



ANNALES

DE

L'INSTITUT FOURIER

Francesco BONSANTE & François FILLASTRE

The equivariant Minkowski problem in Minkowski space

Tome 67, n° 3 (2017), p. 1035-1113.

http://aif.cedram.org/item?id=AIF_2017__67_3_1035_0



© Association des Annales de l'institut Fourier, 2017,

Certains droits réservés.



Cet article est mis à disposition selon les termes de la licence
CREATIVE COMMONS ATTRIBUTION – PAS DE MODIFICATION 3.0 FRANCE.
<http://creativecommons.org/licenses/by-nd/3.0/fr/>

L'accès aux articles de la revue « Annales de l'institut Fourier »
(<http://aif.cedram.org/>), implique l'accord avec les conditions générales
d'utilisation (<http://aif.cedram.org/legal/>).

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
<http://www.cedram.org/>

THE EQUIVARIANT MINKOWSKI PROBLEM IN MINKOWSKI SPACE

by Francesco BONSANTE & François FILLASTRE (*)

ABSTRACT. — The classical Minkowski problem in Minkowski space asks, given a positive function ϕ on \mathbb{H}^d , for a convex set \mathbf{K} in Minkowski space with C^2 space-like boundary S , such that $\phi(\eta)^{-1}$ is the Gauss–Kronecker curvature at the point with normal η . Analogously to the Euclidean case, it is possible to formulate a weak version of this problem: given a Radon measure μ on \mathbb{H}^d the generalized Minkowski problem in Minkowski space asks for a convex subset \mathbf{K} such that the area measure of \mathbf{K} is μ .

In the present paper we look at an equivariant version of the problem: given a uniform lattice Γ of isometries of \mathbb{H}^d , a Γ invariant Radon measure μ and an isometry group Γ_τ of Minkowski space with Γ as linear part, there exists a unique convex set with area measure μ , invariant under the action of Γ_τ . The proof uses a functional which is the covolume associated to every invariant convex set.

This result translates as a solution of the Minkowski problem in flat space times with compact hyperbolic Cauchy surface. The uniqueness part, as well as the regularity results, follow from properties of the Monge–Ampère equation. The existence part can be translated as an existence result for Monge–Ampère equation.

The regular version was proved by T. Barbot, F. Béguin and A. Zeghib for $d = 2$ and by V. Oliker and U. Simon for $\Gamma_\tau = \Gamma$. Our method is totally different. Moreover, we show that those cases are very specific: in general, there is no smooth Γ_τ -invariant hypersurface of constant Gauss–Kronecker curvature equal to 1.

Keywords: Minkowski problem, Lorentzian geometry, covolume, Monge–Ampère equation.

Math. classification: 53C50, 52A20, 53C42, 35J60 .

(*) The authors thanks the anonymous referee for his/her valuable comments which helped to improve the presentation of the paper. The second author thanks Guillaume Carlier for useful conversations. The first author was partially supported by FIRB 2010 project RBFR10GHHH_002 “Geometria e topologia delle varietà in bassa dimensione”. The second author was partially supported by the ANR GR-Analysis-Geometry. Most of this work was achieved during stays of the second author in the Department of Mathematics of the University of Pavia. He thanks the institution for its hospitality.

RÉSUMÉ. — La forme classique du problème de Minkowski dans l'espace de Minkowski est la recherche, pour une fonction positive ϕ sur \mathbb{H}^d , d'un ensemble convexe \mathbf{K} dans l'espace de Minkowski, avec un bord S C^2 de type espace, tel que $\phi(\eta)^{-1}$ soit la courbure de Gauss–Kronecker au point dont la normale est η . De façon analogue au cas euclidien, il est possible de formuler une version faible de ce problème : étant donné une mesure de Radon μ sur \mathbb{H}^d , le problème de Minkowski généralisé dans l'espace de Minkowski est la recherche d'un ensemble convexe \mathbf{K} tel que la mesure d'aire de \mathbf{K} est μ .

Dans ce travail nous regardons une version équivariante de ce problème : étant donné un réseau uniforme Γ d'isométries de \mathbb{H}^d , étant donné une mesure de Radon μ invariante pour l'action de Γ , étant donné un groupe d'isométries Γ_τ de l'espace de Minkowski, dont Γ est la partie linéaire, il existe un unique ensemble convexe de mesure d'aire μ , et qui est invariant sous l'action de Γ_τ . La preuve utilise une fonctionnelle, le covolume associé à tout ensemble convexe invariant.

Ce résultat donne une solution du problème de Minkowski dans des espace-temps plats avec des surfaces de Cauchy compactes. L'unicité de la solution, ainsi que les résultats à propos de la régularité de la solution, sont des conséquences de propriétés de l'équation de Monge–Ampère. L'existence de la solution se traduit en un résultat d'existence pour l'équation de Monge–Ampère.

La version régulière de notre résultat a été montrée par T. Barbot, F. Béguin et A. Zeghib dans le cas $d = 2$ et par V. Oliker et U. Simon quand $\Gamma_\tau = \Gamma$. Notre méthode est totalement différente. De plus, nous montrons que ces deux cas sont vraiment spéciaux : en général, il n'existe pas d'hypersurface lisse invariante sous l'action de Γ_τ ayant une courbure de Gauss–Kronecker constante égale à 1.

1. Introduction

In this paper we study an adapted version of classical Minkowski problem in Minkowski space and more generally in globally hyperbolic flat Lorentzian manifolds.

Let us recall that Minkowski space $\mathbb{R}^{d,1}$ is the manifold \mathbb{R}^{d+1} equipped with the flat Lorentzian structure given by Minkowski product:

$$\langle X, Y \rangle_- = \sum_{i=1}^d X_i Y_i - X_{d+1} Y_{d+1}.$$

The pseudo-sphere is the subset of $\mathbb{R}^{d,1}$ of unit future oriented time-like vectors

$$\mathbb{H}^d = \{X \in \mathbb{R}^{d+1} \mid \langle X, X \rangle_- = -1, X_{d+1} > 0\}.$$

It is well-known that it is a space-like (i.e., the restriction of $\langle \cdot, \cdot \rangle_-$ to $T\mathbb{H}^d$ is positive) convex hypersurface in \mathbb{R}^{d+1} isometric to the hyperbolic space.

Let S be the boundary of a strictly convex set \mathbf{K} of $\mathbb{R}^{d,1}$. Suppose that S is C^2 , and space-like. This assumption forces \mathbf{K} to be unbounded: in fact either \mathbf{K} coincides with the future of S or with its past. We will always assume that \mathbf{K} is the future of S .

The normal vectors of S are time-like, so a natural Gauss map is defined $\mathcal{G} : S \rightarrow \mathbb{H}^d$ sending p to the unique future-oriented unit time-like vector

orthogonal to $T_p S$. The map \mathcal{G} is injective but in general is not surjective. We say that \mathbf{K} is an F -convex set if \mathcal{G} is surjective.

Let ϕ be a continuous positive function on \mathbb{H}^d . The classical Minkowski problem in Minkowski space asks for an F -convex set \mathbf{K} in Minkowski space with C^2 space-like boundary S , such that $\frac{1}{\phi(\eta)}$ is the Gaussian curvature at $\mathcal{G}^{-1}(\eta)$. It turns out that $\phi(x)$ corresponds to the density of the direct image of the intrinsic volume measure of S through the map \mathcal{G} with respect to the volume measure of \mathbb{H}^d .

Analogously to the Euclidean case, it is possible to formulate a weak version of this problem. Indeed if \mathbf{K} is any convex set whose support planes are space-like, it is always possible to define a set-valued function \mathcal{G} from $\partial\mathbf{K} = S$ to \mathbb{H}^d . Namely \mathcal{G} sends a point p to the set of unit time-like vectors orthogonal to support planes at p . As in the previous case, the condition that support planes are not time-like implies that \mathbf{K} coincides either with the future or the past of S . We will always assume that \mathbf{K} is the future of S . We say that \mathbf{K} is an F -convex set if moreover \mathcal{G} is surjective in the sense that $\bigcup_{x \in S} \mathcal{G}(x) = \mathbb{H}^d$.

In [17] a Radon measure is defined on \mathbb{H}^d as the first-order variation of the volume of an ϵ -neighborhood of S . If S is C^1 and space-like, it coincides with the direct image of the intrinsic volume measure on S through the map \mathcal{G} . We denote by $A(\mathbf{K})$ this measure, called the *area measure*.

So given a Radon measure μ on \mathbb{H}^d the generalized Minkowski problem in Minkowski space asks for a convex subset \mathbf{K} such that $A(\mathbf{K}) = \mu$.

As \mathbb{H}^d is not compact, a boundary condition must be added for the well-possessedness of this problem. Geometrically this boundary condition corresponds to the choice of the domain of dependence of the hypersurface S , or equivalently the light-like support planes of \mathbf{K} .

Results on this direction are for instance pointed out in [23], where an assumption on the regularity of the boundary condition has been considered.

In a different direction one could look at an equivariant version of the problem. Isometries of Minkowski space are affine transformations of \mathbb{R}^{d+1} whose linear part preserves the Minkowski product. We will restrict to isometries in the connected component of the identity of the isometry group, that is $\sigma \in \text{Isom}(\mathbb{R}^{d,1})$ whose linear part $\gamma \in \text{SO}^+(d, 1)$ preserves the orientation and leaves the future cone from 0 invariant. There is a natural isometric action of $\text{SO}^+(d, 1)$ on \mathbb{H}^d which identifies $\text{SO}^+(d, 1)$ with the group of orientation preserving isometries of \mathbb{H}^d .

Let us fix a group Γ_τ of isometries of Minkowski space. If \mathbf{K} is an F-convex set invariant under the action of Γ_τ , the area measure $A(\mathbf{K})$ is a measure on \mathbb{H}^d invariant by the group $\Gamma < \mathrm{SO}^+(d, 1)$ obtained by taking the linear parts of elements of Γ_τ .

So if a Γ -invariant measure μ in \mathbb{H}^d is fixed, the equivariant version of Minkowski problem asks for a Γ_τ -invariant F-convex set \mathbf{K} such that $A(\mathbf{K}) = \mu$. The geometric interest of this equivariant version relies on the fact that Minkowski problem can be intrinsically formulated in any flat globally hyperbolic space-time (see Section 1.1.2 of the Introduction). By using the theory developed by Mess in [24] and generalized in [3, 11], this intrinsic Minkowski problem on a flat Lorentzian manifold turns out to be equivalent to an equivariant Minkowski problem in Minkowski space in many geometrically interesting cases.

From [11] it is known that if Γ is a uniform lattice in $\mathrm{SO}^+(d, 1)$ (that means that Γ is discrete and \mathbb{H}^d/Γ is compact), there is a maximal Γ_τ -invariant F-convex set, say Ω_τ , which contains all the Γ_τ -invariant F-convex sets. Moreover, if \mathbf{K} is any Γ_τ -invariant F-convex set, then the domain of dependence of its boundary coincides with Ω_τ . So requiring the invariance under Γ_τ of \mathbf{K} automatically fixes the boundary conditions.

On the other hand, the boundary conditions arising in this setting do not satisfy the regularity conditions required in [23], so the existence result of that paper is not helpful in this context.

In the present paper we solve the equivariant generalized Minkowski problem in any dimension, with the only assumption that the group Γ is a uniform lattice of $\mathrm{SO}^+(d, 1)$.

The study of the Minkowski problem in Minkowski space, and its relations with Monge–Ampère equation is not new. But, as far as the authors know, all the results concern the regular Minkowski problem (excepted a polyhedral Minkowski problem in the Fuchsian case $\Gamma_\tau = \Gamma$ in [16]). Two previously known results are particular cases of Theorem 1.1.

The first one is a theorem of V. Oliker and U. Simon [26], solving the Minkowski problem in the C^∞ setting and in the Fuchsian case, in all dimensions. Actually they consider the underlying PDE problem intrinsically on the compact hyperbolic manifold \mathbb{H}^d/Γ .

The second one is a theorem of T. Barbot, F. Béguin and A. Zeghib [5], solving the equivariant Minkowski problem in the C^∞ setting in the $d = 2$ case. It is interesting to note that their proof is totally different from the one presented here, and use a geometric feature of the dimension 2. They show that a sequence of Cauchy surfaces with constant curvature $< -\infty$

and bounded diameter cannot have small systole as a simple consequence of Margulis Lemma. By contrast, in higher dimensions hypersurfaces with constant Gauss–Kronecker curvature does not have any bound on the sectional curvature.

We also discuss the equivariant regular Minkowski problem. Opposite to the $d = 2$ case, we will show that in general there is no smooth Γ_τ -invariant hypersurface of constant Gauss–Kronecker curvature equal to 1. On the other hand, we prove that there is a constant c depending on the group Γ_τ such that if ϕ is a smooth Γ -invariant function with $\phi \geq c$, then the Γ_τ -invariant F-convex set \mathbf{K} such that $A(\mathbf{K}) = \phi \, d\mathbb{H}^d$ is smooth and strictly convex.

We prove that the constant c can be taken equal to 0 if the dual stratification in the sense of [11] does not contain strata of dimension $k < d/2$, that roughly means that the dimension of the initial singularity of Ω_τ is $\leq d/2$. If $d = 2$, then this condition is always true so this gives a new proof of the result in [5]. At least for $d = 3$ this condition seems optimal in the sense that the construction of the example in Section 3.7 works every time the singularity of Γ_τ is simplicial (in the sense of [11]) with at least a stratum of dimension bigger than 1. In fact that construction works in any dimension under the weaker assumption that the initial singularity of Ω_τ contains a $(d - 1)$ -dimensional polyhedron.

Finally, let us mention related results in [18], and also a “dual” problem to the Minkowski problem, known as the Alexandrov problem. Here one prescribes a “curvature measure” rather than an area measure. Curvature measures are defined using a radial function, and the Alexandrov problem was solved in the Fuchsian case in [8]. It is not clear if this problem have an analog in the general non-Fuchsian case.

The trivial $d = 1$ case was considered in [17], so in all the paper, $d \geq 2$.

1.1. Main results

Let us formulate in a more precise way the results of the paper. Let Γ be a uniform lattice in $SO^+(d, 1)$. An affine deformation of Γ is a subgroup of $\text{Isom}(\mathbb{R}^{d,1})$ obtained by adding translations to elements of Γ :

$$\Gamma_\tau = \{\gamma + \tau_\gamma \mid \gamma \in \Gamma\}.$$

Notice that Γ_τ is determined by the function $\tau : \Gamma \rightarrow \mathbb{R}^{d+1}$ which assigns to each $\gamma \in \Gamma$ the translation part of the corresponding element of Γ_τ . The

condition that Γ_τ is a subgroup forces τ to verify a *cocycle* relation:

$$\tau_{\alpha\beta} = \tau_\alpha + \alpha\tau_\beta.$$

A τ -convex set is a proper convex set (globally) invariant under the action of Γ_τ . Here we also require the τ -convex sets to be future: they are disjoint from the past of their support planes. When $\tau = 0$, the term Fuchsian is sometimes used.

THEOREM 1.1. — *Let μ be a Γ -invariant Radon measure on \mathbb{H}^d . Then for any cocycle τ there exists a unique τ -convex set with area measure μ .*

Moreover the correspondence between the τ -convex sets and the measures is continuous for a natural Hausdorff topology on the space of τ -convex sets, see Section 3.3.

The proof of the existence part Theorem 1.1 is variational. It is similar to the proofs of the Minkowski problem for convex bodies, where the functional involves the volume of the convex bodies [1, 13]. Here the group action allows to define a *covolume* for any τ -convex set. The set of τ -convex sets is convex (for linear combinations given by homotheties and Minkowski addition of convex sets), and the main ingredient in the variational approach is the following.

THEOREM 1.2. — *The covolume is convex on the set of τ -convex sets.*

1.1.1. Monge–Ampère equation and regular Minkowski problem

An important point widely used in the paper is that the Minkowski problem can be formulated in terms of a Monge–Ampère equation on the unit ball \mathbb{B} in \mathbb{R}^d . Indeed the boundary of any F-convex set \mathbf{K} is the graph of a 1-Lipschitz function \tilde{u} on \mathbb{R}^d . The Legendre–Fenchel transform of \tilde{u} is a function h defined on \mathbb{B} (since \tilde{u} is 1-Lipschitz). This correspondence is in fact bijective, as any convex function h on \mathbb{B} defines dually a 1-Lipschitz convex function over the “horizontal” plane \mathbb{R}^d .

Recall that for a convex function h there is a well-defined Monge–Ampère measure $\text{MA}(h)$ on \mathbb{B} such that $\text{MA}(h)(\omega)$ is the Lebesgue measure of the set of sub-differentials at points in ω . If h is C^2 , then $\text{MA}(h) = \det(\text{Hess } h)\mathcal{L}$ where \mathcal{L} is the Lebesgue measure on \mathbb{B} .

There is a very direct relation between the area measure $A(\mathbf{K})$ and the Monge–Ampère measure $\text{MA}(h)$ of the corresponding function h on \mathbb{B} . Regarding \mathbb{R}^d as an affine chart in \mathbb{RP}^d , \mathbb{B} is identified to the set of time-like directions. This gives an identification $\mathbf{v} : \mathbb{B} \rightarrow \mathbb{H}^d$ and we simply have $\mathbf{v}_*^{-1}(A(\mathbf{K})) = (1 - \|x\|^2)^{1/2}\text{MA}(h)$.

The correspondence between F-convex sets and convex functions on \mathbb{B} allows to define an action of $\text{Isom}(\mathbb{R}^{d,1})$ on the space of convex functions on \mathbb{B} . Namely, if $\sigma \in \text{Isom}(\mathbb{R}^{d,1})$ and h is a convex function on \mathbb{B} corresponding to the F-convex set \mathbf{K} , then $\sigma \cdot h$ is the convex function on \mathbb{B} corresponding to the F-convex set $\sigma(\mathbf{K})$.

This action is made explicit in Lemma 3.4. It turns out that if σ is a pure translation then $\sigma \cdot h$ differs by h by an affine function, whereas if σ is linear, then $\sigma \cdot h$ is explicitly related (although not equal) to the pull-back of h by the projective transformation induced by σ on \mathbb{B} .

Using those correspondences, Theorem 1.1 can be also stated in terms of Monge–Ampère equation:

THEOREM 1.3. — *Let \mathbb{B} be the open unit ball of \mathbb{R}^d , and $\|\cdot\|$ be the usual norm. Let Γ be a uniform lattice in $\text{SO}^+(d, 1)$.*

Let μ be a Radon measure on \mathbb{B} such that $\sqrt{1 - \|x\|^2} \mu$ is invariant for the projective action of Γ on \mathbb{B} .

Then, for any convex function h_0 on $\overline{\mathbb{B}}$ such that, for any $\gamma \in \Gamma$, on $\partial\mathbb{B}$, the restriction of $\gamma \cdot h_0 - h_0$ coincides with the restriction of an affine map, there exists a unique convex function h on \mathbb{B} such that

- $\text{MA}(h) = \mu$, where MA is the Monge–Ampère measure,
- $h = h_0$ on $\partial\mathbb{B}$.

The uniqueness part of Theorem 1.1 follows from classical uniqueness result for the Monge–Ampère equation. Conversely, the existence part of Theorem 1.1 can be regarded as a new result about Monge–Ampère equation.

The main result about Monge–Ampère equation, close to the ones considered here, is the existence of a convex function h on a strictly convex open bounded set Ω , with prescribed continuous values on the boundary, satisfying $\text{MA}(h) = \mu$, but for μ with finite total mass, [30], [19, 1.4.6]. Also result in [23] is formulated in terms of existence of a solution of a Monge–Ampère equation.

Under this correspondence, in the Fuchsian case the covolume can be written as the functional introduced by I. Bakelman for the Monge–Ampère equation, see Remark 3.43.

Using again classical result of Monge–Ampère theory, we can get a “regular” version of Theorem 1.1. There are two main points that will allow to use those regularity results. The first one is a result contained in [23] which will imply that convex functions h on \mathbb{B} corresponding to τ -convex sets have a continuous extension g on the boundary of the ball. It turns

out that the convex envelope of g corresponds to the maximal τ -convex set Ω_τ . So g only depends on τ .

The second ingredient is a positive interior upper bound of the difference between the convex envelope of g in \mathbb{B} and the support function h of a τ -convex set with positive area measure. The example in Section 3.7 shows that some assumption must be added to get this bound. In particular we find a bound if the prescribed measure has sufficiently large total mass. This assumption can be weakened if the dual stratification of the initial singularity of Ω_τ contains only strata of dimension $\geq d/2$. In this case Ω_τ is called *simple*. This condition comes out as we notice here that the strata of the dual stratification pointed out in [11] correspond to the maximal regions of \mathbb{B} where the convex envelope of g is affine. Assuming that those regions have dimension $\geq d/2$ a multi-dimensional version of the Alexander Heinz Theorem ensures that h is strictly convex, providing the bound we need. This is stated in details in Theorem 2.34 and Corollary 2.37. See also Theorem 3.44.

1.1.2. Minkowski problem in flat space times with compact hyperbolic Cauchy surfaces

A Cauchy surface in a Lorentzian manifold S is an embedded hypersurface such that any inextensible time-like path meets S exactly at one point. A space-time is said globally hyperbolic (GH) if it contains a Cauchy surface. Topologically a GH space-time M is simply $S \times \mathbb{R}$, where S any Cauchy surface in M .

If S is any space-like hypersurface in a flat space-time N , then it is a Cauchy surface of some neighborhood in N . An extension of S is a GH flat space-time M where a neighborhood of S in N can be isometrically embedded so that the image of S is a Cauchy surface in M . By a very general result in general relativity [15], there exists a unique maximal extension M , where here maximal means that all the other extensions isometrically embeds into M .

A maximal globally hyperbolic compact flat space-time (we will use the acronym MGHCF) is a maximal flat space-time containing a compact Cauchy surface. In [24] Mess studied MGHCF space-times in dimension $2 + 1$, and a generalization of Mess theory was done in any dimension in [3, 11].

By [3] it is known that if S is the Cauchy surface of some MGHCF space-time then up to a finite covering S can be equipped with a metric locally isometric to $\mathbb{H}^k \times \mathbb{R}^{d-k}$ for some $0 \leq k \leq d$. We will be mainly interested

in the case where S is of hyperbolic case (that is $k = d$), as the Minkowski problem is not meaningful in the other cases (see Remark 1.5).

In this hyperbolic case it turns out that any MGHCF space-time (up to reversing the time orientation) is obtained as the quotient under an affine deformation Γ_τ of a uniform lattice Γ in $\text{SO}^+(d, 1)$ of the interior of the maximal Γ_τ -invariant F-convex set Ω_τ .

A subset \mathbf{K} in M is convex if whenever the end-points of a geodesic segment in M lies in \mathbf{K} , then the whole segment is in \mathbf{K} . Any τ -convex set is contained in Ω_τ and projects to a convex subset of M . The boundary of \mathbf{K} projects to a Cauchy surface in M , provided that \mathbf{K} does not meet the boundary of Ω_τ . Conversely if \mathbf{K} is a convex subset of M , its pre-image in Ω_τ is a τ -convex set.

By the flatness of the manifold M , the tangent bundle TM is a \mathbb{R}^{d+1} flat bundle with holonomy Γ . This implies that if S is any C^1 -space-like hypersurface, its normal field define a Gauss map

$$\mathcal{G}_S : S \rightarrow \mathbb{H}^d/\Gamma$$

which lifts to the usual Gauss map $\mathcal{G} : \tilde{S} \rightarrow \mathbb{H}^d$. If S is a C^1 -strictly convex Cauchy surface whose future is convex, then \mathcal{G}_S is a homeomorphism.

We say that a Cauchy surface in M is convex, if its future is convex. For any convex Cauchy surface it is possible to define a Gauss map as the set-valued function that sends a point $p \in S$ to the set of unit time-like direction orthogonal to local support planes of S at p . It turns out that \mathcal{G}_S is always surjective.

The area measure can be defined for every convex Cauchy surface in the same way as in the local model. If S is C^1 , the corresponding area measure is a measure on \mathbb{H}^d/Γ that coincides with the intrinsic volume form via the Gauss map. So the Minkowski problem in M asks for a convex Cauchy surface in M whose area measure is μ , where μ is a fixed measure of \mathbb{H}^d/Γ .

If S is smooth with positive Gauss–Kronecker curvature, then $\mu = f^{-1} dV$, where dV is the volume measure on \mathbb{H}^d/Γ and $f(x)$ is the Gauss–Kronecker curvature of S at $\mathcal{G}^{-1}(x)$.

Lifting the problem to the universal covering, it can be rephrased as a τ -equivariant Minkowski problem, with the additional requirement that the solution of the equivariant problem must be contained in the interior of Ω_τ . Even assuming that μ is a strictly positive measure (in the sense that it is bigger than $c dV$, where dV is the intrinsic measure on \mathbb{H}^d/Γ_τ), Theorem 1.1 does not ensure that the equivariant solution does not meet the boundary of Ω_τ . To this aim we need to require that μ has sufficiently big mass.

THEOREM 1.4. — *For any MGHCF space-time M , there is a constant c_0 such that if μ is a measure on \mathbb{H}^d/Γ such that $\mu > c_0 dV$, then there is a unique Cauchy surface S in M with area measure equal to μ . If Ω_τ is simple we may assume $c_0 = 0$.*

Remark 1.5. — By [3] if M is a MGHCF space-time whose Cauchy surfaces are not of hyperbolic-type, then for any convex Cauchy surface S in M the Gauss map on the universal covering $\mathcal{G} : \tilde{S} \rightarrow \mathbb{H}^d$ is never a proper map, as its image has dimension $k < d$ and the fibers are $n - k$ affine subspaces. Notice that in this case the Gauss–Kronecker curvature of S is zero, so the Minkowski problem is not meaningful in this case.

The results about the regular equivariant Minkowski problem allows to state the following theorem.

THEOREM 1.6. — *Let M be a flat MGHCF spacetime, let f be a positive function on \mathbb{H}^d/Γ of class C^{k+1} , $k \geq 2$.*

There exists $c(M) \geq 0$, such that if $f < c(M)$, then there exists a unique smooth strictly convex Cauchy surface S of class C^{k+2} in M such that $f(x)$ is the Gauss–Kronecker curvature of S at $\mathcal{G}_S^{-1}(x)$.

If Ω_τ is simple, then $c(M) = +\infty$.

1.2. Outline of the paper

In Section 2 we state some basic facts we need on F-convex sets in Minkowski space. In particular, we describe the connection between Minkowski problem in Minkowski space and the Monge–Ampère equation on the disc. As already mentioned this is achieved by considering the Legendre transform h of the function $\tilde{u} : \mathbb{R}^d \rightarrow \mathbb{R}$ whose graph is identified to the boundary of the F-convex set \mathbf{K} . More intrinsically we study the support function of \mathbf{K} , as defined in [17]. This is a 1-homogenous function on $\mathbb{I}^+(0)$. The area measure of \mathbf{K} can be directly computed by the restriction of the support function on \mathbb{H}^d , whereas its restriction on the ball coincides with h .

Although the material of this section is already known, we give a detailed exposition of the theory as we widely use this correspondence. Most of the results are only stated and some references are given for the proof. One of the results we prove in details is the multidimensional version of the Alexandrov–Heinz Theorem, Theorem 2.34. Although the proof follows closely the proof of the standard Alexandrov–Heinz Theorem given in [35],

we did not find a reference so details are given for the convenience of the reader.

In Section 3 we fix an affine deformation Γ_τ of a uniform lattice Γ and study τ -convex sets, i.e. F-convex sets invariant by Γ_τ .

After stating some simple properties on the action of the isometry group of Minkowski space on F-convex sets and dually on the space of support functions, we point out a structure of affine cone on the space of τ -convex sets. By using a result of [23] we show that the support function of a τ -convex set extends to the boundary of $I^+(0)$ and the value of the support function on the boundary g_τ only depends on the cocycle. Observing that the convex envelope of g_τ is the support function of the maximal τ -convex set Ω_τ , we will recover the strata of \mathbb{H}^d studied in [11] as maximal regions of \mathbb{H}^d where g_τ is an affine function.

We put a distance on the space of τ -convex sets that basically corresponds to the L^∞ -distance on the corresponding support functions. A compactness property of this distance is proved, Lemma 3.17, that can be regarded as the analog of Blaschke Selection Theorem for convex bodies. We then study area measures of τ -convex sets. They are Γ -invariant so can be regarded as measures on \mathbb{H}^d/Γ , in particular for any τ -convex set \mathbf{K} , its total area is the total mass of the induced measure on \mathbb{H}^d/Γ . We prove a monotonicity result for the total area Lemma 3.24, which implies that a τ -convex set with sufficiently big area does not meet the boundary of the maximal τ -convex set.

After this preliminary part we introduce the covolume and study its main properties. Covolume of \mathbf{K} is defined as the volume of the complement of \mathbf{K}/Γ_τ in Ω_τ/Γ_τ . In particular we prove that it is a convex functional on the set of τ -convex set. The proof of this result relies in a concavity result for the volume of particular convex bodies Lemma 3.30 and its Lorentzian version Corollary 3.31. A technical point in the proof is that we need to use a convex fundamental domain for the action of Γ_τ on a τ -convex set \mathbf{K} . Actually we are able to construct this fundamental domain only if \mathbf{K} lies in the interior of Ω_τ , but this is in fact sufficient for our aims. Using this convex fundamental domain we prove that the covolume is strictly convex on the convex subspace of τ -convex sets lying in the interior of Ω_τ . Then by approximation the convexity is proved in general.

Let us explain how convexity of the covolume is related to the Minkowski problem. In the Fuchsian case ($\tau = 0$), similarly to the Euclidean case for convex bodies [13], the derivative of the covolume along a family of support functions $t\bar{h}_1 + (1-t)\bar{h}_0$ at \bar{h}_0 is just $-\int(\bar{h}_1 - \bar{h}_0) dA(\bar{h}_0)$. Here \bar{h}

denotes the *hyperbolic support function* obtained by restricting on \mathbb{H}^d the 1-homogenous extension of the support function h on \mathbb{B} over $\mathbb{I}^+(0)$.

So if μ is a Γ -invariant measure and \bar{h}_0 is a Fuchsian solution of the Minkowski problem $A(\bar{h}_0) = \mu$, \bar{h}_0 turns to be a critical point of the functional $L_\mu(\bar{h}) = \text{covol}(\bar{h}) - \int \bar{h} d\mu$, and by convexity of L_μ it must be the minimum of \bar{h}_0 . So to find out the solution one looks at minima of the functional L_μ .

