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CORRIGENDUM TO "HOMOLOGY OF ORIGAMIS WITH SYMMETRIES"

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As it was kindly pointed out to us by Simion Filip, in Proposition 3.18 of [1] we gave a wrong description of the group $Sp(W_a)$ when a is quaternionic.

More precisely, all facts claimed in [1] during the proof of this proposition (from pages 1149 to 1153) are correct *except* for the penultimate phrase of the argument:

> "Moreover A preserves the symplectic form on W_a iff $\iota^{-1} \circ A \circ \iota$ preserves the hermitian form
>
> $$\sum_{1}^{p} \langle v_m, v'_m \rangle - \sum_{p+1}^{p+q} \langle v_m, v'_m \rangle$$
>
> on V^{ℓ_a}."

As we are going to see in Section 1 below, after correcting the error pointed out above, one has that, when a is quaternionic, the group $Sp(W_a)$ is not isomorphic to the unitary group $U_{\mathbf{H}}(p,q)$ (as it was previously claimed in [1]), but rather to the group $O(\ell_a, \mathbf{H})$ of matrices (with coefficients in $\mathbf{H}$) satisfying $A^\sharp A = I$, where $A^\sharp$ is the transpose matrix of $\sigma(A)$ and $\sigma(a+bi+cj+dk) = a - bi + cj + dk$ is a reversion on $\mathbf{H}$.

In particular, besides the discussion of $Sp(W_a)$ when a is quaternionic undertaken in pages 1149 to 1153 in [1], all statements in [1] which were based on the description of $Sp(W_a)$ in Proposition 3.18 in [1] must be changed. More concretely:

— at page 1135, lines 6 and 7, the phrase "a quaternionic unitary group $U_{\mathbf{H}}(p,q)$ in the quaternionic case" must be replaced by "a quaternionic orthogonal group $O(\ell,\mathbf{H})$ in the quaternionic case";

— at page 1135, the second item of Theorem 1.4 must be replaced by "When V_a is complex, resp. quaternionic, and $Sp(W_a)$ is isomorphic to an unitary group $U_{\mathbf{C}}(p,q)$, resp. $O(\ell,\mathbf{H})$ with ℓ odd, the multiplicity in W_a of the exponent 0 is at least $|q-p|\dim_{\mathbf{R}}(V_a)$, resp. $\dim_{\mathbf{R}}(V_a)$;

— at page 1158, lines −5 to −3 must be replaced by: "Assume that a is *complex*, resp. *quaternionic*. From Propositions 3.17, resp. the new version of Proposition 3.18, there exists nonnegative integers p,q with $p+q=\ell_a$ such that $Sp(W_a)$ is isomorphic to $U_{\mathbf{C}}(p,q)$, resp. $Sp(W_a)$ is isomorphic to $O(\ell_a,\mathbf{H})$.";

— at page 1158, the statement of Proposition 4.11 must be replaced by "If a is complex, the multiplicity of the exponent 0 in W_a is at least $|q-p|\dim_{\mathbf{R}}(V_a)$. If a is quaternionic and ℓ_a is odd, the multiplicity of the exponent 0 in W_a is at least $\dim_{\mathbf{R}}(V_a)$.";

— at page 1159, the proof of Proposition 4.11 remains unchanged in the complex case, but in the case of a quaternionic and ℓ_a odd, the new argument is the following. "By Corollary 4.10, the list of Lyapunov exponents in W_a has the form $\theta_1\geqslant\cdots\geqslant\theta_{\ell_a}$ where each θ_m appears with multiplicity $\dim_{\mathbf{R}}(V_a)$. By symplecticity, the list $\theta_1\geqslant\cdots\geqslant\theta_{\ell_a}$ is symmetric with respect to 0. Therefore, $\theta_{(\ell_a+1)/2}=0$ when ℓ_a is odd.";

— at page 1173, the end of the phrase in Remark 5.13 must be simply "… isotypical components of complex type" (instead of "… isotypical components of complex or quaternionic type") because there is no need for discussing the signature of Hermitian form in the quaternionic case.

