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MAXIMALITY OF HYPERSPECIAL COMPACT SUBGROUPS AVOIDING BRUHAT–TITS THEORY

by Marco MACULAN

ABSTRACT. — Let k be a complete non-archimedean field (non trivially valued). Given a reductive k -group G , we prove that hyperspecial subgroups of $G(k)$ (i.e. those arising from reductive models of G) are maximal among bounded subgroups. The originality resides in the argument: it is inspired by the case of GL_n and avoids all considerations on the Bruhat–Tits building of G .

RÉSUMÉ. — Soit k un corps non-archimédien complet et non trivialement valué. Étant donné un k -groupe réductif G , nous démontrons que les sous-groupes hyperspéciaux de $G(k)$ (c'est-à-dire ceux qui proviennent des modèles réductifs de G) sont maximaux parmi les sous-groupes bornés. La nouveauté réside dans l'argument suivant : inspiré par le cas de GL_n , il n'utilise pas la théorie de Bruhat–Tits.

1. Introduction

1.1. Background

Over the complex numbers, a connected linear algebraic group G is reductive if and only if it contains a Zariski-dense compact subgroup. If G is semi-simple such a subgroup corresponds to a maximal real Lie subalgebra of $\mathrm{Lie} G$ on which the Killing form is negative definite.

If one replaces the field of complex numbers by the field of p -adic ones (or, more generally, any finite extension of it) an analogue characterisation holds: a connected linear algebraic group G is reductive if and only if $G(\mathbb{Q}_p)$ contains a maximal compact subgroup [18, Propositions 3.15–16]. In this case, a maximal compact subgroup is of the form $\mathcal{G}(\mathbb{Z}_p)$ for a suitable integral model \mathcal{G} of G .

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Reversing the logic one might wonder, given an integral model \mathcal{G} of G , whether the compact subgroup $\mathcal{G}(\mathbb{Z}_p)$ is maximal. It has to be (according to work of Bruhat, Hijikata, Rousseau, Tits among others) if the special fibre of \mathcal{G} is a reductive group over \mathbb{F}_p – the associated compact subgroup is then called *hyperspecial*, whence the title of the article. The purpose of the present paper is to expound a proof of this result without using the theory of Bruhat–Tits building (and the combinatorics needed to construct it).

1.2. Statement of the results

In order to be more precise and to state the main theorem in its full generality, let k be a non-archimedean field (that we suppose complete and non-trivially valued), k° its ring of integers and \tilde{k} its residue field. Let \mathcal{G} be a reductive k° -group⁽¹⁾ and G its generic fibre. The main result is the following:

THEOREM 1.1. — *The subgroup $\mathcal{G}(k^\circ)$ is a maximal bounded subgroup of $G(k)$.*

When G is split this theorem can be deduced from [7, §3.3 and §8.2] taking in account Exemple 6.4.16(b), loc. cit.. Note that, under the hypothesis of G being split, Theorem 1.1 is due to Bruhat over a p -adic field [6, 5]. The quasi-split case, i.e. when G contains a Borel subgroup defined over k , is covered by [8, Théorème 4.2.3] and the general case, by [20, Théorème 5.1.2] (the existence of the reductive model \mathcal{G} implies that G splits over a non-ramified extension [11, Theorem 6.1.16]).

The subgroups of $G(k)$ of the form $\mathcal{G}(k^\circ)$ are called hyperspecial. When the residue field \tilde{k} is finite, the existence of a k° -reductive model \mathcal{G} of G is equivalent to G being quasi-split over k and being split over a non-ramified extension [10, Theorem 2.6]. In particular, although maximal compact subgroups always exist for an arbitrary reductive group over a locally compact field, hyperspecial subgroups do not.

Hyperspecial subgroups are anyway crucial objects in the study of representations of p -adic groups and, even though Theorem 1.1 is a basic result,

⁽¹⁾ Let G be a group S -scheme. We say that G is *reductive* (resp. *semi-simple*) if it verifies the following conditions:

- (1) G is affine and smooth over S ;
- (2) for all $s \in S$, the \bar{s} -algebraic group $G_{\bar{s}} := G \times_S \bar{s}$ is connected and reductive (resp. connected and semi-simple).

Here \bar{s} denotes the spectrum of an algebraic closure of the residue field $\kappa(s)$ at s . See [12, XIX, Définition 2.7].

all the proofs I am aware of rely on the deep knowledge of the combinatorics of $G(k)$ which comes at the end of Bruhat–Tits theory. More precisely, one sees $\mathcal{G}(k^\circ)$ as the stabiliser of a (hyperspecial) vertex of the Bruhat–Tits building $\mathcal{B}(G, k)$, which is a maximal bounded subgroup. This exploits implicitly that the integral model \mathcal{G} induces a Tits system on $G(k)$ (when k is discretely valued) and a valued root datum of $G(k)$ (when the valuation is dense).

Instead, the proof of Theorem 1.1 presented here elaborates on the argument for the case GL_n , using tools from algebraic geometry involving flag manifolds of G . When k is a p -adic field the advantage of the present approach is that it avoids all the computations contained in [15] needed in order to show that $G(k)$ admits a Tits system. Let us recall that for the general linear group the proof of Theorem 1.1 goes as follows:

- (1) Consider the norm $\|(x_1, \dots, x_n)\| = \max\{|x_1|, \dots, |x_n|\}$ on k^n . The subgroup of $\mathrm{GL}_n(k)$ of elements letting $\|\cdot\|$ invariant is $\mathrm{GL}_n(k^\circ)$.
- (2) If H is a bounded subgroup containing $\mathrm{GL}_n(k^\circ)$ consider the norm $\|\cdot\|_H$ defined for every $x \in k^n$ by $\|x\|_H := \sup_{h \in H} \|h(x)\|$. The ratio of the norms $\|\cdot\|_H / \|\cdot\|$ gives rise to a well-defined function $\phi: \mathbb{P}^{n-1}(k) \rightarrow \mathbb{R}_+$ which is clearly $\mathrm{GL}_n(k^\circ)$ -invariant.
- (3) Since the group $\mathrm{GL}_n(k^\circ)$ acts transitively on $\mathbb{P}^{n-1}(k)$, ϕ must be constant. In particular, H is contained in $\mathrm{GL}_n(k^\circ)$.

The problem with passing from GL_n to an arbitrary reductive k -group G is that the latter does not have a canonical representation on which one can consider norms. We prefer to interpret \mathbb{P}^{n-1} as a flag variety of GL_n and the norm $\|\cdot\|$ as the metric that it induces on the line bundle $\mathcal{O}(1)$. Moreover, we think at the latter as the metric naturally induced by the line bundle $\mathcal{O}(1)$ on \mathbb{P}^{n-1} over the ring of integers k° .