In the non Fuchsian case some difficulties arise. First in the Fuchsian case one has a simple representation formula for the covolume

$$\text{covol}(\bar{h}) = -\frac{1}{d+1} \int \bar{h} dA(\bar{h})$$

where \bar{h} is the support function [16]. In the affine case there is no similar explicit and simple formula.

Actually this is not a major problem, as the main point is a formula for the first order variation of the covolume, and not for the covolume itself. In fact by using the convexity of the covolume its derivative can be computed along a family of support functions $t\bar{h}_1 + (1-t)\bar{h}_0$ with $\bar{h}_1 \leq \bar{h}_0$ and this turns out to be sufficient to show that the minimum point of L_μ is a solution of the Minkowski problem.

Another difference with respect to the Fuchsian case is that the strategy to find the solution is slightly different. In fact first it was proved that the minimum \bar{h} of the covolume on some level set $\{\bar{h} \mid \int \bar{h} d\mu = 1\}$ satisfies the equation $A(\bar{h}) = c\mu$ where c is a some positive. Then by scaling \bar{h} by $c^{-1/d}$ one finds the solution. A similar strategy could be also followed in the case we consider in the present paper, but, due to the fact that the set of τ -convex sets is not stable under homotheties, the argument would become more intricate and the strategy we propose seems in this case more direct and simple.

Indeed, using the convexity of the covolume one sees that the difference between the covolume of two τ -convex sets $\mathbf{K}_0 \subset \mathbf{K}_1$ can be estimated in terms of the integrals of the difference of the corresponding support functions with respect to respective area measures, see formula (3.16). Then refining the expected arguments in the Fuchsian case one sees that Γ_τ -equivariant solutions of the equations $A(\bar{h}_0) = \mu$ corresponds to minima of the functional $L_\mu(\bar{h}) = \text{covol}(\bar{h}) - \int (\bar{h} - \bar{h}_\tau) d\mu$. The existence of the minimum is then simply achieved using the compactness property of the support functions.

Once the general Minkowski problem is proved, using the correspondence with Monge–Ampère equation we see that if the measure μ is regular with

respect to the volume form of \mathbb{H}^d with smooth and strictly positive density, the corresponding solution is smooth provided that the τ -convex set lies in the interior of Ω_τ . A counterexample is then given to show that in general even if μ coincides with the Lebesgue measure up to a constant factor the corresponding domain can meet the boundary of Ω_τ .

Finally in the appendix we prove that the space of C^2_+ τ -convex sets is dense in the space of τ -convex sets (where C^2 can be replaced by smooth). Here a C^2_+ τ -convex set is a C^2 τ -convex set such that the Gauss map is a C^1 -diffeomorphism, or analogously with positive Gauss–Kronecker curvature is strictly positive at any point. This basically implies that any convex hypersurface in a maximal globally hyperbolic flat space-time can be approximated by smooth strictly convex hypersurfaces. Although the fact is not surprising, there was no proof at our knowledge, so we give some details.

1.3. Notations

$\mathbb{R}^{d,1}$	Minkowski space
\mathbb{R}^d	Euclidean space identified with the <i>horizontal</i> hyperplane in $\mathbb{R}^{d,1}$
\mathbb{B}	unit ball in \mathbb{R}^d centred at 0
$\hat{\mathbb{B}}$	intersection of $I^+(0)$ with the affine plane $(0, \dots, 0, 1) + \mathbb{R}^d$
$I^+(0)$	future cone of 0 in $\mathbb{R}^{d,1}$
\mathbb{H}^d	hyperboloid model of hyperbolic space
$J^+(\mathbb{H})$	convex region bounded by the hyperboloid in $\mathbb{R}^{d,1}$
$\langle \cdot, \cdot \rangle_-$	Minkowski product
$P_H(X)$	hyperplane defined by the equation $\langle z, X \rangle = H(X)$
$\pi: \mathbb{R}^{d,1} \rightarrow \mathbb{R}^d$	orthogonal projection
\hat{x}	$(x, 1) \in \hat{\mathbb{B}}$ for $x \in \mathbb{B}$
$\lambda(x)$	$\sqrt{1 - \ x\ ^2}$ for $x \in \mathbb{B}$
$\mathbf{v}: \mathbb{B} \rightarrow \mathbb{H}^d$	radial identification
grad^-	Lorentzian gradient in $\mathbb{R}^{d,1}$
$\text{grad}^{\mathbb{H}}$	hyperbolic gradient in \mathbb{H}^d
grad	Euclidean gradient in \mathbb{R}^d
∇	Levi-Civita connection on \mathbb{H}^d
∇^2	covariant Hessian in \mathbb{H}^d
Hess	Euclidean Hessian in \mathbb{R}^d
\mathbf{K}	F-convex set
$\partial_s \mathbf{K}$	space-like boundary of \mathbf{K}
$\partial_{\text{reg}} \mathbf{K}$	regular part of the space-like boundary
$\mathcal{G}: \partial_s \mathbf{K} \rightarrow \mathbb{H}^d$	Gauss map

$\chi: \mathbb{H}^d \rightarrow \partial_s \mathbf{K}$	inverse of the Gauss map
C_+^2	F-convex set with $\partial_s \mathbf{K} \in C^2$ and \mathcal{G} a C^1 diffeomorphism
$H: I^+(0) \rightarrow \mathbb{R}$	extended support function
$\bar{h}: \mathbb{H}^d \rightarrow \mathbb{R}$	hyperbolic support function
$h: \mathbb{B} \rightarrow \mathbb{R}$	ball support function
∂h	subdifferential of h
$\mathcal{P}(x)$	Lorentzian Dirichlet polyhedron
\mathcal{L}	Lebesgue measure of \mathbb{R}^d
$d\mathbb{H}^d$	volume measure of \mathbb{H}^d
$\text{MA}(h)$	Monge–Ampère measure of h
$A(\mathbf{K}) = A(\bar{h})$	area measure on \mathbb{H}^d
$A(h)$	area measure on \mathbb{B} , $A(h) = (\mathbf{v}^{-1})_* (A(\bar{h}))$
ϕ	curvature function, density of $A(\bar{h})$ for $\bar{h} \in C^2$
$\text{Area}(\mathbf{K})$	$A(K)(\mathbb{H}^d/\Gamma)$
$V_\epsilon(\mathbf{K})$	measure on \mathbb{H}^d introduced in Definition 2.10
$S_i(\mathbf{K})$	area measure of order i
$\bar{V}_\epsilon(\mathbf{K})$	measure on \mathbb{H}^d/Γ induced by $V_\epsilon(\mathbf{K})$
$\bar{S}_i(\mathbf{K})$	area measure of order i on \mathbb{H}^d/Γ induced by $S_i(\mathbf{K})$
covol	covolume
$\tilde{u}: \mathbb{R}^d \rightarrow \mathbb{R}$	function whose graph is $\partial \mathbf{K}$
$\tilde{h}: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$	Legendre–Fenchel transform of \tilde{u}
u	restriction of \tilde{u} on $\pi(\partial_s \mathbb{B}) = \nabla u(\mathbb{B})$
g	continuous function on $\partial \mathbb{B}$
h_g	convex envelope of g
Ω_g	F-convex set with support function h_g
$F_g(x)$	maximal convex set of \mathbb{B} containing x on which h_g is affine
Γ	uniform lattice in $\text{SO}_0(2, 1)$
$\tau: \Gamma \rightarrow \mathbb{R}^{d+1}$	cocycle
Γ_τ	affine deformation of Γ
γ	element of Γ
γ_τ	affine transformation $p \rightarrow \gamma(p) + \tau_\gamma$
$\tilde{\gamma}$	projective transformation on \mathbb{B} induced by γ
Ω_τ	maximal τ -convex set
$H_\tau, \bar{h}_\tau, h_\tau$	support functions of Ω_τ
g_τ	boundary value of the support function of any τ -convex set
T	cosmological time of \mathbf{K}
Σ_t	level set $T^{-1}(t)$

2. Area measure of F-convex sets

In this section, we recall some facts about particular convex sets in Minkowski space and the Monge–Ampère equation.

2.1. Background on convex sets

Notations. For a set S in $\mathbb{R}^{d,1}$, $I^+(S)$ is the set of the final points of curves directed by a future time-like vector and starting from S . In particular $I^+(0)$ is the future cone of the origin.

We will consider \mathbb{R}^d as a subset of $\mathbb{R}^{d,1}$ through the isometric inclusion $x \mapsto (x, 0)$. We denote by P the affine hyperplane $(0, \dots, 0, 1) + \mathbb{R}^d$: for $x \in \mathbb{R}^d$ we denote by $\hat{x} = (x, 1)$ the corresponding point on P . The intersection of P with $I^+(0)$ is $(0, \dots, 0, 1) + \mathbb{B}$, where \mathbb{B} is the unit ball centered in 0 in \mathbb{R}^d . We denote the translated ball by $\hat{\mathbb{B}} = (0, \dots, 0, 1) + \mathbb{B}$.

Let us introduce the following notations

- for $X \in \mathbb{R}^{d,1}$ we put $\|X\|_- = \sqrt{|\langle X, X \rangle|_-}$
- $\langle \cdot, \cdot \rangle$ is the usual scalar product on \mathbb{R}^d ,
- $\|\cdot\|$ is the associated norm,
- $\lambda(x) = \sqrt{1 - \|x\|^2} = \|\hat{x}\|_-$, $x \in \mathbb{B}$,
- $\pi : \mathbb{R}^{d,1} \rightarrow \mathbb{R}^d$ is the orthogonal projection sending (x, x_{d+1}) to x
- \mathcal{L} is the Lebesgue measure on \mathbb{R}^d .

The radial map \mathbf{v} from \mathbb{B} to \mathbb{H}^d is $\mathbf{v}(x) = \frac{1}{\lambda} \hat{x}$, while \mathbf{v}^{-1} maps $(x, x_{d+1}) \in \mathbb{H}^d$ to $x/x_{d+1} = x/\sqrt{1 + \|x\|^2} \in \mathbb{B}$. It can be useful to consider on \mathbb{B} the pull-back of the hyperbolic metric (we then get the Klein model of the hyperbolic space). If $d\mathbb{H}^d$ is the volume element for the hyperbolic metric on \mathbb{B} , an explicit computation gives

$$(2.1) \quad d\mathbb{H}^d = \lambda^{-d-1} \mathcal{L}.$$

F-convex sets. Let \mathbf{K} be a closed convex set which is defined as an intersection of the future side of space-like hyperplanes of $\mathbb{R}^{d,1}$. Note that it may have light-like support planes but no time-like support plane. We will denote by $\partial_s \mathbf{K}$ the part of the boundary $\partial \mathbf{K}$ of \mathbf{K} which meets space-like support planes. The Gauss map of \mathbf{K} is a set-valued map from $\partial_s \mathbf{K}$ to \mathbb{H}^d , which associates to a point of $\partial_s \mathbf{K}$ all the unit future time-like support vectors of \mathbf{K} at x . We say that \mathbf{K} is an *F-convex set* if its Gauss map is surjective. In other terms, any unit future time-like vector is a support vector of \mathbf{K} . An F-convex set \mathbf{K} is called C^2_+ if its boundary is a C^2 hypersurface and its Gauss map is a C^1 diffeomorphism. We say that \mathbf{K} is C^∞_+ if its boundary is smooth and its Gauss map is a diffeomorphism.

Support functions. Let \mathbf{K} be an F-convex set. The (extended) support function H of \mathbf{K} is the map from $I^+(0)$ onto \mathbb{R} given by

$$H(X) = \sup\{\langle p, X \rangle_- \mid \forall p \in \mathbf{K}\}.$$

By definition the support function is sublinear: it is homogeneous of degree 1 and subadditive, or equivalently 1-homogenous and convex. Conversely any sublinear function on $I^+(0)$ is the support function of a unique F -convex set [17].

By homogeneity, H is determined by its restriction on any hypersurface of $I^+(0)$ that meets all the radial lines, like for instance \mathbb{H}^d or the translated ball \mathbb{B} . We will denote by \bar{h} the restriction of H on \mathbb{H}^d . It will be called the *hyperbolic support function* of \mathbf{K} . Analogously we set $h : \mathbb{B} \rightarrow \mathbb{R}$ by $h(x) = H(\hat{x})$, and refer to h as the *ball support function* of \mathbf{K} . By the homogeneity of H we get

$$h(x) = \lambda(x)\bar{h}(\mathbf{v}(x)).$$

Of course the support function h is convex on \mathbb{B} . The following lemma says that all convex functions on \mathbb{B} are obtained in this way.

LEMMA 2.1. — *A convex function h on \mathbb{B} extends (in the sense above) to a unique sublinear function on $I^+(0)$, namely*

$$H((x, z)) = zh(x/z),$$

$x \in \mathbb{B}, z \in \mathbb{R}$.

Proof. — The function H is 1-homogeneous by definition. So we only need to prove that it is subadditive. Take $X = (x, x_{d+1}), Y = (y, y_{d+1}) \in I^+(0)$. By definition,

$$H(X + Y) = (x_{d+1} + y_{d+1})h\left(\frac{x + y}{x_{d+1} + y_{d+1}}\right).$$

Notice that the point $\frac{x+y}{x_{d+1}+y_{d+1}}$ can be expressed as the following convex combination of x/x_{d+1} and y/y_{d+1} :

$$\frac{x + y}{x_{d+1} + y_{d+1}} = \frac{x_{d+1}}{x_{d+1} + y_{d+1}} \frac{x}{x_{d+1}} + \frac{y_{d+1}}{x_{d+1} + y_{d+1}} \frac{y}{y_{d+1}}.$$

Since h is convex we get that

$$\begin{aligned} & H(X + Y) \\ & \leq (x_{d+1} + y_{d+1}) \left(\frac{x_{d+1}}{x_{d+1} + y_{d+1}} h\left(\frac{x}{x_{d+1}}\right) + \frac{y_{d+1}}{x_{d+1} + y_{d+1}} h\left(\frac{y}{y_{d+1}}\right) \right) \\ & = H(X) + H(Y). \end{aligned} \quad \square$$

COROLLARY 2.2. — *Any convex function on \mathbb{B} is the support function of a unique F -convex set of $\mathbb{R}^{d,1}$.*

Example 2.3. — The support function of the convex side of the upper-sheet of the hyperboloid with radius t is $H(X) = -t\|X\|_-$, and its restriction to \mathbb{H}^d is the constant function equal to $-t$. Its restriction to \mathbb{B} is $h(x) = H(\hat{x}) = -t\lambda(x)$. Note that $\text{Hess } h = -t \text{Hess } \lambda$, so the convexity of h implies that $\text{Hess } \lambda$ is semi-negative definite.

We will denote by $J^+(\mathbb{H})$ the convex side of the upper-sheet of the hyperboloid with radius 1.

Example 2.4. — The restriction to $I^+(0)$ of the linear functionals of \mathbb{R}^{d+1} are the support functions on $I^+(0)$ of the future cone of points. Their restriction to \mathbb{B} are exactly the restriction to \mathbb{B} of the affine maps of \mathbb{R}^d .

The support function H determines \mathbf{K} : indeed for any $X \in I^+(0)$ the affine plane $P_H(X) = \{p \in \mathbb{R}^{d,1} \mid \langle p, X \rangle_- = H(X)\}$ is the support plane for \mathbf{K} orthogonal to X so

$$\mathbf{K} = \bigcap_{X \in I^+(0)} \overline{I^+(P_H(X))}.$$

A less obvious remark is that $P_H(X)$ meets $\partial_s \mathbf{K}$ at least in one point, see [17]. Under some regularity assumption on H it is possible to determine the intersection between $\partial_s \mathbf{K}$ and $P_H(X)$. Indeed, if H is C^1 , by convexity we have $H(Y) \geq H(X) + \langle \text{grad}_X^- H, Y - X \rangle_-$, where

$$\text{grad}^-(H) = (\partial_1 H, \dots, \partial_d H, -\partial_{d+1} H)$$

is the Minkowski gradient. Moreover, since H is 1-homogeneous we have

$$H(X) = \langle \text{grad}_X^- H, X \rangle_- ,$$

so we deduce that $\langle \text{grad}_X^- H, Y \rangle_- \leq H(Y)$, that implies that $\text{grad}_X^- H$ lies on $I^+(P_H(Y))$. Indeed, $\text{grad}_X^- H$ is the only point in \mathbf{K} with support plane orthogonal to X , that is $\text{grad}_X^- H = \mathcal{G}^{-1}(X)$. In fact the following lemma holds:

LEMMA 2.5 ([17]). — *If the support function H of a F -convex set \mathbf{K} is C^1 , then $\partial_s \mathbf{K}$ is the range of the map $\chi : I^+(0) \rightarrow \mathbb{R}^{d+1}$ sending X to $\text{grad}_X^- H$. Moreover, $\partial_s \mathbf{K}$ is strictly convex, in the sense that each space-like support plane meets \mathbf{K} at one point.*

Conversely if $\partial_s \mathbf{K}$ is strictly convex, the support function H is C^1 .

Assume that H is C^1 , or analogously that $\partial_s \mathbf{K}$ is strictly convex. Denote by $\chi : I^+(0) \rightarrow \mathbb{R}^{d,1}$ the map sending X to $\text{grad}_X^- H$. As H is 1-homogeneous, χ is constant along the radial rays. So we have $\partial_s \mathbf{K} = \chi(\mathbb{H}^d) = \chi(\hat{\mathbb{B}})$. The restriction of χ to \mathbb{H}^d (respectively $\hat{\mathbb{B}}$) can be easily expressed in terms of the support functions \bar{h} (respectively h):

LEMMA 2.6. — Assume that the support function H is C^1 . Then, for $\eta \in \mathbb{H}^d$

$$(2.2) \quad \chi(\eta) = \text{grad}_{\eta}^{\mathbb{H}} \bar{h} - \bar{h}(\eta)\eta$$

where $\text{grad}^{\mathbb{H}}$ is the hyperbolic gradient regarded as a vector in \mathbb{R}^{d+1} by the natural inclusion $T_{\eta}\mathbb{H}^d \subset \mathbb{R}^{d,1}$.

Analogously, for $x \in \mathbb{B}$

$$(2.3) \quad \chi(\hat{x}) = \text{grad}_x h + (\langle x, \text{grad}_x h \rangle - h(x))e_{d+1}$$

where grad is the Euclidean gradient of \mathbb{R}^d .

Proof. — The tangential component of $\text{grad}_{\eta}^{-} H$ is clearly the intrinsic gradient of \bar{h} . The normal component is obtained by imposing that $D_{\eta}H(\eta) = H(\eta) = \bar{h}(\eta)$.

Analogously the tangential component of the gradient of H to \mathbb{R}^d is equal to the gradient of h . To compute the normal component,

$$D_{\hat{x}}H(e_{d+1}) = D_{\hat{x}}H(\hat{x}) - D_{\hat{x}}H(\hat{x} - e_{d+1}) = H(\hat{x}) - D_x h(x)$$

where $x \in \mathbb{R}^d$ is identified with $(x, 0) \in \mathbb{R}^{d+1}$. Formula (2.3) follows. \square

By Lemma 2.5 it turns out that the injectivity of the Gauss map of \mathbf{K} is related to the regularity of H . In particular the support function of a C^2_+ F -convex set must be C^1 . This condition is not sufficient. Indeed it implies that $\chi = \mathcal{G}^{-1}$ is a well-defined map, but still \mathcal{G} could be set valued: e.g., take $\mathbf{K} = I^+(0)$, its support function is $H = 0$ and \mathcal{G} maps 0 to the whole \mathbb{H}^d .

The following characterization of C^2_+ F -convex sets in terms of the hyperbolic support functions is proved in [17].

LEMMA 2.7. — A function \bar{h} on \mathbb{H}^d is the support function of a C^2_+ F -convex set iff it is C^2 and satisfies

$$(2.4) \quad \nabla^2 \bar{h} - \bar{h}g_{\mathbb{H}} > 0$$

with $g_{\mathbb{H}}$ the hyperbolic metric and ∇^2 the hyperbolic Hessian.

Condition (2.4) can be also expressed in term of the ball support function h , as the next lemma suggests.

LEMMA 2.8. — Let H be the support function of an F -convex set \mathbf{K} . If H is C^2 , then

$$(2.5) \quad \text{Hess}_x h(v, w) = \lambda(x) (\nabla^2 \bar{h} - \bar{h}g) (D_x \mathbf{v}(v), D_x \mathbf{v}(w))$$

where Hess is the Euclidean Hessian on \mathbb{R}^d .

In particular \mathbf{K} is C^2_+ iff h is C^2 with positive Hessian everywhere.

Proof. — By 0-homogeneity of χ and because $\lambda\mathbf{v}$ is the identity

$$h(x) = \langle \chi(\mathbf{v}(x)), (\lambda\mathbf{v})(x) \rangle_- .$$

We compute

$$\begin{aligned} D_x h(v) &= \langle D_x(\chi \circ \mathbf{v})(v), (\lambda\mathbf{v})(x) \rangle_- + \langle \chi(\mathbf{v}(x)), D_x(\lambda\mathbf{v})(v) \rangle_- \\ &= \langle \chi(\mathbf{v}(x)), D_x(\lambda\mathbf{v})(v) \rangle_- \end{aligned}$$

because $\mathbf{v}(x)$ is orthogonal to the tangent plane at the point $\chi(\mathbf{v}(x))$. For the same reason, and also because $\text{Hess}_x(\lambda\mathbf{v}) = 0$ (because $\lambda\mathbf{v}$ is the identity) we compute that

$$\text{Hess}_x h(v, w) = \lambda(x) \langle D_{\mathbf{v}(x)} \chi(D_x \mathbf{v}(v)), D_x \mathbf{v}(w) \rangle_-$$

that together with

$$\text{Hess}_\eta H(v, w) = \nabla^2 \bar{h}(v, w) - \bar{h}g_{\mathbb{H}}(v, w) = \langle D_\eta \chi(v), w \rangle_-$$

gives (2.5). □

We denote by $C_+^2(\mathbb{H}^d)$ the set of hyperbolic support functions satisfying (2.4).

Since $\chi|_{T\mathbb{H}^d}$ is the inverse of the Gauss map, it turns out that if \mathbf{K} is C_+^2 then $D\chi|_{T\mathbb{H}^d}$ is the inverse of the shape operator. Now by Equation (2.2) we have that if $v \in T_\eta \mathbb{H}^d$ then

$$D\chi(v) = D(\text{grad}^{\mathbb{H}}(\bar{h}))(v) - D\bar{h}(v)\eta - \bar{h}v .$$

By Gauss–Weingarten formulas we deduce

$$D(\text{grad}^{\mathbb{H}}(\bar{h})) = \nabla_v(\text{grad}^{\mathbb{H}}(\bar{h})) + \text{II}(\text{grad}^{\mathbb{H}}(\bar{h}), v)\eta$$

where ∇ is the Levi-Civita connection on \mathbb{H}^d and II is the second fundamental form of the immersion $\mathbb{H}^d \subset \mathbb{R}^{d,1}$. Since the second fundamental form is the identity, putting together these formulas we have

$$D\chi|_{T\mathbb{H}^d} = \nabla(\text{grad}^{\mathbb{H}}(\bar{h})) - \bar{h} \text{Id} .$$

It follows that the *radii of curvature* r_i of \mathbf{K} at some point p are the eigenvalues of $\nabla^2 \bar{h} - \bar{h}g_{\mathbb{H}}$ (computed with respect to the hyperbolic metric) at $\mathcal{G}(p)$. The *curvature function* of \mathbf{K} is

$$(2.6) \quad \phi = \prod_{i=1}^d r_i = \det(\nabla^2 \bar{h} - \bar{h}g_{\mathbb{H}}) .$$

The *mean radius of curvature* is defined as $\varphi = \frac{1}{d} \sum_{i=1}^d r_i = \frac{1}{d} \text{tr}(\nabla^2 \bar{h} - \bar{h}g_{\mathbb{H}})$.

Both the curvature function and the mean radius of curvature are defined on \mathbb{H}^d . If \mathcal{G} is the Gauss map of \mathbf{K} , $\frac{1}{\phi} \circ \mathcal{G}$ is the Gauss–Kronecker curvature of $\partial_s \mathbf{K}$.

It is useful to express the curvature function and the mean radius of curvature in terms of the ball support function h .

LEMMA 2.9. — *Let \mathbf{K} be an F -convex set, and h be the corresponding support function on \mathbb{B} . If h is C^2 we have*

$$(2.7) \quad \det \text{Hess}_x h = \lambda^{-d-2}(x)\phi(\mathbf{v}(x))$$

$$(2.8) \quad \varphi(\mathbf{v}(x)) = \frac{\lambda(x)}{d}(\text{tr Hess}_x h - \text{Hess}_x h(x, x))$$

with ϕ the curvature function and φ the mean radius of curvature.

Proof. — We know $\det \nabla^2 \bar{h} - \bar{h}g_{\mathbb{H}} = \phi$ so, by (2.5) $\det \text{Hess } h = \lambda^d \det(D\mathbf{v})^2 \phi$ and we already know $\det(D\mathbf{v})$, see (2.1).

Let \square be the wave operator on \mathbb{R}^{d+1} :

$$\square_{\hat{x}} H = \text{tr Hess}_x h - \text{Hess}_{\hat{x}} H(e_{d+1}, e_{d+1}).$$

Writing

$$e_{d+1} = e_{d+1} - \hat{x} + \hat{x},$$

as by homogeneity $\text{Hess}_X H(X, \cdot) = 0 \forall X$, we have $\text{Hess}_{\hat{x}} H(e_{d+1}, e_{d+1}) = \text{Hess}_x h(x, x)$. (2.8) follows because $\varphi(\mathbf{v}(x)) = \frac{1}{d} \square_{\hat{x}/\lambda} H$ and $\square_{\hat{x}/\lambda} H = \lambda \square_{\hat{x}} H$ by (-1) -homogeneity. □

Area measures. Let $\omega \subset \mathbb{H}^d$ be a Borel set.

DEFINITION 2.10. — *We will denote by $V_\epsilon(\mathbf{K})(\omega)$ the volume of the the set of points of \mathbf{K} which are at Lorentzian distance at most ϵ from $\partial_s \mathbf{K}$ and such that the direction of their orthogonal projection onto $\partial_s \mathbf{K}$ is colinear to an element of ω .*

The (Lorentzian) area measure of \mathbf{K} is a Radon measure on \mathbb{H}^d defined by

$$A(\mathbf{K})(\omega) = \lim_{\epsilon \downarrow 0} \frac{V_\epsilon(\mathbf{K})(\omega) - V_0(\mathbf{K})(\omega)}{\epsilon} = \lim_{\epsilon \downarrow 0} \frac{V_\epsilon(\mathbf{K})(\omega)}{\epsilon}.$$

Remark 2.11. — In the convex body case, the area measure was introduced under the name area function by A.D. Alexandrov. A difference with respect to the Euclidean case is that the Lorentzian projection on $\partial_s \mathbf{K}$ is well-defined in the interior of \mathbf{K} (see [11] for a discussion on this point). For this reason is convenient to define $V_\epsilon(\mathbf{K})(\omega)$ as the volume of points on \mathbf{K} at distance ϵ from the boundary instead of considering the exterior of \mathbf{K} .

If \mathbf{K} has support function \bar{h} on \mathbb{H}^d , we will denote $A(\mathbf{K})$ by $A(\bar{h})$. If $\bar{h} \in C^2_+(\mathbb{H}^d)$, then

$$(2.9) \quad A(\bar{h}) = \det(\nabla^2 \bar{h} - \bar{h}g_{\mathbb{H}}) d\mathbb{H}^d = \phi d\mathbb{H}^d,$$

where $d\mathbb{H}^d$ is the volume form on \mathbb{H}^d given by the hyperbolic metric and ϕ the curvature function. Notice that in this case $A(\bar{h})$ coincides with the direct image of the intrinsic volume form on $\partial_s \mathbf{K}$ through the Gauss map.

We need a result of weak convergence of the area measure. For completeness we will consider all kinds of area measures. It is proved in [17] that there exist Radon measures $S_0(\mathbf{K}), \dots, S_d(\mathbf{K})$ on \mathbb{H}^d such that, for any Borel set ω of \mathbb{H}^d and any $\epsilon > 0$,

$$(2.10) \quad V_\epsilon(\mathbf{K})(\omega) = \frac{1}{d+1} \sum_{i=0}^d \epsilon^{d+1-i} \binom{d+1}{i} S_i(\mathbf{K})(\omega).$$

$S_i(\mathbf{K})$ is called the *area measure of order i* of \mathbf{K} . Note that $V_\epsilon(\mathbf{K})$ is itself a Radon measure. We have that $S_0(\mathbf{K})$ is given by the volume form of \mathbb{H}^d . The area measure $A(\mathbf{K})$ introduced above coincides with the area measure of order d , that is $S_d(\mathbf{K})$.

LEMMA 2.12. — *Let $(\bar{h}_n)_n$ be support functions on \mathbb{H}^d of F-convex sets converging uniformly on a compact set ω to a support function \bar{h} . Then, for any continuous function $f : \mathbb{H}^d \rightarrow \mathbb{R}$ with compact support contained in the interior of ω , for any $\epsilon > 0$,*

$$\int_\omega f dV_\epsilon(\mathbf{K}_n) \rightarrow \int_\omega f dV_\epsilon(\mathbf{K}) \quad \text{as } n \rightarrow \infty,$$

and, for $0 \leq i \leq s$,

$$\int_\omega f dS_i(\mathbf{K}_n) \rightarrow \int_\omega f dS_i(\mathbf{K}) \quad \text{as } n \rightarrow \infty.$$

Proof. — The convergence for the measures $S_i(\mathbf{K})$ follows from the convergence for $V_\epsilon(\mathbf{K})$ and the linearity of the integral, using a polynomial interpolation from (2.10). We have to prove the convergence result for $V_\epsilon(\mathbf{K})$. This follows directly from the following facts, which are explicitly proved in [17]:

- the result is true if the F-convex sets are invariant under the action of a same uniform lattice Γ , provided that ω is contained in a fundamental domain for the action of Γ ;
- there exist a uniform lattice Γ and Γ -invariant convex sets $\tilde{\mathbf{K}}_n$ and $\tilde{\mathbf{K}}$ such that the set of points on \mathbf{K}_n (resp. \mathbf{K}) with support normals in ω is, up to a translation, the set of points on $\tilde{\mathbf{K}}_n$ (resp. $\tilde{\mathbf{K}}$) with support normals in ω . As for any subset b , $V_\epsilon(\mathbf{K})(b)$ depends only

on the subset of $\partial\mathbf{K}$ of points with support normals in b , and not on the whole \mathbf{K} , it follows that $V_\epsilon(\tilde{\mathbf{K}}_n)$ (resp. $V_\epsilon(\tilde{\mathbf{K}})$) coincide with $V_\epsilon(\mathbf{K}_n)$ (resp. $V_\epsilon(\mathbf{K})$) on ω ;

- as (\bar{h}_n) uniformly converges to \bar{h} on ω , the same is true for the support functions of $\tilde{\mathbf{K}}_n$ and $\tilde{\mathbf{K}}$. □

2.2. F-convex sets as graphs

Let \mathbf{K} be an F-convex set. Since \mathbf{K} has no time-like support plane, each time-like line must meet \mathbf{K} . Moreover since \mathbf{K} is future complete ($\mathbf{K} = \mathbb{I}^+(\mathbf{K})$) the intersection must be a future-complete half-line. So each time-like line meets $\partial\mathbf{K}$ exactly at one point. This shows that $\partial\mathbf{K}$ is a graph of a function \tilde{u} defined on the horizontal plane $\mathbb{R}^d \subset \mathbb{R}^{d+1}$. Convexity of \mathbf{K} implies that \tilde{u} is a convex function.

