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ON COTANGENT MANIFOLDS, COMPLEX STRUCTURES AND GENERALIZED GEOMETRY

by Liana DAVID (*)

Abstract. — We develop various properties of symmetric generalized complex structures (in connection with their holomorphic space and B-field transformations), which are analogous to the well-known results of Gualtieri on skew-symmetric generalized complex structures. Given an adapted (symmetric or skew-symmetric) generalized complex structure $\mathcal{J}$ and a linear connection D on a manifold M, we construct an almost complex structure $J^{\mathcal{J},D}$ on the cotangent manifold T^*M and we study its integrability. For $\mathcal{J}$ skew-symmetric, we relate the Courant integrability of $\mathcal{J}$ with the integrability of $J^{\mathcal{J},D}$. We consider in detail the case when $M=G$ is a Lie group and $\mathcal{J}$, D are left-invariant. We also show that our approach unifies and generalizes various known results from special complex geometry.

Résumé. — Nous développons plusieurs propriétés des structures complexes généralisées symétriques (en relation avec leurs espaces holomorphes et les "B-field" transformations), qui sont similaires aux résultats bien connus de Gualtieri sur les structures complexes généralisées anti-symétriques. Étant données une structure complexe généralisée adaptée (symétrique ou anti-symétrique) $\mathcal{J}$ et une connexion D sur une variété M, nous construisons une structure presque-complexe $J^{\mathcal{J},D}$ sur la variété cotangente T^*M et nous étudions son intégrabilité. Pour $\mathcal{J}$ anti-symétrique, nous relions l'intégrabilité de $\mathcal{J}$ au sens de Courant avec l'intégrabilité de $J^{\mathcal{J},D}$. Nous considérons en détail le cas où $M=G$ est un groupe de Lie et $\mathcal{J}, D$ sont invariants à gauche. Nous montrons aussi que notre approche unifie et généralise plusieurs résultats connus en géométrie spéciale complexe.

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1. Introduction

1.1. Motivation

The starting point of this note is a result proved in [1], which states that the cotangent manifold of a special symplectic manifold (M, J, ∇, ω) inherits, under some additional conditions, a hyper-Kähler structure. Recall that a manifold M with a complex structure J, a flat, torsion-free connection ∇ and a symplectic form ω is special symplectic if $d^{\nabla} J = 0$ (i.e. $\nabla_X(J)(Y) = \nabla_Y(J)(X)$, for any $X, Y \in TM$) and $\nabla\omega = 0$. The connection ∇, acting on the cotangent bundle $\pi : T^*M \to M$, induces a decomposition

$$T(T^*M) = H^{\nabla} \oplus \pi^*(T^*M) = \pi^*(TM \oplus T^*M) \tag{1.1}$$

into horizontal and vertical subbundles. Assume now that the $(1,1)$-part of ω (with respect to J) is non-degenerate and satisfies $\nabla\omega^{1,1} = 0$. Under these additional conditions, the hyper-Kähler structure on T^*M mentioned above is given, by means of (1.1), by (the pull-back of)

$$J_1 := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix}, \; J_2 := \begin{pmatrix} 0 & -(\omega^{1,1})^{-1} \\ \omega^{1,1} & 0 \end{pmatrix}, \; g := \begin{pmatrix} g^{1,1} & 0 \\ 0 & (g^{1,1})^{-1} \end{pmatrix}$$

where $g^{1,1} := \omega^{1,1}(J\cdot, \cdot)$. A key fact in the proof that (J_1, J_2, g) is hyper-Kähler is the integrability of J_1 and J_2. The integrability of J_2 follows from a local argument, which uses ∇-flat coordinates and $\nabla\omega^{1,1} = 0$. For the integrability of J_1, one notices, using the special complex condition $d^{\nabla} J = 0$, that $H^{\nabla} \subset T(T^*M)$ is invariant with respect to the canonical complex structure J_{can} of T^*M induced by J. Hence, J_1 coincides with J_{can} and is integrable. These arguments were developed in [1].

With special geometry as a motivation, in this note we consider the following setting: a manifold M with a linear connection D and a smooth field of endomorphisms $\mathcal{J}$ of the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, such that $\mathcal{J}^2 = -\mathrm{Id}$. Following [10] (rather than the usual terminology from generalized geometry), we call $\mathcal{J}$ a generalized complex structure. Motivated by J_1 and J_2 above, we assume that $\mathcal{J}$ is adapted, *i.e.* either symmetric or skew-symmetric with respect to the canonical metric of neutral signature of $\mathbb{T}M$. From D and $\mathcal{J}$ we construct an almost complex structure $J^{\mathcal{J},D}$ on the cotangent manifold T^*M and we study its integrability. This provides a new insight, from the generalized complex geometry point of view, on the above arguments from [1]. Along the way, we prove various properties we need on symmetric generalized complex structures. The relation with the Courant integrability is also discussed. As a main application, we construct a large class of complex structures on cotangent manifolds of real semisimple Lie groups.

In the remaining part of the introduction we describe in detail the results and the structure of the paper.

1.2. Structure of the paper

In Section 2 we prove basic facts we need from generalized geometry. While skew-symmetric generalized complex structures are well-known (see e.g. [6] for basic facts), the symmetric ones do not seem to appear in the literature. We begin by studying symmetric generalized complex structures on (real) vector spaces (see Definition 2.1). We find the general form of their holomorphic space (see Proposition 2.4) and we show that any symmetric generalized complex structure on a vector space is, modulo a B-field transformation, the direct sum of one determined by a complex structure and another determined by a pseudo-Euclidean metric (see Example 2.6 and Theorem 2.7). Therefore, there is an obvious analogy with the theory of skew-symmetric generalized complex structures developed by Gualtieri in [6]. We discuss this analogy in Subsection 2.2. For our purposes it is particularly relevant the common description of the holomorphic space $L^{\tau}(E,\alpha)$ of a symmetric or, respectively, skew-symmetric generalized complex structure on a vector space V, in terms of a complex subspace $E\subset V^{\mathbb{C}}$ and a skew-Hermitian, respectively skew-symmetric 2-form α on E, satisfying some additional conditions (see Corollary 2.8). These results extend pointwise to manifolds (see Subsection 2.3). Despite the above analogies, there is an important difference between symmetric and skew-symmetric generalized complex structures on manifolds: unlike the skew-symmetric ones, the symmetric generalized complex structures are never Courant integrable (see Lemma 2.13).

In Section 3 we consider a manifold M together with a connection D and an adapted generalized complex structure $\mathcal{J}$ (see Definition 2.9). Using D and $\mathcal{J}$ we define an almost complex structure $J^{\mathcal{J},D}$ on $T^{*}M$ (see Definition 3.1) and we discuss its integrability. It turns out that the integrability of $J^{\mathcal{J},D}$ imposes obstructions on the curvature of D and the data (E,α) which defines the holomorphic bundle $L=L^{\tau}(E,\alpha)$ of $\mathcal{J}$. In particular, the complex subbundle $E\subset T^{\mathbb{C}}M$ must be involutive and α must satisfy a differential equation involving D (see Theorem 3.3). As a straightforward application of Theorem 3.3, we relate the Courant integrability of a skew-symmetric generalized complex structure $\mathcal{J}$ with the integrability of $J^{\mathcal{J},D}$ (see Corollary 3.4). In particular, we deduce that a left-invariant, skew-symmetric, generalized complex structure $\mathcal{J}$ on a Lie group G is Courant integrable, if and only if the almost complex structure $J^{\mathcal{J},D^c}$ on T^*G is integrable, where D^c is the left-invariant connection on G which on left-invariant vector fields is the Lie bracket (see Example 3.5). A systematic description of Courant integrable, left-invariant, skew-symmetric generalized complex structures on real semisimple Lie groups was developed in [2]. This is the motivation for our treatment from the next section.

Section 4 is devoted to applications of Theorem 3.3 to Lie groups. Our main goal here is to describe a large class of left-invariant symmetric (rather than skew-symmetric) generalized complex structures $\mathcal{J}$ on a semisimple Lie group, which, together with a suitably chosen left-invariant connection D^0, determine an integrable complex structure $J^{\mathcal{J},D^0}$ on the cotangent group. (The connection D^0 plays the role of D^c above). In the first part of Section 4, intended to fix notation, we briefly recall the basic facts we need on the structure theory of semisimple Lie algebras. We follow closely [9], Chapter VI. In Subsection 4.2 we develop an infinitesimal description, in terms of the so-called admissible triples $(\mathfrak{k}, \mathcal{D}, \epsilon)$, of pairs $(\mathcal{J}, D)$ formed by a left-invariant adapted generalized complex structure $\mathcal{J}$ and a left-invariant connection D on a (not necessarily semisimple) Lie group G, such that the associated almost complex structure $J^{\mathcal{J},D}$ on T^*G is integrable (see Definition 4.2 and Proposition 4.3). In this description, the pair $(\mathfrak{k}, \epsilon)$ defines the fiber $L^{\tau}(\mathfrak{k}, \epsilon)$ at $e \in G$ of the holomorphic bundle of $\mathcal{J}$ and $\mathcal{D}$ is the restriction of D to the space of left-invariant vector fields. The notion of admissible triple generalizes the notion of admissible pair, defined in [2] to encode the Courant integrability of left-invariant skew-symmetric generalized complex structures on Lie groups. When G is semisimple, we define the notion of regularity for the structures involved (see Definition 4.2); in the above notation, this means that $\mathfrak{k}$ is a regular subalgebra of $\mathfrak{g}^{\mathbb{C}}$, normalized by a maximally compact Cartan subalgebra of $\mathfrak{g}$. The preferred connection D^0 is introduced in Definition 4.8 and our motivation for its choice is explained before Lemma 4.7. Our main result in this section is Theorem 4.9, which provides a description (in terms of admissible triples) of regular symmetric generalized complex structures $\mathcal{J}$ on a semisimple Lie group G, which, together with the connection D^0, determine an (integrable) complex structure on T^*G. The description from Theorem 4.9 requires further clarifications: one needs to construct the constants $\{\nu_{\alpha},\ \alpha \in R_0^{\mathrm{sym}}\}$, which are subject to conditions (4.20), (4.21) and to study the non-degeneracy of the (symmetric) bilinear form g_{Δ}. A method to construct the ν_{α}'s is provided by Lemma 4.10. When the root system R_0 of the regular subalgebra $\mathfrak{k}$ is not only σ-parabolic, as required by Theorem 4.9, but σ-positive (see Definition 4.5), the non-degeneracy of g_{Δ} is straightforward (see Remark 4.11) and we obtain, on any semisimple Lie group G, a large class of regular symmetric generalized complex structures $\mathcal{J}$, such that $J^{\mathcal{J},D^0}$ is integrable. In the special case when G is (semisimple) of inner type, the root system R_0 of $\mathfrak{k}$ is always a positive root system and we obtain a full explicit description of all regular symmetric generalized complex structures $\mathcal{J}$, such that $J^{\mathcal{J},D^0}$ is integrable (see Theorem 4.14).

In Section 5 we use Theorem 3.3 in order to reobtain and generalize various well-known results from special complex geometry, with emphasis on those from [1], already mentioned at the beginning of this introduction.

2. Symmetric generalized complex structures

In this section we study symmetric generalized complex structures. Subsections 2.1 and 2.2 are algebraic, while in Subsection 2.3 we discuss the Courant integrability.