When treating the case of an arbitrary reductive k° -group \mathcal{G} of generic fibre G , this suggests to replace:

- the projective space \mathbb{P}_k^{n-1} by the variety $X = \mathrm{Bor}(G)$ of Borel subgroups of G ;
- the line bundle $\mathcal{O}(1)$ by the anti-canonical bundle $L = -K_X$ of X ;
- the norm $\|\cdot\|$ by the metric $\|\cdot\|_{\mathcal{L}}$ on L induced by the line bundle

$$\mathcal{L} = (\det \Omega_{\mathcal{X}/k^\circ}^1)^\vee \otimes \alpha^*(\det \mathrm{Lie} \mathcal{G})^\vee$$

on the k° -scheme of Borel subgroups $\mathcal{X} = \mathrm{Bor}(\mathcal{G})$ of \mathcal{G} , where α is the structural morphism of \mathcal{X} and $\mathrm{Lie} \mathcal{G}$ the Lie algebra of \mathcal{G} .

Note that, when $G = \mathrm{GL}_n$, these new choices do not correspond to the original ones so that even in this case we get a new (but slightly more complicated) proof.

The construction of the metric $\|\cdot\|_{\mathcal{L}}$ is inspired by the embedding of the Bruhat–Tits building in the flag varieties defined by Berkovich and Rémy–Thuillier–Werner [19].

The anti-canonical bundle L of X has a natural structure of G -linearised sheaf, that is, G acts linearly on the fibres on L respecting the action on X . We can therefore consider the stabiliser $\mathrm{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}})$ in $G(k)$ of the metric $\|\cdot\|_{\mathcal{L}}$ (see also Paragraph 2.3).

THEOREM 1.2. — *Let us suppose that G is semi-simple and quasi-split. Then,*

$$\mathrm{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}}) = \mathcal{G}(k^\circ).$$

Theorem 1.2 is the critical result that we need to prove Theorem 1.1 when G is quasi-split. It corresponds indeed to step (1) in the proof in the case $\mathcal{G} = \mathrm{GL}_{n,k^\circ}$, whereas step (2) is trivial and step (3) is a standard fact in the theory of reductive k° -groups (see Proposition 4.2).

To show Theorem 1.2 we reduce the problem to studying the intersection of the stabiliser with the unipotent radical $\mathrm{rad}^u(B)$ of a Borel subgroup B of G . Then, identifying $\mathrm{rad}^u(B)$ with the open subset $\mathrm{Opp}(B)$ of Borel subgroups opposite to B , it remains to understand the behaviour of the metric $\|\cdot\|_{\mathcal{L}}$ on $\mathrm{Opp}(B, k)$: this boils down to a basic fact in the theory of Schubert varieties over the residue field \tilde{k} (see Proposition 4.4).

Instead if G is not quasi-split (hence the residue field infinite), then $X(k)$ is empty by definition and the metric $\|\cdot\|_{\mathcal{L}}$ gives no information. To get round this problem we remark that, for every analytic extension⁽²⁾ K of k , $\mathcal{G}(K^\circ)$ is the K -holomorphically convex envelope of $\mathcal{G}(k^\circ)$ (see Definition 6.1). The key point here is that, the residue field being infinite, the \tilde{k} -valued points of \mathcal{G} are Zariski-dense in the special fibre of \mathcal{G} .

Then, choosing an analytic extension K that splits G , we deduce the maximality of $\mathcal{G}(k^\circ)$ from the maximality of $\mathcal{G}(K^\circ)$, which holds by the quasi-split case.

⁽²⁾ i.e. a complete valued field endowed with an isometric embedding $k \rightarrow K$.

To conclude let us remark that the construction of the metric can be generalised to any type of parabolic subgroups of G . When \mathcal{G} is semi-simple and the type is k -rational⁽³⁾ and non-degenerate⁽⁴⁾, the stabiliser is still $\mathcal{G}(k^\circ)$. Since these facts are of no use in the present paper we do not treat them.

1.3. Organisation of the paper

In Section 2 we introduce the notations that we use throughout the paper and we recall some basic facts on reductive groups. In Section 3 we show how Theorem 1.1 follows from Theorem 1.2 when G is quasi-split. The proof of Theorem 1.1 for G quasi-split is given in Section 5, based on some preliminary facts established in Section 4. Finally in Section 6 we show how to reduce to the quasi-split case.

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2. Notations, reminders and definitions

2.1. Notations and conventions

Let us list some notations that we use throughout the paper:

- k is a non-archimedean field, k° its ring of integers and \tilde{k} its residue field;

⁽³⁾ Namely, the corresponding connected component of the variety of parabolic subgroups $\text{Par}(G)$ has a k -rational point.

⁽⁴⁾ That is, the restriction of a parabolic subgroup of type t to every quasi-simple factor H of G is not the whole H (see [19, 3.1]).

⁽⁵⁾ In a previous version of the paper I proved Theorem 1.2 only when the residue field was finite and used Berkovich geometry when the residue field was infinite.

- \mathcal{G} is a reductive k° -group;
- \mathcal{X} is the k° -scheme of Borel subgroups $\text{Bor}(\mathcal{G})$, that is, the k° -scheme representing the functor that associates to a k° -scheme S the set of Borel subgroups of the reductive S -group $\mathcal{G} \times_{k^\circ} S$ (cf. [12, XXII, Corollaire 5.8.3]);
- \mathcal{L} is the invertible sheaf $(\det \Omega_{\mathcal{X}/k^\circ})^\vee \otimes \alpha^*(\det \text{Lie } \mathcal{G})^\vee$ on \mathcal{X} (it is a line bundle because \mathcal{X} is smooth by loc. cit.), where α is the structural morphism of \mathcal{X} over $\text{Spec } k^\circ$;
- G, X, L are respectively the generic fibre of $\mathcal{G}, \mathcal{X}, \mathcal{L}$ and by $\tilde{G}, \tilde{X}, \tilde{L}$ their special fibre;
- $\|\cdot\|_{\mathcal{L}}$ is the metric on L associated to \mathcal{L} , that we consider as a continuous function $\|\cdot\|_{\mathcal{L}}: \mathbb{V}(L, k) \rightarrow \mathbb{R}_+$ (see definition in Paragraph 2.3).
- for every Borel subgroup \mathcal{B} we denote by $\text{Opp}(\mathcal{B})$ the k° -scheme of Borel subgroups opposite to \mathcal{B} , that is, the k° -scheme representing the functor that associates to a k° -scheme S the set of Borel subgroups of $\mathcal{G}_S := \mathcal{G} \times_{k^\circ} S$ such that the intersection with $\mathcal{B} \times_{k^\circ} S$ is a maximal S -torus of \mathcal{G}_S . A similar notation is also used for Borel subgroups of G and \tilde{G} (cf. [12, XXII, Proposition 5.9.3(ii)]).
- In this paper we refer to [14, Corollaire 1.11] as “Hensel’s Lemma”.