Let h be the support function of \mathbf{K} on \mathbb{B} . In this section we investigate the relation between h and \tilde{u} . To this aim it is convenient to consider the extension of h to \mathbb{R}^d defined as

$$\tilde{h} : \mathbb{R}^d \rightarrow \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}, \quad \tilde{h}(x) = \sup \{ \langle k, \hat{x} \rangle_- \mid k \in \mathbf{K} \}.$$

It is a proper convex lower semi-continuous function (as a sup of continuous functions). It takes values $+\infty$ out of $\bar{\mathbb{B}}$ and is equal to h on \mathbb{B} . Its value on $\partial\mathbb{B}$ may be finite or infinite. In other terms, \tilde{h} is the lower semi-continuous hull of h . If \tilde{h} has a finite value at $\ell \in \partial\mathbb{B}$, then \mathbf{K} has a light-like support plane “at infinity” directed by ℓ . This means that any parallel displacement of the hyperplane in the future direction will meet the interior of \mathbf{K} . The hyperplane may or may not meet \mathbf{K} .

By properties of lower semi-continuous convex functions, for any $\ell \in \partial\mathbb{B}$ and any $x \in \mathbb{B}$, we have

$$(2.11) \quad \tilde{h}(\ell) = \lim_{t \downarrow 0} h(\ell + t(x - \ell)).$$

Let us write

$$\begin{aligned} \tilde{h}(x) &= \sup_{(p, p_{d+1}) \in \mathbf{K}} \left\langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} p \\ p_{d+1} \end{pmatrix} \right\rangle_- \\ &= \sup_{(p, p_{d+1}) \in \mathbf{K}} \{ \langle x, p \rangle - p_{d+1} \} \\ &= \sup_{p \in \mathbb{R}^d} \{ \langle x, p \rangle - \tilde{u}(p) \} \end{aligned}$$

where \tilde{u} is the graph of $\partial\mathbf{K}$. From this simple relation we immediately get

DEFINITION 2.13. — *The function \tilde{h} is the dual of \tilde{u} . The correspondence $\tilde{u} \mapsto \tilde{h}$ is the Legendre–Fenchel transform.*

Conversely by the involution property of the Legendre–Fenchel transform we have that \tilde{u} is the dual of \tilde{h} , that is

$$\tilde{u}(p) = \sup_{x \in \mathbb{R}^d} (\langle p, x \rangle - \tilde{h}(x))$$

for all $p \in \mathbb{R}^d$.

Since the dual of \tilde{u} (which is \tilde{h}) takes finite values only on \mathbb{B} , it turns out that \tilde{u} is 1-Lipschitz. Indeed a general fact is that proper future convex sets are exactly the graphs of 1-Lipschitz convex functions [11, Lemma 3.11]. As \tilde{u} is convex and 1-Lipschitz, by Rademacher theorem, $\text{grad } \tilde{u}$ exists and is continuous almost everywhere, and its norm is less than 1.

Remark 2.14. — Given a convex 1-Lipschitz function $\tilde{u} : \mathbb{R}^d \rightarrow \mathbb{R}$, its graph bound a convex set \mathbf{K} that coincides with its future. For $x \in \mathbb{B}$ the point $\mathbf{v}(x)$ lies in the image of the Gauss map of \mathbf{K} iff \tilde{h} is finite at x . So \mathbf{K} is an F-convex set iff the function \tilde{h} , dual of \tilde{u} , takes finite values on \mathbb{B} .

For a convex function f defined on some convex domain U of \mathbb{R}^d , the *normal mapping*, or *subdifferential* of f at $x \in U$, denoted by $\partial f(x)$, is a subset of \mathbb{R}^d defined as follow: it is the horizontal projection onto \mathbb{R}^d of the set of inward (Lorentzian) normals with last coordinate equal to one of the support hyperplanes of the graph of f at the point $(x, f(x))$. In other terms, $p \in \partial f(x)$ if and only if, for all $y \in U$, $\langle \begin{pmatrix} p \\ 1 \end{pmatrix}, \begin{pmatrix} y-x \\ f(y)-f(x) \end{pmatrix} \rangle_- \leq 0$, i.e. $f(y) \geq f(x) + \langle p, y-x \rangle$. It turns out that $\partial f(x)$ is a convex set in \mathbb{R}^d , [19]. If ω is a Borel set of U , we set $\partial f(\omega) := \cup_{x \in \omega} \partial f(x)$.

Let h be a convex function on \mathbb{B} and \tilde{h} be its extension to \mathbb{R}^d . Note that for $x \in \mathbb{B}$, $\partial \tilde{h}(x) = \partial h(x)$.

If \tilde{u} is the conjugate of \tilde{h} , we have [32, 11.3]

$$(2.12) \quad \partial \tilde{u} = (\partial \tilde{h})^{-1}$$

that has to be understood as: $p \in \partial \tilde{h}(x)$ if and only if $x \in \partial \tilde{u}(p)$

From (2.12) we simply deduce the following relation between ∂h and the Gauss map \mathcal{G} of the domain supported by h .

LEMMA 2.15. — *Let \mathcal{G} be the Gauss map of the F-convex set supported by h . Let π be the orthogonal projection from \mathbb{R}^{d+1} onto the horizontal plane. Then for any Borel set ω of \mathbb{H}^d*

$$(2.13) \quad \partial h(\mathbf{v}^{-1}(\omega)) = \pi(\mathcal{G}^{-1}(\omega)).$$

Proof. — By (2.12) $p \in \partial h(x)$ if and only if $p \in \partial \tilde{h}(x)$ if and only if $x \in \partial \tilde{u}(p)$ if and only if \hat{x} is an inward unit normal to \mathbf{K} at $(p, \tilde{u}(p))$ if and only if $(p, \tilde{u}(p)) \in \mathcal{G}^{-1}(\mathbf{v}(x))$, if and only if $p \in \pi(\mathcal{G}^{-1}(\mathbf{v}(x)))$. \square

As a corollary the projection to \mathbb{R}^d of $\partial_s \mathbf{K}$ is $\partial h(\mathbb{B}) = \bigcup_{x \in \mathbb{B}} \partial h(x)$. We will denote by u the restriction of \tilde{u} on $\partial h(\mathbb{B})$, so that $\partial_s \mathbf{K}$ is the graph of u .

If h is C^1 , then ∂h is single-valued and coincides with the gradient map, so it is a continuous mapping from \mathbb{B} to $\partial \mathbf{K}$. If h is also strictly convex, then the gradient map is injective so $\partial_s \mathbf{K}$ is open in $\partial \mathbf{K}$. However there are examples of smooth strictly convex functions h , so that $\partial_s \mathbf{K} \neq \partial \mathbf{K}$. A criterion to have $\text{grad } h(\mathbb{B}) = \mathbb{R}^d$ is that $\partial_s \mathbf{K}$ is C^1 and the induced Riemannian metric is complete [11, Lemma 3.1]. Space-like hypersurfaces which are graph over \mathbb{R}^d are usually called *entire*. Entire hypersurfaces are not necessarily graphs of F-convex sets. A condition for a C^2 entire space-like hypersurface to bound an F-convex set can be found in [23].

If h is C^1 , strictly convex, and $\text{grad } h(\mathbb{B}) = \mathbb{R}^d$, then h is of *Legendre type*, and the gradient of u is the inverse of the gradient of h at the corresponding points [31, 26.5]. In particular, if moreover the Hessian of h is positive definite, then, looking the Hessian as the Jacobian of the gradient, the Hessian matrix of h is the inverse of the Hessian matrix of u at the corresponding points. So from (2.7) we recover the well-known formula for the Gauss–Kronecker curvature of $\partial_s \mathbf{K}$:

$$\phi^{-1} = (1 - \|\text{grad } u\|^2)^{-1-d/2} \det \text{Hess } u.$$

LEMMA 2.16. — *Let H be a support function of an F-convex \mathbf{K} set on $I^+(0)$. Then there exists a sequence of C_+^∞ support functions converging to H , uniformly on any compact set.*

Proof. — Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be the convex function whose graph is $\partial \mathbf{K}$. Let ϕ_n be smooth functions on \mathbb{R}^d , with compact support in a ball of radius $1/n$, and with $\int_{\mathbb{R}^d} \phi_n \, dx = 1$. By the convolution properties,

$$u_n(x) = (u * \phi_n)(x) = \int_{\mathbb{R}^d} u(x - y) \phi_n(y) \, dy,$$

are smooth functions converging uniformly to u . More precisely since u is 1-Lipschitz, it is straightforward to check that $|u(x) - u_n(x)| < 1/n$.

Fact 1: The graph of u_n bounds an F-convex set. — Notice that u_n is convex. Indeed, by convexity of u

$$\begin{aligned} u_n((1-t)x_1 + tx_2) &= \int_{\mathbb{R}^d} u((1-t)(x_1 - y) + t(x_2 - y))\phi_n(y) \, dy \\ &\leq \int_{\mathbb{R}^d} (1-t)u(x_1 - y)\phi_n(y) \, dy + \int_{\mathbb{R}^d} tu(x_2 - y)\phi_n(y) \, dy \\ &= (1-t)u_n(x_1) + tu_n(x_2). \end{aligned}$$

Let $y \in \mathbb{B}$. As the graph of u is an F-convex set, there exists $\alpha \in \mathbb{R}$ such that for any $x \in \mathbb{R}^d$, $\langle \binom{x}{u(x)}, \binom{y}{1} \rangle_- = \langle x, y \rangle - u(x) \leq \alpha$. Since $|u(x) - u_n(x)| < 1/n$ we deduce $\langle \binom{x}{u_n(x)}, \binom{y}{1} \rangle_- \leq \alpha + 1/n$, i.e. there is a support plane orthogonal to $(y, 1)$.

The function u_n is smooth but its Hessian is not strictly positive, so in principle the inverse of the Gauss map of the corresponding graph is not differentiable. To fix this problem we consider the functions

$$v_n = \left(1 - \frac{1}{n}\right)u_n + \frac{1}{n}f(x)$$

with $f(x) = \sqrt{1 + \|x\|^2}$ the graph function of the upper hyperboloid. Clearly the Hessian of v_n is strictly positive. On the other hand v_n converges to u uniformly only on compact sets, so we cannot use the simple argument above to show that the graph of v_n bounds an F-convex set. This problem is balanced by the following observation.

Fact 2: $(u_n - f)$ is uniformly bounded from above. — The F-convex set bounded by the graph of u_n contains the future cone of any of its points, so there exists a constant M with $u_n(x) < M + \|x\|$, and $f(x) > \|x\|$, and the result follows.

Fact 3: v_n is the graph of the boundary of an F-convex set. — We want to prove that, given $y \in \mathbb{B}$, there exists α such that for any $x \in \mathbb{R}^d$,

$$\left\langle \binom{x}{v_n(x)}, \binom{y}{1} \right\rangle_- \leq \alpha.$$

The left-hand side of the above equation is

$$\langle x, y \rangle - u_n(x) + \frac{1}{n}(u_n - f)(x).$$

As u_n bounds an F-convex set by Fact 1, there exists α' such that $\langle x, y \rangle - u_n(x) < \alpha'$. The result follows by Fact 2.

Fact 4: The support functions h_n of the graph of u_n converge pointwise to the support function h of the graph of u . — Let $y \in \mathbb{B}$, and let $\begin{pmatrix} x \\ u(x) \end{pmatrix}$ be a point on the intersection of \mathbf{K} and its support plane orthogonal to $(y, 1)$. Let $\epsilon > 0$. First we show that for n sufficiently large, $h_n(y) \geq h(y) - \epsilon$. Notice that

$$h_n(y) \geq \left\langle \begin{pmatrix} x \\ v_n(x) \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right\rangle_- = \left\langle \begin{pmatrix} x \\ u(x) \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right\rangle_- + (u(x) - v_n(x)).$$

As

$$h(y) = \left\langle \begin{pmatrix} x \\ u(x) \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right\rangle_-$$

we need

$$v_n(x) - u(x) < \epsilon.$$

This is true for n big as v_n converges to u .

Let us prove that for $\epsilon > 0$ and $\forall x \in \mathbb{R}^d$,

$$\left\langle \begin{pmatrix} x \\ v_n(x) \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right\rangle_- < h(y) + \epsilon.$$

We have

$$\left\langle \begin{pmatrix} x \\ u(x) \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right\rangle_- < h(y) + \epsilon,$$

that gives, adding $v_n(x) - v_n(x)$ on the left-hand side and reordering the terms

$$\langle x, y \rangle_- - v_n(x) < h(y) + u(x) - v_n(x).$$

But $u(x) - v_n(x) = u(x) - u_n(x) + \frac{1}{n}(u_n - f)(x)$, and the result follows by Fact 2 and because u_n uniformly converge to u .

By homogeneity, we obtain the convergence on the support functions on $\Gamma^+(0)$. A classical property of convex functions says that the convergence is actually uniform on each compact set [20, 31]. □

2.3. Area measure and Monge–Ampère measure

Let h be a convex function on \mathbb{B} . The *Monge–Ampère measure* $\text{MA}(h)$ of h is a Borel measure over \mathbb{B} defined by

$$\text{MA}(h)(\omega) = \mathcal{L}(\partial h(\omega)),$$

where ω is any Borel subset of \mathbb{B} . The Monge–Ampère measure of h is finite on compact sets [19, 1.1.13]. As \mathbb{B} is σ -compact (i.e., a countable union of compact sets), the Monge–Ampère measure is a Radon measure on \mathbb{B} .

Let us list some immediate properties.

- If h_1 and h_2 are convex functions, and coincide over ω , then $\text{MA}(h_1)(\omega) = \text{MA}(h_2)(\omega)$.
- For $c > 0$, $\partial(ch)(x) = c\partial h(x)$, so

$$(2.14) \quad \text{MA}(ch) = c^d \text{MA}(h).$$

- If A is an affine function, then $\partial(h + A)(\omega)$ is a translate of $\partial(h)(\omega)$, so

$$(2.15) \quad \text{MA}(h + A) = \text{MA}(h).$$

- Let $h = \max\{h_1, h_2\}$ with h_i a convex function on \mathbb{B} . Denote by D_1 the set where $h_1 > h_2$ and D_2 the set where $h_2 > h_1$. Notice that D_i are open and for each $x \in D_i$ the set of subdifferentials of h at x coincides with the set of subdifferentials of h_i . Now let E be where h_1 coincides with h_2 . If $x \in E$ each subdifferential of h_i at x is also a subdifferential of h at x . It follows that

$$(2.16) \quad \text{MA}(\max\{h_1, h_2\})(\omega) \geq \min\{\text{MA}(h_1)(\omega), \text{MA}(h_2)(\omega)\}.$$

- If h is C^1 , then $\partial h(x) = \text{grad}_x h$, and then

$$\text{MA}(h)(\omega) = \mathcal{L}(\text{grad } h(\omega)),$$

for which it can be deduced that [19, 1.1.14], if h is C^2

$$(2.17) \quad \text{MA}(h)(\omega) = \int_{\omega} \det \text{Hess}_x h \, dx.$$

- If h is a convex function on \mathbb{B} , by the Alexandrov Theorem, it is twice differentiable almost everywhere and then the right-hand side of (2.17) is still meaningful. It is actually the regular part of the Lebesgue decomposition of $\text{MA}(h)$ [35, Lemma 2.3]. So, for a general convex function h we get

$$(2.18) \quad \text{MA}(h)(\omega) \geq \int_{\omega} \det \text{Hess}_x h \, dx.$$

Let h be a convex function on \mathbb{B} and \mathbf{K} be the corresponding F-convex set. There is a precise relation between the Monge–Ampère measure of h and the area measure $\mathbf{A}(\mathbf{K})$. To this aim it is convenient to push forward the area measure $\mathbf{A}(\mathbf{K})$ to \mathbb{B} by the radial map $\mathbf{v}^{-1} : \mathbb{H}^d \rightarrow \mathbb{B}$. We denote by $\mathbf{A}(h)$ the corresponding area measure on \mathbb{B} :

$$\mathbf{A}(h)(\omega) = \mathbf{A}(\mathbf{K})(\mathbf{v}(\omega)).$$

PROPOSITION 2.17. — *Let h be a convex function on \mathbb{B} . Then $A(h)$ and $MA(h)$ are equivalent. More precisely,*

$$A(h) = \lambda MA(h), \quad MA(h) = \frac{1}{\lambda} A(h).$$

Proof. — The second relation will follow from the first one, because λ is positive on \mathbb{B} . Let $h \in C^2(\mathbb{B})$ with positive definite Hessian. From (2.9), (2.7) and (2.1) we obtain

$$A(h) = \lambda \det(\text{Hess } h) \mathcal{L}$$

and the result follows from (2.17).

Now let h be convex on \mathbb{B} . From Lemma 2.16, we can find a sequence (h_n) of elements of $C^2(\mathbb{B})$ with positive definite Hessian, converging uniformly to h on compact sets. From [19, 1.2.3], $MA(h_n)$ weakly converges to $MA(h)$. From Lemma 2.12, $A(h_n)$ also weakly converge to $A(h)$. Since $A(h_n) = \lambda MA(h_n)$, the result follows by the uniqueness part of the Riesz Representation Theorem. \square

COROLLARY 2.18. — *Let $c > 0$. Then on \mathbb{B}*

$$A(h) \geq c(\mathbf{v}^{-1})_*(d\mathbb{H}^d) \Rightarrow MA(h) \geq c\mathcal{L}.$$

Conversely if $MA(h) \geq c\mathcal{L}$ then on any ball of radius $r < 1$, $A(h) \geq c(1 - r^2)^{(d+2)/2} d\mathbb{H}^d$.

Let us denote by $\partial_{\text{reg}}\mathbf{K}$ the regular part of the space-like boundary of \mathbf{K} , that is the set of points such that the Gauss map \mathcal{G} of \mathbf{K} is well defined. Equivalently $\partial_{\text{reg}}\mathbf{K}$ is the differentiable part of $\partial_s\mathbf{K}$, i.e. the set of points where \mathbf{K} admits a unique space-like support plane. Notice that $\partial_{\text{reg}}\mathbf{K}$ is the intersection of the region of differentiable points of $\partial\mathbf{K}$ and $\partial_s\mathbf{K}$. In general it can be strictly smaller than the the region of differentiable points in the boundary of \mathbf{K} . For instance if $\mathbf{K} = \mathbb{I}^+(0)$ then $\partial_{\text{reg}}\mathbf{K}$ is empty.

PROPOSITION 2.19. — *Let \mathbf{K} be an F -convex set. The Gauss image of $\partial_s\mathbf{K} \setminus \partial_{\text{reg}}\mathbf{K}$ has zero area measure.*

Proof. — Let \hat{E} be the Gauss image in \mathbb{H}^d of the non-regular points of $\partial_s\mathbf{K}$, and denote by E its radial projection to \mathbb{B} . By Proposition 2.17 and the definition of the Monge–Ampère measure, it suffices to show that the Lebesgue measure of $\partial h(E)$ is zero. From (2.13), $\partial h(E)$ is the projection onto \mathbb{R}^d of the non-differentiable points of $\partial_s\mathbf{K}$. As $\partial\mathbf{K}$ is the graph of a convex function, the set of non-differentiable points has zero measure. \square

Notice that Proposition 2.19 implies that if $\partial_{\text{reg}}\mathbf{K}$ is empty, the area measure is 0. In Section 2.4 we will give a precise characterization of F-convex sets with empty regular boundary.

Another simple remark implied by Proposition 2.19 is the following:

COROLLARY 2.20. — *Let \mathbf{K} be an F-convex set, with Gauss map \mathcal{G} and $\partial\mathbf{K}$ the graph of a function \tilde{u} . Then*

$$A(\mathbf{K})(\omega) = \int_{\pi(\mathcal{G}^{-1}(\omega))} \sqrt{1 - \|\text{grad } \tilde{u}\|^2} \, dx,$$

where $\pi : \mathbb{R}^{d,1} \rightarrow \mathbb{R}^d$ is the orthogonal projection.

Proof. — Suppose that \tilde{u} is C^1 and h is the support function of \mathbf{K} on \mathbb{B} . Let $b = \mathbf{v}^{-1}(\omega)$. Then

$$MA(h)(b) = \mathcal{L}(\partial h(b)) = \mathcal{L}(\partial \tilde{h}(b)) \stackrel{(2.12)}{=} \mathcal{L}((\text{grad } \tilde{u})^{-1}(b)) = (\text{grad } \tilde{u})_* \mathcal{L}(b),$$

so

$$\begin{aligned} A(\mathbf{K})(\omega) &= \int_b \lambda \, d(\text{grad } \tilde{u})_* \mathcal{L} = \int_{\partial h(b)} (\lambda \circ \text{grad } \tilde{u}) \, dx \\ &= \int_{\pi(\mathcal{G}^{-1}(\omega))} \sqrt{1 - \|\text{grad } \tilde{u}\|^2} \, dx. \end{aligned}$$

Now for any convex h , \tilde{u} is convex and by Rademacher Theorem, $\|\text{grad } \tilde{u}\|$ is continuous except on a zero measure set N . Let us define the following measure on \mathbb{H}^d :

$$\mu(\omega) = \int_{\pi(\mathcal{G}^{-1}(\omega))} \sqrt{1 - \|\text{grad } \tilde{u}\|^2} \, dx.$$

By Proposition 2.19 $A(\mathbf{K})(N) = 0$, so $A(\mathbf{K})$ and μ coincide on N . Out of N they also coincide by the argument above. \square

COROLLARY 2.21. — *Let \mathbf{K} be an F-convex set with $\partial_s \mathbf{K}$ a C^1 hypersurface. Then for any Borel set ω of \mathbb{H}^d , $A(\mathbf{K})(\omega)$ is the measure of the inverse image of ω by the Gauss map, for the measure given by the induced Riemannian metric on $\partial_s \mathbf{K}$.*

Proof. — Let $\partial_s \mathbf{K}$ be the graph of u . For two tangent vectors U, V in \mathbb{R}^d , the first fundamental form of $\partial_s \mathbf{K}$ is given by

$$\begin{aligned} I(U, V) &= \left\langle \left(\begin{array}{c} U \\ \langle \text{grad}_x u, U \rangle \end{array} \right), \left(\begin{array}{c} V \\ \langle \text{grad}_x u, V \rangle \end{array} \right) \right\rangle_- \\ &= \langle U, V \rangle - \langle \text{grad}_x u, U \rangle \langle \text{grad}_x u, V \rangle \end{aligned}$$

so $I = \text{Id}_d - du \otimes du$. So the area element is $\sqrt{\det I} \mathcal{L} = \sqrt{1 - \|\text{grad } u\|^2} \mathcal{L}$ and the result follows from Corollary 2.20. \square

Remark 2.22 (Euclidean area). — To define the Monge–Ampère measure, we used the Lorentzian normal vectors. Classically the definition is given by taking the Euclidean outward unit normal vectors with last coordinate equal to -1 . Actually, if $\hat{x} = \begin{pmatrix} x \\ 1 \end{pmatrix}$ is a Lorentzian inward normal vector of some space-like hypersurface S in \mathbb{R}^{d+1} , then $\begin{pmatrix} x \\ -1 \end{pmatrix}$ is a Euclidean outward normal vector. So the definition of the Monge–Ampère measure is independent of this choice.

Considering a convex function h on \mathbb{B} , it can be seen as the restriction to $\mathbb{B} \times \{1\}$ of the Lorentzian extended support function of an F-convex set \mathbf{K} , or the restriction to $\mathbb{B} \times \{-1\}$ of the Euclidean extended support function (defined on the past cone) of an unbounded convex set \mathbf{K}' . Actually $\mathbf{K} = \{p \in \mathbb{R}^{d+1} \mid \langle p, \begin{pmatrix} x \\ 1 \end{pmatrix} \rangle_- \leq h(x), \forall x \in \mathbb{B}\}$ and $\mathbf{K}' = \{p \in \mathbb{R}^{d+1} \mid \langle p, \begin{pmatrix} x \\ -1 \end{pmatrix} \rangle_{d+1} \leq h(x), \forall x \in \mathbb{B}\}$, with $\langle \cdot, \cdot \rangle_{d+1}$ the usual scalar product of \mathbb{R}^{d+1} , so $\mathbf{K} = \mathbf{K}'$.

If we denote by $A^e(h)$ the Euclidean area measure of \mathbf{K} , we have in the same manner as above (or see e.g. [14])

$$A^e(h) = \lambda^e \text{MA}(h), \quad \text{where } \lambda^e(x) = \sqrt{1 + \|x\|^2}.$$

In particular the Euclidean area measure and the Lorentzian area measure are equivalent. Heuristically, this is not surprising: if ω is a Borel set of \mathbb{B} , the pre-image by the Lorentzian or the Euclidean Gauss map of ω gives the same set on $\partial\mathbf{K}$. This because the hyperplane defined by the equation $\langle p, \hat{x} \rangle_- = h(x)$ coincides with the hyperplane defined by the equation $\langle p, \hat{x} \rangle_{d+1} = h(x)$.

The fact that area measure is null on ω means that for any small parallel displacement of the support planes defined by ω , the first order variation of the volume is zero. Parallel displacement and volume determined on \mathbb{R}^{d+1} by the Euclidean metric and by the Minkowski metric coincide (although the parallel displacement used in the definition of the area measure is not in the same direction in each case). It also follows that the Lorentzian area measure and the Hausdorff measure are equivalent [34, Theorem 4.2.5].

2.4. Support functions with continuous extension on the sphere

Let $g \in C^0(\partial\mathbb{B})$. A g -convex set is an F-convex set \mathbf{K} with support function \tilde{h} on \mathbb{R}^d such that $\tilde{h}|_{\partial\mathbb{B}} = g$. Not all F-convex sets are g -convex sets for some g as $h = \lambda^{-1}$ shows. The following result gives a sufficient condition.

PROPOSITION 2.23 ([23]). — *Let \mathbf{K} be a C^2_{\pm} F-convex set with mean radius of curvature bounded from above. Then $\tilde{h}|_{\partial\mathbb{B}}$ is continuous.*

For any $g \in C^0(\partial\mathbb{B})$, there is a distinguished g -convex set, that can be introduced in many equivalent ways. They are listed in the following proposition.

PROPOSITION 2.24. — *Let \mathbf{K} be an F -convex set with support function $h \in C^0(\mathbb{B})$, Gauss map \mathcal{G} , and set $g = h|_{\partial\mathbb{B}}$. The following are equivalent:*

- (1) h is the convex envelope of g , i.e. $h(x) = \sup\{l(x) \mid l \text{ affine and } l \leq g \text{ on } \partial\mathbb{B}\}$;
- (2) $A(\mathbf{K}) = 0$;
- (3) $\partial_{\text{reg}}\mathbf{K}$ is empty;
- (4) $\forall p \in \partial_s\mathbf{K}$, $\mathbf{v}^{-1}(\mathcal{G}(p))$ is the convex hull of points of $\partial\mathbb{B}$;
- (5) $\mathbf{K} = \bigcap_{\ell \in \partial\mathbb{B}} \overline{I^+(P_g(\ell))}$, with $P_g(\ell)$ are the null hyperplane defined by $\langle \cdot, \ell \rangle_- = g(\ell)$;
- (6) any g -convex set is contained in \mathbf{K} ;
- (7) the interior of \mathbf{K} union $\partial_s\mathbf{K}$ is the domain of dependence (or Cauchy development) of the boundary S of any g -convex set: it is the set of points p of $\mathbb{R}^{d,1}$ such that all inextensible causal curve through p meet S .

Proof.

(1 \Leftrightarrow 2) By Proposition 2.17, $A(\mathbf{K}) = 0$ is equivalent to $MA(h) = 0$, which happens exactly when h is the convex envelope of $h|_{\partial\mathbb{B}}$ (see [19, 1.5.2]).

(1 \Rightarrow 4) Let $x \in \mathbf{v}^{-1}(\mathcal{G}(p)) =: C(p)$. Up to composing h by an affine function we may suppose that $h = 0$ on $C(p)$ and $g \geq 0$ (it suffices to translate p to the origin). Suppose that $g \geq c > 0$. Then by definition of h , $h \geq c$, that contradicts the assumption. So $g^{-1}(0) =: \Lambda$ is not empty. As $C(p)$ is convex, its closure contains the convex hull $C(\Lambda)$ of Λ . We want to prove that the closure of $C(p)$ is contained in $C(\Lambda)$. Let us suppose that this is false: let $x \in C(p)$ and $x \notin C(\Lambda)$. As $C(\Lambda)$ is convex, the Separating Hyperplane Theorem implies the existence of a hyperplane separating x from $C(\Lambda)$, that is there exists a vector v such that $\langle v, \eta \rangle < a$ and $\langle v, x \rangle > a$. Take the affine function $l(y) = \langle v, y \rangle - a$. Note that $l(x) > 0$, and that g is strictly positive on $l^{-1}[0, +\infty) \cap \partial\mathbb{B}$, so there is $k > 0$ such that $g \geq k$ whenever $l > 0$. It follows that there is $\epsilon > 0$ such that $\epsilon l < g$, so $h \geq \epsilon l$ on \mathbb{B} , but evaluating the inequality at p we get a contradiction.

(4 \Rightarrow 3) is clear.