2.1. Linear symmetric generalized complex structures

Let V be a real vector space. We denote by

$$g_{\mathrm{can}}(X+\xi, Y+\eta) = \frac{1}{2}\left(\xi(Y)+\eta(X)\right), \quad X+\xi, Y+\eta \in V \oplus V^* \tag{2.1}$$

the canonical pseudo-Euclidian metric of neutral signature on $V \oplus V^*$.

Definition 2.1. — *A generalized complex structure on V is an endomorphism $\mathcal{J} \in \mathrm{End}(V \oplus V^*)$, such that $\mathcal{J}^2 = -\mathrm{Id}$. The generalized complex structure $\mathcal{J}$ is called symmetric (respectively, skew-symmetric) if it is symmetric (respectively, skew-symmetric) with respect to g_{can}. The generalized complex structure $\mathcal{J}$ is called adapted if it is symmetric or skew-symmetric.*

Remark 2.2. — In the classical terminology of generalized geometry (see e.g. [6, 8]), a generalized complex structure is, by definition, skew-symmetric. In this note we prefer the language of [10], where generalized complex structures are not assumed, a priori, to be compatible in any way with g_{can}.

In the following proposition we describe the holomorphic space of symmetric generalized complex structures. Before we need to introduce a notation which will be used along the paper.

Notation 2.3. — For a complex subspace $E \subset V^{\mathbb{C}}$, we denote by $\bar{E}$ the image of E through the antilinear conjugation $V^{\mathbb{C}} \ni X \to \bar{X} \in V^{\mathbb{C}}$ with respect to the real form V of $V^{\mathbb{C}}$. In particular, $\bar{E}$ is a complex subspace of $V^{\mathbb{C}}$ (not to be confused with the conjugate vector space of E).

Proposition 2.4. — *A complex subspace L of $(V \oplus V^*)^{\mathbb{C}}$ is the holomorphic space of a symmetric generalized complex structure on V if and only if it is of the form*

$$L = L^-(E, \alpha) := \{X + \xi \in E \oplus (V^{\mathbb{C}})^*, \quad \xi|_{\bar{E}} = i_X \alpha\}, \tag{2.2}$$

where E is any complex subspace of $V^{\mathbb{C}}$, such that $E + \bar{E} = V^{\mathbb{C}}$, and $\alpha \in E^* \otimes \bar{E}^*$ is any complex bilinear form satisfying the following two conditions:

i) it is skew-Hermitian, i.e.

$$\alpha(X, \bar{Y}) + \overline{\alpha(Y, \bar{X})} = 0, \quad \forall X, Y \in E; \tag{2.3}$$

ii) $\mathrm{Im}(\alpha|_{\Delta})$ *is non-degenerate. Here $\Delta \subset V$ is the real part of $E \cap \bar{E}$, i.e. $\Delta^{\mathbb{C}} = E \cap \bar{E}$.*

Proof. — Let $\mathcal{J}$ be a symmetric generalized complex structure on V, with holomorphic space L. Thus L is a complex subspace of $(V \oplus V^*)^{\mathbb{C}}$, with $L \oplus \bar{L} = (V \oplus V^*)^{\mathbb{C}}$, and L is g_{can}-orthogonal to $\bar{L}$ (from the symmetry of $\mathcal{J}$). We denote by

$$\pi_1 : (V \oplus V^*)^{\mathbb{C}} \to V^{\mathbb{C}}, \quad \pi_2 : (V \oplus V^*)^{\mathbb{C}} \to (V^{\mathbb{C}})^*$$

the natural projections. We define $E := \pi_1(L)$ and we let

$$\alpha : E \to \bar{E}^*, \quad \alpha(X) := \pi_2 \circ (\pi_1|_L)^{-1}(X)|_{\bar{E}}. \tag{2.4}$$

We claim that $\alpha \in E^* \otimes \bar{E}^*$ is well defined. To prove this claim, we use

$$\xi(\bar{Y}) + \bar{\eta}(X) = 0, \quad \forall X + \xi, Y + \eta \in L, \tag{2.5}$$

(which holds because L is g_{can}-orthogonal to $\bar{L}$). Thus, if $X + \xi_1, X + \xi_2 \in (\pi_1|_L)^{-1}(X)$, i.e. $X + \xi_1, X + \xi_2 \in L$, then, from (2.5), $\xi_1 = \xi_2$ on $\bar{E}$ and we obtain that α is well-defined, as required. From the very definition of α, $L \subset L^-(E, \alpha)$ and, being of the same dimension, we deduce that $L = L^-(E, \alpha)$. Since L is g_{can}-orthogonal to $\bar{L}$, α is skew-Hermitian. Moreover, $L \oplus \bar{L} = (V \oplus V^*)^{\mathbb{C}}$ implies that $E + \bar{E} = V^{\mathbb{C}}$. We now claim that $L \cap \bar{L} = \{0\}$ implies that $\mathrm{Im}(\alpha|_{\Delta})$ is non-degenerate. To prove this claim, we assume, by absurd, that there is $X \neq 0$ in the kernel of $\mathrm{Im}(\alpha|_\Delta)$. Define $\xi \in (V^{\mathbb{C}})^*$ by

$$\xi(Z) = \overline{\alpha(X, \bar{Z})}, \quad \xi(\bar{Z}) = \alpha(X, \bar{Z}), \quad \forall Z \in E.$$

Using that $X \in \mathrm{Ker}(\mathrm{Im}(\alpha|_\Delta))$, one can check that ξ is well-defined and $X + \xi \in L \cap \bar{L}$, which is a contradiction. We proved that the holomorphic space L of $\mathcal{J}$ is of the required form.

Conversely, it may be shown that any subspace $E \subset V^{\mathbb{C}}$, which satisfies $E + \bar{E} = V^{\mathbb{C}}$, together with a skew-Hermitian form $\alpha \in E^* \otimes \bar{E}^*$, which satisfies the non-degeneracy property ii), define, by (2.2), the holomorphic space of a symmetric generalized complex structure on V. □

Corollary 2.5. — *Let $\mathcal{J}$ be a symmetric generalized complex structure on V, with holomorphic space $L^-(E, \alpha)$. Then* $\mathrm{Re}(\alpha|_\Delta)$ *is a 2-form and* $\mathrm{Im}(\alpha|_\Delta)$ *is a pseudo-Euclidean metric on Δ (the real part of $E \cap \bar{E}$).*

Proof. — Straightforward, from (2.3) and the non-degeneracy of $\mathrm{Im}(\alpha|_\Delta)$. □

The second example below shows that symmetric generalized complex structures exist on vector spaces of arbitrary dimension.

Example 2.6.

i) A complex structure J on V defines a symmetric generalized complex structure

$$\mathcal{J} := \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix},$$

where $J^*\xi := \xi \circ J$, for any $\xi \in V^*$. Its holomorphic space is $L^-(V^{1,0}, 0) = V^{1,0} \oplus \mathrm{Ann}(V^{0,1})$, where $V^{1,0}$ and $V^{0,1}$ are the holomorphic and anti-holomorphic spaces of J.

ii) A pseudo-Euclidean metric on V, seen as an isomorphism $g : V \to V^*$, defines a symmetric generalized complex structure

$$\mathcal{J} := \begin{pmatrix} 0 & g^{-1} \\ -g & 0 \end{pmatrix}.$$

Its holomorphic space is $L^-(V^{\mathbb{C}}, ig^{\mathbb{C}})$, where $g^{\mathbb{C}} \in (V^{\mathbb{C}} \otimes V^{\mathbb{C}})^*$ is the complex linear extension of g.

iii) If $\mathcal{J}$ is a symmetric generalized complex structure on V, then so is its B-field transformation $\exp(B) \cdot \mathcal{J} := \exp(B) \circ \mathcal{J} \circ \exp(-B)$, where $B \in \Lambda^2(V^*)$ and the B-field action is defined by

$$\exp(B) : V \oplus V^* \to V \oplus V^*, \quad X + \xi \to X + i_X B + \xi.$$

If $L^{-}(E,\alpha)$ is the holomorphic space of $\mathcal{J}$, then $L^{-}(E,\alpha+B^{\mathbb{C}}|_{E\otimes\bar{E}})$ is the holomorphic space of $\exp(B)\cdot\mathcal{J}$, where $B^{\mathbb{C}}\in\Lambda^{2}(V^{\mathbb{C}})^{*}$ is the complex linear extension of B.

In following theorem we show that any symmetric generalized complex structure can be (non-canonically) obtained from a complex structure, a pseudo-Euclidean metric and a B-field transformation.

Theorem 2.7. — *Any symmetric generalized complex structure on a vector space V is a B-field transformation of the direct sum of one determined by a complex structure and another determined by a pseudo-Euclidean metric (as in Example 2.6).*

Proof. — Let $\mathcal{J}\in\mathrm{End}(V\oplus V^{*})$ be a symmetric generalized complex structure, with holomorphic space $L=L^{-}(E,\alpha)$. Let Δ be the real part of $E\cap\bar{E}$ (i.e. $\Delta\subset V$ and $\Delta^{\mathbb{C}}=E\cap\bar{E}$) and N a complement of Δ in V. Thus

$$V=\Delta\oplus N,\quad E=\Delta^{\mathbb{C}}\oplus(E\cap N^{\mathbb{C}}),\quad \bar{E}=\Delta^{\mathbb{C}}\oplus(\bar{E}\cap N^{\mathbb{C}}). \tag{2.6}$$

We notice that Δ comes with a pseudo-Euclidean metric, namely $g_{\Delta}:=\mathrm{Im}(\alpha|_{\Delta})$, and N with a complex structure J^{N}, with holomorphic space $E\cap N^{\mathbb{C}}$ (and anti-holomorphic space $\bar{E}\cap N^{\mathbb{C}}$). We claim that there is $B\in\Lambda^{2}(V^{*})$ such that (as vector spaces with symmetric generalized complex structures)

$$(V,\exp(B)\cdot\mathcal{J})=(\Delta,g_{\Delta})\oplus(N,J^{N}), \tag{2.7}$$

or, in terms of their holomorphic spaces,

$$L^{-}(E,\alpha+B^{\mathbb{C}}|_{E\otimes\bar{E}})=L^{-}(\Delta^{\mathbb{C}},i(g_{\Delta})^{\mathbb{C}})\oplus\big(E\cap N^{\mathbb{C}}\oplus\mathrm{Ann}(\bar{E}\cap N^{\mathbb{C}})\big). \tag{2.8}$$

From the second and third relation (2.6), we obtain that (2.8) holds if and only if, for any $X\in E$, the covector $i_{X}(\alpha+B^{\mathbb{C}})\in\bar{E}^{*}$ is given by

$$i_{X}(\alpha+B^{\mathbb{C}})|_{\Delta^{\mathbb{C}}}=i(g_{\Delta})^{\mathbb{C}}(\mathrm{pr}_{\Delta^{\mathbb{C}}}(X),\cdot),\quad i_{X}(\alpha+B^{\mathbb{C}})|_{\bar{E}\cap N^{\mathbb{C}}}=0, \tag{2.9}$$

where $\mathrm{pr}_{\Delta^{\mathbb{C}}}:V^{\mathbb{C}}\to\Delta^{\mathbb{C}}$ and $\mathrm{pr}_{N^{\mathbb{C}}}:V^{\mathbb{C}}\to N^{\mathbb{C}}$ are the natural projections determined by the decomposition $V^{\mathbb{C}}=\Delta^{\mathbb{C}}\oplus N^{\mathbb{C}}$. Moreover, it is easy to see that (2.9) is equivalent to

$$\begin{aligned}&(\mathrm{Re}(\alpha)+B)|_{\Delta\otimes\Delta}=0,\quad(\alpha+B^{\mathbb{C}})|_{(E\cap N^{\mathbb{C}})\otimes\Delta^{\mathbb{C}}}=0,\\&(\alpha+B^{\mathbb{C}})|_{E\otimes(\bar{E}\cap N^{\mathbb{C}})}=0.\end{aligned} \tag{2.10}$$