2.2. Reminders

- For a reductive group over a general base, the notion of quasi-split is fairly involved [12, XXIV, 3.9]. Nonetheless, thanks to [12, XXIV, Proposition 3.9.1], the k° -reductive group \mathcal{G} is quasi-split if and only if G is.
- The k° -scheme \mathcal{X} is projective and smooth (see [11, Theorem 5.2.11] or [12, XXII, 5.8.3(i)]) and the invertible sheaf \mathcal{L} is ample. Indeed, \mathcal{L} can also be constructed as follows: if $\mathcal{U} \rightarrow \mathcal{X}$ is the universal Borel subgroup and $\text{Lie } \mathcal{U}$ is the Lie algebra of \mathcal{U} , then \mathcal{L} is the dual of $\det \text{Lie } \mathcal{U}$ [11, Theorem 2.3.6 and Remark 2.3.7].

This construction also shows that the adjoint action of \mathcal{G} induces a natural equivariant action of \mathcal{G} on \mathcal{L} [12, I, Définition 6.5.1]. The equivariant action on \mathcal{L} induces for all integer n a linear action of \mathcal{G} on the global sections $H^0(\mathcal{X}, \mathcal{L}^{\otimes n})$ [12, I, Lemme 6.6.1]. We always consider these actions as tacitly understood.

- For a Borel subgroup \mathcal{B} of \mathcal{G} the scheme $\text{Opp}(\mathcal{B})$ of Borel subgroups of \mathcal{G} opposite to \mathcal{B} is an open affine subscheme of $\mathcal{X} = \text{Bor}(\mathcal{G})$ [12, XXVI, Corollaire 4.3.4 and Corollaire 4.3.5].
- The total space of L is the k -scheme $\mathbb{V}(L)$ representing the functor that associates to a k -scheme S the set of couples (x, s) made of a S -valued point $x: S \rightarrow X$ and a section $s \in H^0(S, x^*L)$ [13, 1.7.10].⁽⁶⁾

2.3. Definitions

- A subset $S \subset G(k)$ is said to be *bounded* if there exists a closed embedding $G \subset \mathbb{A}_k^n$ such that S is contained in $\mathbb{A}^n(k^\circ)$ (this generalises [3, 1.1, Definition 2] when k is not discretely valued).
- A *metric* on L is a function $\|\cdot\|: \mathbb{V}(L, k) \rightarrow \mathbb{R}_+$, $(x, s) \mapsto \|s\|(x)$ verifying the following properties for all k -points (x, s) of $\mathbb{V}(L)$:
 - $\|s\|(x) = 0$ if and only if $s = 0$;
 - $\|\lambda s\|(x) = |\lambda| \|s\|(x)$ for all $\lambda \in k$.
- The metric $\|\cdot\|_{\mathcal{L}}$ is defined as follows. A k -point of $\mathbb{V}(L)$ corresponds to the data of a point $x \in X(k)$ and a section $s \in x^*L$. By the valuative criterion of properness, the point x lifts to a unique morphism of k° -schemes $\varepsilon_x: \text{Spec } k^\circ \rightarrow \mathcal{X}$ and the k° -module $\varepsilon_x^*\mathcal{L}$ is free of rank 1 (thus it is a lattice the K -line x^*L). Pick a generator s_0 of the k° -module $\varepsilon_x^*\mathcal{L}$ and set, for all $s = \lambda s_0$ with $\lambda \in k$,

$$\|s\|_{\mathcal{L}}(x) := |\lambda|.$$

The real number $\|s\|_{\mathcal{L}}(x)$ does not depend on the chosen generator s_0 , so this gives a well-defined function $\|\cdot\|_{\mathcal{L}}: \mathbb{V}(L, k) \rightarrow \mathbb{R}_+$,

$$\|(x, s)\|_{\mathcal{L}} := \|s\|_{\mathcal{L}}(x).$$

It is easily seen that $\|\cdot\|_{\mathcal{L}}$ is continuous on $\mathbb{V}(L, k)$ and bounded on bounded subsets. Similarly, for every integer n , one constructs the metric $\|\cdot\|_{\mathcal{L}^{\otimes n}}$ on $L^{\otimes n}$ associated to $\mathcal{L}^{\otimes n}$.

- The group $G(k)$ acts on the set of metrics on L . Indeed, given a metric $\|\cdot\|$ and $g \in G(k)$, the function $(x, s) \mapsto \|g^{-1} \cdot (x, s)\|$ is again a metric (because $G(k)$ acts linearly on the fibres of L).

⁽⁶⁾ In loc. cit. the k -scheme $\mathbb{V}(L)$ is denoted $\mathbb{V}(L^\vee)$.

We denote by $\text{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}})$ the stabiliser of the metric $\|\cdot\|_{\mathcal{L}}$ with respect to this action. More explicitly, $\text{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}})$ is the set of points $g \in G(k)$ such that, for all k -points (x, s) of $\mathbb{V}(L)$, we have

$$\|g \cdot s\|_{\mathcal{L}}(g \cdot x) = \|s\|_{\mathcal{L}}(x).$$

3. Proof of Theorem 1.1 in the quasi-split case

In this section we admit temporarily Theorem 1.2 and we prove the following:

THEOREM 3.1. — *Let us suppose \mathcal{G} quasi-split. Then, $\mathcal{G}(k^\circ)$ is a maximal bounded subgroup of $G(k)$.*

Proof. — We may assume that G is semi-simple. Indeed, if it is not the case, we consider the derived group \mathcal{D} of \mathcal{G} (which is a semi-simple k° -group scheme [12, XXII, Théorème 6.2.1(iv)]) and the identity component of the center \mathcal{Z} of \mathcal{G} (which is a k° -torus). The map $\pi: \mathcal{D} \times_{k^\circ} \mathcal{Z} \rightarrow \mathcal{G}$ given by multiplication is an isogeny [12, XXII, Proposition 6.2.4]. If H is a bounded subgroup of $G(k)$ containing $\mathcal{G}(k^\circ)$, then the subgroup $\pi^{-1}(H)$ contains $\mathcal{D}(k^\circ) \times \mathcal{Z}(k^\circ)$ and is bounded because π is a finite morphism⁽⁷⁾. Since $\mathcal{Z}(k^\circ)$ is the maximal bounded subgroup of $Z(k)$ (Proposition 4.6), we are left with proving that $\mathcal{D}(k^\circ)$ is a maximal bounded subgroup of $D(k)$.

Let us henceforth suppose that G is semi-simple. Let H be a bounded subgroup containing $\mathcal{G}(k^\circ)$ and let us consider the metric $\|\cdot\|_H$ on $L|_{X(k)}$ defined, for every point $x \in X(k)$ and every section $s \in x^*L$, by

$$\|s\|_H(x) := \sup_{h \in H} \|h \cdot s\|_{\mathcal{L}}(h \cdot x).$$

Note that $\|\cdot\|_H$ takes real values because H is bounded and $\|\cdot\|_{\mathcal{L}}$ is continuous and bounded. The ratio of the metrics $\|\cdot\|_H$ and $\|\cdot\|_{\mathcal{L}}$ defines a function

$$\phi = \frac{\|\cdot\|_H}{\|\cdot\|_{\mathcal{L}}} : X(k) \longrightarrow \mathbb{R}_+,$$

⁽⁷⁾If V, W are k -schemes of finite type and $f: V \rightarrow W$ is a finite morphism, then the inverse image of a bounded subset of $W(k)$ is bounded. Since finite morphisms are projective, in order to prove this statement, one is immediately led back to prove it when $V = \mathbb{P}^n \times_k W$ and f is the projection on the second factor. This latter statement is clear because $\mathbb{P}^n(k)$ is bounded (the proof given in [3, 1.1, Proposition 6] when k is discretely valued generalises without problems to the non-discretely valued case).

which is invariant under the action of $\mathcal{G}(k^\circ)$. Since $\mathcal{G}(k^\circ)$ acts transitively on $X(k)$ (Proposition 4.2 (2)), the function ϕ must be constant. Thus H is contained in $\text{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}})$ and, according to Theorem 1.2, we conclude. \square

4. Some preliminary facts

In this section we collect some facts that will be used during the proof of Theorem 1.2. Some of them are standard facts but we included their proof for the sake of completeness.