After these preliminaries, we complete this note by explaining how to change the statement and proof of Proposition 3.18 in [1] to get a correct description of $Sp(W_a)$ when a is quaternionic.

1. Corrections to the statement and proof of Proposition 3.18 in [1]

For the sake of convenience of the reader, instead of just giving a list of punctual changes, we indicated below how to completely rewrite the content of pages 1149 to 1153 in [1] in order to obtain a correct description of $Sp(W_a)$ when a is quaternionic.

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* *

Consider finally the case where a is quaternionic. We fix an isomorphism between D_a and $\mathbf{H}$. We equip V_a with the structure of a right vector space over $\mathbf{H}$ by setting

$$vz = \bar{z}v, \quad z \in \mathbf{H},\, v \in V_a.$$

Here, $\bar{z} = a - bi - cj - dk$ is the conjugate of the quaternion $z = a + bi + cj + dk$.

Recall that an hermitian form on a right vector space V over $\mathbf{H}$ is a map $H : V \times V \to \mathbf{H}$ which satisfies

$$\begin{aligned} H(v, v_1z_1 + v_2z_2) &= H(v, v_1)z_1 + H(v, v_2)z_2, & \forall\, v, v_1, v_2 \in V,\, z_1, z_2 \in \mathbf{H}, \\ H(v, v') &= \overline{H(v', v)}, & \forall\, v, v' \in V. \end{aligned}$$

Writing $H = H_0 + H_ii + H_jj + H_kk$, the $\mathbf{R}$-bilinear form H_0 is symmetric, and the $\mathbf{R}$-bilinear forms H_i, H_j, H_k are alternate. They are related by

$$H_0(v, v') = H_i(v, v'i) = H_j(v, v'j) = H_k(v, v'k).$$

For $v, v' \in V$, $z \in \mathbf{H}$, we have $H(vz, v'z) = \bar{z}H(v, v')z$, hence

$$\begin{aligned} H_i(vi, v'i) &= H_i(v, v'), \\ H_i(vj, v'j) &= -H_i(v, v'), \\ H_i(vk, v'k) &= -H_i(v, v'). \end{aligned}$$

Equivalently, one has, for $v, v' \in V, z \in \mathbf{H}$

$$H_i(v, v'z) = H_i(v\sigma(z), v'),$$

where σ is the reversion $\sigma(a + bi + cj + dk) = a - bi + cj + dk$.

Conversely, if a $\mathbf{R}$-bilinear alternate form H_i satisfy these relations, one defines an hermitian form by

$$H(v, v') = H_i(v, v'i) + H_i(v, v')i + H_i(v, v'k)j - H_i(v, v'j)k.$$

On the irreducible $\mathrm{Aut}(M)$-module V_a, there exists, up to a positive real scalar, a unique positive definite $\mathrm{Aut}(M)$-invariant hermitian form $\langle , \rangle$ (obtained as usual by averaging over the group an arbitrary positive definite hermitian form). The space of $\mathrm{Aut}(M)$-invariant $\mathbf{R}$-bilinear forms on V_a is 4-dimensional, generated by the four components of $\langle\, ,\rangle$.

Proposition 3.18. — *There exists an isomorphism of* Aut(M)*-modules* $\iota : V_a^{\ell_a} \to W_a$ *such that the bilinear form* $\{\iota(v_1, \dots, v_{\ell_a}), \iota(v'_1, \dots, v'_{\ell_a})\}$ *is the* i*-component of the hermitian form* $\sum_1^{\ell_a} \langle v_m, v'_m \rangle$.