(3 \Rightarrow 2) By Proposition 2.19 that if $\partial_{\text{reg}}\mathbf{K}$ is empty then $A(\mathbf{K}) = 0$. Here have that (1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4).

(7 \Leftrightarrow 5) It is proved in [4].

(5 \Rightarrow 6) If H is a g -convex set then H is contained in $\overline{I^+(P_g(\ell))}$ for every $\ell \in \partial\mathbb{B}$ so H is contained in \mathbf{K} .

(6⇒1) Let h' be the convex envelope of g . Then $h \leq h'$ so $\mathbf{K} \subset \mathbf{K}(h')$. On the other hand by hypothesis $\mathbf{K}(h') \subset \mathbf{K}$ so $\mathbf{K} = \mathbf{K}(h')$, that is $h = h'$.

(1⇒5) Let $\mathbf{K}_0 = \bigcap_{\ell \in \partial \mathbb{B}} \overline{I^+(P_g(\ell))}$. Notice that if $p \in \mathbf{K}_0$ then the affine function on \mathbb{B} , $l_p(x) = \langle \hat{x}, p \rangle_-$ satisfies $l_p \leq g$ on $\partial \mathbb{B}$. Conversely if $l(x) = \langle x, y \rangle + c$ is an affine function on \mathbb{B} with $f \leq g$ then the point $p_l = \begin{pmatrix} y \\ -c \end{pmatrix}$ lies on \mathbf{K}_0 . That is there is a 1-to-1 correspondence between points of \mathbf{K}_0 and affine functions l with $l \leq g$. Now, as the support function of \mathbf{K}_0 is defined as $h_0(x) = \sup_{p \in \mathbf{K}_0} \langle x, p \rangle$, the correspondence above shows that h_0 is the convex envelope of g . □

The unique g -convex set satisfying the properties above will be denoted by Ω_g . Its support function on \mathbb{B} will be denoted by h_g .

Remark 2.25. — If g is the restriction to $\partial \mathbb{B}$ of an affine map (in particular if g is constant), then Ω_g is the future cone of a point.

Actually, any $g \in C^0(\partial \mathbb{B})$ has a maximal value m . Then Ω_g is in the future side of light-like hyperplanes $\langle \cdot, \hat{\ell} \rangle_- = m \ \forall \ell \in \partial \mathbb{B}$. They meet at the point $(0, \dots, 0, -m)$: Ω_g is always contained in the future cone of a point.

Remark 2.26. — With the terminology of [11], from Property 2.24, Ω_g is a *regular domain*. Moreover, the space-like boundary $\partial_s \Omega_g$ corresponds to the *initial singularity*. Given an element $p \in \partial_s \Omega_g$ then $\mathcal{G}(p)$ is a ideal convex subset of \mathbb{H}^d . Since on $\mathcal{G}(p)$ the support function coincides with the scalar product by p , the restriction of the support function to the radial projection of $\mathcal{G}(p)$ to \mathbb{B} is an affine function. So the dual stratification pointed out in [11] gives a decomposition of \mathbb{H}^d into ideal convex regions where the support function is affine.

For $x \in \mathbb{B}$, we will denote by $F_g(x)$ the maximal convex set of \mathbb{B} containing x on which h_g is affine. In other words, $F_g(x) = \mathbf{v}^{-1}(\mathcal{G}(\mathcal{G}^{-1}(\mathbf{v}(x))))$.

DEFINITION 2.27. — We say that Ω_g is simple if for any $x \in \partial_s \Omega_g$, $F_g(x)$ has codimension k , with $k \leq d/2$, $k \in \mathbb{N}$.

Sometimes for brevity we will say that the support function h_g is simple to mean that Ω_g is simple.

Note that when $d = 2$, Ω_g is always simple. In higher dimensions $d > 2$, Ω_g is not necessarily simple. However those simple domains are of some interest in the study of MGHCF space-times (see Section 3.7).

LEMMA 2.28. — If \mathbf{K} is a g -convex set contained in the interior of Ω_g , then $\partial_s \mathbf{K} = \partial \mathbf{K}$. In particular, $\partial_s \mathbf{K}$ is an entire space-like hypersurface.

Proof. — Suppose that the statement is false. Then there would be a point $p \in \partial \mathbf{K}$ such that $\langle p, \hat{x} \rangle_- < h(x)$ for every $x \in \mathbb{B}$. Since p is in the

boundary of \mathbf{K} and

$$\mathbf{K} = \{q \in \mathbb{R}^{d+1} \mid \langle q, \hat{x} \rangle_- \leq h(x) \forall x \in \overline{\mathbb{B}}\},$$

there is $\ell \in \partial\mathbb{B}$ such that $\langle p, \hat{\ell} \rangle_- = h(\ell)$.

On the other hand, on $\partial\mathbb{B}$ the function h coincides with h_g , so if the point p satisfies $\langle p, \hat{\ell} \rangle_- = h(\ell) = h_g(\ell)$, it cannot lie in the interior of Ω_g , contradicting the assumption on \mathbf{K} . \square

We will also need the following definition, adapted from a base concept of general relativity.

DEFINITION 2.29. — *A Cauchy g -convex set is a g -convex set with support function satisfying $h < h_g$.*

Clearly, a g -convex set contained in the interior of Ω_g is Cauchy. The converse does not hold, as the convex set with support function $h(x) = \frac{1}{2}\|x\|^2$ shows. In this case $g = h|_{\partial\mathbb{B}}$ is constant equal to $1/2$ so $h_g = 1/2$ and $h < 1/2$ on \mathbb{B} . On the other hand $\text{grad } h(\mathbb{B}) = \mathbb{B} \neq \mathbb{R}^d$. Note that h is the only convex function equal to its conjugate u [31, p. 106]. This example also shows that for a g -convex set \mathbf{K} , $\partial_s \mathbf{K}$ can be a non-complete C^1 hypersurface.

In this case Ω_g is the future of the point $(0, \dots, 0, -1/2)$, whereas the boundary of the domain \mathbf{K} corresponding to h is the graph of \tilde{u} , where $\tilde{u}(x) = u(x) = h(x)$ if $x \in \mathbb{B}$ and $\tilde{u}(x) = \|x\| - \frac{1}{2}$ otherwise.

The g -convex sets that we will consider in the next section will have the property that Cauchy implies contained in the interior of Ω_g .

2.5. Some classical results of Monge–Ampère equation

We quote some results on Monge–Ampère equation with prescribed continuous data on the boundary of the ball that will be used in the next section. The first one is about uniqueness.

THEOREM 2.30 (Comparison Principle, Rauch–Taylor, [30], [19, 1.4.6]). *Let O be a bounded open convex set of \mathbb{R}^d . If $u, v \in C(\overline{O})$ are convex functions such that for any Borel set $E \subset O$, $\text{MA}(u)(E) \leq \text{MA}(v)(E)$, then*

$$\min\{u(x) - v(x) \mid x \in \overline{O}\} = \min\{u(x) - v(x) \mid x \in \partial O\}.$$

COROLLARY 2.31. — *A g -convex set is uniquely determined by its area measure among g -convex sets.*

Now we will quote some regularity results. We will need the following assumption: for a bounded open convex set O , h convex on O and for any Borel set E of O ,

$$(2.19) \quad m\mathcal{L}(E) \leq \text{MA}(h)(E) \leq M\mathcal{L}(E), \quad \text{for some fixed } M > m > 0.$$

Remark 2.32. — The condition $m\mathcal{L}(E) \leq \text{MA}(h)(E)$ for any Borel set E is equivalent to require that $\det \text{Hess } h(x) \geq m$ \mathcal{L} -almost everywhere. Indeed as the determinant of the Hessian is the density of the regular part of $\text{MA}(h)$ with respect to the Lebesgue measure, if it is bounded from below by m , then $m\mathcal{L}(E) \leq \text{MA}(h)(E)$ by definition of density. Conversely suppose that $m\mathcal{L}(E) \leq \text{MA}(h)(E)$ for any Borel set and let E_0 be the support of the singular part of $\text{MA}(h)$ with respect to \mathcal{L} . Consider $E_1 = \{x \mid \det \text{Hess } h(x) < m\}$. Notice that $\text{MA}(h)(E_1 \setminus E_0) \leq m\mathcal{L}(E_1 \setminus E_0)$, but by the assumption the opposite inequality holds. As a consequence $\mathcal{L}(E_1 \setminus E_0) = 0$. As $\mathcal{L}(E_0) = 0$, we conclude that $\det \text{Hess } h(x) \geq m$ \mathcal{L} -almost everywhere.

On the other hand the condition $\text{MA}(h)(E) \leq M\mathcal{L}(E)$ is equivalent to require that $\text{MA}(h)$ is absolutely continuous with respect to \mathcal{L} with density bounded from above by M .

THEOREM 2.33 (Caffarelli, [19, Theorem 5.2.1]). — *Let O be a bounded open convex set of \mathbb{R}^d . Let h be a convex function in O satisfying (2.19). Assume that $h \geq 0$ on O and let $C = \{x \in O \mid h(x) = 0\}$. If C is non-empty and contains more than one point, then C has no extremal point in O .*

An extremal point of a convex set C is a point of ∂C which is not a convex combination of other points of the closure \overline{C} of C . A classical result of convex geometry states that if \overline{C} is compact then \overline{C} is the convex hull of the extremal points of C .

THEOREM 2.34 (Multidimensional Alexandrov–Heinz Theorem). — *Let h be a convex function on the d dimensional unit ball \mathbb{B}^d in \mathbb{R}^d such that the Monge–Ampère measure of h satisfies $\text{MA}(h) \geq c_0\mathcal{L}$ for a positive c_0 .*

Let W be an affine subspace of \mathbb{R}^d passing through 0, of dimension $d - k$, for $k \leq d/2$. Suppose that $h \equiv 0$ on the boundary of a $(d - k)$ ball $\mathbb{B}^{d-k} = W \cap \mathbb{B}^d$. Then

$$h(0) < 0.$$

The theorem says that, if the Monge–Ampère measure of h is bounded from below by a positive constant (times the Lebesgue measure), then h cannot be affine on a $(d - k)$ -plane. In particular, h cannot be affine on a hyperplane. If $d = 2$, this says that if the Monge–Ampère measure of h is

bounded from below by a positive constant, then h is strictly convex. This is the usual formulation of Alexandrov–Heinz Theorem, see [35, Remark 3.2]. For completeness, we give a proof of Theorem 2.34 at the end of this section. The proof closely follows the proof in the 2-d case given in [35].

Remark 2.35. — The assumption on the dimension on k in the Theorem is optimal. In fact a famous example due to Pogorelov [28, pp. 81–86] shows that for $d \geq 3$ there exists a convex function f on \mathbb{B}^d with strictly positive Monge–Ampère measure (in the sense above) which is constant on a segment.

Actually the example can be easily generalized: let $k > d/2$ and for any vector $x \in \mathbb{R}^d$ let $r(x) = \sqrt{x_{d-k+1}^2 + \cdots + x_d^2}$ and $t(x) = \sqrt{x_1^2 + \cdots + x_{d-k}^2}$. Consider the function

$$f(x) = \beta r(x)^\alpha (1 + \beta t(x)^2)$$

with $\alpha = 2k/d$ and $\beta > 1$. This function is 0 on the ball \mathbb{B}^{d-k} of dimension $d-k$ defined by the equation $x_{d-k+1} = \cdots = x_d = 0$. Moreover f is smooth on $\mathbb{B}^d \setminus \mathbb{B}^{d-k}$ and C^1 on \mathbb{B}^d (here the assumption on k is needed). A direct computation shows that for a suitable choice of β the function f satisfies $\det \text{Hess } f \geq c_0$ on $\mathbb{B}^d \setminus \mathbb{B}^{d-k}$ and some positive constant c_0 . From this estimates and (2.18) one deduces that $\text{MA}(f)$ is bigger than $c_0 \mathcal{L}$.

As we will not use this result in the paper we omit the computation leaving it to the reader (basically it can be done following the same line as in [28, pp. 81–86]).

THEOREM 2.36 ([35, Theorem 3.1]). — *Let h be a strictly convex solution of*

$$\text{MA}(h) = \varphi \mathcal{L}$$

on a ball O . Suppose $\varphi > 0$ and $\varphi \in C^{1,1}(O)$. Then $h \in C^{3,\alpha}(O)$ for any $\alpha \in (0, 1)$. If furthermore $\varphi \in C^{k,\alpha}(O)$ for some $k \geq 2$ and $\alpha \in (0, 1)$, then $h \in C^{k+2,\alpha}(O)$.

COROLLARY 2.37. — *Let \mathbf{K} be a g -convex set with support function h and such that $\text{A}(\mathbf{K}) = f(\mathbf{v}^{-1})_*(d\mathbb{H}^d)$ with $f \in C^{k+1}(\mathbb{H}^d)$, $k \geq 2$ and $c_2 \geq f \geq c_1 > 0$.*

If moreover

- (1) h_g is simple, or
- (2) $h < h_g$ on \mathbb{B} ,

then $\partial_s \mathbf{K}$ is a strictly convex C^{k+2} hypersurface.

Proof. — First notice that (1) implies (2)

Indeed suppose that $h = h_g$ at some point x . By Proposition 2.24, $F := F_g(x)$ is a convex set with vertices on the boundary and h_g is affine on F . Thus since h coincides with h_g on $\partial\mathbb{B} \cap F$ and at some interior point we get that those functions coincide on the whole F . Up to compose by an affine transformation we may assume that $h = h_g$ is zero on F .

As h_g is simple Theorem 2.34 implies that h is strictly negative on F , giving a contradiction,

So we suppose that (2) is fulfilled.

Let us first check that h is strictly convex. By contradiction suppose that it is not strictly convex. There exists an affine function l such that its graph is a support plane for the graph of h , and the convex region $C = \{x \in \mathbb{B} \mid h(x) = l(x)\}$ contains more than two points.

Notice that the convex function $h'(x) = h(x) - l(x)$ is non negative and

$$C = \{x \in \mathbb{B} \mid h'(x) = 0\}.$$

Moreover $\text{MA}(h) = \text{MA}(h')$ by (2.15).

Let \mathbb{B}_ϵ be the open ball centered at the origin of radius $0 < \epsilon < 1$, and let $C_\epsilon = C \cap \mathbb{B}_\epsilon$. Choose ϵ so that C_ϵ is not empty. On \mathbb{B}_ϵ , λ is bounded from below by a positive constant depending on ϵ , hence by the previous paragraph, $\text{MA}(h')$ satisfies (2.19). So Theorem 2.33 applies, and the extremal points of C_ϵ are on the boundary of \mathbb{B}_ϵ . This is true for any $\epsilon' > \epsilon$, so the extremal points of C are on $\partial\mathbb{B}$. Clearly, C has at least two extremal points x_0, x_1 in $\partial\mathbb{B}$, and $h = l$ on the segment joining x_0 to x_1 . As $h = h_g$ on $\partial\mathbb{B}$, by definition of h_g , $h = h_g$ on the segment, that contradicts the hypothesis $h < h_g$ on \mathbb{B} . Hence h is strictly convex.

For any $0 < \epsilon < 1$, $f\lambda^{-d-2}$ has the same regularity than f on the ball of radius ϵ , so from Theorem 2.36, $h \in C^{k+2}$ on this ball, hence $h \in C^{k+2}(\mathbb{B})$. The determinant of the Hessian of h is equal to $f\lambda^{-d-2} > 0$, so the principal radii or curvature of \mathbf{K} are positive. \mathbf{K} is then a C^2_+ F-convex set, and in this case $\partial_s \mathbf{K}$ has the same regularity than h [17]. □

Remark 2.38. — Another result of Caffarelli says that if h is the support function of a g -convex set, (2.19) is satisfied, and $h(x) < h_g(x)$ at $x \in \mathbb{B}$, then h is $C^{1,\alpha}$ at x , for $0 < \alpha < 1$, see [19, 5.4.6].

For the particularities of the $d = 2$ case, see also [25], [27] for different viewpoints.

The g affine case has also particularities, see [19, 35] or [14, Theorem 7].

Finally we give a proof of Theorem 2.34.

Proof of Theorem 2.34. — Without loss of generality, we set $\mathbb{B}^{d-k} = \{x \in \mathbb{B}^d \mid x_{d-k+1} = \dots = x_d = 0\}$. For $x \in \mathbb{B}^d$, let us denote $\tilde{x} = (x_1, \dots, x_{d-k}, 0, \dots, 0)$ and $\bar{x} = (0, \dots, 0, x_{d-k+1}, \dots, x_d)$, so that if $x \in \mathbb{B}^{d-k}$, then $x = \tilde{x}$.

Since $h \equiv 0$ on $\partial\mathbb{B}^{d-k}$, by convexity, $h \leq 0$ on \mathbb{B}^{d-k} . Using this and again convexity, we have actually on \mathbb{B}^{d-k} :

$$(2.20) \quad 0 \geq h(\tilde{x}) \geq -2|h(0)|.$$

Let

$$\mathbb{D} = \frac{1}{4}\mathbb{B}^{d-k} \times \frac{1}{4}\mathbb{B}^k = \left\{ x \mid \|\tilde{x}\| < \frac{1}{4}, \|\bar{x}\| < \frac{1}{4} \right\}.$$

First we will prove that for almost every $x \in \mathbb{D}$ the derivatives of h along the first $d - k$ directions are estimated as follows:

$$(2.21) \quad |\partial_i h(x)| < 8(|h(0)| + C\|\bar{x}\|) \quad \text{for } i \leq d - k,$$

where C is the Lipschitz constant of the restriction of h to $\frac{3}{4}\mathbb{B}^d$. For sake of simplicity we prove the formula for $i = 1$.

As h is C -Lipschitz on $\frac{3}{4}\mathbb{B}^d$, using (2.20), if $x \in \frac{1}{2}\mathbb{B}^d$ we have

$$\begin{aligned} h(x) &\geq h(\tilde{x}) - C\|\bar{x}\| \geq -2|h(0)| - C\|\bar{x}\|, \\ h(x) &\leq h(\tilde{x}) + C\|\bar{x}\| \leq C\|\bar{x}\|. \end{aligned}$$

If $x \in \mathbb{D} \subset \frac{1}{2}\mathbb{B}$, then $(1/2, x_2, \dots, x_d) \in \frac{3}{4}\mathbb{B}$. so we have for almost all x

$$\begin{aligned} \partial_1 h(x) &\leq \left(\frac{1}{2} - x_1\right)^{-1} \left[h\left(\frac{1}{2}, x_2, \dots, x_d\right) - h(x) \right] \\ &\leq 4(2|h(0)| + 2C\|\bar{x}\|) \leq 8(|h(0)| + C\|\bar{x}\|), \end{aligned}$$

where we have applied the general estimate $f'(a)(b - a) \leq f(b) - f(a)$ to the convex function $f(t) = h(t, x_2, \dots, x_d)$.

Analogously

$$\partial_1 h(x) \geq \left(x_1 + \frac{1}{2}\right)^{-1} \left[h(x) - h\left(-\frac{1}{2}, x_2, \dots, x_d\right) \right] \geq -8(|h(0)| + C\|\bar{x}\|),$$

and (2.21) follows.

Let us denote $D_{[d-k, d-k]}(x)$ (resp. $D_{[k, k]}(x)$) the determinant of the $(d - k) \times (d - k)$ (resp. $k \times k$) submatrix of the Hessian of h at x made by first $d - k$ lines and columns (resp. the last k lines and columns). Notice that $D_{[d-k, d-k]}$ and $D_{[k, k]}$ are function defined almost everywhere on \mathbb{B} . As the Hessian is positive definite, by Fischer inequality [21, 7.8.3],

$$\det \text{Hess } h \leq D_{[d-k, d-k]} D_{[k, k]}.$$

Now by (2.21) for any $\bar{x} \in \frac{1}{4}\mathbb{B}^k$ we have that the function

$$\tilde{x} \mapsto \widetilde{\text{grad } h}(\tilde{x} + \bar{x}) = (\partial_1 h, \dots, \partial_{d-k} h, 0, \dots, 0)(\tilde{x} + \bar{x})$$

sends almost all points in $\frac{1}{4}\mathbb{B}^{d-k}$ into a $(d - k)$ -dimensional disk of radius $8(d - k)(|h(0)| + C\|\bar{x}\|)$. In particular with (2.18):

$$\int_{\frac{1}{4}\mathbb{B}^{d-k}} D_{[d-k, d-k]}(x) \, d\tilde{x} \leq C(d, k)(|h(0)| + C\|\bar{x}\|)^{d-k}.$$

Moreover as $D_{[d-k, d-k]}D_{[k, k]} \geq c_0$, for almost any $\bar{x} \in \frac{1}{4}\mathbb{B}^k$,

$$\begin{aligned} \int_{\frac{1}{4}\mathbb{B}^{d-k}} \frac{1}{D_{[k, k]}(x)} \, d\tilde{x} &\leq c_0^{-1} \int_{\frac{1}{4}\mathbb{B}^{d-k}} D_{[d-k, d-k]}(x) \, d\tilde{x} \\ &\leq C(d, k)c_0^{-1}(|h(0)| + C\|\bar{x}\|)^{d-k}. \end{aligned}$$

Cauchy–Schwarz inequality gives

$$\begin{aligned} \int_{\frac{1}{4}\mathbb{B}^{d-k}} D_{[k, k]}(x) \, d\tilde{x} &\geq \frac{\text{Vol}(\frac{1}{4}\mathbb{B}^{d-k})^2}{\int_{\frac{1}{4}\mathbb{B}^{d-k}} \frac{1}{D_{[k, k]}(x)} \, d\tilde{x}} \\ &\geq C'(d, k)c_0(|h(0)| + C\|\bar{x}\|)^{-d+k}. \end{aligned}$$

Now as h is C -Lipschitz on $\frac{3}{4}\mathbb{B}^d$, for any $\tilde{x} \in \frac{3}{4}\mathbb{B}^{d-k}$ the function

$$\bar{x} \mapsto \overline{\text{grad } h}(\tilde{x} + \bar{x}) = (0, \dots, 0, \partial_{d-k+1} h, \dots, \partial_d h)(\tilde{x} + \bar{x})$$

sends almost all points in $\frac{1}{4}\mathbb{B}_+^k$ —the part of $\frac{1}{4}\mathbb{B}^k$ with vectors of positive entries— into a k -dimensional disc of radius C , and again by (2.18), $\int_{\frac{1}{4}\mathbb{B}_+^k} D_{[k, k]}(x) \, d\bar{x}$ is less than the volume of this disc, so

$$(2.22) \quad \int_{\frac{1}{4}\mathbb{B}^{d-k}} \int_{\frac{1}{4}\mathbb{B}_+^k} D_{[k, k]}(x) \, d\bar{x} \, d\tilde{x} \leq C''(d, k)C^k.$$

Putting the inequalities together, by Fubini,

$$\int_{\frac{1}{4}\mathbb{B}_+^k} (|h(0)| + C\|\bar{x}\|)^{-d+k} \, d\bar{x} \leq C'''(d, k)C^k c_0^{-1}.$$

Now integrating in polar coordinates on \mathbb{B}_+^k we get that

$$\int_0^{1/4} \frac{t^{k-1}}{(|h(0)| + Ct)^{d-k}} \, dt \leq C''''(d, k)C^k c_0^{-1}.$$

The function $F(y) = \int_0^{1/4} \frac{t^{k-1}}{(y+Ct)^{d-k}} \, dt$ is decreasing and by Beppo-Levi Theorem, as $k \leq d/2$, $\lim_{y \rightarrow 0^+} F(y) = +\infty$.

This shows that $|h(0)| > c$ where c depends only on C, c_0, k , and d . \square

3. Equivariant convex sets

3.1. Definition

We fix a uniform lattice Γ in $\mathrm{SO}^+(d, 1)$. Recall that an affine deformation of Γ is a subgroup Γ_τ of $\mathrm{Isom}(\mathbb{R}^{d,1})$ obtained by adding a translation part to elements of Γ . Thus an affine deformation is defined by the translation function $\tau : \Gamma \rightarrow \mathbb{R}^{d+1}$. Given $\gamma \in \Gamma$ we denote by γ_τ the corresponding element in Γ_τ , i.e. the affine transformation $\gamma_\tau(x) = \gamma(x) + \tau_\gamma$.

The condition that Γ_τ is a subgroup forces τ to verify the cocycle relation

$$\tau_{\alpha\beta} = \tau_\alpha + \alpha\tau_\beta.$$

The set of cocycles is a vector space of finite dimension denote here by $Z^1(\Gamma, \mathbb{R}^{d+1})$. The subgroup of $\mathrm{Isom}(\mathbb{R}^{d,1})$ obtained by conjugating Γ by a translation in \mathbb{R}^{d+1} of a vector t_0 is an affine deformation Γ_τ of Γ . We say that such deformation is trivial. The corresponding cocycle is $\tau_\alpha = \alpha(t_0) - t_0$. A cocycle obtained in this way is called a coboundary. The set of coboundary is a subspace of $Z^1(\Gamma, \mathbb{R}^{d+1})$ of dimension $d + 1$. The quotient $H^1(\Gamma, \mathbb{R}^{d+1}) = Z^1(\Gamma, \mathbb{R}^{d+1})/B^1(\Gamma, \mathbb{R}^{d+1})$ is a geometrically meaningful object under different points of view.

If Γ is torsion free, it parameterizes the MGHCF structures on the manifold $\mathbb{R} \times (\mathbb{H}^d/\Gamma)$, but, if $d > 2$, it is also the Zarinski tangent space at the identity of the character variety of representations of Γ into $\mathrm{SO}(d + 1, 1)$ (see [33]).

Recall from the introduction that a τ -convex set is a closed future convex set invariant under the action of Γ_τ .

If Γ is not torsion-free, then by Selberg Lemma it contains a torsion free normal subgroup Γ' of finite index. By [11, 3], there exists a unique maximal convex subset invariant under Γ'_τ . By uniqueness it is also Γ_τ -invariant. That is, τ -convex sets always exist.

Remark 3.1. — The argument to show the existence of τ -convex set for general uniform lattices is quite general. It turns out that if Theorems 1.1 and 1.3 hold for Γ' , then they also hold for Γ . So there is no loss of generality to assume that Γ is torsion-free.

In the sequel we will always make such assumption, leaving to the reader the details in the general case.

First we want to show that the Gauss map of a τ -convex set is surjective.

LEMMA 3.2. — *A τ -convex set \mathbf{K} is an F -convex set.*

Proof. — We need to prove that the support function $\tilde{H} : \mathbb{I}^+(0) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\tilde{H}(X) = \sup_{p \in \mathbf{K}} \langle X, p \rangle_-$$

is finite over $\mathbb{I}^+(0)$.

First we notice that \tilde{H} is convex and 1-homogenous. Thus, the region where \tilde{H} is finite is a convex cone with vertex at 0: its intersection with \mathbb{H}^d is a hyperbolic convex set. If we prove that this region contains an orbit of Γ in \mathbb{H}^d , we conclude that \tilde{H} is finite over $\mathbb{I}^+(0)$. Indeed as Γ is a uniform lattice, the convex hull in \mathbb{H}^d of the orbit of any point is the whole \mathbb{H}^d .

In order to prove that \tilde{H} is finite on some orbit of Γ , we remark that \tilde{H} is finite on the image of the Gauss map. Indeed, if $\eta = \mathcal{G}(p)$, for some $p \in \partial_s \mathbf{K}$, $\tilde{H}(p) = \langle \eta, p \rangle_-$.

By the invariance of \mathbf{K} , if η is a support vector for \mathbf{K} at p , then $\gamma(\eta)$ is a support vector for \mathbf{K} at $\gamma_\tau(p)$. Thus the image of the Gauss map is Γ -invariant, i.e. it is union of some orbits of Γ .

To conclude it is sufficient to observe that $\partial_s \mathbf{K}$ is not empty, that is \mathbf{K} is not a half-space bounded by any lightlike plane P . Otherwise the lightlike direction orthogonal to the tangent space of P would be fixed by Γ , but we know that Γ does not fix any point in $\partial \mathbb{H}^d$. □

As stated in the introduction, there is a natural action of $\text{Isom}(\mathbb{R}^{d,1})$ on the set of the support functions of F-convex sets. Basically $\sigma \cdot H$ is by definition the support function of $\sigma(\mathbf{K})$. It turns out that \mathbf{K} is a τ -convex set if and only if its support function H is fixed by the action of Γ_τ .

Another simple remark is that the action of $\text{Isom}(\mathbb{R}^{d,1})$ on the set of support functions preserves the natural partial order. Indeed since we have $H \leq H'$ if and only if the corresponding F-convex set satisfy $\mathbf{K} \subset \mathbf{K}'$, it turns out that

$$(3.1) \quad H \leq H' \Rightarrow \sigma \cdot H \leq \sigma \cdot H'.$$

The function $\sigma \cdot H$ can be explicitly described in terms of σ and H .

LEMMA 3.3. — *If H is a support function of an F-convex set, and $\sigma \in \text{Isom}(\mathbb{R}^{d,1})$ has linear part γ and translational part τ , then*

$$(3.2) \quad (\sigma \cdot H)(X) = H(\gamma^{-1}X) + \langle \tau, X \rangle_-.$$

Proof. — Let \mathbf{K} be the F-convex set corresponding to H . We have

$$\begin{aligned} (\sigma \cdot H)(X) &= \sup_{p \in \sigma \mathbf{K}} \langle X, p \rangle_- = \sup_{q \in \mathbf{K}} \langle X, \sigma q \rangle_- = \sup_{q \in \mathbf{K}} \langle X, \gamma q \rangle_- + \langle X, \tau_\gamma \rangle_- \\ &= H(\gamma^{-1}(X)) + \langle X, \tau_\gamma \rangle_- . \end{aligned} \quad \square$$

In other words the linear part of σ acts on H by pull-back, whereas the translation τ adds a linear factor to H .

Notice that if H is a τ -support function, then $\gamma_\tau \cdot H = H$ so we have

$$H(X) = H(\gamma^{-1}X) + \langle \tau_\gamma, X \rangle_-$$

that is, putting $Y = \gamma^{-1}X$

$$(3.3) \quad H(\gamma Y) = H(Y) + \langle \gamma^{-1}\tau_\gamma, Y \rangle_- = H(Y) - \langle \tau_{\gamma^{-1}}, Y \rangle_- .$$

Notice that equations (3.3) are linear but, if $\tau \neq 0$, not homogeneous. So the set of functions satisfying (3.3) is an affine space whose tangent space is the space of Γ -invariant functions.