4.1. On the scheme of Borel subgroups

A perfect field F is said to be of cohomological dimension ≤ 1 if every homogeneous space under a connected linear algebraic group has a F -rational point [21, §2.2, III, Théorème 1 and §2.3, Corollaire 1]. The only examples of fields of cohomological dimension ≤ 1 we are interested in are finite fields (“Lang’s theorem” [1, Corollary 16.5(i)]).

PROPOSITION 4.1. — *Let us suppose that the residue field \tilde{k} is perfect of cohomological dimension ≤ 1 and let \mathcal{G} be a k° -reductive group. Then, its generic fibre G is quasi-split.*

Proof. — The special fibre \tilde{X} of \mathcal{X} is a homogeneous space under the action of the connected group \tilde{G} . Therefore, by definition of field of cohomological dimension ≤ 1 , it admits a \tilde{k} -rational point. Thanks to the smoothness of \mathcal{X} and Hensel’s lemma, such a rational point can be lifted to a k° -valued point of \mathcal{X} , that is, to a Borel subgroup \mathcal{B} of \mathcal{G} . The generic fibre of \mathcal{B} does the job. \square

PROPOSITION 4.2. — *Let us suppose \tilde{k} arbitrary and G quasi-split. Then,*

- (1) every Borel subgroup \mathcal{B} of \mathcal{G} contains a maximal torus of \mathcal{G} ;
- (2) $\mathcal{G}(k^\circ)$ acts transitively on $X(k)$;
- (3) (Iwasawa decomposition) for every Borel subgroup B of G , we have

$$G(k) = \mathcal{G}(k^\circ) \cdot B(k).$$

Proof.

- (1) [12, XII, Corollaire 5.9.7].

- (2) We can apply [12, XXVI, Corollaire 5.2]. Indeed, if \mathcal{B} is a Borel subgroup of \mathcal{G} , by (1) it contains a maximal torus \mathcal{T} and we can consider the Borel subgroup \mathcal{B}' opposite to \mathcal{B} with respect to \mathcal{T} [12, XXII, Proposition 5.9.2].
- (3) Since \mathcal{X} is proper, the valuative criterion of properness entails the equality $\mathcal{X}(k^\circ) = X(k)$, which, according to (2), gives

$$\mathcal{G}(k^\circ)/\mathcal{B}(k^\circ) = G(k)/B(k).$$

The result follows immediately. \square

4.2. Size of global sections of the anti-canonical bundle

Let us start by recalling a basic fact in the theory of Schubert varieties over a field.

LEMMA 4.3. — *Let F be a field. Let H be a quasi-split reductive F -group and P a Borel subgroup of H . Let Y be the variety of Borel subgroups of H and M the anti-canonical bundle of Y . Then,*

- (1) *there exists a unique (up to scalar factor) non-zero eigenvector in $H^0(Y, M)$ for P ;*
- (2) *the locus where such an eigenvector does not vanish is the open subset $\text{Opp}(P) \subset Y$ made of Borel subgroups opposite to P .*

Proof. — When H is split, Y is the Schubert variety associated to the maximal element w_0 of the Weyl group of H (with respect to the Bruhat order) and $\text{Opp}(P)$ is the corresponding Bruhat cell – see, for instance, [4, Proposition 1.4.5], [22, §8.5.7] or [16, §4] for a thorough discussion of these aspects. The quasi-split case follows by Galois descent. \square

Let us go back to the general notation introduced in Section 2.

PROPOSITION 4.4. — *Let us suppose G quasi-split and let B be a Borel subgroup. Let $s \in H^0(\mathcal{X}, \mathcal{L})$ be an eigenvector for B such that its reduction \bar{s} is non-zero. Then,*

$$\{x \in X(k) : \|s\|_{\mathcal{L}}(x) = 1\} = \text{Opp}(\mathcal{B}, k^\circ),$$

where \mathcal{B} is the Borel subgroup of \mathcal{G} lifting B and $\text{Opp}(\mathcal{B})$ is the open subset of $\text{Bor}(\mathcal{G})$ made of Borel subgroups opposite to \mathcal{B} .

Remark 4.5. — This statement is a “coordinate-free” analogue of [19, Proposition 2.18(i)] (in the sense that we do not need to consider a maximal split k -torus of G and the corresponding roots).

Proof. — Let $x \in X(k)$. First of all, applying Lemma 4.3 with $F = k$ and $H = G$, let us remark that we have $\|s\|_{\mathcal{L}}(x)$ does not vanish exactly when x belongs to $\text{Opp}(B, k)$. Furthermore, the equality $\|s\|_{\mathcal{L}}(x) = 1$ is equivalent to say that the reduction $\tilde{s} \in H^0(\tilde{X}, \tilde{L})$ of s does not vanish at the reduction $\tilde{x} \in \tilde{X}(\tilde{k})$ of x .

If \tilde{B} denotes special fibre of B , then \tilde{s} is a non-zero eigenvector for \tilde{B} . Therefore, applying again Lemma 4.3 to $F = \tilde{k}$ and $H = \tilde{G}$, we obtain that \tilde{s} does not vanishes precisely on the open subset $\text{Opp}(\tilde{B})$ of \tilde{X} made of Borel subgroups of \tilde{G} opposite to \tilde{B} . Let \mathcal{B}_x the Borel subgroup of \mathcal{G} associated to x . Summing up we have:

$$\begin{aligned} \|s\|_{\mathcal{L}}(x) \neq 0 &\iff \text{the generic fibre of } \mathcal{B}_x \text{ is opposite to } B, \\ \|s\|_{\mathcal{L}}(x) = 1 &\iff \begin{array}{l} \text{the generic and the special fibre of } \mathcal{B}_x \text{ are} \\ \text{respectively opposite to } B \text{ and } \tilde{B}. \end{array} \end{aligned}$$

In other words, $\|s\|_{\mathcal{L}}(x) = 1$ if and only if $x \in \text{Opp}(B, k^\circ)$. □

4.3. Compact subgroups of tori

Given a torus T over a complete non-archimedean field k , the set of its k -rational points contains a unique maximal bounded subgroup U_T .⁽⁸⁾