Hence, we are looking for a (real) 2-form $B\in\Lambda^{2}(V^{*})$ such that (2.10) is satisfied. In order to define B, we use $V=\Delta\oplus N$ and $N^{\mathbb{C}}=(E\cap N^{\mathbb{C}})\oplus$

$(\bar{E}\cap N^{\mathbb{C}})$. For any $X, Y\in\Delta$ and $Z, W\in N$, let

$$B(X,Y) := -\mathrm{Re}(\alpha)(X,Y),\quad B(Z,W) := -2\mathrm{Re}(\alpha)(z,\bar{w})$$

and

$$B(X,Z) = -B(Z,X) := 2\mathrm{Re}(\alpha)(z,X),$$

where $z, w\in E\cap N^{\mathbb{C}}$ (uniquely determined) are such that $Z = z+\bar{z}$ and $W = w+\bar{w}$. Since $\alpha\in E^*\otimes\bar{E}^*$ is skew-Hermitian, B is skew-symmetric and its complexification satisfies (2.10) (easy check). This concludes our claim. □

2.2. Analogy with skew-symmetric generalized complex structures

The theory of symmetric generalized complex structures from the previous section is similar to the theory of skew-symmetric generalized complex structures developed by Gualtieri in [6] and owing to this, one can treat these two types of structures in a unified way, using the notion of adapted generalized complex structure (see Definition 2.1). More precisely, it is well-known (see e.g. [6]) that complex and symplectic structures define skew-symmetric generalized complex structures and this corresponds to Example 2.6 i) and ii) from the previous section. In the same framework, Theorem 2.7 above is analogous to Theorem 4.13 from [6], which states that any skew-symmetric generalized complex structure, is, modulo a B-field transformation, the direct sum of a skew-symmetric generalized complex structure of symplectic type and of one of complex type.

The following unified description of the holomorphic space of symmetric and skew-symmetric generalized complex structures on vector spaces is a rewriting of Proposition 2.4 from the previous section and of Propositions 2.6 and 4.4 from [6]. We shall use it in the statement of Theorem 3.3.

Corollary 2.8. — *A complex subspace $L\subset (V\oplus V^*)^{\mathbb{C}}$ is the holomorphic space of an adapted generalized complex structure $\mathcal{J}$ if and only if it is of the form*

$$L = L^{\tau}(E,\alpha) = \{X+\xi\in E\oplus (V^{\mathbb{C}})^*,\quad \xi|_{\tau(E)} = i_X\alpha\} \tag{2.11}$$

where $E\subset V^{\mathbb{C}}$ is a complex subspace with $E+\bar{E} = V^{\mathbb{C}}$ and $\alpha\in E^*\otimes\tau(E)^*$ is complex bilinear, such that

$$\alpha(X,\tau(Y)) + \tau(\alpha(Y,\tau(X))) = 0,\quad \forall X, Y\in E \tag{2.12}$$

and $\mathrm{Im}(\alpha|_{\Delta})$ is non-degenerate (where $\Delta\subset V$, $\Delta^{\mathbb{C}} = E\cap\bar{E}$).

In (2.11) and (2.12) the maps $\tau : V^{\mathbb{C}} \to V^{\mathbb{C}}$ and $\tau : \mathbb{C} \to \mathbb{C}$ are both the standard conjugations ($\mathcal{J}$ symmetric), respectively both the identity maps ($\mathcal{J}$ skew-symmetric).

2.3. Remarks on integrability

The generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ of a smooth manifold M has a canonical metric of neutral signature, defined like in (2.1), and the theory developed in the previous sections extends pointwise to manifolds, in an obvious way.

Definition 2.9. — *A (symmetric, respectively skew-symmetric) generalized complex structure on a manifold M is a smooth field of endomorphisms $\mathcal{J}$ of $\mathbb{T}M$, which, at any $p \in M$, is a (symmetric, respectively skew-symmetric) generalized complex structure on T_pM. An adapted generalized complex structure on M is a generalized complex structure which is either symmetric or skew symmetric.*

Remark 2.10. — As opposed to the usual terminology, we do not assume that generalized complex structures on manifolds are Courant integrable (see Definition 2.11 below). In fact, the generalized complex structures we are mainly interested in, namely, the symmetric ones, turn out not to be Courant integrable (see Lemma 2.13).

Definition 2.11. — *A generalized complex structure $\mathcal{J}$ on a manifold M is called Courant integrable if the space of sections of its holomorphic bundle $L \subset \mathbb{T}^{\mathbb{C}}M$ (the i-eigenbundle of $\mathcal{J}$) is closed under the Courant bracket, defined by*

$$[X + \xi, Y + \eta] = [X, Y] + L_X \eta - L_Y \xi + \frac{1}{2}\left(\xi(Y) - \eta(X)\right),$$

for any vector fields X, Y and 1-forms ξ, η.

The holomorphic bundle L of an adapted generalized complex structure on M may be described in terms of a complex subbundle $E \subset T^{\mathbb{C}}M$ (the image of L through the natural projection $\mathbb{T}^{\mathbb{C}}M \to T^{\mathbb{C}}M$) and a smooth section $\alpha \in \Gamma(E^* \otimes \tau(E)^*)$, with the algebraic properties from Corollary 2.8 (we assume that all points are regular, i.e. E is a genuine complex vector bundle). There is a basic result of Gualtieri (see [6, Propostition 4.19]) which expresses the Courant integrability of a skew-symmetric generalized complex structure in terms of its holomorphic bundle. Since we shall use it repeatedly, we state it here:

Proposition 2.12. — [6] *A skew-symmetric generalized complex structure on a manifold M, with holomorphic bundle $L = L(E,\alpha)$, is Courant integrable, if and only if the subbundle $E \subset T^{\mathbb{C}}M$ is involutive and $d_E\alpha = 0$, where $d_E\alpha \in \Gamma(\Lambda^3 E^*)$ is the exterior differential of $\alpha \in \Gamma(\Lambda^2 E^*)$, defined by*

$$\begin{aligned}(d_E\alpha)(X,Y,Z) := X\left(\alpha(Y,Z)\right) + Z\left(\alpha(X,Y)\right) + Y\left(\alpha(Z,X)\right)\\ + \alpha(X,[Y,Z]) + \alpha(Z,[X,Y]) + \alpha(Y,[Z,X]),\end{aligned}$$

for any $X, Y, Z \in \Gamma(E)$.

The following simple lemma holds.

Lemma 2.13. — *A symmetric generalized complex structure is never Courant integrable.*

Proof. — As proved in Proposition 3.26 of [6], a Courant integrable subbundle of $\mathbb{T}^{\mathbb{C}}M$ is either g_{can}-isotropic or of the form $(\Delta \oplus T^*M)^{\mathbb{C}}$, where $\Delta \subset TM$ is involutive (and non-trivial). Hence, it cannot be the holomorphic bundle L of a symmetric generalized complex structure (recall that L is g_{can}-orthogonal to $\bar{L}$, $L \oplus \bar{L} = \mathbb{T}^{\mathbb{C}}M$ and g_{can} is non-degenerate). □

3. Integrable complex structures on cotangent manifolds

Let $(M, \mathcal{J}, D)$ be a manifold with a generalized complex structure $\mathcal{J}$ and a linear connection D. The connection D acts on the cotangent bundle $\pi : T^*M \to M$ and induces a decomposition

$$T(T^*M) = H^D \oplus T^{\mathrm{vert}}(T^*M) = \pi^*(\mathbb{T}M) \tag{3.1}$$

into horizontal and vertical subbundles. Above, we identified the horizontal bundle H^D with $\pi^*(TM)$ and the vertical bundle $T^{\mathrm{vert}}(T^*M)$ of the projection π with $\pi^*(T^*M)$. From now on, we shall use systematically, without mentioning explicitly, the identification (3.1) between $T(T^*M)$ and $\pi^*(\mathbb{T}M)$.

Definition 3.1. — *The almost complex structure $J^{\mathcal{J},D} := \pi^*(\mathcal{J})$ on the cotangent manifold T^*M is called the almost complex structure defined by $\mathcal{J}$ and D.*

In this section we study the integrability of $J^{\mathcal{J},D}$, under the assumption that $\mathcal{J}$ is adapted. We begin by fixing notation.

Notation 3.2. — In computations, we shall use the notation $\tilde{X} \in \mathcal{X}(T^*M)$ for the D-horizontal lift of a vector field $X \in \mathcal{X}(M)$. Forms of degree one on M will be considered as constant vertical vector fields on the cotangent manifold T^*M. With these conventions, the Lie brackets $[\cdot,\cdot]_{\mathcal{L}}$ of various vector fields on T^*M are given by: at any $\gamma \in T^*M$,

$$(3.2)\quad [\tilde{X},\tilde{Y}]_{\mathcal{L}}(\gamma) = \widetilde{[X,Y]}(\gamma) + R^D_{X,Y}(\gamma), \quad [\tilde{X},\xi]_{\mathcal{L}} = D_X\xi, \quad [\xi,\eta]_{\mathcal{L}} = 0$$

where $X, Y \in \mathcal{X}(M)$, $\xi,\eta \in \Omega^1(M)$ and

$$R^D_{X,Y} := -D_X D_Y + D_Y D_X + D_{[X,Y]}$$

is the curvature of D.

The main result from this section is the following.

Theorem 3.3. — *Let $(M,\mathcal{J},D)$ be a manifold with a generalized complex structure $\mathcal{J}$ and a linear connection D. Assume that $\mathcal{J}$ is adapted and let $L^\tau(E,\alpha)$ be its holomorphic bundle, where $E \subset T^{\mathbb{C}}M$ and $\alpha \in \Gamma(E^* \otimes \tau(E)^*)$ satisfy the algebraic properties from Corollary 2.8. The almost complex structure $J^{\mathcal{J},D}$ from Definition 3.1 is integrable, if and only if the following conditions hold:*

i) E is an involutive subbundle of $T^{\mathbb{C}}M$;

ii) the complex linear extensions of D and R^D satisfy

$$(3.3)\qquad D_{\Gamma(E)}\Gamma(\tau(E)) \subset \Gamma(\tau(E)), \quad R^D|_{E\times E}(\tau(E)) = 0.$$

iii) for any $X,Y,Z \in \Gamma(E)$,

$$(3.4)\qquad (D_X\alpha)(Y,\tau(Z)) - (D_Y\alpha)(X,\tau(Z)) + \alpha(T^D_X Y,\tau(Z)) = 0,$$

where T^D is the complex linear extension of the torsion of the connection D.