It follows that a convex combination of τ -support function is still a τ -support function. If \mathbf{K}_1 and \mathbf{K}_2 are τ -convex set then for any $0 \leq \lambda \leq 1$, $\lambda\mathbf{K}_1 + (1 - \lambda)\mathbf{K}_2$ is a τ -convex set. In other words, for those operations, the set of τ -convex sets is convex.

Moreover if \mathbf{K}_1 is a τ -convex set and \mathbf{K}_2 is a Γ -invariant set, then $\mathbf{K}_1 + \mathbf{K}_2$ is again a τ -convex set.

As explained in Section 2.1, a support function H is determined by its restriction on \mathbb{H}^d , denoted here by \bar{h} and by its restriction on the ball \mathbb{B} , say h , where more precisely we embed \mathbb{B} into $I^+(0)$ by the affine map $x \mapsto \hat{x} = \begin{pmatrix} x \\ 1 \end{pmatrix}$ and put $h(x) = H(\hat{x})$.

Notice that since \mathbb{H}^d is left invariant by the action of linear isometries of $\mathbb{R}^{d,1}$, the action of $\text{Isom}(\mathbb{R}^{d,1})$ on the space of hyperbolic support functions can be easily deduced by (3.2). In fact for $\eta \in \mathbb{H}^d$ we have $(\sigma \cdot \bar{h})(\eta) = \bar{h}(\gamma^{-1}\eta) + \langle \tau, \eta \rangle_-$, so if \bar{h} is the hyperbolic support function of a τ -convex set we deduce $\bar{h}(\gamma\zeta) = \bar{h}(\zeta) - \langle \tau_{\gamma^{-1}}, \zeta \rangle_-$, for $\zeta \in \mathbb{H}^d$.

The action of $\text{Isom}(\mathbb{R}^{d,1})$ on the ball support functions (that are the convex functions on \mathbb{B}) is more complicated, due to the fact that the immersion of \mathbb{B} into $I^+(0)$ is not invariant by the action of $\text{Isom}(\mathbb{R}^{d,1})$. The translation part contributes by adding an affine function, whereas the linear part of σ does not act simply by pull-back. More precisely, notice that \mathbb{B} is naturally identified to the projective set of the lines contained in $I^+(0)$, that is a subset of \mathbb{RP}^{d+1} . As $\text{SO}^+(d, 1)$ preserves $I^+(0)$ there is a natural projective action of $\text{SO}^+(d, 1)$ on \mathbb{B} . Given $\gamma \in \text{SO}^+(d, 1)$ let us denote by $\bar{\gamma}$ the corresponding projective transformation of \mathbb{B}

$$\begin{pmatrix} \bar{\gamma}(x) \\ 1 \end{pmatrix} = \frac{1}{(\gamma(\hat{x}))_{d+1}} \gamma(\hat{x}) .$$

Using (3.2) and the 1-homogeneity of H we deduce how $\text{Isom}(\mathbb{R}^{d,1})$ acts on the ball support functions.

LEMMA 3.4. — For $\sigma \in \text{Isom}(\mathbb{R}^{d,1})$ and h a convex function on \mathbb{B} ,

$$\begin{aligned} (\sigma \cdot h)(x) &= (\gamma^{-1}(\hat{x}))_{d+1} h(\bar{\gamma}^{-1}(x)) + \langle \hat{x}, \tau_\gamma \rangle_- \\ &= \frac{\lambda(x)}{\lambda(\bar{\gamma}^{-1}x)} h(\bar{\gamma}^{-1}(x)) + \langle \hat{x}, \tau_\gamma \rangle_- . \end{aligned}$$

Proof. — We have $(\sigma \cdot h)(x) = (\sigma \cdot H)(\hat{x}) = H(\gamma^{-1}\hat{x}) + \langle \hat{x}, \tau_\gamma \rangle_-$. On the other hand,

$$H(\gamma^{-1}\hat{x}) = (\gamma^{-1}(\hat{x}))_{d+1} H\left(\begin{pmatrix} \bar{\gamma}^{-1}x \\ 1 \end{pmatrix}\right) = (\gamma^{-1}(\hat{x}))_{d+1} h(\bar{\gamma}^{-1}x) .$$

So the first part of the formula is proved.

Remark now that

$$\lambda(x) = \|\hat{x}\|_- = \|\gamma^{-1}\hat{x}\|_- = (\gamma^{-1}(\hat{x}))_{d+1} \|\widehat{\bar{\gamma}^{-1}x}\|_- = (\gamma^{-1}(\hat{x}))_{d+1} \lambda(\bar{\gamma}^{-1}x)$$

and so also the second part of the formula holds. □

Since isometries of Minkowski space preserve the volume, the area measure associated to any F-convex set is preserved by the action: $A(\sigma\mathbf{K}) = \gamma_*A(\mathbf{K})$. In particular if \mathbf{K} is a τ -convex set, then $A(\mathbf{K})$ is a Γ -invariant measure.

Using the correspondence between $A(\mathbf{K})$ and the Monge–Ampère measure associated to the ball support function stated in Proposition 2.17, we deduce that the action of $\text{Isom}(\mathbb{R}^{d,1})$ on the set of convex functions of \mathbb{B} effects in a very controlled way the Monge–Ampère measure.

LEMMA 3.5. — If h is a convex function of \mathbb{B} and $\sigma \in \text{Isom}(\mathbb{R}^{d,1})$ then

$$\text{MA}(\sigma \cdot h) = \frac{\sqrt{1 - \|\bar{\gamma}^{-1}(x)\|^2}}{\sqrt{1 - \|x\|^2}} \bar{\gamma}_* \text{MA}(h) .$$

Proof. — If $\mathbf{v} : \mathbb{B} \rightarrow \mathbb{H}^d$ is the radial map, we have

$$\text{MA}(\sigma \cdot h) = \lambda^{-1}(\mathbf{v}^{-1})_*(A(\sigma\mathbf{K})) = \lambda^{-1}(\mathbf{v}^{-1})_*(\gamma_*A(\mathbf{K})) .$$

Since $\mathbf{v}^{-1}\gamma = \bar{\gamma}\mathbf{v}^{-1}$ we conclude that $\text{MA}(\sigma \cdot h) = \lambda^{-1}(\bar{\gamma}_*)(\lambda\text{MA}(h))$. □

COROLLARY 3.6. — For any $\sigma \in \text{Isom}(\mathbb{R}^{d,1})$, $c_0 \in \mathbb{R}$ and C compact subset of \mathbb{B} , there is a constant $c = c(\sigma, c_0, C)$ such that if $\text{MA}(h) \geq c_0\mathcal{L}$ on \mathbb{B} then $\text{MA}(\sigma \cdot h) \geq c\mathcal{L}$ on C .

Moreover c is uniformly bounded if the linear part of σ lies in a compact subset of $\text{SO}^+(d, 1)$.

3.2. Extension of τ -support functions to the boundary

PROPOSITION 3.7. — *Let $\tau \in Z^1(\Gamma, \mathbb{R}^{d+1})$. Then there exists $g_\tau \in C^0(\partial\mathbb{B})$ such that τ -convex sets are g_τ -convex sets.*

See also [3].

Proof. — As there exists a C^2_+ τ -convex set with constant mean radii of curvature [17], from Proposition 2.23, the corresponding support function on the ball, say h_0 , extends to the sphere $\partial\mathbb{B}$. Denote by g_τ the extension of h_0 to $\partial\mathbb{B}$.

Let us take any other τ -convex set \mathbf{K} . We want to prove that its support function on the ball, say h , extends g_τ . By (3.3) we have that the difference $H - H_0$ is a Γ -invariant function on $\Gamma^+(0)$.

So, its restriction to \mathbb{H}^d reaches a minimum a and a maximum b . Using that on the ball $h(x) = H(x, 1) = \lambda(x)H(\mathbf{v}(x))$, we deduce that

$$a\lambda(x) < h(x) - h_0(x) < b\lambda(x)$$

As $\lambda(x) = \sqrt{1 - \|x\|^2}$ goes uniformly to 0 as $\|x\| \rightarrow 1$ we get the result.

So the value of the extension on the sphere depends only on τ . □

Remark 3.8. — The argument of the proof of Proposition 3.7 shows that if \bar{f} is any bounded function on \mathbb{H}^d , then the restriction of its 1-homogeneous extension to the ball, say f , is continuous on $\bar{\mathbb{B}}$ and in fact $f|_{\partial\mathbb{B}}$ is zero.

This remark can be applied for instance if \bar{f} is a Γ -invariant function and we will often use it in the sequel.

We will denote Ω_{g_τ} by Ω_τ . If τ is a coboundary, then Ω_τ is the closure of the future cone of a point, and g_τ is the restriction of an affine map. Despite the fact that in the literature Ω_τ is often considered as an open subset, it is more convenient to consider its closure in the present paper.

DEFINITION 3.9. — *For any cocycle τ , h_τ will be the support function of Ω_τ (that is the convex envelope of g_τ).*

By Proposition 2.24 any τ -convex set is contained in Ω_τ .

For $x \in \Omega_\tau$, the *cosmological time* is the Lorentzian distance from x to the boundary of Ω_τ . By [11] the cosmological time is a Γ_τ -invariant $C^{1,1}$ -function of the interior of Ω_τ with value on $[0, +\infty)$. Moreover, the level set Σ_t is the boundary of the set $\Omega_\tau + tJ^+(\mathbb{H})$. In particular the support function of the future of Σ_t on \mathbb{H}^d is $\bar{h}_\tau - t$.

For a given Γ_τ -invariant hypersurface S , we will denote by $T_{\min}(S)$ (resp. $T_{\max}(S)$) the minimum (resp. the maximum) of the cosmological time on S .

LEMMA 3.10. — *Let \mathbf{K} be a τ -convex set with hyperbolic support function \bar{h} , and let S be its boundary. Then $T_{\min}(S)$ (resp. $T_{\max}(S)$) is equal to the minimum (resp. maximum) of $\bar{h}_\tau - \bar{h}$ on \mathbb{H}^d .*

Proof. — Let Σ_m be the level set of T corresponding to the level $m = T_{\min}(S)$. As $\mathbf{K} \subset \mathbb{I}^+(\Sigma_m)$, then $h \leq h_\tau - m$, i.e. $h_\tau - h \geq m$. On the other hand, if p is the point of \mathbf{K} realizing the minimum of the cosmological time on $\partial\mathbf{K}$, then the tangent plane of Σ_m at p is a support plane of \mathbf{K} at p . If \hat{x} , $x \in \mathbb{B}$, is the corresponding normal vector, then $\bar{h}(x) = \bar{h}_\tau(x) - m$. \square

Recall from Definition 2.29 that a τ -convex set is *Cauchy* if $h < h_\tau$.

LEMMA 3.11. — *Cauchy τ -convex sets are exactly the τ -convex sets contained in the interior of Ω_τ . For such a \mathbf{K} , $\partial\mathbf{K} = \partial_s\mathbf{K}$.*

Proof. — If \mathbf{K} is in the interior of Ω_τ , it is clearly Cauchy. Now suppose that \mathbf{K} is Cauchy and τ -convex. By Lemma 3.10, \mathbf{K} is in the future of the level set of the cosmological time Σ_m , for $m > 0$ the minimum of $\bar{h}_\tau - \bar{h}$. But Σ_m is contained in the interior of Ω_τ ([11]) so the same holds for \mathbf{K} .

The last assertion comes from Lemma 2.28. \square

Remark 3.12. — The existence of a maximal τ -convex subset was already proved in [11] and in [3] by different arguments. We remark here some consequences of the theory developed in [11] that we will use here.

A first important property is that the action of Γ_τ on the interior of Ω_τ is proper. The quotient $(\text{int } \Omega_\tau)/\Gamma_\tau$ is foliated by space-like convex compact hypersurfaces. Moreover if \mathbf{K} is any Cauchy τ -convex set then $\partial\mathbf{K}/\Gamma_\tau$ is compact.

Since any point in the interior of Ω_τ is contained in some space-like convex hypersurface we get that the action of Γ_τ in $\text{int } \Omega_\tau$ is space-like in the sense that the segment joining two points on the same orbit is always space-like.

Remark 3.13. — Proposition 3.7 implies that for any light-like vector $\ell \in \partial\mathbb{B}$, the plane $P_{g_\tau}(\ell)$ of equation $\langle x, \ell \rangle_- = g_\tau(\ell)$ does not meet the interior of Ω_τ .

Notice however that this does not mean that any light-like direction is orthogonal to a light-like support plane of Ω_τ . Actually if $d = 2$ and if τ is not a coboundary, then the set of directions of light-like support plane has zero measure in $\partial\mathbb{H}^2$, [12, Proposition 4.15]. For most of the light-like directions $P_{g_\tau}(\ell)$ is a support plane at infinity: it does not meet Ω_τ but any parallel displacement of the hyperplane in the future direction will meet the interior of Ω_τ .

We say that a continuous function $\bar{h} : \mathbb{H}^d \rightarrow \mathbb{R}$ is τ -equivariant if it satisfies the equivariance relation (3.3). Notice that τ -equivariant functions are precisely those functions \bar{h} that can be written as $\bar{h} = \bar{h}_\tau + \bar{f}$, with $\bar{f} = \bar{h} - \bar{h}_\tau$ a continuous Γ -invariant function.

For any Γ -invariant function \bar{f} , we define the following convex set:

$$\mathbf{K}_\tau(\bar{f}) = \{p \in \mathbb{R}^{d+1} \mid \langle p, \eta \rangle_- \leq \bar{h}_\tau(\eta) + \bar{f}(\eta), \forall \eta \in \mathbb{H}^d\}.$$

It is immediate to see that if $p \in \Omega_\tau$ and $a > \sup_{\mathbb{H}^d} \bar{f}$, then $p+a\eta \in \mathbf{K}_\tau(\bar{f})$ for any $\eta \in \mathbb{H}^d$. So this set is not empty. Moreover, a simple computation shows that it is a τ -convex set.

If f denotes the restriction of the 1-homogeneous extension of \bar{f} to the ball \mathbb{B} , the support function h of $\mathbf{K}_\tau(\bar{f})$ is the convex envelope of $f + h_\tau$ (because the convex envelope corresponds to the pointwise supremum of all affine functions majored by $f + h_\tau$ [31, 12.1.1], that is exactly the definition of the support function). Notice that the restriction of f to $\partial\mathbb{B}$ is 0 (see Remark 3.8), so if f is non negative, then h equals h_τ , that is $\mathbf{K}_\tau(f) = \Omega_\tau$.

Many of the functionals we will introduce below on the set of τ -support functions can be then extended to the space of τ -equivariant continuous functions, but we will only need the following.

LEMMA 3.14. — *Let \bar{f} be any Γ -invariant function on \mathbb{H}^d , and \bar{h} be the support function of $\mathbf{K} = \mathbf{K}_\tau(\bar{f})$, and G be the Gauss map of \mathbf{K} . Then on $\partial_{\text{reg}}\mathbf{K}$,*

$$(\bar{h}_\tau + \bar{f}) \circ \mathcal{G} = \bar{h} \circ \mathcal{G}.$$

In particular, the set of points such that $\bar{h} < \bar{h}_\tau + \bar{f}$ (i.e. such that $\bar{h}_\tau + \bar{f} \neq \bar{h}$) has zero area measure.

Proof. — Denote by f the restriction of the 1-homogeneous extension of \bar{f} to \mathbb{B} . Clearly $h < h_\tau + f$ at $x \in \mathbb{B}$ if and only if $\bar{h} < \bar{h}_\tau + \bar{f}$ at $\mathbf{v}(x)$.

Now let $p \in \partial_s\mathbf{K}$ and \hat{x} , $x \in \mathbb{B}$, be a support vector at p , so that $\langle \hat{x}, p \rangle_- = h(x)$. Suppose that $h(x) < (h_\tau + f)(x)$.

By Remark 3.8 and Proposition 3.7, the function $h_\tau + f$ is continuous on $\overline{\mathbb{B}}$. Notice that \mathbf{K} coincides with the set $\{p \in \mathbb{R}^{d+1} \mid \langle p, \hat{x} \rangle_- \leq h_\tau(x) + f(x), \forall x \in \overline{\mathbb{B}}\}$. Since $p \in \partial_s\mathbf{K}$, by compactness of $\overline{\mathbb{B}}$, there must exist $y \in \overline{\mathbb{B}}$ such that $\langle p, \hat{y} \rangle_- = (h_\tau + f)(y)$ (otherwise, p would be an interior point of \mathbf{K}).

Hence x and y are two different support vectors at p , so p is not regular. The Gauss image of the set of regular points has full area measure by Proposition 2.19. □

3.3. Hausdorff distance and convergence

The difference of two τ -equivariant maps is Γ -invariant. This gives a continuous function on the compact manifold \mathbb{H}^d/Γ . The set of τ -convex sets can be endowed with the following distance.

DEFINITION 3.15 (Hausdorff distance). — *The τ -Hausdorff distance between two τ -convex sets is the uniform norm of the difference of their support functions.*

Remark 3.16. — Recall that the support function of $J^+(\mathbb{H})$ on \mathbb{H}^d equal to -1 . So if \mathbf{K} is any F-convex set with support function h , the support function of $\mathbf{K} + \lambda J^+(\mathbb{H})$ is $h - \lambda$.

It follows that the distance between two τ -convex sets \mathbf{K} and \mathbf{K}' is the minimum of the $\lambda \geq 0$ such that $\mathbf{K} + \lambda J^+(\mathbb{H}) \subset \mathbf{K}'$ and $\mathbf{K}' + \lambda J^+(\mathbb{H}) \subset \mathbf{K}$.

The main tool will be the following convergence result.

LEMMA 3.17. — *Let Σ_t be a level set of the cosmological time, and let (\mathbf{K}_n) be a sequence of τ -convex sets which meet the past of Σ_t . Then — up to passing to a subsequence — (\mathbf{K}_n) converges to a τ -convex set with respect to the τ -Hausdorff distance.*

Proof. — Let (S_n) be the sequence of boundaries of the \mathbf{K}_n . Notice that S_n is the graph of some 1-Lipschitz function \tilde{u}_n defined on the horizontal plane \mathbb{R}^d . By a standard use of Ascoli–Arzelà theorem, if there is a compact subset in \mathbb{R}^{d+1} which meets all the hypersurfaces S_n , then up to passing to a subsequence the functions \tilde{u}_n converge to a 1-Lipschitz function u uniformly on compact subsets of \mathbb{R}^d .

To check that S_n meets a compact region of \mathbb{R}^{d+1} , notice that by the assumption there is a point $x_n \in \Sigma_t$ whose past meets S_n . By the invariance of S_n , any point in the orbit of x_n works as well. Since the quotient Σ_t/Γ_τ is compact [11], then there is a compact fundamental region C of Σ_t for the action of Γ_τ . So we may assume that $x_n \in C$. Thus, S_n meets $\overline{I^-(C)} \cap \Omega_\tau$ which is a compact region.

Up to passing to a subsequence, we can deduce that \tilde{u}_n converges to a 1-Lipschitz function \tilde{u} . The graph of \tilde{u} is the boundary of a τ -convex set \mathbf{K} . It remains to prove that \mathbf{K}_n converges to \mathbf{K} with respect to the τ -Hausdorff distance.

Suppose by contradiction that (\mathbf{K}_n) does not converge to \mathbf{K} in the sense of Definition 3.15. That means that there exists a $\epsilon > 0$ such that, for arbitrarily large n , there exists $x_n \in \mathbf{K}_n + \epsilon J^+(\mathbb{H})$ with $x_n \notin \mathbf{K}$, or $x_n \in \mathbf{K} + \epsilon J^+(\mathbb{H})$ and $x_n \notin \mathbf{K}_n$.

Let us suppose that we are in the former case.

Let P be any fundamental region of Σ_t and let \mathcal{P} the union of the time-like rays in Ω_τ orthogonal to P . By [11] it turns out that \mathcal{P} is a fundamental region of the action of Γ_τ on Ω_τ , and the intersection of \mathcal{P} with the boundary of any τ -convex set is compact. It turns out that there is a point in the orbit of x_n , say y_n , that lies in $\overline{I^-(\mathcal{P} \cap \partial\mathbf{K})} \cap \Omega_\tau$ that is a compact region.

Notice that y_n lies in $\mathbf{K}_n + \epsilon J^+(\mathbb{H})$, so $y_n = z_n + \epsilon \eta_n$ for some $\eta_n \in \mathbb{H}^d$ and $z_n \in \mathbf{K}_n$. Since $z_n \in I^-(y_n) \cap \Omega_\tau \subset \overline{I^-(\mathcal{P} \cap \partial\mathbf{K})} \cap \Omega_\tau$, we can also suppose that z_n converges to z . By the convergence of \tilde{u}_n to \tilde{u} we deduce that $z \in \mathbf{K}$. Moreover notice that $\langle y - z, y - z \rangle_- = -\epsilon^2$ so y lies in the future of \mathbf{K} contradicting the assumption that it is a limit of a sequence of points in the past of \mathbf{K} .

The latter case is analogous. □

We can deduce the analogue of the Blaschke Selection theorem for convex bodies.

COROLLARY 3.18. — *From each sequence of τ -convex sets uniformly bounded in the future, one can extract a sequence converging to a τ -convex set.*

Another simple consequence of Lemma 3.17 is that the cosmological time is uniformly bounded for any τ -invariant hypersurface that meets a fixed level of the cosmological time, see also [6, Proposition 7.4.14] and [7, Proposition 4.1].

COROLLARY 3.19. — *Let $\epsilon > 0$. There exists $T = T(\epsilon, \tau)$ such that if \mathbf{K} is any τ -convex set whose boundary S meets the level set of the cosmological time Σ_ϵ then $T_{\max}(S) \leq T$, where $T_{\max}(S)$ is the maximum of the cosmological time of S .*

Proof. — By contradiction suppose that there is a sequence (\mathbf{K}_n) of τ -convex sets such that the boundary of \mathbf{K}_n , say S_n , meets Σ_ϵ and $T_{\max}(S_n) \geq n$. By Lemma 3.17, up to passing to a subsequence, (\mathbf{K}_n) converges to an F-convex set \mathbf{K} in the τ -Hausdorff sense. This implies that the corresponding hyperbolic support functions $\bar{h}_\tau - \bar{h}$ are uniformly bounded, but this contradicts Lemma 3.10. □

LEMMA 3.20. — *There is a constant $\delta = \delta(\tau) > 0$ such that if $T_{\min}(S) \geq 1$ then*

$$T_{\max}(S)/T_{\min}(S) < \delta.$$

Proof. — Suppose that there is a sequence of τ -convex sets \mathbf{K}_n with boundaries S_n such that $T_{\min}(S_n) \geq 1$ and $T_{\max}(S_n)/T_{\min}(S_n) > n$. From

Corollary 3.19, $T_{\min}(S_n) \rightarrow +\infty$. Let us set $t_n = T_{\min}(S_n)$ and consider the sequence of rescaled hypersurfaces $S'_n = (1/t_n)S_n$. Notice that S'_n is the boundary of the $(1/t_n)\tau$ -convex set $(1/t_n)\mathbf{K}_n$ and satisfies $T_{\min}(S'_n) = 1$.

So for any n , S'_n has a common point with the level set 1 for the cosmological time T_n of $\frac{1}{t_n}\Omega_\tau = \Omega_{\frac{1}{t_n}\tau}$. By [11, 6], the graph functions of those level sets of T_n converge in the compact open topology to the graph function of the cosmological time T_0 of $\Omega_0 = \mathbb{I}^+(0)$. Moreover for every c there is a compact region U_c of \mathbb{R}^{d+1} (independent of n) that contains a fundamental region of the level set $T_n^{-1}(c)$ for the action of $\Gamma_{\frac{1}{t_n}\tau}$ for every n .

As for each n , the intersection point of S'_n with $T_n^{-1}(1)$ can be chosen in U_1 , we deduce that S'_n meets U_1 for every n . As S'_n are graphs of 1-Lipschitz functions \tilde{v}_n , we deduce that — up to extracting a subsequence — \tilde{v}_n converges to a convex 1-Lipschitz function \tilde{v} uniformly on compact subsets as a consequence of Arzelà–Ascoli theorem. The graph of \tilde{v} , is the boundary of a convex subset \mathbf{K} which is invariant by the action of Γ by construction.

Since S'_n is contained in the closure of the future of $T_n^{-1}(1)$ then $\partial\mathbf{K}$ is contained in $\mathbb{I}^+(\mathbb{H}^d) = \mathbb{J}^+(\mathbb{H})$ in \mathbb{R}^{d+1} . In particular, it is a Cauchy surface in $\mathbb{I}^+(0)$.

As U_1 contains a fundamental region of the action of $\Gamma_{\frac{1}{t_n}\tau}$ over $T_n^{-1}(1)$, there is a sequence of points $x_n \in U_1$ such that $x_n \in S'_n \cap T_n^{-1}(1)$. Taking the limit (up to a subsequence) of (x_n) , we deduce that $\partial\mathbf{K}$ meets the level set $T_0^{-1}(1)$.

By Corollary 3.19 $\partial\mathbf{K}$ is contained in the past of the level set $T_0^{-1}(c)$, for some fixed constant c . Since the graph functions of S'_n converge to the graph function of $\partial\mathbf{K}$ uniformly on compact subsets of \mathbb{R}^d and the same holds for the graph functions of $T_n^{-1}(c)$, we deduce that for any compact region V of \mathbb{R}^{d+1} there is n_0 (a priori depending on V) such that $S'_n \cap V$ is in the past of $T_n^{-1}(c) \cap V$ for $n \geq n_0$.

Since the compact region U_c contains a fundamental region of $T_n^{-1}(c)$ for every n , we deduce that for n big S'_n is contained in the past of $T_n^{-1}(c)$.

But by our starting assumption, one also has that $T_{\max}(S'_n) = T_{\max}(S_n)/t_n \geq n$, and this gives a contradiction. □

3.4. Area and total area

In the sequel, except if explicitly mentioned, the integrals will be on the compact manifold \mathbb{H}^d/Γ .

Let us recall notations from Subsection 2.1. If \mathbf{K} is a τ -convex set, then $V_\epsilon(\mathbf{K})$ is clearly a Γ -invariant measure, and so are the area measures $S_i(\mathbf{K})$.

Any Γ -invariant Radon measure μ on \mathbb{H}^d defines a Radon measures in \mathbb{H}^d/Γ : it is the only measure $\bar{\mu}$ on \mathbb{H}^d/Γ such that if $U \subset \mathbb{H}^d/\Gamma$ is a Borel set and $\psi : U \rightarrow \mathbb{H}^d$ is a measurable section of the projection $\pi : \mathbb{H}^d \rightarrow \mathbb{H}^d/\Gamma$, then $\psi_*(\bar{\mu}) = \mu$.

In practice, if E is a Borel subset of \mathbb{H}^d which meets every orbit of Γ exactly once, then the measure of any Borel set ω of \mathbb{H}^d/Γ is $\mu(\pi^{-1}(\omega) \cap E)$.

In our case, we obtain measures $\bar{V}_\epsilon(\mathbf{K})$ and $\bar{S}_i(\mathbf{K})$ on \mathbb{H}^d/Γ .

Using (2.10) we deduce that

$$(3.4) \quad \bar{V}_\epsilon(\mathbf{K}) = \frac{1}{d+1} \sum_{i=0}^d \epsilon^{d+1-i} \binom{d+1}{i} \bar{S}_i(\mathbf{K}).$$

Remark 3.21. — Notice that for any Borel subset $\omega \subset \mathbb{H}^d/\Gamma$, $\bar{S}_d(\mathbf{K})(\omega)$ is the derivative of $\bar{V}_\epsilon(\mathbf{K})(\omega)$ for $\epsilon = 0$.

As $\bar{V}_{\epsilon+h}(\mathbf{K}) = \bar{V}_\epsilon(\mathbf{K}) + \bar{V}_h(\mathbf{K} + \epsilon J^+(\mathbb{H}))$ we get that for any Borel subset $\omega \subset \mathbb{H}^d/\Gamma$

$$\frac{d\bar{V}_\epsilon(\mathbf{K})(\omega)}{d\epsilon}(\epsilon) = \bar{S}_d(\mathbf{K} + \epsilon J^+(\mathbb{H}))(\omega),$$

or equivalently

$$(3.5) \quad \bar{V}_\epsilon(\mathbf{K})(\omega) = \int_0^\epsilon \bar{S}_d(\mathbf{K} + tJ^+(\mathbb{H})) dt.$$

We also get an invariant formulation of the weak-convergence stated in Lemma 2.12.

LEMMA 3.22. — *If the support functions of τ -convex sets converge for the sup norm to the support function of a τ -convex set, then the associated measures \bar{V}_ϵ and \bar{S}_i on \mathbb{H}^d/Γ weakly converge.*

Proof. — As any continuous function on \mathbb{H}^d/Γ can be expressed as the sum of continuous functions with supports in small open sets by means of a partition of the unity, it is sufficient to prove that

$$\int f d\bar{V}(\mathbf{K}_n) \rightarrow \int f d\bar{V}(\mathbf{K})$$

assuming that the support of f is contained in a small chart U where a section $\psi : U \rightarrow \mathbb{H}^d$ is defined. But, by definition we have $\int f d\bar{V}(\mathbf{K}_n) = \int_U f d\bar{V}(\mathbf{K}_n) = \int_{\psi(U)} f \circ \psi^{-1} dV(\mathbf{K}_n)$ and so the convergence stated directly follows from Lemma 2.12. □

We still denote by $A(\mathbf{K})$ the measure $\bar{S}_d(\mathbf{K})$ on \mathbb{H}^d/Γ , and by $\text{Area}(\mathbf{K})$ the total area of \mathbf{K} , i.e.

$$\text{Area}(\mathbf{K}) = A(\mathbf{K})(\mathbb{H}^d/\Gamma).$$

COROLLARY 3.23. — *If the support functions of τ -convex sets converge for the sup norm to the support functions of a τ -convex set, then the corresponding area measures weakly converge. In particular, the total area is continuous for this topology.*

If $\partial_s \mathbf{K}$ is C^1 , then the total area is the total volume of the manifold $\partial_s \mathbf{K} / \Gamma_\tau$ for the induced metric by Corollary 2.21.

We prove below that the total area is monotonic.