It is not true in general that U_T is the group of k° -valued points of a k° -torus \mathcal{T} . When k is discretely valued, U_T coincides with the set of k° -valued points of the identity component \mathcal{T} of the Néron model of T [8, 4.4.12] but, if the splitting extension of T is ramified, then the special fibre of \mathcal{T} may not be a torus [8, 4.4.13]. Anyway, this is true if T is already the generic fibre of k° -torus :

PROPOSITION 4.6. — *Let \mathcal{T} be a k° -torus and T its generic fibre. Then, $\mathcal{T}(k^\circ)$ is the unique maximal bounded subgroup of $T(k)$.*

Proof. — If $\mathcal{T} \simeq \mathbb{G}_{m, k^\circ}^r$ is split, then $\mathbb{G}_m^r(k^\circ)$ is the unique maximal bounded subgroup of $\mathbb{G}_m^r(k)$. In general there exists a finite unramified extension K of k such that $\mathcal{T}_K := \mathcal{T} \times_{k^\circ} K^\circ$ is split and, by the split case, $\mathcal{T}(K^\circ)$ is the unique maximal bounded subgroup of $T(K)$. It follows that $\mathcal{T}(K^\circ) \cap T(k) = \mathcal{T}(k^\circ)$ is the unique maximal bounded subgroup of $T(k)$. □

⁽⁸⁾ Indeed, if $T \simeq \mathbb{G}_{m, k}^r$ is split one takes $U_T = \mathbb{G}_m^r(k^\circ)$. If T is not split, let k' be a finite separable extension splitting T and let $T' = T \times_k k'$. Then, $U_T = U_{T'} \cap T(k)$.

4.4. Boundedness of the stabiliser

In this section we establish that the stabiliser $\text{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}})$ is a bounded subset of $G(k)$. Let us begin with two results that we need in the proof.

LEMMA 4.7. — *Let $n \geq 1$ be such that $L^{\otimes n}$ is very ample. If G is semi-simple, then the natural representation $\rho: G \rightarrow \text{GL}(H^0(X, L^{\otimes n}))$ is finite as a morphism of k -schemes.*

Proof. — We may assume that k is algebraically closed. We prove that $\text{Ker } \rho$ is finite, which clearly implies the statement.

Since X embeds G -equivariantly in $\mathbb{P}(H^0(X, L^{\otimes n})^\vee)$, then $\text{Ker } \rho$ is contained in the stabiliser of every point of X . That is, $\text{Ker } \rho$ is contained in the intersection of all Borel subgroups. In other words, the identity component of $\text{Ker } \rho$ is the radical of G , which is trivial since G is semi-simple [1, §11.21]. \square

LEMMA 4.8. — *Let V be a finite dimensional k -vector space and let $\|\cdot\|$ be a norm on V . Then the following subgroup of $\text{GL}(V, k)$,*

$$\text{Stab}_{\text{GL}(V, k)}(\|\cdot\|) := \{g \in \text{GL}(V, k) : \|g \cdot v\| = \|v\| \text{ for all } v \in V\},$$

is bounded.

Proof. — Let us see $\text{GL}(V)$ as a closed subscheme of the affine scheme $\text{End}(V) \times_k \text{End}(V)$ through the closed embedding $g \mapsto (g, g^{-1})$. If we consider the subset

$$E = \{\phi \in \text{End}(V, k) : \|\phi(v)\| \leq \|v\| \text{ for all } v \in V\},$$

then we have $\text{Stab}_{\text{GL}(V, k)}(\|\cdot\|) = (E \times E) \cap \text{GL}(V, k)$. Therefore it suffices to show that the subset E is bounded. Let $\mathcal{V}_1, \mathcal{V}_2$ be k° -lattices of V such that the associated norms on V satisfy, for all $v \in V$,

$$\|v\|_1 \leq \|v\| \leq \|v\|_2,$$

(they exist because the norms on V are all equivalent). It follows, through the canonical isomorphism $\text{End}(V) = \text{Hom}_k(V, k) \otimes_k V$, that E is a subset of

$$\text{Hom}_{k^\circ}(\mathcal{V}_2, k^\circ) \otimes_{k^\circ} \mathcal{V}_1.$$

In particular E is bounded by definition. \square

PROPOSITION 4.9. — *If \mathcal{G} is semi-simple and quasi-split, then the stabiliser $\text{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}})$ is bounded.*

Proof. — Let $n \geq 1$ be an integer such that $L^{\otimes n}$ is very ample, let $V := H^0(X, L^{\otimes n})$ and let $\rho: G \rightarrow \text{GL}(V)$ be the representation induced by the equivariant action of G on $L^{\otimes n}$. According to Lemma 4.7, ρ is a finite morphism.

For every global section $s \in V$ let us set

$$\|s\|_{\text{sup}} := \sup_{x \in X(k)} \|s\|_{\mathcal{L}^{\otimes n}(x)}.$$

Remark that $\|\cdot\|_{\text{sup}}$ is a norm on V because $X(k)$ is non-empty and thus, by the Zariski-density of $G(k)$ in G , Zariski-dense in X [9, Theorem 1.1].

The subgroup $S = \text{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}})$ fixes the norm $\|\cdot\|_{\text{sup}}$, therefore its image in $\text{GL}(V, k)$ through ρ is bounded (Lemma 4.8). Since ρ is a finite morphism, S must be bounded too. \square

5. Proof of Theorem 1.2

In this section we prove Theorem 1.2 and therefore we suppose that the group G is semi-simple and quasi-split.

In order to prove Theorem 1.2, we start by noticing that the metric $\|\cdot\|_{\mathcal{L}}$ is invariant under $\mathcal{G}(k^\circ)$. Indeed, let $x \in X(k)$, $g \in \mathcal{G}(k^\circ)$ and let us denote by $\varepsilon_x, \varepsilon_{g \cdot x}$ the unique k° -valued points of \mathcal{X} that lift, by valuative criterion of properness, respectively the points x and $g \cdot x$. Since \mathcal{G} acts equivariantly on \mathcal{L} , the multiplication by g induces an isomorphism of k° -modules

$$\varepsilon_x^* \mathcal{L} \xrightarrow{\sim} \varepsilon_{g \cdot x}^* \mathcal{L},$$

extending the isomorphism of k -vector spaces $x^* L \rightarrow (g \cdot x)^* L$. For a section $s \in x^* L$ let us write $g \cdot s$ its image in $(g \cdot x)^* L$. Since the isomorphism is defined at the level of k° -modules, if s_0 is a generator of the k° -module $\varepsilon_x^* \mathcal{L}$, then $g \cdot s_0$ generates the k° -module $\varepsilon_{g \cdot x}^* \mathcal{L}$. In particular, for every section $s \in x^* L$, we have

$$\|g \cdot s\|_{\mathcal{L}}(g \cdot x) = \|s\|_{\mathcal{L}}(x).$$

We are thus left with proving the inclusion

$$(5.1) \quad \text{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}}) \subset \mathcal{G}(k^\circ).$$

Since G is supposed to be quasi-split, it contains a Borel subgroup B and by the Iwasawa decomposition (Proposition 4.2 (3)), we have

$$G(k) = \mathcal{G}(k^\circ) \cdot B(k).$$

Therefore, in order to prove the inclusion (5.1), it suffices to prove the following :

LEMMA 5.1. — *With the notations just introduced, let \mathcal{B} be the unique Borel subgroup of \mathcal{G} lifting B . Then, we have*

$$\mathrm{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}}) \cap B(k) = \mathcal{B}(k^{\circ}).$$

Proof of Lemma 5.1. — Let us simplify the notation by writing S instead of $\mathrm{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}})$. Let \mathcal{T} be a maximal k° -torus of \mathcal{B} (it exists by Proposition 4.2 (1)) and $\mathrm{rad}^u(\mathcal{B})$ be the unipotent radical of \mathcal{B} . Let T and $\mathrm{rad}^u(B)$ be their generic fibres.