An element of $Sp(W_a)$ *is of the form* $\iota^{-1} \circ A \circ \iota(v_1, \dots, v_{\ell_a}) = (V_1, \dots, V_{\ell_a})$ *with* $V_m = \sum_n v_n a_{n,m}$, $a_{n,m} \in \mathbf{H}$. *The map* $A \mapsto (a_{m,n})$ *is an isomorphism from* $Sp(W_a)$ *onto the group* $O(\ell_a, \mathbf{H})$ *of matrices satisfying* $A^\sharp A = I$, *where* $A^\sharp$ *is the transpose matrix of* $\sigma(A)$.

The proof of this proposition relies on the following three lemmas:

Lemma 3.19. — *Let* G *be a finite group, and let* W *be an isotypic* G*-module of quaternionic type. Let* B *be an alternate non-degenerate* G*-invariant bilinear form on* W. *Then, there exists a non-zero vector* $v \in W$ *and* $g \in G$ *such that* $B(v, g.v) \neq 0$.

Proof. — Assume that the conclusion of the lemma does not hold. Then, one has $B(v, g.v') + B(v', g.v) = 0$ for all $v, v' \in W$. As B is alternate and G-invariant, one has $B(v, g^2.v') = B(v, v')$ for all $v, v' \in W$, $g \in G$. As B is non-degenerate, this implies that $g^2.v' = v'$ for all $v' \in W$, $g \in G$. Thus G acts through a group where all non trivial elements are of order 2. Such a group is abelian and W cannot be quaternionic. □

Lemma 3.20. — *Under the hypotheses of the lemma above, one can write*

$$W = V_1 \oplus \dots \oplus V_\ell,$$

where $V_1, \dots, V_\ell$ *are irreducible* G*-modules which are orthogonal for* B.

Proof. — This is an immediate induction on the multiplicity ℓ of W. From the lemma above, one can find $v \in W$ such that the restriction of B to the irreducible G-module V_1 generated by v is nonzero. Because V_1 is irreducible and B is G-invariant, this restriction is non-degenerate. Then, the B-orthogonal W' of V_1 in W is G-invariant and satisfies $W = V_1 \oplus W'$. We conclude by applying to W' the induction hypothesis. □

Lemma 3.21. — *Let* b *be an alternate* Aut(M)*-invariant nonzero* $\mathbf{R}$*-bilinear form on* V_a. *There exists* $u \in \mathbf{H}^*$ *such that the form* $b_u(v, v') := b(vu, v'u)$ *is the* i*-component* B_i *of* $\langle \, , \rangle$.

Proof. — Any nonzero alternate Aut(M)-invariant $\mathbf{R}$-bilinear form b on V_a is non-degenerate. This allows to define an adjoint map $\sigma_b : \mathbf{H} \to \mathbf{H}$ through $b(v, v'a) = b(v\sigma_b(a), v')$ (the $\mathbf{R}$-endomorphism $\sigma_b(a)$ of V_a belongs to $\mathbf{H}$ as it commutes with the action of Aut(M)). The map σ_b is a $\mathbf{R}$-linear involution satisfying $\sigma_b(aa') = \sigma_b(a')\sigma_b(a)$ and $\sigma_b(a) = a$ for

$a \in \mathbf{R}$. Therefore σ_b preserves the set of quaternions a such that $a^2 = -1$, which is nothing else than the purely imaginary quaternions of norm 1. Observe that, for the i-component B_i of the hermitian scalar product on V_a, one has that σ_{B_i} is the reversion σ.

As any nonzero alternate $\mathrm{Aut}(M)$-invariant $\mathbf{R}$-bilinear form b on V_a is a linear combination of the imaginary components B_i, B_j, B_k of $\langle\,,\rangle$ and thus can be deformed to B_i through nonzero alternate $\mathrm{Aut}(M)$-invariant $\mathbf{R}$-bilinear forms, we conclude that:

— there exists a unique (up to sign) quaternion a_0 of norm 1 such that $\sigma_b(a_0) = -a_0$ and, moreover, a_0 is purely imaginary;
— σ_b is the identity on a hyperplane H of $\mathbf{H}$ containing $\mathbf{R}$ and the purely imaginary quaternions in H are those satisfying $za_0 + a_0 z = 0$.

