the above lemma to $G = lL$, s_i, E and $f = s$, we see that there are $u_i \in H^0(M,(n+k)G+E+K_M)$ such that

$$\int_M \frac{|u_i|^2_{h^{\otimes(n+k)l}\otimes h_1}}{(\Sigma_{i=0}^N |s_i|^2_{h^{\otimes l}})^{n+k}}\,\mathrm{dv}_g \leqslant \frac{n+k}{k}\int_M \frac{|s|^2_{h^{\otimes(n+k+1)l}\otimes h_1}}{(\Sigma_{i=0}^N |s_i|^2_{h^{\otimes l}})^{n+k+1}}\,\mathrm{dv}_g.$$

The above is equivalent to

$$\int_M \frac{|u_i|^2_{h^{\otimes m-l}}}{(\Sigma_{i=0}^N |s_i|^2_{h^{\otimes l}})^{n+k}} e^{-u}\,\mathrm{dv}_g \leqslant \frac{n+k}{k}\int_M \frac{|s|^2_{h^{\otimes m}}}{(\Sigma_{i=0}^N |s_i|^2_{h^{\otimes l}})^{n+k+1}} e^{-u}\,\mathrm{dv}_g.$$

By (A.1), it follows

$$\frac{1}{e^{2A}c'^{n+k}}\int_M |u_i|^2_{h^{\otimes m-l}}\,\mathrm{dv}_g \leqslant \frac{n+k}{kc^{n+k+1}}\int_M |s|^2_{h^{\otimes m}}\,\mathrm{dv}_g,$$

which implies (A.2) immediately. □

The following Perelman version of pseudolocality theorem for modified Kähler–Ricci flow (7.2) is proved in [31]. The result is an analogy of Theorem 11.2 in [18], Proposition 3.1 in [26] for Ricci flow.

Theorem A.3. — *For any $\alpha, r \in [0,1]$, there exist $\tau = \tau(n,\alpha)$, $\eta = \eta(n,\alpha)$, $\epsilon = \epsilon(n,\alpha)$, $\delta = \delta(n,\alpha)$, such that if $g(\cdot,t) = g_t$ $(0 \leqslant t \leqslant (\epsilon r)^2)$ is a solution of* (7.2) *whose initial metric $g(\cdot,0) = g_0$ satisfies*

$$\begin{aligned} &\text{(i)}\ \ \mathrm{Ric}(g_0) + L_X g_0 \geqslant -(2n-1)r^{-2}\tau^2 g_0,\\ &\text{(ii)}\ \ |X|_{g_0}(x) \leqslant r^{-1}\eta,\\ &\text{(iii)}\ \ \mathrm{vol}(B_q(r,g_0)) \geqslant (1-\delta)c_{2n}r^{2n}, \end{aligned}$$

where c_{2n} is the volume of unit ball in $\mathbb{R}^{2n}$, then for any $x \in B_q(\epsilon r, g_0)$ and $t \in (0,(\epsilon r)^2]$, we have

$$|\,\mathrm{Rm}(x,t)| < \alpha t^{-1} + (\epsilon r)^{-2}, \tag{A.3}$$

Moreover,

$$\mathrm{vol}(B_x(\sqrt{t})) \geqslant \kappa(n) t^{\frac{n}{2}}. \tag{A.4}$$

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Wenshuai JIANG
School of Mathematical Sciences, Zhejiang University, Zheda Road 38, Hangzhou, Zhejiang, 310027, P.R. China

Feng WANG
School of Mathematical Sciences, Zhejiang University, Zheda Road 38, Hangzhou, Zhejiang, 310027, P.R. China

Xiaohua ZHU
School of Mathematical Sciences and BICMR, Peking University, Beijing, 100871, P.R. China
xhzhu@math.pku.edu.cn