The inclusion $\mathcal{T} \subset \mathcal{B}$ induces an isomorphism $\mathcal{T} \simeq \mathcal{B}/\mathrm{rad}^u(\mathcal{B})$ [12, XXVI, Proposition 1.6]. Thanks to this identification, let us write $\pi: \mathcal{B} \rightarrow \mathcal{T}$ the quotient map.

CLAIM. — *We have the following equalities:*

$$S \cap T(k) = \mathcal{T}(k^{\circ}), \quad \pi(S \cap B(k)) = \mathcal{T}(k^{\circ}).$$

Proof of the Claim. — Since the stabiliser S is a bounded subgroup (Proposition 4.9), the subgroups $S \cap T(k)$, $\pi(S \cap B(k))$ of $T(k)$ are bounded too. Therefore they must be contained in $\mathcal{T}(k^{\circ})$ because the latter is the unique maximal bounded subgroup of $T(k)$ (Proposition 4.6).

On the other hand S contains $\mathcal{G}(k^{\circ})$ by hypothesis and thus it contains $\mathcal{T}(k^{\circ})$. So both $S \cap T(k)$ and $\pi(S \cap B(k))$ contain $\mathcal{T}(k^{\circ})$, whence the claim. \square

Since $B(k)$ is the semi-direct product of $\mathrm{rad}^u(B, k)$ and $T(k)$, in order to conclude the proof of Lemma 5.1, it is sufficient to prove the following:

LEMMA 5.2. — *With the notations introduced above, we have*

$$\mathrm{Stab}_{G(k)}(\|\cdot\|_{\mathcal{L}}) \cap \mathrm{rad}^u(B, k) = \mathrm{rad}^u(\mathcal{B}, k^{\circ}).$$

Proof of Lemma 5.2. — Let $s \in H^0(\mathcal{X}, \mathcal{L})$ be an eigenvector for B whose reduction \tilde{s} is non-zero. Then, by Proposition 4.4 we have

$$\mathrm{Opp}(\mathcal{B}, k^{\circ}) = \{x \in X(k) : \|s\|_{\mathcal{L}}(x) = 1\}.$$

Since the subgroup $S \cap \mathrm{rad}^u(B, k)$ fixes the metric, $\mathrm{Opp}(\mathcal{B}, k^{\circ})$ is stable under the action of $S \cap \mathrm{rad}^u(B, k)$.

On the other hand, we can identify in a \mathcal{B} -equivariant way $\mathrm{Opp}(\mathcal{B})$ with the unipotent radical of \mathcal{B} . To do this, let $\mathcal{B}^{\mathrm{op}}$ be the Borel subgroup of \mathcal{G} opposite to \mathcal{B} relatively to \mathcal{T} . Then the map $\mathrm{rad}^u(\mathcal{B}) \rightarrow \mathrm{Opp}(\mathcal{B})$ defined by $b \mapsto b\mathcal{B}^{\mathrm{op}}b^{-1}$ is an isomorphism [12, XXVI, Corollaire 4.3.5].

Through this identification, the action of $\mathrm{rad}^u(B)$ on $\mathrm{Opp}(\mathcal{B})$ becomes the action of $\mathrm{rad}^u(B)$ on itself by left multiplication. Moreover, saying that $\mathrm{Opp}(\mathcal{B}, k^{\circ})$ is stable under the action of $S \cap \mathrm{rad}^u(B)$ translates into

the the fact that the unipotent radical $\text{rad}^u(\mathcal{B}, k^\circ)$ is stable under the left multiplication by $S \cap \text{rad}^u(B)$. This obviously implies that $S \cap \text{rad}^u(B)$ is contained in $\text{rad}^u(\mathcal{B}, k^\circ)$, which concludes the proof of Lemma 5.2, thus of Lemma 5.1 and Theorem 1.2. \square

6. Reduction to the quasi-split case

In this section we deduce Theorem 1.1 when G is not quasi-split from Theorem 3.1. The reduction to the quasi-split case makes an essential use of the concept of holomorphically convex envelope, that we pass in review in the first paragraph.

6.1. Holomorphically convex envelopes

We briefly discuss holomorphically convex envelopes. The naive point of view we opt for, far from being well-suited to study holomorphically convex spaces, will suffice to draw the result that we are interested in (cf. Proposition 6.3).

DEFINITION 6.1. — *Let V be a affine k -scheme of finite type, $S \subset V(k)$ a bounded subset and K be an analytic extension of k . Let $K[V]$ be the K -algebra of regular functions on $V \times_k K$. Then, for every $f \in K[V]$, let us set*

$$\|f\|_S := \sup_{s \in S} |f(s)|.$$

The K -holomorphically convex envelope of S is the subset

$$\hat{S}_K := \{x \in V(K) : |f(x)| \leq \|f\|_S \text{ for all } f \in K[V]\}.$$

PROPOSITION 6.2. — *Let $f: V \rightarrow W$ be a closed immersion between affine k -schemes of finite type. Let $S \subset V(k)$ be a bounded subset and K an analytic extension of k . Then,*

$$f(\hat{S}_K) = \widehat{f(S)}_K.$$

The proof is left to reader as a direct consequence of the definitions.

PROPOSITION 6.3. — *Let \mathcal{H} be a smooth affine group k° -scheme with connected geometric fibres. Let us suppose that its special fibre \tilde{H} is unirational and that the residue field \tilde{k} is infinite.*

Then, for every analytic extension K of k , the K -holomorphically convex envelope of $\mathcal{H}(k^\circ)$ is $\mathcal{H}(K^\circ)$.

We will use the previous Proposition only when \mathcal{H} is a reductive k° -group: over a field reductive groups are indeed unirational varieties [9, Theorem 1.1] so that the hypotheses are fulfilled.