Proof. — We need to prove that the holomorphic bundle $\pi^*L^\tau(E,\alpha) \subset T^{\mathbb{C}}(T^*M)$ of $J^{\mathcal{J},D}$ is involutive if and only if the conditions i), ii) and iii) from the statement of the theorem hold. For this, we will compute the Lie brackets of basic sections of $\pi^*L^\tau(E,\alpha)$. (By a basic section of $\pi^*L^\tau(E,\alpha)$ we mean a vector field on the cotangent manifold T^*M, of the form $\tilde{X}+\xi$, where $X+\xi$ is a section of $L^\tau(E,\alpha)$). Therefore, let $X+\xi, Y+\eta \in \Gamma(L^\tau(E,\alpha))$. Then

$$(3.5)\quad X,Y \in \Gamma(E), \quad \xi,\eta \in \Gamma(T^{\mathbb{C}}M)^*, \quad \xi|_{\tau(E)} = i_X\alpha, \quad \eta|_{\tau(E)} = i_Y\alpha.$$

From (3.2), at any $\gamma \in T^*M$,

$$(3.6)\qquad [\tilde{X}+\xi,\tilde{Y}+\eta]_{\mathcal{L}}(\gamma) = \widetilde{[X,Y]}(\gamma) + R^D_{X,Y}(\gamma) + D_X\eta - D_Y\xi.$$

We obtain that $[\tilde{X}+\xi, \tilde{Y}+\eta]_{\mathcal{L}}$ is a section of $\pi^* L^{\tau}(E,\alpha)$ if and only if

$$[X,Y] + R^D_{X,Y}(\gamma) + D_X\eta - D_Y\xi$$

belongs to the fiber of $L^{\tau}(E,\alpha)$ at $\pi(\gamma)$, for any $\gamma \in T^*M$, i.e.

$$[X,Y] \in \Gamma(E), \quad R^D_{X,Y}(\gamma)|_{\tau(E)} = 0 \tag{3.7}$$

and

$$(D_X\eta - D_Y\xi)(\tau(Z)) = \alpha([X,Y],\tau(Z)), \quad \forall Z \in \Gamma(E). \tag{3.8}$$

We now rewrite (3.8). From (3.5), the left hand side of (3.8) is equal to

$$X\alpha(Y,\tau(Z)) - Y\alpha(X,\tau(Z)) - \eta(D_X(\tau(Z))) + \xi(D_Y(\tau(Z)))$$

and (3.8) becomes

$$\begin{aligned} X\alpha(Y,\tau(Z)) - Y\alpha(X,\tau(Z)) - \eta(D_X(\tau(Z)))&+\xi(D_Y(\tau(Z))) \\ &= \alpha([X,Y],\tau(Z)), \end{aligned} \tag{3.9}$$

for any $Z \in \Gamma(E)$. From (3.5) again, $\xi|_{\tau(E)} = i_X\alpha$, but ξ can take any values on a complement of $\tau(E)$ in $T^{\mathbb{C}}M$. Similarly, the only obstruction on η is $\eta|_{\tau(E)} = i_Y\alpha$. Thus, if (3.9) holds for any sections $X+\xi$ and $Y+\eta$ of $L^{\tau}(E,\alpha)$, then

$$D_X(\tau(Z)) \in \Gamma(\tau(E)), \quad \forall X, Z \in \Gamma(E)$$

and relation (3.9) becomes (3.4). We proved that $\pi^* L^{\tau}(E,\alpha)$ is involutive if and only if

$$\begin{aligned} &[\Gamma(E),\Gamma(E)] \subset \Gamma(E), \quad R^D|_{E\times E}\tau(E) = 0, \\ &D_{\Gamma(E)}\Gamma(\tau(E)) \subset \Gamma(\tau(E)) \end{aligned} \tag{3.10}$$

and relation (3.4) holds. Our claim follows. $\square$

We end this section by relating the Courant integrability of a skew-symmetric generalized complex structure $\mathcal{J}$ with the integrability of the almost complex structure $J^{\mathcal{J},D}$. This is a straightforward application of Theorem 3.3.

Corollary 3.4. — *Let $\mathcal{J}$ be a skew-symmetric generalized complex structure, with holomorphic bundle $L(E,\alpha)$, and D a linear connection on M. Suppose that E is involutive, $D_{\Gamma(E)}\Gamma(E) \subset \Gamma(E)$, $R^D_{E,E}E = 0$ and the relation*

$$(D_Z\alpha)(X,Y) + \alpha(T^D_Z X, Y) + \alpha(X, T^D_Z Y) = 0, \quad \forall X,Y,Z \in \Gamma(E) \tag{3.11}$$

holds. Then $J^{\mathcal{J},D}$ is integrable if and only if $\mathcal{J}$ is Courant integrable.

Proof. — From Proposition 2.12 and Theorem 3.3, we need to prove that $d_E\alpha = 0$ is equivalent to (3.4) (with $\tau : T^{\mathbb{C}}M \to T^{\mathbb{C}}M$ the identity map). This is a consequence of (3.11) and the following general identity: for any 2-form β and vector fields X, Y, Z,

$$\begin{aligned} &(D_X\beta)(Y,Z) - (D_Y\beta)(X,Z) + \beta(T^D_X Y, Z) \\ &\quad = (d\beta)(X,Y,Z) - \left((D_Z\beta)(X,Y) + \beta(T^D_Z X, Y) + \beta(X, T^D_Z Y)\right). \end{aligned} \tag{3.12}$$

□

Example 3.5. — Let $\mathcal{J}$ be a left-invariant skew-symmetric generalized complex structure on a Lie group G and D^c the (flat) left-invariant connection on G given by $D^c_X Y = [X, Y]$, for any left-invariant vector fields X, Y. Then D^c satisfies (3.11), for any left-invariant 2-form α. We obtain that $\mathcal{J}$ is Courant integrable if and only if $J^{\mathcal{J},D^c}$ is integrable.

4. Complex structures on cotangent manifolds of Lie groups

We begin by recalling basic facts we need about semisimple Lie algebras.

4.1. Semisimple Lie algebras

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} + \mathfrak{g}(R) = \mathfrak{h} + \sum_{\alpha\in R} \mathfrak{g}_\alpha \tag{4.1}$$

a Cartan decomposition. We identify $\mathfrak{h}$ with $\mathfrak{h}^*$, using the restriction of the Killing form B of $\mathfrak{g}^{\mathbb{C}}$ to $\mathfrak{h}$. By means of this identification, we denote by $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$ the real span of the set of roots $R \subset \mathfrak{h}^*$ of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}$ and by $H_\alpha \in \mathfrak{h}_{\mathbb{R}}$ the vector which corresponds to the root $\alpha \in R$. Recall that a Weyl basis of the root part $\mathfrak{g}(R) := \sum_{\alpha\in R} \mathfrak{g}_\alpha$ consists of root vectors $\{E_\alpha, \alpha \in R\}$, satisfying the following conditions:

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \quad B(E_\alpha, E_{-\alpha}) = 1, \quad N_{-\alpha,-\beta} = -N_{\alpha\beta}, \quad N_{\alpha\beta} \in \mathbb{R},$$

where the structure constants $N_{\alpha\beta}$ are defined by

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}, \quad \forall \alpha, \beta, \alpha+\beta \in R.$$

A simple argument which uses the Jacobi identity for E_α, E_β, E_γ shows that for any $\alpha, \beta, \gamma \in R$, such that $\alpha + \beta + \gamma = 0$,

$$N_{\alpha\beta} = N_{\beta\gamma} = N_{\gamma\alpha} \tag{4.2}$$

(see e.g. [7], page 146).

Recall now that a real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ is the fixed point set of an antilinear involution

$$\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}, \quad x \to \sigma(x) = \bar{x},$$

i.e. an automorphism of real Lie algebras, which is complex antilinear and satisfies $\sigma^2 = \mathrm{Id}$. We review, following Theorem 6.88 of [9], the structure of such real forms. The idea is that $\mathfrak{g}$ is determined (up to isomorphism) by its Vogan diagram, which is the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ (representing a set of simple roots Π relative to a chosen Cartan subalgebra $\mathfrak{h}$) together with two pieces of data: an involutive automorphism $\theta : \Pi \to \Pi$ of the Dynkin diagram and some painted nodes, in the fixed point set of θ. Choose a Weyl basis $\{E_\alpha\}$ of $\mathfrak{g}(R)$, where $R = [\Pi]$ is the set of roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}$. The action of θ on Π extends by linearity to $\mathfrak{h}^*_{\mathbb{R}} \cong \mathfrak{h}_{\mathbb{R}}$ and this action preserves R. The antiinvolution σ preserves $\mathfrak{h}$ and it acts on R by

$$\sigma : R \to R, \quad \sigma(\alpha) := \overline{\alpha \circ \sigma}.$$

This action coincides, up to a minus sign, with the action of θ: $\sigma|_R = -\theta|_R$. On root vectors from the chosen Weyl basis, σ acts as

$$\sigma(E_\alpha) = -a_\alpha E_{\sigma(\alpha)}, \quad \forall \alpha \in R, \tag{4.3}$$

where $\{a_\alpha, \alpha \in R\}$ (determined by the painted nodes in the Vogan diagram) is a set of constants, satisfying

$$a_\alpha = a_{-\alpha} = a_{\sigma(\alpha)} \in \{\pm 1\}, \quad \forall \alpha \in R \tag{4.4}$$

and

$$a_{\alpha+\beta} = -a_\alpha a_\beta N^{-1}_{\alpha\beta} N_{\sigma(\alpha)\sigma(\beta)}, \quad \forall \alpha, \beta, \alpha + \beta \in R. \tag{4.5}$$

The real form $\mathfrak{h}_{\mathfrak{g}} = \mathfrak{h}^\sigma = \mathfrak{h}^+ + \mathfrak{h}^-$, where

$$\mathfrak{h}^+ := \langle i(H_\alpha + H_{-\sigma(\alpha)}),\ \alpha \in R\rangle, \quad \mathfrak{h}^- := \langle H_\alpha + H_{\sigma(\alpha)},\ \alpha \in R\rangle \tag{4.6}$$

(the sign $\langle \cdots \rangle$ means the real span of the respective vectors) is a Cartan subalgebra of $\mathfrak{g}$. Up to isomorphism, $\mathfrak{g}$ can be recovered from its Vogan diagram as

$$\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^\sigma = \mathfrak{h}_{\mathfrak{g}} + \sum_{\alpha \in R} \mathbb{R}(E_\alpha - a_\alpha E_{\sigma(\alpha)}) + \sum_{\alpha \in R} \mathbb{R} i(E_\alpha + a_\alpha E_{\sigma(\alpha)}). \tag{4.7}$$

Remark 4.1. — Since θ permutes Π, there is no root $\alpha \in R$ such that $\sigma(\alpha) = \alpha$. This means that $\mathfrak{h}_{\mathfrak{g}}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}$ (see [9, Proposition 6.70]). The real form $\mathfrak{g}$ (and any Lie group G with Lie algebra $\mathfrak{g}$) is called of inner type if $\sigma(\alpha) = -\alpha$ for any $\alpha \in R$ (the automorphism θ of the Vogan diagram is the identity). Any compact real form is of inner type, with $a_\alpha = 1$, for any $\alpha \in R$. A real form $\mathfrak{g}$ (and any Lie group G with Lie algebra $\mathfrak{g}$) which is not of inner type is called of outer type. For more details on real semisimple Lie algebras, Vogan diagrams, maximally compact Cartan subalgebras, see e.g. [9], Chapter VI.

4.2. Admissible triples on Lie groups

Let G be a Lie group. We identify T_eG with the space of left-invariant vector fields on G and with the Lie algebra $\mathfrak{g}$ of G, in the usual way. The following definition encodes the conditions from Theorem 3.3, when $M = G$ and $\mathcal{J}$, D are left-invariant. Recall that a connection D is left-invariant if D_XY is left-invariant, when X and Y are so.