LEMMA 3.24. — *Let $\mathbf{K}_0, \mathbf{K}_1$ be two τ -convex sets, with \mathbf{K}_1 contained in the interior of \mathbf{K}_0 . Then*

$$\text{Area}(\mathbf{K}_0) \leq \text{Area}(\mathbf{K}_1).$$

Proof. — Let us first suppose that \mathbf{K}_0 and \mathbf{K}_1 are C^2_+ , with respective hyperbolic support functions \bar{h}_0 and \bar{h}_1 on \mathbb{H}^d . Note that $\bar{h}_1 < \bar{h}_0$. For $\lambda \in (0, 1)$, let $\mathbf{K}_\lambda = (1 - \lambda)\mathbf{K}_0 + \lambda\mathbf{K}_1$. It has hyperbolic support function $(1 - \lambda)\bar{h}_0 + \lambda\bar{h}_1$, so \mathbf{K}_λ is C^2_+ . We can use the inverse of the Gauss map of \mathbf{K}_λ , say χ_λ , to get a smooth family of parameterizations of $\partial\mathbf{K}_\lambda$. Recall by (2.2) that $\chi_\lambda(\eta) = \text{grad}_\eta^{\mathbb{H}} \bar{h}_\lambda - \bar{h}_\lambda(\eta)\eta$, so the variational field $X = \partial_\lambda \chi_\lambda$ at a point $\chi_\lambda(\eta)$ is given by

$$X(\chi_\lambda(\eta)) = \text{grad}_\eta^{\mathbb{H}}(\bar{h}_1 - \bar{h}_0) - (\bar{h}_1 - \bar{h}_0)(\eta)\eta.$$

The variation of the area of $\partial\mathbf{K}_\lambda$ is given by

$$\frac{d}{d\lambda} \text{Area}(\mathbf{K}_\lambda) = - \int_{\partial\mathbf{K}_\lambda / \Gamma_\tau} s_\lambda \langle X(x), \mathcal{G}_\lambda(x) \rangle_- \, d\mu_\lambda$$

where μ is the intrinsic volume form and s_λ is the sum of the principal eigenvalues computed with respect the normal \mathcal{G} (so s_λ is positive).

Using the parameterization given by χ_λ and the fact that $\chi_\lambda^*(\mu_\lambda) = dA(\mathbf{K}_\lambda)$ we conclude that

$$\frac{d}{d\lambda} \text{Area}(\mathbf{K}_\lambda) = \int s_\lambda \langle X(\chi_\lambda(\eta)), \eta \rangle_- \, dA(\mathbf{K}_\lambda) = \int s_\lambda (\bar{h}_0 - \bar{h}_1) \, dA(\mathbf{K}_\lambda) \geq 0.$$

The general case follows by a density argument, see Appendix A and Corollary 3.23. □

PROPOSITION 3.25. — *There exists a non-negative number $c(\tau)$ such that for every τ -convex set \mathbf{K} , if $\text{Area}(\mathbf{K}) > c(\tau)$ then \mathbf{K} is a Cauchy convex set (that is, it is contained in the interior of Ω_τ).*

As already noted, if τ is a coboundary, i.e. Ω_τ is the future cone of a point, and \mathbf{K} is a τ -convex set with $\text{Area}(\mathbf{K}) \neq 0$, then $\mathbf{K} \neq \Omega_\tau$ that implies that \mathbf{K} is Cauchy. So in this case $c(\tau) = 0$. On the other hand, the example constructed in Subsection 3.7 implies that in general $c(\tau) \neq 0$.

Proof. — For any $n > 0$, let \mathbf{K}_n be such that $\text{Area}(\mathbf{K}_n) > n$ and suppose that \mathbf{K}_n touches the boundary of Ω_τ . By Corollary 3.19, \mathbf{K}_n is in the past of Σ_T for some T independent of n , hence by Lemma 3.24, $\text{Area}(\mathbf{K}_n) \leq \text{Area}(\Sigma_T)$, that is again a contradiction if n is sufficiently large. \square

3.5. The covolume

Let \mathbf{K} be a τ -convex set. The *covolume* of \mathbf{K} , denoted by $\text{covol}(\mathbf{K})$ is the volume of the complementary of \mathbf{K}/Γ_τ in the manifold Ω_τ/Γ_τ . As this complementary is open, the covolume is well defined. It is also finite. This can be seen for example for level sets of the cosmological time, then using the monotonicity of the covolume. Equivalently, it is the volume (the Lebesgue measure) of $\Omega_\tau \setminus \mathbf{K}$ in a fundamental domain for Γ_τ (note that Minkowski isometries preserve the Lebesgue measure).

It can be convenient to consider the covolume as a function on the set of τ -equivariant hyperbolic support functions. For this reason we often will write $\text{covol}(\bar{h})$ to denote the covolume of the corresponding convex set. In the same way we will denote by $A(\bar{h})$ the area measure on \mathbb{H}^d induced by the domain \mathbf{K} .

LEMMA 3.26. — *The function covol is continuous on the set of τ -convex sets.*

Proof. — Let \mathbf{K}_n be at distance $1/n$ from \mathbf{K} . From Remark 3.16

$$\mathbf{K} + \frac{1}{n}J^+(\mathbb{H}) \subset \mathbf{K}_n \text{ and } \mathbf{K}_n + \frac{1}{n}J^+(\mathbb{H}) \subset \mathbf{K},$$

so

$$\text{covol}\left(\mathbf{K} + \frac{1}{n}J^+(\mathbb{H})\right) \geq \text{covol}(\mathbf{K}_n) \text{ and } \text{covol}\left(\mathbf{K}_n + \frac{1}{n}J^+(\mathbb{H})\right) \geq \text{covol}(\mathbf{K}).$$

But clearly $\text{covol}\left(\mathbf{K} + \frac{1}{n}J^+(\mathbb{H})\right) = \text{covol}(\mathbf{K}) + \bar{V}_{1/n}(\mathbf{K})(\mathbb{H}^d/\Gamma)$, so

$$\text{covol}(\mathbf{K}_n) - \text{covol}(\mathbf{K}) \leq \bar{V}_{1/n}(\mathbf{K})(\mathbb{H}^d/\Gamma).$$

By (3.4) $\bar{V}_{1/n}(\mathbf{K})(\mathbb{H}^d/\Gamma)$ goes to 0 when n goes to infinity. In the same way

$$\text{covol}(\mathbf{K}) - \text{covol}(\mathbf{K}_n) \leq \bar{V}_{1/n}(\mathbf{K}_n)(\mathbb{H}^d/\Gamma),$$

so to conclude it remains to prove that $\bar{V}_{1/n}(\mathbf{K}_n)(\mathbb{H}^d/\Gamma)$ goes to 0 when n goes to infinity. But by (3.5) we have

$$(3.6) \quad \bar{V}_{1/n}(\mathbf{K}_n)(\mathbb{H}^d/\Gamma) = \int_0^{1/n} \text{Area}(\mathbf{K}_n + tJ^+(\mathbb{H})) dt.$$

By Lemma 3.24, $\text{Area}(\mathbf{K}_n + tJ^+(\mathbb{H})) \leq \text{Area}(\mathbf{K}_n + 1/nJ^+(\mathbb{H})) \leq \text{Area}(\mathbf{K}_n + J^+(\mathbb{H}))$. As \mathbf{K}_n converges to \mathbf{K} , $\mathbf{K}_n + J^+(\mathbb{H})$ are in the past of a τ -convex set C , so again by Lemma 3.24, $\text{Area}(\mathbf{K}_n + tJ^+(\mathbb{H})) \leq \text{Area}(C)$ and $\bar{V}_{1/n}(\mathbf{K}_n)(\mathbb{H}^d/\Gamma) \leq 1/n \text{Area}(C)$. \square

As we noted in Section 3.1 the set of τ -convex sets is convex. So speaking about convexity of the covolume is meaningful. We will first prove that it is convex on the convex subset of Cauchy τ -convex sets. The convexity on the set of τ -convex sets will follow by continuity. The idea is to write the covolume in terms of a volume of a convex body, obtained by cutting some convex fundamental domain by a hyperplane. Such convex fundamental is described in the following.

PROPOSITION 3.27. — *Let S be any convex Γ_τ -invariant C^1 space-like hypersurface in Ω_τ . Given $x, y \in \mathbb{R}^{d+1}$, let us set $\psi(x, y) = \langle x - y, x - y \rangle_-$. Let us fix a point $x \in S$ and consider the region*

$$\mathcal{P}(x) = \{y \in I^+(S) \mid \psi(x, y) \leq \psi(\gamma x + \tau_\gamma, y) \ \forall \gamma \in \Gamma\}.$$

Then $\mathcal{P}(x)$ is a convex fundamental domain of $I^+(S)$ for the action of Γ_τ .

Moreover the interior of $\mathcal{P}(x)$ is the set

$$(3.7) \quad \text{int } \mathcal{P} = \{y \in I^+(S) \mid \psi(x, y) < \psi(\gamma x + \tau_\gamma, y) \ \forall \gamma \in \Gamma \setminus \{1\}\}.$$

DEFINITION 3.28. — *The domain $\mathcal{P}(x)$ is the Lorentzian Dirichlet polyhedron centered at x .*

Note that in the Fuchsian case it is possible to construct a convex fundamental domain for the whole domain Ω (the future cone of the origin), as it suffices to consider the cone in \mathbb{R}^{d+1} of a Dirichlet polyhedron for Γ in \mathbb{H}^d . Moreover, if $x \in \mathbb{H}^d$, the polyhedron $\mathcal{P}(x)$ defined in Proposition 3.27 coincides with the cone on the Dirichlet polyhedron centered at x .

To prove the Proposition we need the following lemma:

LEMMA 3.29. — *For any divergent sequence $\gamma_n \in \Gamma$ and any compact set C in $I^+(S)$*

$$(3.8) \quad \inf_{y \in C} \psi(\gamma_n x + \tau_{\gamma_n}, y) \rightarrow +\infty.$$

Proof. — This property is easily checked if $C \subset S$. Indeed for $y \in S$, $\sqrt{\psi(\gamma_n(x) + \tau_{\gamma_n}, y)}$ is bigger than the intrinsic distance $d_S(\gamma_n(x) + \tau_{\gamma_n}, y)$.

On the other hand, since S/Γ_τ is compact, the intrinsic distance on S is complete and since the action of Γ_τ is proper we conclude that $d_S(\gamma_n(x) + \tau_{\gamma_n}, C)$ diverges as $n \rightarrow +\infty$.

Now if $C \subset I^+(S)$, we can consider its projection onto S . For $y \in I^+(S)$, there is a unique point y' on S such that $y = y' + e$ with e time-like supporting vector of S at y' ([11]). Then for any $z \in S$ we have that

$$\psi(z, y) = \psi(z, y') + \langle e, e \rangle_- + 2\langle y' - z, e \rangle_-.$$

Notice that since the plane through y' and orthogonal to e is a support plane for S , we have that $\langle y' - z, e \rangle_- \geq 0$.

Then putting $c = \sup_{y' \in \mathbf{K}'} |\langle e, e \rangle_-|$, then $\psi(z, y) \geq \psi(z, y') - c$ for any $z \in S$ and $y \in C$. Then (3.8) follows. \square

Proof of Proposition 3.27. — If $x_1, x_2 \in \mathbb{R}^{d+1}$ are two space-like related points, then the set of points y such that $\psi(x_1, y) \leq \psi(x_2, y)$ is the closed half-space in \mathbb{R}^{d+1} bounded by the plane orthogonal to the segment $[x_1, x_2]$ and passing through its middle point $(x_1 + x_2)/2$. Since the orbit of any point $x \in \Omega_\tau$ is space-like (see Remark 3.12), $\mathcal{P}(x)$ is convex.

In order to show that the Γ_τ -orbit of any point $y \in I^+(S)$ meets $\mathcal{P}(x)$, notice that by Lemma 3.29, for any $y \in I^+(S)$, there is a point x_1 on the orbit of x where the function $\psi(\bullet, y)$ attains the minimum. If $x_1 = \gamma_1 x + \tau_{\gamma_1}$, then putting $y_1 = \gamma_1^{-1} y + \tau_{\gamma_1^{-1}}$ we have that on the orbit of x the minima of the function $\psi(\bullet, y_1)$ is attained at x so $y_1 \in \mathcal{P}(x)$.

Finally proving (3.7) we will have the interior of $\mathcal{P}(x)$ contains at most 1 point for each orbit, that concludes the proof that $\mathcal{P}(x)$ is a fundamental region.

So, it remains to prove that

$$\text{int } \mathcal{P} = \{y \in I^+(S) \mid \psi(x, y) < \psi(\gamma x + \tau_\gamma, y) \ \forall \gamma \in \Gamma \setminus \{1\}\}.$$

The inclusion \subset is immediate. Suppose by contradiction that there is a point y on the boundary of $\mathcal{P}(x)$ such that $\psi(y, x) < \psi(y, \gamma x + \tau_\gamma) \ \forall \gamma \in \Gamma \setminus \{1\}$.

There exists a sequence $y_n \in \overline{I^+(S)}$ converging to y and γ_n such that

$$\psi(\gamma_n(x) + \tau_{\gamma_n}, y_n) < \psi(x, y_n).$$

As y_n converges to y we may apply (3.8) and deduce that γ_n is not diverging. Since Γ is discrete, up to passing to a subsequence we may assume that $\gamma_n = \gamma$ is constant different from 1.

But then passing to the limit in the above inequality we get

$$\psi(\gamma(x) + \tau_\gamma, y) \leq \psi(x, y),$$

that contradicts the assumption on y . \square

The following lemma in the Euclidean setting will be the main ingredient of the convexity of the covolume. Although it is classical, see e.g. [10, 50.], we reproduce the proof for completeness.

LEMMA 3.30. — *In the Euclidean space \mathbb{R}^{d+1} let R be a hyperplane orthogonal (with respect to the Euclidean structure) to a vector u . Let D be a convex body with non-empty interior in R , and C_0 and C_1 be two convex bodies in \mathbb{R}^{d+1} contained in the same side of R as u , such that their orthogonal projection onto R is D . Then, for $\lambda \in [0, 1]$,*

$$V((1 - \lambda)C_0 + \lambda C_1) \geq (1 - \lambda)V(C_0) + \lambda V(C_1),$$

where V is the volume.

Equality holds if and only if either $C_0 = C_1 + U$ or $C_1 = C_0 + U$, where U is some segment directed by u .

Convex bodies with the same image for the orthogonal projection onto a given hyperplane are said to form a *canal class* [34].

Proof. — For any $x \in D$, let $L(x)$ be the line orthogonal to R and passing through x . Take any $k_i \in C_i \cap L(x)$, $i = 1, 2$. The convex combination of k_0 and k_1 belongs to $((1 - \lambda)C_0 + \lambda C_1) =: C_\lambda$ and also to $L(x)$. So the convex combination of $C_i \cap L(x)$, for $i = 0, 1$, is a subset of $C_\lambda \cap L(x)$. Denoting by $l_\lambda(x)$ the length of the segment $C_\lambda \cap L(x)$, and using the linearity of the length, we get

$$(3.9) \quad l_\lambda(x) \geq (1 - \lambda)l_1(x) + \lambda l_2(x).$$

Of course C_λ projects orthogonally onto D , and by Fubini Theorem,

$$(3.10) \quad V(C_\lambda) = \int_D l_\lambda(x) dx \geq (1 - \lambda) \int_D l_0(x) dx + \lambda \int_D l_1(x) dx \\ = (1 - \lambda)V(C_0) + \lambda V(C_1).$$

Suppose that $C_1 = C_0 + U$ where U is a segment parallel to u of length c . Then $C_\lambda = C_0 + \lambda U$, so $l_\lambda(x) = l_0(x) + \lambda c$, and equality holds in (3.9) and hence in (3.10).

Now suppose that equality holds in (3.10). Then equality holds in (3.9) and so for every $x \in D$ and every $\lambda \in [0, 1]$

$$(3.11) \quad (1 - \lambda)(C_0 \cap L(x)) + \lambda(C_1 \cap L(x)) = C_\lambda \cap L(x).$$

For $x \in D$, let $y_{1/2}$ be an end-point of the vertical segment $L(x) \cap C_{1/2}$, and let v be a (Euclidean) support vector at $y_{1/2}$.

By (3.11) there are end-points y_0, y_1 on $L(x) \cap C_0$ and $L(x) \cap C_1$ respectively such that $y_{1/2} = \frac{1}{2}y_0 + \frac{1}{2}y_1$.

Denoting by H_λ^e the (Euclidean) support functions of C_λ , we have

$$H_{1/2}^e(v) = \langle y_{1/2}, v \rangle = \frac{1}{2}\langle y_0, v \rangle + \frac{1}{2}\langle y_1, v \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the positive product of \mathbb{R}^{d+1} .

On the other hand,

$$H_{1/2}^e(v) = \frac{1}{2}H_0^e(v) + \frac{1}{2}H_1^e(v),$$

that, together with the previous equation and with $\langle y_i, v \rangle \leq H_i^e(v)$, leads to $H_i^e(v) = \langle y_i, v \rangle$.

In other words, C_0 and C_1 have parallel support hyperplanes along the lines directed by u . On any segment in the interior of D , part of the boundaries of C_0 and C_1 are graph of convex or concave functions with parallel tangents, so they have the same derivative almost everywhere, hence they differ by a constant c_1 for the convex part and c_2 for the concave part.

Let $C'_0 = C_0 + c_1u$. Notice that the convex part of the boundary of C'_0 coincides with with the convex part of the boundary of C_1 , whereas the concave part differ by $c_2 - c_1$. If $c_2 - c_1 > 0$ then $C_1 = C'_0 + U'$, where U is the segment joining the origin to $|c_2 - c_1|u$. On the other hand if $c_2 - c_1 < 0$, then $C'_0 = C_1 + U'$. In the former case $C_1 = C_0 + [c_1u, c_2u]$, in the latter case $C_0 = C_1 + [-c_1u, -c_2u]$ □

From 3.30 we can simply deduce an analogous statement in the Lorentzian setting.

COROLLARY 3.31. — *In Minkowski space $\mathbb{R}^{d,1}$ let R be a space-like hyperplane orthogonal (for the Minkowski product) to a vector u . Let D be a convex body with non-empty interior in R , and C_0 and C_1 be two convex bodies in \mathbb{R}^{d+1} contained in the same side of R as u , such that their orthogonal projection (in Minkowski sense) onto R is D . Then, for $\lambda \in [0, 1]$,*

$$V((1 - \lambda)C_0 + \lambda C_1) \geq (1 - \lambda)V(C_0) + \lambda V(C_1),$$

where V is the volume.

Equality holds if and only if either $C_0 = C_1 + U$ or $C_1 = C_0 + U$, where U is some segment directed by u .

Proof. — If R is the horizontal plane $\mathbb{R}^d \subset \mathbb{R}^{d+1}$, then the Lorentzian orthogonal projection coincides with the Euclidean projection, so the statement is an immediate consequence of Lemma 3.30.

If R is not horizontal, there is a Lorentzian isometry f sending R to the horizontal plane $\mathbb{R}^d \subset \mathbb{R}^{d+1}$. Notice that the orthogonal projections of

$f(C_i)$ to \mathbb{R}^d coincide with $f(D)$. So we deduce

$$V((1 - \lambda)f(C_0) + \lambda f(C_1)) \geq (1 - \lambda)V(f(C_0)) + \lambda V(f(C_1)).$$

Since f is volume preserving and $(1 - \lambda)f(C_0) + \lambda f(C_1) = f((1 - \lambda)C_0 + \lambda C_1)$ we obtain the result. \square

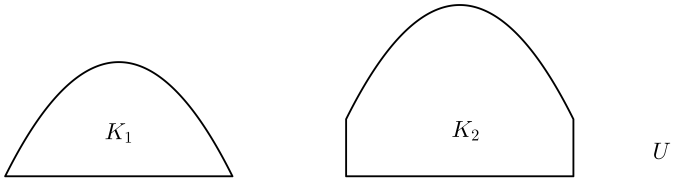


Figure 3.1. \mathbf{K}_1 is a convex cap, and $\mathbf{K}_2 = \mathbf{K}_1 + U$. For $\lambda \in [0, 1]$, the volume is linear along $(1 - \lambda)\mathbf{K}_1 + \lambda\mathbf{K}_2$, but for $\lambda \neq 0$, this is not a convex cap.

Let D be a convex body contained in a space-like hyperplane R with non-empty relative interior. A (Lorentzian) convex cap based on D is a convex body in \mathbb{R}^{d+1} with no empty interior such that

- R is a support plane of \mathbf{K} and $\mathbf{K} \cap R = D$
- For each $x \in \partial D$ the line $L(x)$ through x orthogonal in the Minkowski sense to R meets \mathbf{K} only at x .

Notice that if \mathbf{K} is a convex cap based on D then the second condition implies that D is the orthogonal projection of \mathbf{K} to R , but the condition is not equivalent (see Figure 3.1).

Since two caps based on D cannot differ by a segment, Corollary 3.31 immediately gives the following consequence.

COROLLARY 3.32. — *The volume is strictly concave on the set of convex caps (of same base).*

Notice that the same result holds as well for caps in Euclidean setting. We can prove now the convexity of the covolume.

PROPOSITION 3.33. — *The covolume is strictly convex on the set of Cauchy τ -convex hypersurfaces.*

First we prove the following technical lemma that is needed in the proof of Proposition 3.33.

LEMMA 3.34. — Let S be a τ -invariant strictly convex C^1 hypersurface, and fix $s \in S$. Let $\mathcal{P} = \mathcal{P}(s)$ be the Lorentzian Dirichlet polyhedron centered at s of $I^+(S)$. For any support vector u of S at s and for any $y \in \mathcal{P}$, the point $y + u$ lies in the interior of \mathcal{P} .

Proof. — By a direct computation we have

$$(3.12) \quad \psi(y+u, \gamma_\tau(s)) - \psi(y+u, s) = \psi(y, \gamma_\tau(s)) - \psi(y, s) + 2\langle u, s - \gamma_\tau(s) \rangle_- .$$

Notice that the last term is strictly positive since u is a support vector of S at s , whereas $\psi(y, \gamma_\tau(s)) - \psi(y, s) \geq 0$ by the assumption on y . By (3.7) it follows that $y + u$ is in the interior of \mathcal{P} . \square

Proof of Proposition 3.33. — Let \mathbf{K}_1 and \mathbf{K}_2 be two Cauchy τ -convex hypersurfaces, and for $\lambda \in [0, 1]$ let

$$\mathbf{K}_\lambda = (1 - \lambda)\mathbf{K}_1 + \lambda\mathbf{K}_2 .$$

Let us choose a strictly convex space-like C^1 τ -invariant convex hypersurface S such that the \mathbf{K}_λ are in the future of S . Let $s \in S$ and $\mathcal{P} := \mathcal{P}(s)$ be the Lorentzian Dirichlet polyhedron of $I^+(S)$ centered at s . Let us fix a support vector u of S at s . Let R be an affine hyperplane orthogonal to u chosen so that $\mathbf{K}_\lambda \cap \mathcal{P}$ is in the past side R^- of R for every $\lambda \in [0, 1]$. Note that $D = R \cap \mathcal{P}$ is compact, as it is part of the boundary of the compact convex set $R^- \cap \Omega_\tau$.

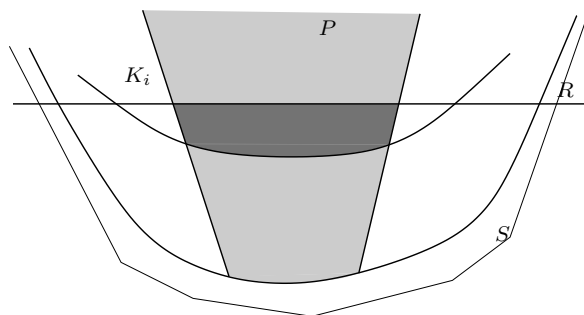


Figure 3.2. The light-grey region represents \mathcal{P} whereas the dark-grey region represents C_i .

Finally let us denote $C_i = \mathbf{K}_i \cap \mathcal{P} \cap R^-, i = 1, 2$. By the choice of R , we have that $C_i \cap R = \mathcal{P} \cap R = D$. Lemma 3.34 shows that if $x - \epsilon u \in \mathcal{P}$ then $x \in \text{int } \mathcal{P}$. So for $x \in R \cap \partial \mathcal{P}$, there is no $\epsilon > 0$ such that the point $x - \epsilon u$ is contained in \mathcal{P} . It follows that C_i is a convex cap (see Figure 3.2).

Define $C_\lambda = (1 - \lambda)C_1 + \lambda C_2$. For $x \in C_\lambda$, there are $x_i \in C_i$ such that x is a convex combination of x_1 and x_2 . By convexity, x belongs to \mathcal{P} and R^- , and by definition x belongs to \mathbf{K}_λ . So $C_\lambda \subset \mathbf{K}_\lambda \cap \mathcal{P} \cap R^-$.

If $\mathbf{K}_1 \neq \mathbf{K}_2$, then C_1 and C_2 , are different caps so Corollary 3.32 gives

$$(3.13) \quad \begin{aligned} (1 - \lambda)V(\mathbf{K}_1 \cap \mathcal{P} \cap R^-) + \lambda V(\mathbf{K}_2 \cap \mathcal{P} \cap R^-) \\ = (1 - \lambda)V(C_1) + \lambda V(C_2) < V(C_\lambda) \leq V(\mathbf{K}_\lambda \cap \mathcal{P} \cap R^-). \end{aligned}$$

By definition, for any $\lambda \in [0, 1]$,

$$(3.14) \quad \text{covol}(\mathbf{K}_\lambda) = -V(\mathbf{K}_\lambda \cap \mathcal{P} \cap R^-) + c$$

where c is a constant depending only on S, \mathcal{P} and H (actually $c = V(\mathcal{P} \cap R^-) + \text{covol}(S)$). □

Since each τ -convex set is limit of Cauchy τ -convex sets, and the covolume is continuous, Proposition 3.33 implies that the covolume is convex on the set of τ -convex sets. Then, Theorem 1.2 is proved.

Remark 3.35. — One could also consider the covolume as a function on the set of all τ -convex sets, for all $\tau \in Z^1(\Gamma, \mathbb{R}^{d+1})$. This set is a convex cone. But the covolume is certainly not convex on this bigger set. Otherwise one should have

$$\text{covol}((1 - \lambda)\Omega_\tau + \lambda\Omega_{\tau'}) \leq (1 - \lambda) \text{covol}(\Omega_\tau) + \lambda \text{covol}(\Omega_{\tau'})$$

that implies that the covolume of $(1 - \lambda)\Omega_\tau + \lambda\Omega_{\tau'}$ is zero. But $(1 - \lambda)\Omega_\tau + \lambda\Omega_{\tau'} \subset \Omega_{(1-\lambda)\tau + \lambda\tau'}$, and in general the inclusion is strict. Then contradiction follows because for any τ , Ω_τ is the only τ -convex set with zero covolume.

For Γ -invariant hypersurfaces, there is a representation formula for the covolume, namely

$$(3.15) \quad \text{covol}(\bar{h}) = -\frac{1}{d+1} \int \bar{h} \, dA(\bar{h})$$

where \bar{h} is the hyperbolic support function of \mathbf{K} . We cannot hope such a simple formula holds in general, but we have the following relation.

LEMMA 3.36. — *If $\mathbf{K}_1 \subset \mathbf{K}_0$ are τ -convex sets with hyperbolic support functions $\bar{h}_1 \leq \bar{h}_0$, then*

$$(3.16) \quad \int (\bar{h}_0 - \bar{h}_1) \, dA(\bar{h}_0) \leq \text{covol}(\bar{h}_1) - \text{covol}(\bar{h}_0) \leq \int (\bar{h}_0 - \bar{h}_1) \, dA(\bar{h}_1).$$

Proof. — First assume that \bar{h}_0 and \bar{h}_1 are C_+^2 and $\bar{h}_1 < \bar{h}_0$. Notice that the region $\mathbf{K}_0 \setminus \mathbf{K}_1$ is foliated by the convex combinations \mathbf{K}_t of \mathbf{K}_0 and

\mathbf{K}_1 . Thus $\text{covol}(\bar{h}_1) - \text{covol}(\bar{h}_0)$ is the volume of the quotient of the region that is image of the equivariant map from $[0, 1] \times \mathbb{H}^d$ to Ω_τ defined by $\chi(t, \eta) = \text{grad}_\eta^{\mathbb{H}} \bar{h}_t - \bar{h}_t(\eta)\eta$ where $\bar{h}_t = (1 - t)\bar{h}_0 + t\bar{h}_1$ (compare with (2.2)).

Notice that $\text{covol}(\bar{h}_1) - \text{covol}(\bar{h}_0) = \int_{[0,1] \times \mathbb{H}^d / \Gamma} \chi^*(dV)$ where dV is the $(d + 1)$ -volume form on \mathbb{R}^{d+1} . Since for each t the map $\chi(t, \bullet)$ is inverse of the Gauss map of the boundary of $\mathbf{K}_t = (1 - t)\mathbf{K}_0 + t\mathbf{K}_1$, and since the area form of a C^1 -space-like hypersurface is defined by contracting the volume form by the normal vector, we deduce that

$$\begin{aligned} \chi^*(dV)(\partial_t, e_1, \dots, e_d) &= dV(\chi_*(\partial_t), \chi_*(e_1), \dots, \chi_*(e_d)) \\ &= -\langle \chi_*(\partial_t), \eta \rangle - dV(\eta, \chi_*(e_1), \dots, \chi_*(e_d)) \\ &= -\langle \chi_*(\partial_t), \eta \rangle - dA(\mathbf{K}_t)(e_1, \dots, e_d). \end{aligned}$$

In other words, $\chi^*(dV) = -\langle \chi_*(\partial_t), \eta \rangle - dt \wedge dA(\bar{h}_t)$. Now

$$D\chi(\partial_t) = \text{grad}^{\mathbb{H}}(\bar{h}_1 - \bar{h}_0) - (\bar{h}_1 - \bar{h}_0)\eta$$

so we deduce that $\chi^*(dV) = (\bar{h}_0 - \bar{h}_1) dt \wedge dA(\bar{h}_t)$. Finally integrating, we obtain

$$\text{covol}(\bar{h}_1) - \text{covol}(\bar{h}_0) = \int_0^1 \int (\bar{h}_0 - \bar{h}_1) dA(\bar{h}_t) dt.$$

As for a convex function $f : [0, 1] \rightarrow \mathbb{R}$

$$f'(0) \leq f(1) - f(0) \leq f'(1),$$

by Theorem 1.2 we deduce the formula above.