In order to prove Proposition 6.3, let H be the generic fibre of \mathcal{H} and let $K[H]$, $K^\circ[\mathcal{H}]$ be respectively the k -algebra of regular functions of $H \times_k K$ and the k° -algebra of regular functions on $\mathcal{H} \times_{k^\circ} K^\circ$. For every $f \in K[H]$ let us set

$$\|f\|_{K^\circ[\mathcal{H}]} := \inf\{|\lambda| : f/\lambda \in K^\circ[\mathcal{H}], \lambda \in K^\times\}.$$

The function $\|\cdot\|_{K^\circ[\mathcal{H}]}$ is a semi-norm on the K -algebra $K[H]$ and it takes values in $|K|$. This very last property is crucial for us and it is trivial if the valuation of K is discrete, while, when the valuation is dense, it is known to experts in non-archimedean geometry. A proof of this is given in Appendix A (cf. Proposition A.1) as I cannot point out a suitable reference. Coming back to the proof of Proposition 6.3, let us remark that we have

$$\mathcal{H}(K^\circ) = \{h \in H(K) : |f(h)| \leq \|f\|_{K^\circ[\mathcal{H}]} \text{ for all } f \in K[H]\},$$

so that it suffices to prove the following:

LEMMA 6.4. — *For every $f \in K[H]$ we have*

$$\|f\|_{K^\circ[\mathcal{H}]} = \|f\|_{\mathcal{H}(k^\circ)} := \sup_{h \in \mathcal{H}(k^\circ)} |f(h)|.$$

Proof of the Lemma. — Since the norm $\|\cdot\|_{K^\circ[\mathcal{H}]}$ takes values in $|K|$, we may assume $\|f\|_{K^\circ[\mathcal{H}]} = 1$. With this hypothesis for every point $h \in \mathcal{H}(k^\circ)$ we have $|f(h)| \leq 1$, thus proving the lemma amounts to find a k° -valued point h of \mathcal{H} such that $|f(h)| = 1$.

Let \tilde{H} be the special fibre of \mathcal{H} and let $\tilde{K}[\tilde{H}]$ be the \tilde{K} -algebra of regular functions on $\tilde{H}_{\tilde{K}} := \tilde{H} \times_{\tilde{k}} \tilde{K}$. With this notation f belongs to $K^\circ[\mathcal{H}]$ and its reduction $\tilde{f} \in \tilde{K}[\tilde{H}]$ is non-zero. Since the field \tilde{k} is infinite and \tilde{H} is supposed to be unirational, the set of \tilde{k} -rational points $\tilde{H}(\tilde{k})$ is Zariski-dense in $\tilde{H}_{\tilde{K}}$. Therefore there exists a \tilde{k} -rational point of \tilde{H} on which \tilde{f} does not vanish. Since \mathcal{H} is smooth, we can lift such a point to a point $h \in \mathcal{H}(k^\circ)$ by means of Hensel's Lemma. Clearly h is the point that we were looking for. \square

Remark 6.5. — In the proof of the preceding proposition we showed that $k^\circ[\mathcal{H}]$ is the k° -subalgebra of $k[H]$ made of regular functions f such that $|f(g)| \leq 1$ for all $h \in \mathcal{H}(k^\circ)$. Adopting the terminology of Bruhat–Tits [8, Définition 1.7.1], one would say that the k° -scheme \mathcal{H} is *étouffé*.

6.2. Proof of the Theorem

Let us complete the proof of Theorem 1.1 when G is not quasi-split. Let us recall that if \mathcal{G} is not quasi-split then the residue field \tilde{k} is necessarily infinite (see Proposition 4.1). Let us begin with the following technical result:

LEMMA 6.6. — *Let H be a bounded subgroup of $G(k)$. Then, there are an analytic extension K of k and a faithful representation $\rho: G_K \rightarrow \mathrm{GL}_{n,K}$ such that*

$$\rho(H) \subset \mathrm{GL}_n(K^\circ).$$

Moreover, if the valuation of k is discrete one can take $K = k$.

We postpone the proof of the previous Lemma to the end of the proof of Theorem 1.1. Let H be a bounded subgroup of $G(k)$ containing $\mathcal{G}(k^\circ)$ and let K and ρ be as in the statement of the previous lemma. Up to extending K we may suppose that \mathcal{G}_{K° is split.

Since ρ is a closed immersion, the K -holomorphically convex envelope of $\rho(H)$ coincides with $\rho(\hat{H}_K)$ (see Proposition 6.2). Therefore, by the preceding Lemma,

$$\rho(\hat{H}_K) \subset \widehat{\mathrm{GL}_n(K^\circ)}_K = \mathrm{GL}_n(K^\circ),$$

where the last equality follows from Proposition 6.3 applied to the group $\mathcal{H} = \mathrm{GL}_{n,K^\circ}$. We have therefore the following chain of inclusions :

$$\mathcal{G}(K^\circ) = \widehat{\mathcal{G}(k^\circ)}_K \subset \hat{H}_K \subset \rho^{-1}(\mathrm{GL}_n(K^\circ)),$$

where the first equality is given by Proposition 6.3 applied with $\mathcal{H} = \mathcal{G}$. Now we can conclude thanks to Theorem 1.1 in the split case: indeed, $\rho^{-1}(\mathrm{GL}_n(K^\circ))$ is a bounded subgroup containing $\mathcal{G}(K^\circ)$ and since \mathcal{G}_{K° is split by hypothesis, we have

$$\mathcal{G}(K^\circ) = \hat{H}_K = \rho^{-1}(\mathrm{GL}_n(K^\circ)),$$

which concludes the proof of Theorem 1.1. □

Let us finally prove Lemma 6.6:

Proof of Lemma 6.6. — Let us first suppose that the valuation of k is discrete. Let $\rho_0: G \rightarrow \mathrm{GL}_{n,k}$ be any faithful representation and let us consider $\mathcal{E}_0 := (k^\circ)^n$. Then, the k° -submodule of k^n ,

$$\mathcal{E} := \sum_{h \in H} h \cdot \mathcal{E}_0,$$

is bounded (as a subset of K^n) because H is bounded. In particular, there exists $\lambda \in k^\times$ such that $\mathcal{E} \subset \lambda \mathcal{E}_0$. Since k° is noetherian, every submodule of \mathcal{E}_0 is finitely generated. Thus \mathcal{E} is a torsion-free, finitely generated k° -module such that $\mathcal{E} \otimes_{k^\circ} k = k^n$ (it contains \mathcal{E}_0). In other words, \mathcal{E} is a lattice of k^n and thus there exists $g \in \mathrm{GL}_n(k)$ such that $g \cdot \mathcal{E} = \mathcal{E}_0$. One concludes by setting $\rho := g\rho_0g^{-1}$.