Definition 4.2. — *A (symmetric or skew-symmetric) $\mathfrak{g}$-admissible triple is a triple $(\mathfrak{k}, \mathcal{D}, \epsilon)$, with the following properties:*

i) $\mathfrak{k}$ is a complex subalgebra of $\mathfrak{g}^{\mathbb{C}}$, such that $\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{g}^{\mathbb{C}}$;

ii) $\mathcal{D} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y) \to \mathcal{D}_X(Y)$, is a bilinear map whose complex linear extension satisfies

$$\mathcal{D}_{\mathfrak{k}}\tau(\mathfrak{k}) \subset \tau(\mathfrak{k}) \tag{4.8}$$

and

$$R^{\mathcal{D}}_{X,Y}Z := -\mathcal{D}_X\mathcal{D}_Y(Z) + \mathcal{D}_Y\mathcal{D}_X(Z) + \mathcal{D}_{[X,Y]}(Z) = 0, \quad \forall X, Y \in \mathfrak{k}, \quad \forall Z \in \tau(\mathfrak{k}). \tag{4.9}$$

iii) $\epsilon \in \mathfrak{k}^* \otimes \tau(\mathfrak{k})^*$ satisfies

$$\epsilon(X, \tau(Y)) + \tau(\epsilon(Y, \tau(X))) = 0, \quad \forall X, Y \in \mathfrak{k}$$

and

$$\epsilon(X, \mathcal{D}_Y(\tau(Z))) - \epsilon(Y, \mathcal{D}_X(\tau(Z))) = \epsilon([X, Y], \tau(Z)), \quad \forall X, Y, Z \in \mathfrak{k} \tag{4.10}$$

and, moreover, $g_\Delta := \mathrm{Im}(\epsilon|_\Delta)$ is non-degenerate on $\Delta = (\mathfrak{k} \cap \bar{\mathfrak{k}})^\sigma$.

Above, the maps $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ and $\tau : \mathbb{C} \to \mathbb{C}$ are both standard conjugations (symmetric case) or both the identity maps (skew-symmetric case).

The following correspondence holds and will play a key role in our treatment from the next subsection.

Proposition 4.3. — *There is a one to one correspondence between:*

i) pairs $(\mathcal{J}, D)$ formed by a left-invariant adapted generalized complex structure $\mathcal{J}$ and a left-invariant connection D on G, such that the associated almost complex structure $J^{\mathcal{J},D}$ on T^*G is integrable;

ii) $\mathfrak{g}$-admissible triples $(\mathfrak{k}, \mathcal{D}, \epsilon)$.

In this correspondence $\mathcal{D}$ is the restriction of D to the space of left-invariant vector fields, $\mathfrak{k} := E_e$ and $\epsilon := \alpha|_{\mathfrak{k}\times\tau(\mathfrak{k})}$, where $L^\tau(E, \alpha)$ is the holomorphic bundle of $\mathcal{J}$.

Proof. — Using the left-invariance of E and α, one may check that the conditions from Theorem 3.3, on the integrability of $J^{\mathcal{J},D}$, become the conditions for $(\mathfrak{k}, \mathcal{D}, \epsilon)$ to be a $\mathfrak{g}$-admissible triple. For example, to prove the equivalence between (3.4) and (4.10), we notice that (3.4) holds if and only if it holds for any $X, Y, Z \in \Gamma(E)$ left-invariant, and for such arguments, $\alpha(Y, \tau(Z))$ and $\alpha(X, \tau(Z))$ are constant (because α is left-invariant). □

4.3. Regular admissible triples and regular generalized complex structures

Here and until the end of Section 4 we fix a complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$, a real form $\mathfrak{g} = (\mathfrak{g}^{\mathbb{C}})^\sigma$ given by (4.7), and a Lie group G with Lie algebra $\mathfrak{g}$. A (complex) subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ is called regular, if it is normalized by the (maximally compact) Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of $\mathfrak{g}$. It is known (see e.g. [11], Proposition 1.1, page 183) that such a subalgebra is of the form

$$\mathfrak{k} = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{g}(R_0) = \mathfrak{h}_{\mathfrak{k}} + \sum_{\alpha\in R_0} \mathfrak{g}_\alpha \tag{4.11}$$

where $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{k} \cap \mathfrak{h}$ and $R_0 \subset R$ is a closed subset of roots (i.e. if $\alpha, \beta \in R_0$ and $\alpha + \beta \in R$, then $\alpha + \beta \in R_0$). Remark that

$$\bar{\mathfrak{k}} = \sigma(\mathfrak{k}) = \bar{\mathfrak{h}}_{\mathfrak{k}} + \sum_{\alpha\in R_0} \mathfrak{g}_{\sigma(\alpha)}, \quad \mathfrak{k} \cap \bar{\mathfrak{k}} = \mathfrak{h}_{\mathfrak{k}} \cap \bar{\mathfrak{h}}_{\mathfrak{k}} + \sum_{\alpha\in R_0\cap\sigma(R_0)} \mathfrak{g}_\alpha. \tag{4.12}$$

Definition 4.4. — *Let $\mathcal{J}$ be a left-invariant adapted generalized complex structure on G and $L^\tau(\mathfrak{k}, \epsilon)$ the fiber at $e \in G$ of its holomorphic bundle. Then $\mathcal{J}$ is called regular if the subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ is regular. Similarly, a $\mathfrak{g}$-admissible triple $(\mathfrak{k}, \mathcal{D}, \epsilon)$ is called regular if the subalgebra $\mathfrak{k} \subset \mathfrak{g}^{\mathbb{C}}$ is regular.*

We need to recall the notions of σ-parabolic and σ-positive systems [2]. They reduce, when $\mathfrak{g}$ is of inner type, to the usual notions of parabolic and positive root systems, respectively (see e.g. [3]).

Definition 4.5. — *A closed set of roots $R_0 \subset R$ is called a σ-parabolic system, if $R_0 \cup \sigma(R_0) = R$. If, moreover, $R_0 \cap \sigma(R_0) = \emptyset$, then R_0 is called a σ-positive system.*

The following simple lemma holds.

Lemma 4.6. — *If a regular subalgebra $\mathfrak{k}$ as in (4.11) belongs to a $\mathfrak{g}$-admissible triple, then its root part R_0 is a σ-parabolic system and its Cartan part $\mathfrak{h}_{\mathfrak{k}}$ satisfies $\mathfrak{h}_{\mathfrak{k}} + \bar{\mathfrak{h}}_{\mathfrak{k}} = \mathfrak{h}$. If, moreover, R_0 is a σ-positive system, then $\mathfrak{k} \cap \bar{\mathfrak{k}} = \mathfrak{h}_{\mathfrak{k}} \cap \bar{\mathfrak{h}}_{\mathfrak{k}}$.*

Proof. — From the definition of $\mathfrak{g}$-admissible triples, $\mathfrak{k} + \bar{\mathfrak{k}} = \mathfrak{h}$. This relation, together with (4.12), implies the statement of the lemma. □

4.4. Complex structures on T^*G

Our aim in this section is to define a natural left-invariant connection D^0 on G and to determine all regular symmetric generalized complex structures $\mathcal{J}$, with the property that the almost complex structure $J^{\mathcal{J},D^0}$ on T^*G is integrable, or, equivalently (from Proposition 4.3), the associated triple $(\mathfrak{k}, \mathcal{D}^0, \epsilon)$ is $\mathfrak{g}$-admissible (and regular, symmetric). From Definition 4.2, a bilinear map $\mathcal{D} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ can belong to a symmetric $\mathfrak{g}$-admissible triple $(\mathfrak{k}, \mathcal{D}, \epsilon)$ only if its complex linear extension $\mathcal{D} : \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ satisfies (4.8) and (4.9) (with $\tau = \sigma$, hence $\tau(\mathfrak{k}) = \bar{\mathfrak{k}}$). Recall now that $\mathfrak{k}$ is of the form (4.11) and $\bar{\mathfrak{k}}$ of the form (4.12). From these relations, it is immediate that if

$$\begin{aligned} &\mathcal{D}_{\mathfrak{g}_\alpha}(\mathfrak{g}_\beta) \subset \mathfrak{g}_{\sigma(\alpha)+\beta}, \quad && \mathcal{D}_{\mathfrak{h}}(\mathfrak{g}_\beta) \subset \mathfrak{g}_\beta, \\ &\mathcal{D}_{\mathfrak{g}_\beta}(\mathfrak{h}) \subset \mathfrak{g}_{\sigma(\beta)}, && \mathcal{D}_{\mathfrak{h}}(\mathfrak{h}) = 0, \end{aligned} \tag{4.13}$$

for any $\alpha, \beta \in R$ (with $\mathfrak{g}_{\sigma(\alpha)+\beta} := 0$ if $\alpha + \sigma(\beta) \notin R$), then (4.8) is satisfied. A map whose complex linear extension satisfies (4.13) and (4.9) is provided by the following lemma.

Lemma 4.7. — *Let $\mathcal{D}^0 : \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be a complex bilinear map given by*

$$\begin{aligned} &\mathcal{D}^0_{E_\alpha}(E_\beta) = -a_\alpha [E_{\sigma(\alpha)}, E_\beta], \quad && \mathcal{D}^0_H(E_\beta) = \sigma(\beta)(H) E_\beta, \\ &\mathcal{D}^0_{E_\beta}(H) = \sigma(\beta)(H) a_\beta E_{\sigma(\beta)}, && \mathcal{D}^0_H(\tilde{H}) = 0 \end{aligned}$$

for any $\alpha, \beta \in R$ and $H, \tilde{H} \in \mathfrak{h}$. Then $\mathcal{D}^0$ satisfies

$$\mathcal{D}^0_{\mathfrak{k}}(\bar{\mathfrak{k}}) \subset \bar{\mathfrak{k}},\ \mathcal{D}^0_{\mathfrak{g}}(\mathfrak{g}) \subset \mathfrak{g},\ R^{\mathcal{D}^0} = 0. \tag{4.14}$$

Proof. — We already explained that $\mathcal{D}^0_{\bar{\mathfrak{k}}}(\bar{\mathfrak{k}}) \subset \bar{\mathfrak{k}}$. We now prove $\mathcal{D}^0_{\mathfrak{g}}(\mathfrak{g}) \subset \mathfrak{g}$. For any $\alpha \in R$, let $A_\alpha := E_\alpha - a_\alpha E_{\sigma(\alpha)}$ and $B_\alpha := i(E_\alpha + a_\alpha E_{\sigma(\alpha)})$. By a straightforward computation, which uses (4.4), we obtain:

$$\begin{aligned}
\mathcal{D}^0_{A_\alpha}(A_\beta) &= - a_\alpha \left([E_{\sigma(\alpha)}, E_\beta] + a_\alpha a_\beta [E_\alpha, E_{\sigma(\beta)}]\right) \\
&\quad + \left([E_\alpha, E_\beta] + a_\alpha a_\beta [E_{\sigma(\alpha)}, E_{\sigma(\beta)}]\right), \\
\mathcal{D}^0_{B_\alpha}(B_\beta) &= \left([E_\alpha, E_\beta] + a_\alpha a_\beta [E_{\sigma(\alpha)}, E_{\sigma(\beta)}]\right) \\
&\quad + a_\alpha \left([E_{\sigma(\alpha)}, E_\beta] + a_\alpha a_\beta [E_\alpha, E_{\sigma(\beta)}]\right), \\
\mathcal{D}^0_{A_\alpha}(B_\beta) &= \quad i \left([E_\alpha, E_\beta] - a_\alpha a_\beta [E_{\sigma(\alpha)}, E_{\sigma(\beta)}]\right) \\
&\quad + a_\beta i \left([E_\alpha, E_{\sigma(\beta)}] - a_\alpha a_\beta [E_{\sigma(\alpha)}, E_\beta]\right), \\
\mathcal{D}^0_{B_\alpha}(A_\beta) &= - i \left([E_\alpha, E_\beta] - a_\alpha a_\beta [E_{\sigma(\alpha)}, E_{\sigma(\beta)}]\right) \\
&\quad - i a_\alpha \left([E_{\sigma(\alpha)}, E_\beta] - a_\alpha a_\beta [E_\alpha, E_{\sigma(\beta)}]\right).
\end{aligned}$$