Now for the general case, take sequences of C_+^2 τ -equivariant support functions $(\bar{h}_0(n))_n$ and $(\bar{h}_1(n))_n$ converging respectively to \bar{h}_0 and \bar{h}_1 , see Appendix A. For large n , up to add a suitable constant, we can suppose that $\bar{h}_1(n) < \bar{h}_0(n)$. By continuity of covolume we have that

$$\text{covol}(\bar{h}_1(n)) - \text{covol}(\bar{h}_0(n)) \xrightarrow{n \rightarrow +\infty} \text{covol}(\bar{h}_1) - \text{covol}(\bar{h}_0).$$

So in order to prove (3.16) it is sufficient to prove that

$$\int (\bar{h}_0(n) - \bar{h}_1(n)) dA(\bar{h}_0(n)) \rightarrow \int (\bar{h}_0 - \bar{h}_1) dA(\bar{h}_0),$$

as well as the convergence for the analog right hand terms. As $\bar{h}_0(n) - \bar{h}_1(n)$ uniformly converges to $\bar{h}_0 - \bar{h}_1$, by Corollary 3.23, $A(\bar{h}_0(n))$ weakly converge

to $A(\bar{h}_0)$. Putting $f_n = \bar{h}_0(n) - \bar{h}_1(n)$ and $f = \bar{h}_0 - \bar{h}_1$ we have

$$\begin{aligned} & \left| \int f_n \, dA(\bar{h}_0(n)) - \int f \, dA(\bar{h}_0) \right| \\ & \leq \|f_n - f\|_\infty \text{Area}(\bar{h}_0(n)) + \left| \int f \, dA(\bar{h}_0(n)) - \int f \, dA(\bar{h}_0) \right|. \end{aligned}$$

As $\text{Area}(\bar{h}_0(n))$ are uniformly bounded we get the result. □

Remark 3.37. — With $\bar{h}_1 \leq \bar{h}_0$, putting $\bar{h}_t = \bar{h}_0 + t(\bar{h}_1 - \bar{h}_0)$, the previous lemma and the weak convergence of the area measure implies that

$$\lim_{t \rightarrow 0} \frac{\text{covol}_\tau(\bar{h}_t) - \text{covol}_\tau(\bar{h}_0)}{t} = \int (\bar{h}_1 - \bar{h}_0) \, dA(\bar{h}_0).$$

More generally, adapting the Euclidean argument [13], one can prove that if $\bar{h} < \bar{h}_\tau$, the area measure is the Gâteaux gradient of the covolume.

Remark 3.38. — In the C^2_+ case, the area measure is the determinant of a symmetric matrix (2.6). The determinant of $d \times d$ symmetric matrix can be polarized as a d -linear form. If moreover we are in the Fuchsian case, it follows from the representation formula (3.15) that the covolume can be polarized as a $(d + 1)$ -linear form. Using a density argument, the covolume can be polarized on the set of all Γ -invariant convex sets. This leads to a theory similar to the mixed-volume theory in the convex bodies case [16]. There is no such similar theory in the general τ -convex case. The point is that the set of Γ invariant convex set is a (convex) cone, and we can take a convex cone as fundamental domain in $I^+(0)$. Actually, a mixed-covolume theory can be done for suitable convex sets in any convex cone — without mention of an ambient metric, see [22].

Although some questions remain, for example to find an isoperimetric inequality in the general τ -convex case, see [16] for the Fuchsian case.

3.6. Proof of the theorems

We fix a Γ -invariant positive Radon measure μ on \mathbb{H}^d and a cocycle τ . The aim of this section is to construct a τ -convex set with area measure μ .

Consider the functional on the space of hyperbolic τ -equivariant support functions

$$L_\mu(\bar{h}) = \text{covol}(\bar{h}) - \int (\bar{h}_\tau - \bar{h}) \, d\mu.$$

Note that L_μ is continuous (and even convex) by Lemma 3.26.

LEMMA 3.39. — L_μ is coercive: if $\|\bar{h}_n - \bar{h}_\tau\|_\infty \rightarrow +\infty$, then $L_\mu(\bar{h}_n) \rightarrow +\infty$.

Proof. — By Lemma 3.10, the maximum of the cosmological time on the corresponding convex sets goes to infinity. By Lemma 3.20, the minimum of the cosmological time also goes to infinity. Let a_n be the minimum of the cosmological time on the convex set with support function h_n . Again by Lemma 3.20, there is a constant δ such that $(\bar{h}_\tau - \bar{h}_n) \leq \delta a_n$. So the linear part of L_μ grows (in absolute value) linearly in a_n . On the other hand $\text{covol}(\bar{h}_n) \geq \text{covol}(\bar{h}_\tau - a_n)$.

But the volume of the past of the level set of the cosmological time coincides with $\bar{V}_{a_n}(\bar{h}_\tau)$ that is a polynomial of degree $d + 1$ by (3.4). It follows that $\text{covol}(\bar{h}_\tau - a_n) \geq c'a_n^{d+1}$. □

LEMMA 3.40. — L_μ attains its minimum.

Proof. — Let $\alpha \in [-\infty, +\infty)$ be the infimum of the range of L_μ . Let (\bar{h}_n) be a minimizing sequence for L_μ of τ -invariant hyperbolic support functions. By Lemma 3.39, the L^∞ -norm of $(\bar{h}_n - \bar{h}_\tau)$ are uniformly bounded from above, so from Corollary 3.18 one can extract a subsequence converging to some support function \bar{h} . By continuity, $L_\mu(\bar{h}) = \alpha$. □

LEMMA 3.41. — Let \bar{h}_0 be a support function on which L_μ attains its minimum. Then for any Γ -invariant support function f ,

$$\int (-f) \, dA(\bar{h}_0) \geq \int (-f) \, d\mu.$$

Proof. — If $f = 0$, the result is obvious. If $f \neq 0$, we know that $f < 0$. Let $\bar{h}_a = \bar{h}_0 + af$ for $a \in [0, 1]$. As L_μ attains its minimum at \bar{h}_0 ,

$$0 \leq L_\mu(\bar{h}_a) - L_\mu(\bar{h}_0) = \text{covol}(\bar{h}_a) - \text{covol}(\bar{h}_0) + \int af \, d\mu.$$

As $h_a < h_0$, we can use (3.16), which gives

$$0 \leq \int (-af) \, dA(\bar{h}_a) + \int af \, d\mu$$

so we get

$$\int (-f) \, dA(\bar{h}_a) \geq \int (-f) \, d\mu$$

for any a . The result follows by letting $a \rightarrow 0$ and using the weak convergence of the area measures (Corollary 3.23). □

LEMMA 3.42. — With the notations of Lemma 3.41,

$$\int dA(\bar{h}_0) = \int d\mu.$$

Proof. — From Lemma 3.41, with $f = -1$, we get

$$\text{Area}(h_0) \geq \int d\mu.$$

We have to prove the other inequality.

For a positive a consider the τ -equivariant function $\bar{h}_0 + a$. Notice that in general it is not a support function. However we consider the domain

$$\mathbf{K}_a = \{p \in \mathbb{R}^{d+1} \mid \langle p, \eta \rangle_- \leq \bar{h}_0(\eta) + a, \forall \eta \in \mathbb{H}^d\},$$

and let \bar{h}_a be the support function of \mathbf{K}_a . As $\mathbf{K} \subset \mathbf{K}_a$ we have $\bar{h}_0 \leq \bar{h}_a$. On the other hand by the definition of support function $\bar{h}_a \leq \bar{h}_0 + a$. From (3.16) we get

$$\int (\bar{h}_a - \bar{h}_0) dA(\bar{h}_a) \leq \text{covol}(\bar{h}_0) - \text{covol}(\bar{h}_a) \leq \int (\bar{h}_a - \bar{h}_0) dA(\bar{h}_0)$$

or in a more convenient way

$$(3.17) \quad - \int (\bar{h}_a - \bar{h}_0) dA(\bar{h}_a) \geq \text{covol}(\bar{h}_a) - \text{covol}(\bar{h}_0) \geq - \int (\bar{h}_a - \bar{h}_0) dA(\bar{h}_0).$$

$$\text{As } L_\mu(\bar{h}_a) \geq L_\mu(\bar{h}_0),$$

$$\text{covol}(\bar{h}_a) - \text{covol}(\bar{h}_0) + \int (\bar{h}_a - \bar{h}_0) d\mu \geq 0,$$

and using (3.17)

$$- \int (\bar{h}_a - \bar{h}_0) dA(\bar{h}_a) + \int (\bar{h}_a - \bar{h}_0) d\mu \geq 0$$

that is

$$\int (\bar{h}_a - \bar{h}_0) d\mu \geq \int (\bar{h}_a - \bar{h}_0) dA(\bar{h}_a).$$

Now using that $a \geq \bar{h}_a - \bar{h}_0$ we get

$$a \int d\mu \geq \int (\bar{h}_a - \bar{h}_0) dA(\bar{h}_a).$$

From Lemma 3.14, \bar{h}_a coincides with $\bar{h}_0 + a$ almost everywhere for $A(\bar{h}_a)$, hence

$$\int d\mu \geq \text{Area}(\bar{h}_a).$$

Letting $a \rightarrow 0$ and using the convergence of the total area, we get the estimate we need. □

Remark 3.43. — In the Fuchsian case, using the representation formula (3.15) for the covolume, the functional L_μ has a form very similar to the functional used in the variational approach for the Monge–Ampère equation, see [35, 4.5] and the references therein.

Proof of Theorem 1.1. — Let f be a C^2 Γ -invariant function on \mathbb{H}^d . There is a constant $c > 0$ such that $\nabla^2 f - (f - c)g_{\mathbb{H}} \geq 0$. In particular $f - c$ is a Γ -invariant support function, see (2.4). By Lemma 3.41:

$$\int (f - c) \, d\mu \geq \int (f - c) \, dA(h_0).$$

From Lemma 3.42, we know that the equality holds for constants. Then we can conclude that

$$\int f \, d\mu \geq \int f \, dA(h_0)$$

for any C^2 -function. Considering $-f$, for any C^2 -function we get

$$\int f \, d\mu = \int f \, dA(h_0).$$

By approximation we obtain that this equality is true for any C^0 function, hence $\mu = A(h_0)$, and the theorem is proved. \square

Using the correspondence between the Area measure of \mathbf{K} and the Monge–Ampère measure $MA(h)$ described in Proposition 2.17, we can prove Theorem 1.3 stated in the introduction.

Proof of Theorem 1.3. — Notice that for each γ there is a vector $s_\gamma \in \mathbb{R}^d$ and $a_\gamma \in \mathbb{R}$ such that

$$(\gamma \cdot h_0)(x) - h_0(x) = \langle s_\gamma, x \rangle - a_\gamma = \left\langle \hat{x}, \begin{pmatrix} s_\gamma \\ a_\gamma \end{pmatrix} \right\rangle_-.$$

In particular letting H_0 be the 1-homogeneous extension of h to $\Gamma^+(0)$ we conclude that

$$H_0(\gamma^{-1}\hat{x}) - H_0(\hat{x}) = \left\langle \hat{x}, \begin{pmatrix} s_\gamma \\ a_\gamma \end{pmatrix} \right\rangle_-.$$

Putting $\tau_\gamma := -\begin{pmatrix} s_\gamma \\ a_\gamma \end{pmatrix}$ the formula above easily shows that τ is a cocycle and the equivariance

$$H_0(\gamma Y) - H_0(Y) = -\langle \tau_{\gamma^{-1}}, Y \rangle_-$$

holds. By (3.2) H_0 is a τ -equivariant convex function.

So by Theorem 1.1, there is a τ -equivariant convex function h on \mathbb{B} such that

$$A(h) = \sqrt{1 - \|x\|^2} \mu.$$

By Proposition 2.17 we have $MA(h) = \mu$. As h is τ -equivariant as h_0 , then on the boundary $h = h_0$. □

Finally let us consider the regularity problem.

From Proposition 3.25 there is a constant $c = c(\tau) \geq 0$ such that if $\mu > c d\mathbb{H}^d$, then the τ -convex set \mathbf{K} with $A(\mathbf{K}) = \mu$ is contained in the interior of Ω_τ .

By Corollary 2.37 we get:

THEOREM 3.44. — *Let f be a Γ -invariant C^{k+1} -function, $k \geq 2$, with $f(x) > c(\tau)$ for every $x \in \mathbb{H}^d$. Then the boundary of the τ -convex set \mathbf{K} with $A(\mathbf{K}) = \int f d\mathbb{H}^d$ is a strictly convex space-like hypersurface of class C^{k+2} with Gauss–Kronecker curvature f^{-1} .*

Proof. — Corollary 2.37 immediately gives that $\partial_s \mathbf{K}$ is strictly convex C^{k+2} hypersurface. Notice that its Gauss–Kronecker curvature is $f^{-1} > 0$ so the Gauss map is a C^{k+1} -homeomorphism with inverse map of class C^{k+1} . Moreover, $\partial \mathbf{K} = \partial_s \mathbf{K}$ by Lemma 3.11. □

Theorem 1.4 and Theorem 1.6 are just quotient versions of preceding results.

3.7. Equivariant Pogorelov example

In this section we fix an uniform lattice Γ and a cocycle $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$ such that $\partial_s \Omega_\tau$ contains a $(d - 1)$ Euclidean polyhedron \mathcal{P} .

We will prove that there exists a τ -convex set with positive area measure (in the sense that $A(\mathbf{K}) \geq c_0 d\mathbb{H}^d$ for some c_0) which meets the boundary of Ω_τ .

Remark 3.45. — In dimension $d = 3$ examples of pairs (Γ, τ) such that $\partial_s \Omega_\tau$ contains a 2-polygon were constructed in [12].

It remains open to understand whether there is a lattice $\Gamma < \text{SO}^+(3, 1)$ and a cocycle τ such that the corresponding domain is not simple but $c(\tau) = 0$.

In practice we will construct a τ -support function h on \mathbb{B} with $MA(h) > c_0 \mathcal{L}$, and $h = h_\tau$ on a segment. The latter condition implies that $\partial_s \Omega_\tau$ meets the τ -convex set supported by h . On the other hand, the former condition and Corollary 2.18 show that on a compact fundamental domain $A(\bar{h}) \geq c' d\mathbb{H}^d$, for some constant c' . As the area measure is Γ invariant we will conclude that $A(\bar{h}) \geq c' d\mathbb{H}^d$ everywhere.

Let f_0 be a function given in Remark 2.35 for $k = d - 1$:

$$f_0(x) = \beta^2(1 + \beta^2 x_1^2)r^\alpha(x).$$

where $r(x) = \sqrt{x_2^2 + \dots + x_d^2}$, $\alpha = 2 - 2/d$ and β is a suitable constant ≥ 1 . We have $f_0 \equiv 0$ on $[\ell_-, \ell_+]$, with $\ell_\pm = (\pm 1, 0, \dots, 0)$.

Up to compose by a global isometry, we can suppose that the polyhedron $\mathcal{P} \subset \partial_s \Omega_\tau$ contains 0 in its interior and is contained in the $(d - 1)$ -linear space

$$R = \{x \in \mathbb{R}^{d+1} \mid x_{d+1} = x_1 = 0\}.$$

Recall that H_τ is the support function of Ω_τ on $\mathbb{I}^+(0)$ and h_τ is its restriction to the ball \mathbb{B} . We notice that $h_\tau \geq 0$ and $h_\tau = 0$ exactly on $[\ell_-, \ell_+]$.

LEMMA 3.46. — *There is $\epsilon > 0$ such that $\epsilon f_0 < h_\tau$ on $\overline{\mathbb{B}} \setminus [\ell_-, \ell_+]$.*

Proof. — By the definition of f_0 we have that $f_0(x) \leq cr(x)$ for any $x \in \overline{\mathbb{B}}$ and some c . So it is sufficient to show that there is M such that $h_\tau(x) > Mr(x)$ on $\overline{\mathbb{B}} \setminus [\ell_-, \ell_+]$.

Let $\Omega = \mathbb{I}^+(\mathcal{P})$. It is an F-convex set contained in Ω_τ with support function h on \mathbb{B} . Note that $h = 0$ on $[\ell_-, \ell_+]$. So to prove the claim it is sufficient to check that for $x \in \mathbb{B} \setminus [\ell_-, \ell_+]$ we have $h(x) > cr(x)$ for some c .

But in R , 0 is contained in the interior of the convex body \mathcal{P} , so \mathcal{P} contains in its interior a small ball centered at 0, whose support function is $c\|\cdot\|_R$ some c , where $\|\cdot\|_R$ is the norm associated to $\langle \cdot, \cdot \rangle_R$, the scalar product on R induced by the Minkowski product. For any $x' \in R \setminus \{0\}$

$$\max_{y \in \mathcal{P}} \langle x', y \rangle_R > c\|x'\|_R.$$

But for any $y \in \mathcal{P}$, $\langle y, e_1 \rangle_- = \langle y, e_{d+1} \rangle_- = 0$, so for any $x = (x_1, x') \in \mathbb{B}$

$$h(x) = \max_{y \in \mathcal{P}} \langle \hat{x}, y \rangle_- = \max_{y \in \mathcal{P}} \langle x', y \rangle_R$$

and $\|x'\| = r(x)$. □

Now we take $f = \epsilon f_0$. Notice that

- $\text{MA}(f) \stackrel{(2.14)}{=} \epsilon^{d+1} \text{MA}(f_0) > \epsilon^{d+1} c\mathcal{L}$;
- $f = 0$ on the segment $[\ell_-, \ell_+]$;
- $f < h_\tau$ on $\overline{\mathbb{B}} \setminus [\ell_-, \ell_+]$.

We then define

$$f_\infty(x) = \sup_{\gamma \in \Gamma_\tau} (\gamma \cdot f)(x)$$

where Γ_τ acts on the set of convex functions of \mathbb{B} as explained in Section 3.1.

By definition f_∞ is a convex function on \mathbb{B} . Actually it is the support function of the convex hull of the union of all the $\gamma \cdot \mathbf{K}(f)$, so it is a

fixed point for the action of Γ_τ . Moreover, since $f \leq h_\tau$, we have that $f_\infty(x) \leq h_\tau(x)$ for any $x \in \mathbb{B}$, and $f_\infty = h_\tau = 0$ on the segment $[\ell_-, \ell_+]$

The rest of this section is devoted to prove the following proposition.

PROPOSITION 3.47. — *For any compact subset C of \mathbb{B} , there is $c_C > 0$ such that*

$$M(f_\infty) \geq c_C \mathcal{L}$$

on C .

The first step is to compute the stabilizer of \mathcal{P} in Γ_τ .

LEMMA 3.48. — *The stabilizer of \mathcal{P} in Γ_τ is a \mathbb{Z} -group generated by an element, say μ_τ .*

Proof. — First we prove that the stabilizer of \mathcal{P} is not trivial. Notice that for each $\gamma_\tau \in \Gamma_\tau$ either $\gamma_\tau(\mathcal{P}) = \mathcal{P}$ or $\gamma_\tau(\text{int } \mathcal{P})$ is disjoint from $\text{int } \mathcal{P}$.

If \mathcal{P}_1 denotes the subset of the level set Σ_1 of the cosmological time that retracts in $\text{int } \mathcal{P}$, by the equivariance of the retraction we have that either $\gamma_\tau(\mathcal{P}_1) = \mathcal{P}_1$ or they are disjoint.

By [11] \mathcal{P}_1 is isometric to the Cartesian product $(\text{int } \mathcal{P}) \times \mathbb{R}$, so its volume is infinite. If the stabilizer of \mathcal{P} were trivial the projection $\mathcal{P}_1 \rightarrow \Sigma_1/\Gamma_\tau$ would be an injective isometric immersion of a Riemann manifold of infinite volume into a compact manifold, and this is impossible.

Now if γ_τ is in the stabilizer of \mathcal{P} , then γ stabilizes the image of \mathcal{P} through the Gauss map in \mathbb{H}^d . By [11] this is the geodesic in \mathbb{H}^d orthogonal to the hyperplane R containing \mathcal{P} . Notice that the stabilizer of a geodesic in a uniform lattice can be either trivial or a cyclic group. Since we have proved that it is not trivial, the Lemma follows. □

By the proof of Lemma 3.48, the projective transformation $\bar{\mu}$ stabilizes the segment $[\ell_-, \ell_+]$. Moreover μ_τ acts as an isometry on \mathcal{P} , so it fixes a point of \mathcal{P} . Up to a translation we may suppose that μ_τ fixes 0, that is, $\tau_\mu = 0$ and $\mu_\tau = \mu$.

Consider the function

$$f_1(x) = \sup_{n \in \mathbb{Z}} (\mu^n \cdot f)(x).$$

It is $\langle \mu \rangle$ -invariant by construction.

LEMMA 3.49. — *For any $\delta > 0$, let $C = \{z \in \bar{\mathbb{B}} \mid |x_1(z)| < 1 - \delta\}$. Then there are a finite number of $n_1 \dots n_k \in \mathbb{Z}$ such that*

$$f_1(z) = \max_i (\mu^{n_i} \cdot f)(z)$$

for any $z \in C$.

Proof. — The transformation μ leaves R and \mathcal{P} invariant (in particular $\mu|_R$ is an orthogonal transformation of finite order). On the other hand we have

$$\begin{aligned} \mu^n(1, 0, \dots, 0) &= (\cosh na, 0, \dots, 0, \sinh na), \\ \mu^n_{d+1}(0, 0, \dots, 0, 1) &= (\sinh na, 0, \dots, 0, \cosh na) \end{aligned}$$

for any $n \in \mathbb{Z}$.

Since $\tau_\mu = 0$, by Lemma 3.4,

$$(\mu^n \cdot f)(x) = (\mu^{-n}\hat{x})_{d+1} f\left(\frac{\mu^{-n}(\hat{x})}{(\mu^{-n}\hat{x})_{d+1}}\right)$$

so a simple computation shows that

$$(\mu^n \cdot f)(x) = \frac{\epsilon\beta^2 r^\alpha(x)}{(\cosh an - x_1 \sinh an)^{\alpha-1}} \left(1 + \beta^2 \left(\frac{x_1 - \tanh an}{1 - x_1 \tanh an}\right)^2\right).$$

Since $x_1 \in [-1, 1]$ we have that $\frac{x_1 - \tanh an}{1 - x_1 \tanh an} \in [-1, 1]$.

By the definition of C , we have $|x_1| \leq 1 - \delta$ for every $x \in C$. So we easily see that

$$\frac{1}{(\cosh an - x_1 \sinh an)^{\alpha-1}} \left(1 + \beta^2 \left(\frac{x_1 - \tanh an}{1 - x_1 \tanh an}\right)^2\right) \rightarrow 0$$

uniformly on C both for $n \rightarrow +\infty$ and for $n \rightarrow -\infty$.

It follows that, putting $a_0 = \min f|_C$, there is a finite list $I = \{n_1, \dots, n_k\}$ such that for $n \notin I$

$$\frac{1}{(\cosh an - x_1 \sinh an)^{\alpha-1}} \left(1 + \beta^2 \left(\frac{x_1 - \tanh an}{1 - x_1 \tanh an}\right)^2\right) \leq a_0$$

for every $x \in C$.

So for $x \in C$

$$f_1(x) = \max_i (\mu^{n_i} f)(x). \quad \square$$

The previous lemma, Corollary 3.6 and formula (2.16) imply the following:

COROLLARY 3.50. — *For any compact subset C in \mathbb{B} there is a constant $c_0 > 0$ such that $\text{MA}(f_1) > c_0 \mathcal{L}$ on C .*

Another simple consequence of Lemma 3.49 is the following

COROLLARY 3.51. — *For any $\xi \in \partial\mathbb{B} \setminus \{\ell_\pm\}$ we have $f_1(\xi) < h_\tau(\xi)$.*

Proof. — By Lemma 3.49 there exists $n \in \mathbb{Z}$ such that $f_1(\xi) = (\mu^n \cdot f)(\xi)$. Now, as $\bar{\mu}^{-n}(\xi) \notin \{\ell_{\pm}\}$ (because $\bar{\mu}$ stabilizes $[\ell_-, \ell_+]$) we have that $(\mu^n \cdot f)(\xi) < (\mu^n \cdot h_{\tau})(\xi)$, and then

$$f_1(\xi) < (\mu^n \cdot h_{\tau})(\xi) = h_{\tau}(\xi),$$

where the last equality holds since h_{τ} is Γ_{τ} -invariant. □

Notice now that the function f_{∞} can be obtained as

$$f_{\infty}(x) = \sup_{\gamma_{\tau} \in \Gamma_{\tau}} (\gamma_{\tau} \cdot f_1).$$

Since f_1 is μ -invariant then $\gamma_{\tau} \cdot f_1 = (\gamma_{\tau} \mu^n) \cdot f_1$ so the function $\gamma_{\tau} \cdot f_1$ depends only on the coset of γ in the set of cosets $X = \Gamma / \langle \mu \rangle$. So putting $f_{[\gamma]} = \gamma_{\tau} \cdot f$ we have that

$$f_{\infty}(x) = \sup_{[\gamma] \in X} (f_{[\gamma]}).$$

Proposition 3.47 follows then from the following lemma, together with Corollary 3.50, Corollary 3.6, and (2.16) imply the following:

LEMMA 3.52. — *For any compact subset C in \mathbb{B} there exist a finite number of cosets $[\gamma_1], \dots, [\gamma_k]$ such that on C*

$$f_{\infty}(x) = \max_i f_{[\gamma_i]}.$$

Proof. — We will prove that for any infinite sequence $[\gamma_n]$ of distinct elements in X we have that $f_{[\gamma_n]}$ converges uniformly to $-\infty$ on C .

Suppose by contradiction that there exists a sequence $[\gamma_n]$ such that $\min f_{[\gamma_n]}$ is bounded on C . As Γ is cocompact, its elements are hyperbolic. So, up to pass to a subsequence there is a point $\ell \in \partial\mathbb{B}$ such that $\bar{\gamma}_n^{-1}(x)$ converges uniformly to ℓ on C .

Fact: we may choose γ_n in their cosets so that $\ell \notin \{\ell_-, \ell_+\}$. — Take the disk $\Delta = \{x \in \mathbb{B} \mid x_1 = 0\}$. Its $\langle \bar{\mu} \rangle$ -orbit is a union of disjoint disks $\Delta_i = \bar{\mu}^i(\Delta)$ which converge to ℓ_+ (for $n \rightarrow +\infty$) and to ℓ_- (for $n \rightarrow -\infty$) and whose hyperbolic distance is a .

In particular, for each n we have that $\bar{\gamma}_n^{-1}(C)$ is contained between Δ_{k_n} and Δ_{k_n+h} where h is any integer number less than the hyperbolic diameter of C divided by a . It follows that $\bar{\mu}^{-k_n} \bar{\gamma}_n^{-1}$ sends C into the region bounded by Δ_1 and Δ_h .

In particular changing γ_n with $\gamma_n \mu^{k_n}$ we have that the point ℓ is contained in the region annulus on $\partial\mathbb{B}$ bounded Δ_1 and Δ_h , so it is neither ℓ_- nor ℓ_+ . The fact is proved.

Now by Lemma 3.4 we have

$$(\gamma_n \cdot f_1)(x) = (\gamma_n^{-1}(\hat{x}))_{d+1}(f_1(\bar{\gamma}_n^{-1}x)) + \langle \hat{x}, \tau_{\gamma_n} \rangle_- .$$

Since h_τ is invariant by the action of Γ_τ we deduce that

$$-\langle \hat{x}, \tau_{\gamma_n} \rangle_- = (\gamma_n^{-1}(\hat{x}))_{d+1}(h_\tau(\bar{\gamma}_n^{-1}\hat{x})) - h_\tau(x)$$

so

$$(\gamma_n \cdot f_1)(x) = (\gamma_n^{-1}(\hat{x}))_{d+1}(f_1(\bar{\gamma}_n^{-1}x) - h_\tau(\bar{\gamma}_n^{-1}x)) + h_\tau(x) .$$

On the other hand since $\bar{\gamma}_n^{-1}x \rightarrow \ell$ uniformly on C , and $f_1(\ell) < h_\tau(\ell)$ (see Corollary 3.51) there is ϵ such that $f_1(\bar{\gamma}_n^{-1}(x)) - h_\tau(\bar{\gamma}_n^{-1}(x)) \leq -\epsilon$ for $x \in C$ and for n big.

Notice that $(\gamma_n^{-1}(\hat{x}))_{d+1} \rightarrow +\infty$ on C so

$$(\gamma_n \cdot f_1)(x) \rightarrow -\infty$$

uniformly on C contradicts the assumption. □

Now let C be a compact set of \mathbb{H}^d containing a fundamental domain for the action of Γ . From Proposition 3.47 and Proposition 2.17, on C the area measure of $\mathbf{K}(f_\infty)$ is bounded from below by a positive constant times the hyperbolic volume measure. As the area measure is invariant under the action of Γ , it follows that on \mathbb{H}^d the area measure of $\mathbf{K}(f_\infty)$ is bounded from below by a positive constant times the hyperbolic volume measure. But $\mathbf{K}(f_\infty)$ touches the boundary of Ω_τ , because $f_\infty = h_\tau$ on $[\ell_-, \ell_+]$.

COROLLARY 3.53. — *Let \mathbf{K}_∞ be the τ -convex set corresponding to f_∞ . There is a constant c_0 such that $A(\mathbf{K}_\infty) \geq c_0 \, d\mathbb{H}^d$, but \mathbf{K}_∞ meets $\partial_s \Omega_\tau$ and in particular it is not strictly convex.*

As a corollary we get

COROLLARY 3.54. — *If $\partial_s \Omega_\tau$ contains a $(d-1)$ -dimensional polyhedron \mathcal{P} , then there is no τ -convex set with smooth boundary and with constant curvature function c_0 , where c_0 is the constant in Corollary 3.53.*

Proof. — Suppose by contradiction that such set, say \mathbf{K} , exists and let h its support function on \mathbb{B} . By Proposition 2.17 and (2.1) we should have

$$MA(h) = c_0(1 - \|x\|^2)^{(-d-2)/2} \mathcal{L} .$$

On the other hand by Corollary 3.53 we have

$$MA(f_\infty) \geq c_0(1 - \|x\|^2)^{(-d-2)/2} \mathcal{L} .$$

By the comparison principle (Theorem 2.30)

$$h_\tau \geq h \geq f_\infty \geq 0 .$$

This implies that h coincides with h_τ on $[\ell_-, \ell_+]$, so $\partial\mathbf{K}$ contains the points of $\partial\Omega_\tau$ which satisfy either $\langle p, \ell_- \rangle_- = 0$ or $\langle p, \ell_+ \rangle_- = 0$. In particular for each point p in the interior of \mathcal{P} , the two light-like vectors through p directed as ℓ_- and ℓ_+ are contained in $\partial\mathbf{K}$.