Next we relate, for $\bar{u}u = 1$, σ_{b_u} to σ_b. For $v, v' \in V_a$, we have

$$\begin{aligned} b_u(v, v'a) &= b(vu, v'au) \\ &= b(vu, v'u\bar{u}au) \\ &= b(vu\sigma_b(\bar{u}au), v'u) \\ &= b(vu\sigma_b(\bar{u}au)\bar{u}u, v'u) \\ &= b_u(vu\sigma_b(\bar{u}au)\bar{u}, v'), \end{aligned}$$

and thus $\sigma_{b_u}(a) = u\sigma_b(\bar{u}au)\bar{u}$. Hence, $\sigma_{b_u} = \sigma_{B_i}$ iff $\bar{u}iu = \pm a_0$. As a_0 is purely imaginary of norm 1, it is possible to choose u such that $\bar{u}iu = a_0$. Then we have $\sigma_{b_u}(i) = -i$ and therefore $\sigma_{b_u} = \sigma$.

It follows that there exists $c \in \mathbf{R}^*$ such that b_u is the i-component of $c\langle\,,\rangle$. Replacing u by $c^{-1/2}u$ if $c > 0$, by $|c|^{-1/2}ju$ if $c < 0$, we obtain the required conclusion. □

Proof of Proposition 3.18. — According to Lemma 3.20 above, one can choose an isomorphism of $\mathrm{Aut}(M)$-modules $\iota_0 : V_a^{\ell_a} \to W_a$ such that the symplectic form is written in a diagonal way as

$$\{\iota_0(v_1, \dots, v_{\ell_a}), \iota_0(v'_1, \dots, v'_{\ell_a})\} = \sum_1^{\ell_a} b^{(m)}(v_m, v'_m),$$

for some alternate nonzero $\mathrm{Aut}(M)$-invariant $\mathbf{R}$-bilinear forms $b^{(m)}$ on V_a. According to Lemma 3.21, one can find nonzero quaternions $u_1, \dots, u_{\ell_a}$ such that, setting $\iota(v_1, \dots, v_{\ell_a}) = \iota_0(v_1 u_1, \dots, v_{\ell_a} u_{\ell_a})$, the alternate form

$$\{\iota(v_1, \dots, v_{\ell_a}), \iota(v'_1, \dots, v'_{\ell_a})\}$$

is the i-component $\sum_1^{\ell_a} B_i(v_m, v'_m)$ of the hermitian form $\sum_1^{\ell_a} \langle v_m, v'_m\rangle$, which proves the first assertion of the proposition. For the second assertion, any automorphism A of the $\mathrm{Aut}(M)$-module W_a is of the form $\iota^{-1} \circ A \circ \iota(v_1, \ldots, v_{\ell_a}) = (V_1, \ldots, V_{\ell_a})$ with $V_m = \sum_n v_n a_{n,m}$, $a_{n,m} \in \mathbf{H}$. Moreover we have

$$\begin{aligned}\sum_m B_i(V_m, V'_m) &= \sum_m \sum_n \sum_{n'} B_i(v_n a_{n,m}, v'_{n'} a_{n',m}) \\ &= \sum_n \sum_{n'} B_i(v_n \sum_m a_{n,m} \sigma(a_{n',m}), v'_{n'})\end{aligned}$$

This is equal to $\sum_m B_i(v_m, v'_m)$ for all v_m, v'_m iff one has, for all n, n'

$$\sum_m a_{n,m} \sigma(a_{n',m}) = \delta_{n,n'}.$$

This means that the matrix associated to A satisfies $AA^\sharp = I$. □

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[1] C. Matheus, J.-C. Yoccoz & D. Zmiaikou, “Homology of origamis with symmetries”, *Ann. Inst. Fourier (Grenoble)* **64** (2014), no. 3, p. 1131-1176.

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