Moreover, for any $\alpha, \beta \in R$ and $H \in \mathfrak{h}^+$,

$$\begin{aligned}
&\mathcal{D}^0_{A_\alpha}(H) = i\alpha(H) a_\alpha B_{\sigma(\alpha)}, && \mathcal{D}^0_{B_\alpha}(H) = i\alpha(H) A_\alpha \\
&\mathcal{D}^0_H(A_\alpha) = i\alpha(H) B_\alpha, && \mathcal{D}^0_H(B_\alpha) = -i\alpha(H) A_\alpha,
\end{aligned}$$

while for any $\alpha, \beta \in R$ and $H \in \mathfrak{h}^-$,

$$\begin{aligned}
&\mathcal{D}^0_{A_\alpha}(H) = \alpha(H) a_\alpha A_{\sigma(\alpha)}, && \mathcal{D}^0_{B_\alpha}(H) = \alpha(H) B_\alpha \\
&\mathcal{D}^0_H(A_\alpha) = \alpha(H) A_\alpha, && \mathcal{D}^0_H(B_\alpha) = \alpha(H) B_\alpha.
\end{aligned}$$

We now remark that for any $\alpha, \beta \in R$, the expressions

$$[E_\alpha, E_\beta] + a_\alpha a_\beta [E_{\sigma(\alpha)}, E_{\sigma(\beta)}], \quad i\left([E_\alpha, E_\beta] - a_\alpha a_\beta [E_{\sigma(\alpha)}, E_{\sigma(\beta)}]\right)$$

belong to $\mathfrak{g}$. We also know that $a_\alpha = a_{\sigma(\alpha)} \in \{\pm 1\}$. Moreover, any root takes real values on $\mathfrak{h}^-$ and purely imaginary values on $\mathfrak{h}^+$. Therefore, the above computations show that $\mathcal{D}^0_{\mathfrak{g}}(\mathfrak{g}) \subset \mathfrak{g}$, as required.

It remains to prove that $R^{\mathcal{D}^0}_{X,Y} Z = 0$, for any $X, Y, Z \in \mathfrak{g}^{\mathbb{C}}$. Without loss of generality, we assume that X, Y, Z are either root vectors from the Weyl basis or vectors from the Cartan subalgebra $\mathfrak{h}$. With this assumption, the above relation follows from a long but straightforward computation, which uses the definition of $\mathcal{D}^0$. Consider for example the case when $X := E_\alpha$, $Y := E_\beta$ and $Z := E_\gamma$ are all root vectors from the Weyl basis (the remaining situations, with at least one argument from $\mathfrak{h}$, can be treated similarly). We distinguish three subcases: I) $\sigma(\alpha) + \gamma \neq 0$ and $\sigma(\beta) + \gamma \neq 0$; II) $\sigma(\alpha) + \gamma = 0$ and $\sigma(\beta) + \gamma \neq 0$; III) $\sigma(\alpha) + \gamma = \sigma(\beta) + \gamma = 0$. In subcase III), $R^{\mathcal{D}^0}_{E_\alpha, E_\beta} E_\gamma = 0$ (trivial, $\alpha = \beta$). Let us study I) in more detail. We need to distinguish two further subcases of I), namely $\alpha+\beta\neq 0$ and, respectively, $\alpha+\beta=0$. Suppose first that $\alpha+\beta\neq 0$. Since $\sigma(\beta)+\gamma\neq 0$, $[E_{\sigma(\beta)},E_{\gamma}]$ is either a root vector (when $\sigma(\beta)+\gamma\in R$) or $[E_{\sigma(\beta)},E_{\gamma}]=0$ (when $\sigma(\beta)+\gamma\notin R$). We obtain, from the definition of $\mathcal{D}^0$,

$$\mathcal{D}^0_{E_\alpha}\mathcal{D}^0_{E_\beta}E_\gamma=-a_\beta\mathcal{D}^0_{E_\alpha}([E_{\sigma(\beta)},E_\gamma])=a_\alpha a_\beta[E_{\sigma(\alpha)},[E_{\sigma(\beta)},E_\gamma]]. \tag{4.15}$$

Similarly, since $\alpha+\beta\neq 0$, $[E_\alpha,E_\beta]$ is either a root vector or $[E_\alpha,E_\beta]=0$. We obtain

$$\mathcal{D}^0_{[E_\alpha,E_\beta]}E_\gamma=N_{\alpha\beta}\mathcal{D}^0_{E_{\alpha+\beta}}E_\gamma=-a_{\alpha+\beta}N_{\alpha\beta}[E_{\sigma(\alpha+\beta)},E_\gamma] \tag{4.16}$$

(with $E_{\alpha+\beta}=E_{\sigma(\alpha+\beta)}=0$ and $a_{\alpha+\beta}=N_{\alpha\beta}=0$ when $\alpha+\beta\notin R$). From (4.15), (4.16) and the Jacobi identity, we deduce that

$$R^{\mathcal{D}^0}_{E_\alpha,E_\beta}E_\gamma=a_\alpha a_\beta[E_\gamma,[E_{\sigma(\alpha)},E_{\sigma(\beta)}]]+a_{\alpha+\beta}N_{\alpha\beta}[E_\gamma,E_{\sigma(\alpha+\beta)}]. \tag{4.17}$$

On the other hand, it is easy to check that

$$a_{\alpha+\beta}N_{\alpha\beta}E_{\sigma(\alpha+\beta)}=-a_\alpha a_\beta[E_{\sigma(\alpha)},E_{\sigma(\beta)}].$$

It follows that $R^{\mathcal{D}^0}_{E_\alpha,E_\beta}E_\gamma=0$, as required. Suppose now that I) holds and $\alpha+\beta=0$. Using again the definition of $\mathcal{D}^0$, we obtain

$$R^{\mathcal{D}^0}_{E_\alpha,E_{-\alpha}}E_\gamma=\big(\sigma(\gamma)(H_\alpha)-\gamma(H_{\sigma(\alpha)})\big)\,E_\gamma.$$

But this last expression is zero: using $\sigma(H_\alpha)=H_{\sigma(\alpha)}$ (easy check) and $\gamma(H_{\sigma(\alpha)})\in\mathbb{R}$, we obtain $\sigma(\gamma)(H_\alpha)=\gamma(H_{\sigma(\alpha)})$. The remaining case II) can be treated similarly. Our claim follows. □

The preferred connection we are looking for is defined as follows.

Definition 4.8. — *The connection D^0 is the unique (flat) left-invariant connection on G which on left-invariant vector fields coincides with the map $\mathcal{D}^0$ from Lemma 4.7.*

With the above preliminary considerations, we can now state our main result from this section. Below we denote by $\{\omega_\alpha\in(\mathfrak{g}^{\mathbb{C}})^*,\alpha\in R\}$ the covectors defined by $\omega_\alpha(E_\beta)=\delta_{\alpha\beta}$ for any $\alpha,\beta\in R$ and $\omega_\alpha|_{\mathfrak{h}}=0$. We use the notation $R_0^{\mathrm{sym}}:=R_0\cap(-R_0)$ for the symmetric part of R_0.

Theorem 4.9. — *Consider a triple $(\mathfrak{k},\mathcal{D}^0,\epsilon)$, with $\mathfrak{k}$ the regular subalgebra (4.11), $\mathcal{D}^0$ as in Lemma 4.7 and $\epsilon\in\mathfrak{k}^*\otimes\bar{\mathfrak{k}}^*$ skew-Hermitian. Assume that*

$$(\alpha+\beta)|_{\mathfrak{h}_{\mathfrak{k}}}\neq 0,\quad\forall\alpha,\beta\in R_0\cup\{0\},\ \alpha+\beta\neq 0. \tag{4.18}$$

Then $(\mathfrak{k},\mathcal{D}^0,\epsilon)$ is a (symmetric) $\mathfrak{g}$-admissible triple (and the associated pair $(\mathcal{J},D^0)$ defines a complex structure $J^{\mathcal{J},D^0}$ on T^*G) if and only if the following conditions hold:

i) the root system R_0 of $\mathfrak{k}$ is a σ-parabolic system (see Definition 4.5) and the Cartan part satisfies $\mathfrak{h}_{\mathfrak{k}} + \bar{\mathfrak{h}}_{\mathfrak{k}} = \mathfrak{h}$;

ii) the skew-Hermitian 2-form $\epsilon \in \mathfrak{k}^* \otimes \bar{\mathfrak{k}}^*$ is given by

$$\begin{aligned}\epsilon = \epsilon_0 &+ \sum_{\alpha\in R_0} \mu_\alpha(\alpha \otimes \omega_{\sigma(\alpha)} + a_\alpha \omega_\alpha \otimes \sigma(\alpha))\\ &- \sum_{\alpha,\beta,\alpha+\beta\in R_0} a_\alpha \mu_{\alpha+\beta} N_{\sigma(\alpha)\sigma(\beta)} \omega_\alpha \otimes \omega_{\sigma(\beta)}\\ &+ \sum_{\gamma\in R_0^{\mathrm{sym}}} \nu_\gamma \omega_\gamma \otimes \omega_{-\sigma(\gamma)}\end{aligned} \tag{4.19}$$

where $\epsilon_0 \in \mathfrak{h}_{\mathfrak{k}}^* \otimes \bar{\mathfrak{h}}_{\mathfrak{k}}^*$ is skew-Hermitian (trivially extended to $\mathfrak{k}$), μ_α, ν_γ ($\alpha \in R_0$, $\gamma \in R_0^{\mathrm{sym}}$) are any real constants, such that the ν_α's satisfy

$$\nu_\alpha + \nu_{-\alpha} = 0, \quad \forall \alpha \in R_0^{\mathrm{sym}} \tag{4.20}$$

and, for any $\alpha, \beta, \gamma \in R_0^{\mathrm{sym}}$, with $\alpha + \beta + \gamma = 0$,

$$a_\alpha \nu_\alpha + a_\beta \nu_\beta + a_\gamma \nu_\gamma = 0. \tag{4.21}$$

iii) The pseudo-Riemannian metric $g_\Delta := \mathrm{Im}(\epsilon|_{\mathfrak{k}\cap\mathfrak{g}})$ is non-degenerate and

$$\epsilon_0(H, H_{\sigma(\alpha)}) = 0, \quad \forall H \in \mathfrak{h}_{\mathfrak{k}}, \quad \forall \alpha \in R_0^{\mathrm{sym}}. \tag{4.22}$$

Proof. — From Definition 4.2, Lemma 4.6 and Lemma 4.7, we need to prove that ϵ satisfies

$$\epsilon(X, \mathcal{D}^0_Y(\bar{Z})) - \epsilon(Y, \mathcal{D}^0_X(\bar{Z})) = \epsilon([X,Y], \bar{Z}), \quad \forall X, Y, Z \in \mathfrak{k} \tag{4.23}$$

if and only if it is of the form (4.19) and conditions (4.20), (4.21) and (4.22) are satisfied. In order to prove this statement, we choose various arguments in (4.23). Below, $H, \tilde{H} \in \mathfrak{h}_{\mathfrak{k}}$ and $\alpha, \beta, \gamma \in R_0$. First, let $X := H$, $Y := \tilde{H}$ and $Z := E_\alpha$. With these arguments, (4.23) becomes

$$\alpha(\tilde{H})\epsilon(H, E_{\sigma(\alpha)}) = \alpha(H)\epsilon(\tilde{H}, E_{\sigma(\alpha)}).$$

From (4.18), $\alpha|_{\mathfrak{h}_{\mathfrak{k}}}$ is non-trivial. Choosing $\tilde{H}$ such that $\alpha(\tilde{H}) \neq 0$, we deduce that the above relation is equivalent to

$$\epsilon(H, E_{\sigma(\alpha)}) = \mu_\alpha \alpha(H), \quad \forall H \in \mathfrak{h}_{\mathfrak{k}}, \quad \forall \alpha \in R_0, \tag{4.24}$$

for a constant $\mu_\alpha \in \mathbb{C}$. By letting $X := H$, $Y := E_\alpha$ and $Z := \tilde{H}$ in (4.23), we obtain that $\mu_\alpha \in \mathbb{R}$, for any $\alpha \in R_0$.