If the valuation is not discrete (or, more precisely, if the field k is not maximally complete) some further work is required because of the existence of norms that are not “diagonalisable”. Let $\rho_0: G \rightarrow \mathrm{GL}_{n,k}$ be any faithful representation as before, K a maximally complete extension of k and let us consider the norm on K^n ,

$$\|(x_1, \dots, x_n)\|_0 := \max\{|x_1|, \dots, |x_n|\}.$$

Since the subgroup H is bounded, the function

$$\|x\| := \sup_{h \in H} \|h \cdot x\|_0,$$

is real-valued and it is a norm on K^n verifying the non-archimedean triangle inequality. Since K is maximally complete, there exists a basis v_1, \dots, v_n of K^n and positive real-numbers r_1, \dots, r_n such that

$$\|x_1 v_1 + \dots + x_n v_n\| = \max\{r_1 |x_1|, \dots, r_n |x_n|\},$$

for all $x_1, \dots, x_n \in K$ [2, 2.4.1 Definition 1 and 2.4.4 Proposition 2]. Up to extending further K , we may assume that the real numbers r_1, \dots, r_n belong to the value group of K . Thus, up to rescaling the basis, we may suppose $r_i = 1$ for all i , so that the norm $\|\cdot\|$ is associated with a K° -lattice of K^n . One finishes the proof as in the discretely-valued case. \square

Appendix A. Semi-norm associated to an integral model

Let \mathcal{A} be a torsion-free k° -algebra of finite type and let $A := \mathcal{A} \otimes_{k^\circ} k$. Since \mathcal{A} is torsion-free, it injects in A and we shall freely consider it as a subset of A . For every $f \in A$ we set

$$\|f\|_{\mathcal{A}} := \inf\{|\lambda| : f/\lambda \in \mathcal{A} \text{ for all } \lambda \in k^\times\}.$$

PROPOSITION A.1. — *The semi-norm $\|\cdot\|_{\mathcal{A}}$ takes values in $|k|$.*

Since I am not able to point out a suitable reference, I sketch here a proof. Before giving the argument, let us fix some notation. Let \hat{A} be the completion of A with respect to the semi-norm $\|\cdot\|_{\mathcal{A}}$: we still denote by $\|\cdot\|_{\mathcal{A}}$ the semi-norm induced on \hat{A} . The completion $\hat{\mathcal{A}}$ of \mathcal{A} , seen as a k° -subalgebra of \hat{A} , verifies the following chain of inclusions:

$$\{f \in \hat{A} : \|f\|_{\mathcal{A}} < 1\} \subset \hat{\mathcal{A}} \subset \{f \in \hat{A} : \|f\|_{\mathcal{A}} \leq 1\}.$$

(A posteriori, once we know that the Proposition holds, the second inclusion will be an equality.)

When \mathcal{A} is the ring of polynomials $k^\circ[t_1, \dots, t_n]$, the semi-norm $\|\cdot\|_{\mathcal{A}}$ is the Gauss norm on polynomials: explicitly, for a polynomial f of the form $\sum_{\alpha \in \mathbf{N}^n} f_\alpha t_1^{\alpha_1} \cdots t_n^{\alpha_n}$, we have

$$\|f\|_{\mathcal{A}} = \max_{\alpha \in \mathbf{N}^n} |f_\alpha|.$$

Thus the completion \hat{A} is the so-called *Tate algebra* $k\{t_1, \dots, t_n\}$ and the semi-norm $\|\cdot\|_{\mathcal{A}}$ takes values in $|k|$.

Proof. — The statement is trivial if the valuation is discrete, so let us suppose that the valuation is dense. Let $\phi: \mathcal{T} = k^\circ[t_1, \dots, t_n] \rightarrow \mathcal{A}$ be a surjective homomorphism of k° -algebras. We adopt for \mathcal{T} notations similar to the ones for \mathcal{A} . The homomorphism ϕ induces a surjective⁽⁹⁾ and bounded⁽¹⁰⁾ homomorphism of k -Banach algebras,

$$\hat{\phi}: \hat{\mathcal{T}} \longrightarrow \hat{\mathcal{A}}.$$

The open mapping theorem shows that the norm $\|\cdot\|_{\mathcal{T}}$ attains a minimum on the subset made of elements $g \in \hat{\mathcal{T}}$ such that $\hat{\phi}(g) = f$ [2, 1.1.5 Definition 1 and 5.2.7 Theorem 7]. If such a minimum is attained in g_0 , it suffices to show

$$\|f\|_{\mathcal{A}} = \|g_0\|_{\mathcal{T}}.$$

⁽⁹⁾ Because of the equalities $\hat{\mathcal{T}} = \hat{\mathcal{T}} \otimes_{k^\circ} k$ and $\hat{\mathcal{A}} = \hat{\mathcal{A}} \otimes_{k^\circ} k$, it suffices to show that the induced homomorphism $\phi: \hat{\mathcal{T}} \rightarrow \hat{\mathcal{A}}$ is surjective.

Let $\lambda \in k$ be a non-zero element such that $|\lambda| < 1$. For every positive integer n let us set $\Lambda_n := k^\circ/\lambda^n k^\circ$, $\mathcal{A}_n := \mathcal{A} \otimes_{k^\circ} \Lambda_n$ and $\mathcal{T}_n := \mathcal{T} \otimes_{k^\circ} \Lambda_n$. The completion $\hat{\mathcal{T}}$ (resp. $\hat{\mathcal{A}}$) is naturally identified with the projective limit of the \mathcal{T}_n 's (resp. of the \mathcal{A}_n 's). For every n , let \mathcal{I}_n be the kernel of the surjective homomorphism $\mathcal{T}_n \rightarrow \mathcal{A}_n$ induced by ϕ . Then the exact sequence of projective systems,

$$0 \longrightarrow (\mathcal{I}_n)_n \longrightarrow (\mathcal{T}_n)_n \longrightarrow (\mathcal{A}_n)_n \longrightarrow 0,$$

satisfies the Mittag-Leffler condition (even better, for every n the map $\mathcal{I}_{n+1} \rightarrow \mathcal{I}_n$ is surjective). Therefore, the induced map between projective limits $\hat{\mathcal{T}} \rightarrow \hat{\mathcal{A}}$ is surjective. See [17, Chapter 1, Lemma 3.1 and Exercise 3.15].

⁽¹⁰⁾ That is, for every $f \in \hat{\mathcal{T}}$, we have $\|\phi(f)\|_{\mathcal{A}} \leq \|f\|_{\mathcal{T}}$.

The inequality $\|f\|_{\mathcal{A}} \leq \|g_0\|_{\mathcal{T}}$ is clear because of the boundedness of the homomorphism $\hat{\phi}$. Let us suppose by contradiction $\|f\|_{\mathcal{A}} < \|g_0\|_{\mathcal{T}}$. Up to rescaling g_0 we may suppose $\|g_0\|_{\mathcal{T}} = 1$ (it is crucial here $\|\cdot\|_{\mathcal{T}}$ takes values in $|k|$). By density of the valuation, there exists $\lambda \in k$ such that $|\lambda| > 1$ and $\|\lambda f\|_{\mathcal{A}} < 1$ hence λf belongs to \mathcal{A} . Since $\hat{\phi}$ is surjective, there exists $g_1 \in \hat{\mathcal{T}}$ such that $\phi(g_1) = \lambda f$. Therefore, $\phi(g_1/\lambda) = f$ and

$$\|g_1/\lambda\|_{\mathcal{T}} < \|g_1\|_{\mathcal{T}} \leq 1,$$

contradicting the minimality of g_0 . □

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