Thus the intersection of $\partial\mathbf{K}$ with the time-like 2-plane U passing through x and orthogonal to the plane R containing \mathcal{P} is the union of two geodesic rays with different directions. Since this intersection is transverse, $\partial\mathbf{K}$ cannot be smooth. □

Appendix A. Smooth approximation

In this section we will prove that any τ -convex set \mathbf{K} can be approximated by a sequence (\mathbf{K}_n) of C^2_+ τ -convex sets. We limit ourself to consider C^2 -approximation only for simplicity. In fact the same techniques can be implemented to produce a smooth approximation. In Remark A.5 some details will be given in this direction.

The main idea is to use an average procedure to construct a sequence \bar{h}_n of C^2 - support functions converging to \bar{h} . Basically, given any $r > 0$ we define by the hyperbolic average of any continuous function \bar{h} of radius r the following function

$$\hat{h}_r(x) = \frac{d}{\omega_{d-1}(\sinh r)^d} \int_{B(x,r)} \bar{h}(y) d\mathbb{H}^d(y),$$

where $B(x, r)$ is the hyperbolic ball centered at x of a radius r and ω_{d-1} is the volume of the standard sphere S^{d-1} of constant curvature 1. For brevity we denote by $\zeta(r) = \frac{d}{\omega_{d-1}(\sinh r)^d}$. Note that $\zeta(r)V(r) \rightarrow 1$ when $r \rightarrow 0$, where $V(r)$ is the volume of the hyperbolic ball of radius r [29, 3.4].

We will show the following facts

- $\hat{h}_r \rightarrow \bar{h}$ as $r \rightarrow 0$ uniformly on compact subsets of \mathbb{H}^d .
- if \bar{h} is τ -equivariant, so is \hat{h}_r .
- \hat{h}_r is of class C^1 . More generally, if \bar{h} is of class C^k , then \hat{h}_r is of class C^{k+1} .

A further difficulty is that the average procedure in general does not produce support functions. However if \bar{h} is τ -equivariant, we will prove that there exists a constant $C > 0$ such that $\hat{h}_r - Cr$ is a support function for any r . To prove this result it will suffice the following property about \hat{h}_r .

- If \bar{h} is a support function, for any convex compact domain K of \mathbb{H}^d , there is a constant $C = C(\|\bar{h}\|_{\text{Lip}(K)})$ such that $\hat{h}_r - Cr$ is a support function on K .

(Here being a support function on $K \subset \mathbb{H}^d$ means that its 1-homogeneous extension on the cone over K is convex.)

First let us show how the enlisted properties of the hyperbolic average imply the density statement we want to prove in this section.

THEOREM A.1. — *For a given τ -support function \bar{h} and for any $\delta > 0$, there is a C^2_+ τ -support function \bar{h}_δ such that $\|\bar{h} - \bar{h}_\delta\|_{L^\infty} \leq \delta$.*

Proof. — Let K be a convex domain such that the Γ -translates of the interior of K cover \mathbb{H}^d , and C be the constant appearing in the last property. If r is sufficiently small, $\|\hat{h}_r - Cr - \bar{h}\|_K < \delta/3$. Notice that the 1-homogeneous extension of $\hat{h}_r - Cr$ is convex in the cone over K . Since $\hat{h}_r - Cr$ is τ -equivariant, it turns out that $\bar{h}' = \hat{h}_r - Cr$ is a C^1 τ -support function over \mathbb{H}^d . Moreover, since $\bar{h}' - \bar{h} = \hat{h}_r - Cr - \bar{h}$ is Γ -invariant we get that $\|\bar{h}' - \bar{h}\|_{\mathbb{H}^d} \leq \delta/3$.

Repeating the average procedure on \bar{h}' we obtain a C^2 τ -support function \bar{h}'' whose distance from \bar{h}' is less than $\delta/3$. Notice that $\nabla^2(\bar{h}'') - \bar{h}'' g_{\mathbb{H}}$ is semidefinite positive by Equation (2.5), and $\|\bar{h} - \bar{h}''\| < 2\delta/3$, so $\bar{h}_\delta := \bar{h}'' - \delta/3$ is a C^2_+ τ -support function whose distance from \bar{h} is less than δ . \square

Let us now prove the properties of this average process.

The first property is a direct consequence of the next proposition and the fact that the function $\zeta(r)$ behaves as the inverse of the volume $V(r)$ of the hyperbolic ball of radius r for small r .

PROPOSITION A.2. — *Let \bar{h} be a continuous function of \mathbb{H}^d . Given a compact subset K , denote by*

$$\delta(r, K) = \sup\{|\bar{h}(x) - \bar{h}(y)|, x \in K, y \in B(x, r)\}.$$

Then $\|\hat{h}_r - \bar{h}\|_K \leq |\zeta(r)V(r) - 1| \|\bar{h}\|_{K_r} + \delta(r, K)$.

Proof. — The estimate is obtained by a direct computation. For $x \in K$:

$$\begin{aligned} \left| \hat{h}_r(x) - \bar{h}(x) \right| &= \left| \zeta(r) \int_{B(x,r)} \bar{h}(y) \, d\mathbb{H}^d(y) - \frac{1}{V(r)} \int_{B(x,r)} \bar{h}(y) \, d\mathbb{H}^d(y) \right. \\ &\quad \left. + \frac{1}{V(r)} \int_{B(x,r)} (\bar{h}(y) - \bar{h}(x)) \, d\mathbb{H}^d(y) \right| \\ &\leq |\zeta(r)V(r) - 1| \|\bar{h}\|_{K_r} + \delta(r, K). \quad \square \end{aligned}$$

We now show that if \bar{h} is τ -equivariant so is \hat{h}_r . Notice that if $\gamma \in \Gamma$ then

$$\begin{aligned} \hat{h}_r(\gamma(x)) - \hat{h}_r(x) &= \zeta(r) \left(\int_{B(\gamma(x),r)} \bar{h}(y) \, d\mathbb{H}^d(y) - \int_{B(x,r)} \bar{h}(y) \, d\mathbb{H}^d(y) \right) \\ &= \zeta(r) \left(\int_{B(x,r)} (\bar{h}(\gamma y) - \bar{h}(y)) \, d\mathbb{H}^d(y) \right). \end{aligned}$$

By (3.3) we have that $\bar{f}(y) = \bar{h}(\gamma y) - \bar{h}(y)$ is the restriction of the linear function

$$\bar{f}(x) = \langle x, \gamma^{-1}\tau_\gamma \rangle_- ,$$

and we have that $\hat{h}_r(\gamma x) - \hat{h}_r(x) = \hat{f}_r(x)$.

Now the τ -equivariance of \hat{h}_r is ensured by this remark and the following Lemma.

LEMMA A.3. — *If \bar{f} is the restriction of a linear function on \mathbb{H}^d , then $\hat{f}_r = \bar{f}$ for every r .*

Proof. — Since the 1-homogeneous extension, F , of \bar{f} is linear, we have that its Minkowski gradient is constant. By the computation in Lemma 2.7 the derivative of F at a point \mathbb{H}^d coincides with $\nabla \text{grad}^{\mathbb{H}}(\bar{f}) - \bar{f} \text{Id}$, so we conclude that $\nabla^2 \bar{f} - \bar{f} g_{\mathbb{H}} = 0$ on \mathbb{H}^d .

In particular we have $\Delta \bar{f} = d \cdot \bar{f}$. So we deduce from the divergence theorem that

$$\begin{aligned} \hat{f}_r(x) &= \frac{d}{\omega_{d-1} \sinh(r)^d} \int_{B(x,r)} \frac{1}{d} \Delta \bar{f}(y) \, d\mathbb{H}^d(y) \\ &= \frac{1}{\omega_{d-1} \sinh(r)^d} \int_{S(x,r)} D\bar{f}(y)[\nu] \, dS_r(y) \end{aligned}$$

where $S(x, r)$ is the boundary of $B(x, r)$, ν is the unit exterior normal and dS_r is the area form of the sphere $S(x, r)$.

In order to compute the last integral consider the unit tangent sphere at x ,

$$S^{d-1} = \{v \in \mathbb{R}^{d,1} \mid \langle v, v \rangle_- = 1, \langle v, x \rangle_- = 0\},$$

and the parameterization given by the exponential map:

$$\sigma : S^{d-1} \rightarrow S(x, r), \quad \sigma(v) = \cosh(r)x + \sinh(r)v .$$

We have $\sigma^*(dS_r) = \sinh(r)^{d-1} dS^{d-1}$. From $\nu(\sigma(v)) = \sinh(r)x + \cosh(r)v$ we deduce that

$$\begin{aligned} D\bar{f}(x)[\nu] &= DF(x)[\nu] = F(\nu) = \sinh(r)F(x) + \cosh(r)F(v) \\ &= \sinh(r)\bar{f}(x) + \cosh(r)F(v), \end{aligned}$$

so we get

$$\hat{f}_r(x) = \frac{1}{\omega_{d-1} \sinh(r)^d} \int_{S^{d-1}} (\sinh(r)^d \bar{f}(x) + \sinh(r)^{d-1} \cosh(r) F(v)) \, dS^{d-1}.$$

Since F is linear $\int_{S^{d-1}} F \, dS^{d-1} = 0$, so the previous formula shows that $\hat{f}_r(x) = \bar{f}(x)$. □

Let us consider now the regularizing properties of the average process. Until the end of the section we use the notation $\langle \cdot, \cdot \rangle$ to denote the metric on \mathbb{H}^d .

LEMMA A.4. — *If \bar{h} is a continuous function, then \hat{h}_r is C^1 and given a tangent vector v at x , we have*

$$D\hat{h}_r(x)[v] = \zeta(r) \int_{S(x,r)} \bar{h}(y) \langle \nu(y), \tilde{V}(y) \rangle \, dS_r(y)$$

where \tilde{V} is any Killing vector field such that $\tilde{V}(x) = v$.

Proof. — First we assume that \bar{h} is C^1 . Let γ_t the isometric flow generated by \tilde{V} . We have to compute

$$\lim_{t \rightarrow 0} \frac{\hat{h}_r(\gamma_t(x)) - \hat{h}_r(x)}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \zeta(r) \int_{B(x,r)} (\bar{h}(\gamma_t y) - \bar{h}(y)) \, d\mathbb{H}^d(y).$$

This limit clearly exists and we have

$$D\hat{h}_r(x)[v] = \zeta(r) \int_{B(x,r)} \langle \text{grad}^{\mathbb{H}} \bar{h}(y), \tilde{V}(y) \rangle \, d\mathbb{H}^d(y).$$

Since \tilde{V} is a Killing vector field, it is divergence free, so $\langle \text{grad}^{\mathbb{H}} \bar{h}, \tilde{V} \rangle = \text{div}(\bar{h}V)$. Applying the divergence theorem we obtain a proof of the formula. Notice that $D\hat{h}_r$ continuously depends on the point, so \hat{h}_r is C^1 .

Consider now the case where \bar{h} is only C^0 . Take a sequence of C^1 -functions \bar{h}_n converging to \bar{h} uniformly on compact subsets of \mathbb{H}^d . We have that $((\hat{h}_n)_r)$ converges to \hat{h}_r as $n \rightarrow +\infty$ uniformly on compact subsets. So in order to conclude it is sufficient to notice that $D(\hat{h}_n)_r(x)$ converges to the 1-form

$$\alpha_r(x)[v] = \int_{S(x,r)} u(y) \langle \nu(y), \tilde{V}(y) \rangle \, dS_r(y).$$

It follows that \hat{h}_r is differentiable and $D\hat{h}_r = \alpha$. □

Remark A.5. — In the proof we have showed in particular that if \bar{h} is C^1 then

$$D\hat{h}_r(x)[v] = \zeta(r) \int_{B(x,r)} \langle \text{grad}^{\mathbb{H}} \bar{h}(y), \tilde{V}(y) \rangle \, d\mathbb{H}^d(y).$$

Applying Lemma A.4 to the continuous function $\bar{h}'(x) = \langle \text{grad}^{\mathbb{H}} \bar{h}(x), \tilde{V}(x) \rangle$, we see that for any Killing vector field \tilde{V} the function $x \mapsto D\hat{h}_r(x)(\tilde{V}(x))$ is C^1 . Thus \hat{h}_r is C^2 . More generally by an induction argument, one shows that if \bar{h} is of class C^k , then \hat{h}_r is of class C^{k+1} .

Finally we have to prove the last property. Let $\text{Lip}_K(\bar{h})$ be the best Lipschitz constant of \bar{h} on K , and let $\|\bar{h}\|_{\text{Lip}(K)} = \|\bar{h}\|_K + \text{Lip}_K(\bar{h})$.

PROPOSITION A.6. — *Let \bar{h} be a support function. For any convex compact subset K of \mathbb{H}^d , there is a constant $C = C(\|\bar{h}\|_{\text{Lip}(K)})$ such that $\hat{h}_r - Cr$ is a support function on K .*

The proof of this proposition is first done assuming that \bar{h} is C^2 and then using an approximation argument. The proof in the regular case follows by a computation of the Hessian of \hat{h}_r .

LEMMA A.7. — *Assume that \bar{h} is C^2 . Let v be a tangent vector at $x \in \mathbb{H}^d$. Then*

$$\begin{aligned} &\nabla^2(\hat{h}_r)_x(v, v) \\ &= \zeta(r) \left(\int_{B(x,r)} (\nabla^2 \bar{h})_y(\tilde{V}_y, \tilde{V}_y) \, d\mathbb{H}^d(y) + \int_{B(x,r)} \langle \text{grad}^{\mathbb{H}} \bar{h}_y, \nabla_{\tilde{V}} \tilde{V} \rangle \, d\mathbb{H}^d(y) \right) \end{aligned}$$

where \tilde{V} is the Killing vector field extending v and generating a hyperbolic transformation with axis through x (i.e., $\tilde{V}(x) = v$ and $\nabla_{\tilde{V}} \tilde{V} = 0$ at x).

Proof of Lemma A.7. — Let γ_t be the isometry flow generated by \tilde{V} . The path $c(t) = \gamma_t(x)$ is a geodesic passing through x such that $\dot{c} = v$. In particular, we have

$$\nabla^2(\hat{h}_r)_x(v, v) = \left. \frac{d^2 \hat{h}_r(c(t))}{dt^2} \right|_{t=0}.$$

Differentiating the formula

$$\hat{h}_r(c(t)) = \zeta(r) \int_{B(x,r)} \bar{h}(\gamma_t y) \, d\mathbb{H}^d(y)$$

we get

$$\frac{d\hat{h}_r \circ c}{dt}(t) = \zeta(r) \int_{B(x,r)} \langle \text{grad}^{\mathbb{H}} \bar{h}(\gamma_t y), \tilde{V}(\gamma_t(y)) \rangle \, d\mathbb{H}^d(y).$$

The statement is then proved by differentiating again this formula at $t = 0$. □

We will need the following lemma.

LEMMA A.8. — Let $(u_n)_n$ be a sequence of convex functions on an open set O of the unit ball $B \subset \mathbb{R}^d$ converging uniformly on any compact set of O to u . Let $K \subset O$ be a compact set, and $K' \subset O$ a compact set containing K in its interior. Then

$$\text{Limsup } \|u_n\|_{\text{Lip}(K)} \leq \|u\|_{\text{Lip}(K')}.$$

Proof. — By uniform convergence, $\|u_n\|_K$ converge to $\|u\|_K$, which is less than $\|u\|_{K'}$. So we have to prove that the limit superior of $\text{Lip}_K(u_n)$ is less than $\text{Lip}_{K'}(u)$.

Let x, y be two points in K , and z be the intersection point of the ray from x towards y and $\partial K'$. So $y = (1 - \lambda)x + \lambda z$ with $\lambda = \|y - x\|/\|z - x\|$, then by convexity we have

$$(A.1) \quad \frac{u_n(y) - u_n(x)}{\|y - x\|} < \frac{u_n(z) - u_n(x)}{\|z - x\|}.$$

Let $\epsilon > 0$ and suppose by contradiction that u_n are not $(\text{Lip}_{K'}(u) + \epsilon)$ -Lipschitz on K for n big. This implies that there is two sequences x_n and y_n in K such that

$$u_n(y_n) - u_n(x_n) > (\text{Lip}_{K'}(u) + \epsilon)\|y_n - x_n\|.$$

Let z_n be the intersection point of the ray from x_n towards y_n and $\partial K'$. By (A.1) we have that

$$(A.2) \quad \frac{u_n(z_n) - u_n(x_n)}{\|z_n - x_n\|} > \text{Lip}_{K'}(u) + \epsilon.$$

Now, up to take a subsequence, $z_n \rightarrow z$ in $\partial K'$ and $x_n \rightarrow x$ in K . Notice in particular that $\|z - x\| > 0$. Since $u_n \rightarrow u$ uniformly on K' we have that

$$u_n(z_n) - u_n(x_n) \rightarrow u(z) - u(x)$$

and by (A.2)

$$\frac{u(z) - u(x)}{\|z - x\|} \geq \text{Lip}_{K'}(u) + \epsilon$$

that contradicts that u is $\text{Lip}_{K'}(u)$ -Lipschitz on K' . □

Given a hyperbolic support function \bar{h} let us denote by h the corresponding support function on the ball. Recall that $h(x) = \lambda(x)\bar{h}(\mathbf{v}(x))$ where $\lambda(x) = \sqrt{1 - \|x\|^2}$ and $\mathbf{v} : B \rightarrow \mathbb{H}^d$ is the radial map.

Using that for each compact subset K of \mathbb{H}^d the map $\mathbf{v} : \mathbf{v}^{-1}(K) \rightarrow K$ is bi-Lipschitz we simply check that there are two constants $\alpha < \beta$ depending on K such that for any support function \bar{h}

$$\alpha \|h\|_{\text{Lip}(\mathbf{v}^{-1}(K))} \leq \|\bar{h}\|_{\text{Lip}(K)} \leq \beta \|h\|_{\text{Lip}(\mathbf{v}^{-1}(K))}$$

where the Lipschitz constant of h is computed with respect to the Euclidean metric of the ball while the Lipschitz constant of \bar{h} is computed with respect to the hyperbolic metric.

So Lemma A.8 implies the following corollary.

COROLLARY A.9. — *Let $(\bar{h}_n)_n$ be a sequence of hyperbolic support functions on an open subset $U \subset \mathbb{H}^d$ converging uniformly on any compact set of U to \bar{h} . Let $K \subset U$ be a compact set, and $K' \subset U$ a compact set containing K in its interior. Then there is a constant $C_1 = C_1(K')$ such that*

$$\text{Limsup } \|\bar{h}_n\|_{\text{Lip}(K)} \leq C_1 \|\bar{h}\|_{\text{Lip}(K')} .$$

Proof of Proposition A.6. — There is a constant $C = C(d)$ such that if \tilde{V} is any Killing vector field and x a point in \mathbb{H}^d such that $\nabla_{\tilde{V}} \tilde{V}(x) = 0$ then for any $y \in B(x, 1)$

$$\left| \|\tilde{V}_y\|^2 - \|\tilde{V}_x\|^2 \right| < Cr, \quad \|\nabla_{\tilde{V}} \tilde{V}(y)\| \leq C \|\tilde{V}_x\| r$$

where $r = d_{\mathbb{H}}(x, y)$. This can be seen because as the hyperbolic space has negative curvature, the map $f(y) = \|\tilde{V}_y\|^2$ is convex, hence locally Lipschitz, and $f \text{ grad}^{\mathbb{H}} f = -\nabla_{\tilde{V}} \tilde{V}$ [9]. By applying an isometry, one sees that the constant does not depend on x .

Let us assume that \bar{h} is C^2 . By Cauchy–Schwarz and one of the inequalities above, we get that for every $y \in B(x, r)$

$$\langle \text{grad}^{\mathbb{H}} \bar{h}, \nabla_{\tilde{V}} \tilde{V}(y) \rangle \geq -\|\text{grad}^{\mathbb{H}} \bar{h}\|_{B(x,r)} \|\tilde{V}_x\| Cr .$$

Moreover, $\nabla^2 \bar{h} - \bar{h} g_{\mathbb{H}} \geq 0$, that implies

$$\nabla^2(\bar{h})(\tilde{V}_y, \tilde{V}_y) - \bar{h}(y) \|\tilde{V}_x\|^2 \geq -|\bar{h}(y)| \left| \|\tilde{V}_y\|^2 - \|\tilde{V}_x\|^2 \right| \geq -\|\bar{h}\|_{B(x,r)} Cr .$$

Lemma A.7 implies that

$$\nabla^2(\hat{h}_r) - \hat{h}_r g_{\mathbb{H}}(v, v) \geq -\zeta(r) V(r) C(\|\bar{h}\|_{B(x,r)} + \|\text{grad}^{\mathbb{H}} \bar{h}\|_{B(x,r)} \|v\|) r .$$

So in particular for any $x \in K$, if $\|v\| = 1$, then

$$(\nabla^2(\hat{h}_r) - \hat{h}_r g_{\mathbb{H}})(v, v) \geq -\zeta(r) V(r) C(\|\bar{h}\|_{K_r} + \|\text{grad}^{\mathbb{H}} \bar{h}\|_{K_r}) r$$

and as K_r is convex, $\|\text{grad}^{\mathbb{H}} \bar{h}\|_{K_r} = \text{Lip}_{K_r}(\bar{h})$, and as $V(r) \rightarrow 1$ when $r \rightarrow 0$, there exists C' such that

$$(\nabla^2(\hat{h}_r) - \hat{h}_r g_{\mathbb{H}})(v, v) \geq -C' \|\bar{h}\|_{\text{Lip}(K_r)} r, \quad \|v\| = 1,$$

so $\hat{h}_r - C' \|\bar{h}\|_{\text{Lip}(K_r)} r$ is a support function.

Now consider the case where \bar{h} is only continuous. Let $(H_n)_n$ be a sequence of C_+^2 support functions converging to the extension H of \bar{h} uniformly on compact subset of $I^+(0)$ (Lemma 2.16). From Corollary A.9 for any $\epsilon > 0$ the functions

$$f_n = (\hat{h}_n)_r - C' C_1 (\|\bar{h}\|_{\text{Lip}(K_{2r})} + \epsilon)r$$

are support functions for n big and $r < 1$, where $C_1 = C_1(K_1)$ is the constant of Corollary A.9.

Taking the 1-homogeneous extension F_n of f_n , we have that F_n is convex over the cone on K for n big. Letting n go to $+\infty$, F_n converges to the 1-homogeneous extension of

$$f = \hat{h}_r - C' C_1 (\|\bar{h}\|_{\text{Lip}(K_r)} + \epsilon)r.$$

So f is a support function on K . Since ϵ can be chose arbitrarily we deduce that $\hat{h}_r - C' C_1 \|\bar{h}\|_{\text{Lip}(K_r)} r$ is a support function over K . \square

BIBLIOGRAPHY

- [1] A. D. ALEXANDROV, "On the theory of mixed volumes II", *Mat. Sbornik* **44** (1937), p. 1205-1238, in Russian, translated in [2].
- [2] ———, *Selected works. Part I*, Classics of Soviet Mathematics, vol. 4, Gordon and Breach Publishers, 1996, Selected scientific papers, Translated from the Russian by P. S. V. Naidu, Edited and with a preface by Yu. G. Reshetnyak and S. S. Kutateladze, x+322 pages.
- [3] T. BARBOT, "Globally hyperbolic flat space-times", *J. Geom. Phys.* **53** (2005), no. 2, p. 123-165.
- [4] ———, "Domaines globalement hyperboliques de l'espace de Minkowski et de l'espace anti-de Sitter", in *Algèbre, dynamique et analyse pour la géométrie : aspects récents*, Ellipses, 2010, Proceedings des Écoles de Géométrie et Systèmes dynamiques, Algérie, 2004-2007, p. 101-138.
- [5] T. BARBOT, F. BÉGUIN & A. ZEGHIB, "Prescribing Gauss curvature of surfaces in 3-dimensional spacetimes: application to the Minkowski problem in the Minkowski space", *Ann. Inst. Fourier* **61** (2011), no. 2, p. 511-591.
- [6] M. BELRAOUTI, "Asymptomatic convergence of level sets of quasi-concave times in a space-time of constant curvature", PhD Thesis, Université d'Avignon (France), 2013, <https://tel.archives-ouvertes.fr/tel-00978618>.
- [7] ———, "Asymptotic behavior of Cauchy hypersurfaces in constant curvature space-times", <https://arxiv.org/abs/1503.06343>, 2015.
- [8] J. BERTRAND, "Prescription of Gauss curvature on compact hyperbolic orbifolds", *Discrete Contin. Dyn. Syst.* **34** (2014), no. 4, p. 1269-1284.
- [9] R. L. BISHOP & B. O'NEILL, "Manifolds of negative curvature", *Trans. Am. Math. Soc.* **145** (1969), p. 1-49.
- [10] T. BONNESEN & W. FENCHEL, *Theory of convex bodies*, BCS Associates, Moscow, ID, 1987, Translated from the German and edited by L. Boron, C. Christenson and B. Smith, x+172 pages.
- [11] F. BONSANTE, "Flat spacetimes with compact hyperbolic Cauchy surfaces", *J. Differ. Geom.* **69** (2005), no. 3, p. 441-521.

- [12] F. BONSANTE, C. MEUSBURGER & J.-M. SCHLENKER, “Recovering the geometry of a flat spacetime from background radiation”, *Ann. Henri Poincaré* **15** (2014), no. 9, p. 1733-1799.
- [13] G. CARLIER, “On a theorem of Alexandrov”, *J. Nonlinear Convex Anal.* **5** (2004), no. 1, p. 49-58.
- [14] S.-Y. CHENG & S.-T. YAU, “On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x, u)$ ”, *Commun. Pure Appl. Math.* **30** (1977), no. 1, p. 41-68.
- [15] Y. CHOQUET-BRUHAT & R. GEROCH, “Global aspects of the Cauchy problem in general relativity”, *Commun. Math. Phys.* **14** (1969), p. 329-335.
- [16] F. FILLASTRE, “Fuchsian convex bodies: basics of Brunn-Minkowski theory”, *Geom. Funct. Anal.* **23** (2013), no. 1, p. 295-333.
- [17] F. FILLASTRE & G. VERONELLI, “Lorentzian area measures and the Christoffel problem”, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **16** (2016), no. 2, p. 383-467.
- [18] B. GUAN, H.-Y. JIAN & R. M. SCHOEN, “Entire spacelike hypersurfaces of prescribed Gauss curvature in Minkowski space”, *J. Reine Angew. Math.* **595** (2006), p. 167-188.
- [19] C. GUTIÉRREZ, *The Monge-Ampère equation*, Progress in Nonlinear Differential Equations and their Applications, vol. 44, Birkhäuser, Boston, MA, 2001, xii+127 pages.
- [20] J.-B. HIRIART-URRUTY & C. LEMARÉCHAL, *Convex analysis and minimization algorithms. I*, Grundlehren der Mathematischen Wissenschaften, vol. 305, Springer-Verlag, Berlin, 1993, Fundamentals, xvii+417 pages.
- [21] R. A. HORN & C. R. JOHNSON, *Matrix analysis*, Cambridge University Press, 1990, Corrected reprint of the 1985 original, xiii+561 pages.
- [22] A. KHOVANSKIĬ & V. TIMORIN, “On the theory of coconvex bodies”, *Discrete Comput. Geom.* **52** (2014), no. 4, p. 806-823.
- [23] A.-M. LI, “Spacelike hypersurfaces with constant Gauss-Kronecker curvature in the Minkowski space”, *Arch. Math.* **64** (1995), no. 6, p. 534-551.
- [24] G. MESS, “Lorentz spacetimes of constant curvature”, *Geom. Dedicata* **126** (2007), p. 3-45.
- [25] I. G. NIKOLAEV & S. Z. SHEFEL', “Convex surfaces with positive bounded specific curvature, and a priori estimates for Monge-Ampère equations”, *Sibirsk. Mat. Zh.* **26** (1985), no. 4, p. 120-136, 205.
- [26] V. I. OLIKER & U. SIMON, “Codazzi tensors and equations of Monge-Ampère type on compact manifolds of constant sectional curvature”, *J. Reine Angew. Math.* **342** (1983), p. 35-65.
- [27] A. V. POGORELOV, *Extrinsic geometry of convex surfaces*, Translations of Mathematical Monographs, vol. 35, American Mathematical Society, 1973, Translated from the Russian by Israel Program for Scientific Translations, vi+669 pages.
- [28] ———, *The Minkowski multidimensional problem*, V. H. Winston & Sons, Washington, D.C., 1978, Translated from the Russian by Vladimir Oliker, Introduction by Louis Nirenberg, Scripta Series in Mathematics, 106 pages.
- [29] J. G. RATCLIFFE, *Foundations of hyperbolic manifolds*, second ed., Graduate Texts in Mathematics, vol. 149, Springer-Verlag, New York, 2006, xii+779 pages.
- [30] J. RAUCH & B. A. TAYLOR, “The Dirichlet problem for the multidimensional Monge-Ampère equation”, *Rocky Mt. J. Math.* **7** (1977), no. 2, p. 345-364.
- [31] R. T. ROCKAFELLAR, *Convex analysis*, Princeton Landmarks in Mathematics, Princeton University Press, 1997, Reprint of the 1970 original, Princeton Paperbacks, xviii+451 pages.
- [32] R. T. ROCKAFELLAR & R. J.-B. WETS, *Variational analysis*, Grundlehren der Mathematischen Wissenschaften, vol. 317, Springer-Verlag, Berlin, 1998, xiv+733 pages.

- [33] K. P. SCANNELL, “Infinitesimal deformations of some $SO(3,1)$ lattices”, *Pac. J. Math.* **194** (2000), no. 2, p. 455-464.
- [34] R. SCHNEIDER, *Convex bodies: the Brunn–Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, 1993, xiii+490 pages.
- [35] N. S. TRUDINGER & X.-J. WANG, “The Monge-Ampère equation and its applications”, in *Handbook of geometric analysis. No. 1*, Adv. Lect. Math. (ALM), vol. 7, Int. Press, Somerville, MA, 2008, p. 467-524.

Manuscrit reçu le 4 février 2016,
révisé le 8 septembre 2016,
accepté le 20 septembre 2016.

Francesco BONSANTE
Università degli Studi di Pavia
Via Ferrata, 1
Pavia, 27100 (Italy)
francesco.bonsante@unipv.it

François FILLASTRE
Université de Cergy-Pontoise
UMR CNRS 8088
Cergy-Pontoise, 95000 (France)
francois.fillastre@u-cergy.fr