Next, let $X := E_\alpha$, $Y := H$ and $Z := E_\beta$ in (4.23). We obtain

$$\epsilon(E_\alpha, \mathcal{D}^0_H(\bar{E}_\beta)) - \epsilon(H, \mathcal{D}^0_{E_\alpha}(\bar{E}_\beta)) = \epsilon([E_\alpha, H], \bar{E}_\beta)$$

or

$$(\alpha+\beta)(H)\epsilon(E_\alpha, E_{\sigma(\beta)}) + a_\alpha \epsilon(H, [E_{\sigma(\alpha)}, E_{\sigma(\beta)}]) = 0. \tag{4.25}$$

If $\alpha+\beta\neq 0$, then $(\alpha+\beta)|_{\mathfrak{h}_{\mathfrak{k}}}$ is non-trivial, by (4.18), and the above relation, together with (4.24), gives

$$\begin{aligned}&\epsilon(E_\alpha,E_{\sigma(\beta)})=-a_\alpha\mu_{\alpha+\beta}N_{\sigma(\alpha)\sigma(\beta)},\ \forall\alpha,\beta,\alpha+\beta\in R_0,\\&\epsilon(E_\alpha,E_{\sigma(\beta)})=0,\ \forall\alpha,\beta\in R_0,\ \alpha+\beta\notin R\cup\{0\}.\end{aligned}\tag{4.26}$$

If $\alpha+\beta=0$, relation (4.25) gives (4.22).

We now remark that conditions (4.24) and (4.26) imply that ϵ is of the form (4.19), with $\mu_\alpha\in\mathbb{R}$ ($\alpha\in R_0$) and $\nu_\alpha:=\epsilon(E_\alpha,E_{-\sigma(\alpha)})\in\mathbb{C}$ ($\alpha\in R_0^{\mathrm{sym}}$).

We still need to consider (4.23), with the remaining two types of arguments: $X=E_\alpha$, $Y=E_\beta$, $Z:=H$, and, respectively, $X:=E_\alpha$, $Y:=E_\beta$, $Z:=E_\gamma$ (from the definition of $\mathcal{D}^0$, (4.23) holds when all X, Y, Z belong to the Cartan part $\mathfrak{h}_{\mathfrak{k}}$).

Let $X=E_\alpha$, $Y=E_\beta$, $Z:=H$. Relation (4.23) gives

$$\overline{\beta(H)}a_\beta\epsilon(E_\alpha,E_{\sigma(\beta)})+\overline{\alpha(H)}a_\beta\overline{\epsilon(E_\alpha,E_{\sigma(\beta)})}=\epsilon([E_\alpha,E_\beta],\bar{H}).\tag{4.27}$$

When $\alpha+\beta\neq 0$, relation (4.27) follows from (4.24) and (4.26) (and the skew-Hermitian property of ϵ). When $\alpha+\beta=0$, relation (4.27) implies that $\nu_\alpha\in\mathbb{R}$, for any $\alpha\in R_0^{\mathrm{sym}}$. Since ϵ is skew-Hermitian and $\nu_\alpha\in\mathbb{R}$, relation (4.20) holds.

Finally, let $X:=E_\alpha$, $Y:=E_\beta$, $Z:=E_\gamma$ in (4.23). From (4.24), (4.26) and $\mu_\alpha,\nu_\beta\in\mathbb{R}$, relation (4.23) is automatically satisfied, when $\alpha+\beta+\gamma\neq 0$; when $\alpha+\beta+\gamma=0$, we obtain

$$a_\beta N_{\sigma(\beta)\sigma(\gamma)}\nu_\alpha+a_\alpha N_{\sigma(\gamma)\sigma(\alpha)}\nu_\beta+N_{\beta\alpha}\nu_\gamma=0.\tag{4.28}$$

Using now the relations

$$N_{\sigma(\beta)\sigma(\gamma)}=-a_{\beta+\gamma}a_\beta a_\gamma N_{\beta\gamma},\quad N_{\sigma(\gamma)\sigma(\alpha)}=-a_{\alpha+\gamma}a_\alpha a_\gamma N_{\gamma\alpha}$$

and $N_{\alpha\beta}=N_{\beta\gamma}=N_{\gamma\alpha}$ (because $\alpha+\beta+\gamma=0$; see Subsection 4.1), we obtain that (4.28) is equivalent to (4.21). Our claim follows. □

The statement of Theorem 4.9 requires various comments. First, we need to explain how the constants ν_α can be constructed, such that (4.20) and (4.21) are satisfied. Next, we need to study the non-degeneracy of g_Δ. This will be done in the following paragraphs.

4.4.1. The construction of ν_α

Let R_0 be a σ-parabolic system of R (we remark that the following argument holds for any closed subsystem of R, not necessarily σ-parabolic). In this paragraph, we describe a method to construct real constants ν_α,

$\alpha \in R_0^{\mathrm{sym}}$, such that conditions (4.20) and (4.21) from Theorem 3.3 hold. Since R_0^{sym} is closed and symmetric, it is a root system (see e.g. [3], page 164). Let $\Pi := \{\alpha_0, \cdots, \alpha_k\}$ be a system of simple roots of R_0^{sym}. Define, as usual, the height of $\alpha = n_1\alpha_1 + \cdots + n_k\alpha_k \in R_0^{\mathrm{sym}}$ with respect to Π, by $n(\alpha) := n_1 + \cdots + n_k$.

Lemma 4.10. — *The constants $\nu_\alpha := a_\alpha n(\alpha)$, for any $\alpha \in R_0^{\mathrm{sym}}$, satisfy (4.20) and (4.21).*

Proof. — The hight function $n : R_0^{\mathrm{sym}} \to \mathbb{Z}$ is additive. In particular, $n(-\alpha) = -n(\alpha)$ and if $\alpha + \beta + \gamma = 0$, then $n(\alpha) + n(\beta) + n(\gamma) = 0$. Recall also that $a_\alpha^2 = 1$ and $a_{-\alpha} = a_\alpha$ for any α. The claim follows. $\square$

4.4.2. The non-degeneracy of g_Δ

We begin with the simplest case, when R_0 is a σ-positive system.

Remark 4.11. — We consider a triple $(\mathfrak{k}, \mathcal{D}^0, \epsilon)$ satisfying the conditions i) and ii) of Theorem 4.9. We assume, moreover, that R_0 is a σ-positive system (not only σ-parabolic). Then $\Delta = (\mathfrak{k} \cap \bar{\mathfrak{k}})^\sigma$ reduces to $(\mathfrak{h}_\mathfrak{k} \cap \bar{\mathfrak{h}}_\mathfrak{k})^\sigma$ and the non-degeneracy of $g_\Delta = \mathrm{Im}(\epsilon|_\Delta)$ concerns only the Cartan part ϵ_0 of ϵ. Our aim is to show that, under a mild additional assumption, we can choose the Cartan part ϵ_0 of ϵ such that g_Δ is non-degenerate and (4.22) is satisfied as well. More precisely, assume that the subspace

$$\mathcal{S} := \mathrm{Span}_\mathbb{C}\{H_\alpha,\ \alpha \in R_0^{\mathrm{sym}}\}$$

is transverse to its conjugate

$$\bar{\mathcal{S}} = \sigma(\mathcal{S}) = \mathrm{Span}_\mathbb{C}\{H_{\sigma(\alpha)},\ \alpha \in R_0^{\mathrm{sym}}\}.$$

(We remark that this holds for many σ-positive systems, see Subsections 5.1-5.3 of [2]). A simple argument (see [2], Section 5), then shows that the Cartan subalgebra $\mathfrak{h}_\mathfrak{k}$ of $\mathfrak{k}$ decomposes as a direct sum

$$\mathfrak{h}_\mathfrak{k} = (\mathfrak{h}_\mathfrak{k} \cap \bar{\mathfrak{h}}_\mathfrak{k}) \oplus \mathcal{S} \oplus \mathcal{W} \tag{4.29}$$

where $\mathcal{W} \subset \mathfrak{h}_\mathfrak{k}$ is any complementary subspace of $(\mathfrak{h}_\mathfrak{k} \cap \bar{\mathfrak{h}}_\mathfrak{k}) \oplus \mathcal{S}$. Choose $\epsilon_0 \in \mathfrak{h}_\mathfrak{k}^* \otimes \bar{\mathfrak{h}}_\mathfrak{k}^*$ such that

$$\epsilon_0(\mathcal{S}, \cdot) = \epsilon_0(\cdot, \bar{\mathcal{S}}) = 0$$

(i.e. (4.22) is satisfied) and $g_\Delta = \mathrm{Im}(\epsilon|_{(\mathfrak{h}_\mathfrak{k} \cap \bar{\mathfrak{h}}_\mathfrak{k})^\sigma})$ is non-degenerate. With this choice, all conditions of Theorem 4.9 are satisfied, $(\mathfrak{k}, \mathcal{D}^0, \epsilon)$ is a symmetric $\mathfrak{g}$-admissible triple and the associated pair $(\mathcal{J}, D^0)$ has the property that $J^{\mathcal{J}, D^0}$ is integrable.

In order to study the non-degeneracy of g_Δ in general (i.e. when R_0 is σ-parabolic, not necessarily σ-positive) we choose a preferred basis of Δ and we compute g_Δ in this basis. To simplify the arguments, we assume that $R_0 \cap \sigma(R_0)$ is symmetric (this is always satisfied, when $\mathfrak{g}$ is of inner type). Then $\mathfrak{k} \cap \bar{\mathfrak{k}}$ is reductive. Its real form $\Delta = (\mathfrak{k} \cap \bar{\mathfrak{k}})^\sigma$ is given by

$$\Delta = \mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{h}_{\mathfrak{g}} + \sum_{\alpha \in R_0 \cap \sigma(R_0)} \mathbb{R}A_\alpha + \sum_{\alpha \in R_0 \cap \sigma(R_0)} \mathbb{R}B_\alpha,$$

where, as in the proof of Lemma 4.7, $A_\alpha := E_\alpha - a_\alpha E_{\sigma(\alpha)}$ and $B_\alpha := i(E_\alpha + a_\alpha E_{\sigma(\alpha)})$. Since $R_0 \cap \sigma(R_0)$ is symmetric, $H_\alpha = [E_\alpha, E_{-\alpha}] \in \mathfrak{k} \cap \bar{\mathfrak{k}}$, for any $\alpha \in R_0 \cap \sigma(R_0)$. Define new vectors

$$F_\alpha^+ := H_\alpha + H_{\sigma(\alpha)},\ F_\alpha^- := i(H_\alpha - H_{\sigma(\alpha)}), \quad \forall \alpha \in R_0 \cap \sigma(R_0).$$

They belong to $\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{h}_{\mathfrak{g}}$. It follows that

$$\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{h}_{\mathfrak{g}} = \mathrm{Span}_{\mathbb{R}}\{F_\alpha^+,\ \alpha \in R_0 \cap \sigma(R_0)\} \oplus \mathrm{Span}_{\mathbb{R}}\{F_\alpha^-, \alpha \in R_0 \cap \sigma(R_0)\} \oplus \mathcal{C},$$

where

$$\mathcal{C} = \mathrm{Ann}(R_0 \cap \sigma(R_0))|_{\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{h}_{\mathfrak{g}}}.$$

Let $\{c_1, \cdots, c_s\}$ be a basis of $\mathcal{C}$. Choose a maximal system of linearly independent vectors $\{F_1^+, \cdots, F_p^+\}$ from $\{F_\alpha^+, \alpha \in R_0 \cap \sigma(R_0)\}$ and similarly, a maximal system of linearly independent vectors $\{F_1^-, \cdots, F_q^-\}$ from $\{F_\alpha^-, \alpha \in R_0 \cap \sigma(R_0)\}$. It follows that the system of vectors

$$\mathcal{B} := \{c_k,\ F_r^+,\ F_t^-,\ A_\alpha,\ B_\alpha,\ \alpha \in R_0 \cap \sigma(R_0)\}$$

(where $1 \leqslant k \leqslant s$, $1 \leqslant r \leqslant p$, $1 \leqslant t \leqslant q$) form a basis $\mathcal{B}$ of Δ.

Lemma 4.12. — *Let $\epsilon \in \mathfrak{k}^* \otimes \bar{\mathfrak{k}}^*$ be given by (4.19), such that condition (4.22) is satisfied. Assume, moreover, that $R_0 \cap \sigma(R_0)$ is symmetric. With respect to the basis $\mathcal{B}$ above, all the entries of $g_\Delta = \mathrm{Im}(\epsilon|_\Delta)$ are zero except:*

$$\begin{aligned} g_\Delta(A_\alpha, B_\beta) =& - a_\alpha N_{\sigma(\alpha)\beta}(\mu_{\alpha+\sigma(\beta)} + a_{\sigma(\alpha)+\beta}\mu_{\sigma(\alpha)+\beta}) \\ &+ N_{\alpha\beta}(\mu_{\sigma(\alpha+\beta)} + a_{\alpha+\beta}\mu_{\alpha+\beta}) \\ g_\Delta(F_r^+, B_\alpha) =& (\mu_{\sigma(\alpha)} + a_\alpha\mu_\alpha)\alpha(F_r^+) \\ g_\Delta(F_t^-, A_\alpha) =& i(\mu_{\sigma(\alpha)} + a_\alpha\mu_\alpha)\alpha(F_t^-) \end{aligned}$$

(where $N_{\delta\gamma} = \mu_{\delta+\gamma} = 0$ for $\delta, \gamma \in R_0$, such that $\delta + \gamma \notin R$). In particular, if g_Δ is non-degenerate, then

$$\dim_{\mathbb{R}}\langle \alpha + \sigma(\alpha),\ \alpha \in R_0 \cap \sigma(R_0)\rangle = \dim_{\mathbb{R}}\langle \alpha - \sigma(\alpha),\ \alpha \in R_0 \cap \sigma(R_0)\rangle. \tag{4.30}$$

Proof. — One can check, using (4.19) and (4.22), that the entries of g_Δ have the required form (for example, (4.22) means that F_r^+ and F_t^- belong to the kernel of $g_\Delta|_{\mathfrak{h}_\mathfrak{k}\cap\mathfrak{h}_\mathfrak{g}}$). It is also easy to check that if the matrix which represents g_Δ in the basis $\mathcal{B}$ is non-degenerate, then $p = q$, i.e. relation (4.30) is satisfied. □

Remark 4.13. — We end this discussion by analysing the condition (4.30) from Lemma 4.12. Let $R_0 \subset R$ be a closed subset of roots, such that $R_0' := R_0 \cap \sigma(R_0)$ is symmetric and (4.30) holds. Since R_0' is symmetric and closed, it is the root system of the σ-invariant complex semisimple subalgebra

$$(\mathfrak{g}')^{\mathbb{C}} := \mathfrak{h}' + \sum_{\alpha\in R_0'} \mathfrak{g}_\alpha \subset \mathfrak{g}^{\mathbb{C}},$$

where $\mathfrak{h}' := \mathrm{Span}_{\mathbb{C}}\{H_\alpha,\ \alpha \in R_0'\}$ is a σ-invariant Cartan subalgebra of $(\mathfrak{g}')^{\mathbb{C}}$. The action of σ on the subset of roots $R_0' \subset R$ is induced by an antilinear involution of $(\mathfrak{g}')^{\mathbb{C}}$, namely by the restriction σ' of σ to $(\mathfrak{g}')^{\mathbb{C}}$. Let $\mathfrak{g}' = (\mathfrak{g}')^{\mathbb{C}} \cap \mathfrak{g}$ be the real form of $(\mathfrak{g}')^{\mathbb{C}}$ defined by σ'. Then $\mathfrak{h}'_{\mathfrak{g}'} := (\mathfrak{h}')^{\sigma'}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}'$. If we assume, in addition, that R_0' is irreducible, then $(\mathfrak{g}')^{\mathbb{C}}$ is a simple Lie algebra. It is easy to see that condition (4.30) holds if and only if the automorphism of the Vogan diagram of $\mathfrak{g}'$ has no fixed points. By inspecting the Vogan diagrams of simple, non-complex real Lie algebras (see e.g. [9], Appendix C) we deduce that (4.30) holds if and only if $(\mathfrak{g}')^{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}(2n+1,\mathbb{C})$ and $\mathfrak{g}'$ is the real form $\mathfrak{sl}(2n+1,\mathbb{R}) \subset \mathfrak{sl}(2n+1,\mathbb{C})$.

4.4.3. Symmetric $\mathfrak{g}$-admissible triples of inner type

Theorem 4.9 provides a complete explicit description of symmetric $\mathfrak{g}$-admissible triples $(\mathfrak{k}, \mathcal{D}^0, \epsilon)$ of inner type, as follows.

THEOREM 4.14. — *Let $\mathfrak{g}$ be a real form of inner type of $\mathfrak{g}^{\mathbb{C}}$, given by (4.7) (with $\sigma(\alpha) = -\alpha$ for any $\alpha \in R$). Consider a triple $(\mathfrak{k}, \mathcal{D}^0, \epsilon)$ with $\mathfrak{k}$ the regular subalgebra (4.11), $\mathcal{D}^0$ as in Lemma 4.7 and $\epsilon \in \mathfrak{k}^* \otimes \bar{\mathfrak{k}}^*$ skew-Hermitian. Then $(\mathfrak{k}, \mathcal{D}^0, \epsilon)$ is a (symmetric) $\mathfrak{g}$-admissible triple (and the associated pair $(\mathcal{J}, D^0)$ defines a complex structure $J^{\mathcal{J},D^0}$ on T^*G) if and only if:*

i) the root system R_0 of $\mathfrak{k}$ is a positive root system ($R_0 = R^+$) and the Cartan part satisfies $\mathfrak{h}_\mathfrak{k} + \bar{\mathfrak{h}}_\mathfrak{k} = \mathfrak{h}$;

ii) ϵ is of the form

$$\epsilon = \epsilon_0 + \sum_{\alpha\in R^+} \mu_\alpha(\alpha \otimes \omega_{-\alpha} - a_\alpha\omega_\alpha \otimes \alpha) + \sum_{\alpha,\beta,\alpha+\beta\in R^+} a_\alpha\mu_{\alpha+\beta}N_{\alpha\beta}\omega_\alpha \otimes \omega_{-\beta}$$

where $\epsilon_0 \in \Lambda^2(\mathfrak{h}_{\mathfrak{k}})$ *is trivially extended to* $\mathfrak{k}$*, and* μ_α *(*$\alpha \in R^+$*) are arbitrary real constants;*

iii) $\mathrm{Im}(\epsilon_0|_{\mathfrak{h}_{\mathfrak{k}} \cap i\mathfrak{h}_{\mathbb{R}}})$ *is non-degenerate.*

Proof. — We use Theorem 4.9. Since $\sigma|_R = -\mathrm{Id}$, $R_0 \cap \sigma(R_0)$ is symmetric and relation (4.30) implies that $R_0 \cap (-R_0) = \emptyset$. Since $R_0 \cup (-R_0) = R$, from a result of Bourbaki we obtain that $R_0 = R^+$ is a positive root system. Condition (4.18) is satisfied (this follows from $\mathfrak{h}_{\mathfrak{k}} + \bar{\mathfrak{h}}_{\mathfrak{k}} = \mathfrak{h}$ and $\sigma|_R = -\mathrm{Id}$). Conditions (4.20), (4.21) and (4.22) do not apply (R^+ is skew-symmetric) and the intersection $\mathfrak{k} \cap \bar{\mathfrak{k}}$ reduces to its Cartan part $\mathfrak{h}_{\mathfrak{k}} \cap \bar{\mathfrak{h}}_{\mathfrak{k}}$. □

5. Special complex geometry

In this section we develop further applications of Theorem 3.3, in relation to special complex geometry.

Proposition 5.1. — *Let* (M, J, D) *be a manifold with an almost complex structure* J *and a linear connection* D*. The almost complex structure* $J^{\pm} := J^{\mathcal{J}^{\pm}, D}$ *on* T^*M*, defined by* D *and the generalized complex structure*

$$\mathcal{J}^{\pm} := \begin{pmatrix} J & 0 \\ 0 & \pm J^* \end{pmatrix}$$

is integrable if and only if J *is a complex structure,* $D_X(J) = \pm J D_{JX}(J)$ *and*

$$(R^D_{X,Y} - R^D_{JX,JY})(Z) \pm (R^D_{JX,Y} + R^D_{X,JY})(JZ) = 0, \tag{5.1}$$

for any $X, Y, Z \in TM$.

Proof. — The generalized complex structure $\mathcal{J}^+$ is symmetric, with holomorphic bundle $T^{1,0}M \oplus \mathrm{Ann}(T^{0,1}M)$, while $\mathcal{J}^-$ is skew-symmetric, with holomorphic bundle $T^{1,0}M \oplus \mathrm{Ann}(T^{1,0}M)$. From Theorem 3.3, if $\mathcal{J}^{\pm}$ is integrable, then the bundle $T^{1,0}M$ is involutive, i.e J is an (integrable) complex structure. Also, $D_{\Gamma(T^{1,0}M)}\Gamma(T^{1,0}M) \subset \Gamma(T^{1,0}M)$ if and only if $D_X(J) = -JD_{JX}(J)$, while $D_{\Gamma(T^{1,0}M)}\Gamma(T^{0,1}M) \subset \Gamma(T^{0,1}M)$ if and only if $D_X(J) = JD_{JX}(J)$, for any $X \in TM$. The condition $R^D|_{T^{1,0}M, T^{1,0}M}(\tau(T^{1,0}M)) = 0$ from Theorem 3.3 translates to (5.1). Condition (3.4) from Theorem 3.3 is also satisfied, because $\alpha = 0$ (in both cases). Our claim follows. □

As already mentioned in the introduction, the first statement of the following corollary was proved in [1] using different methods.

Corollary 5.2. — *Consider the setting of Proposition 5.1.*

i) If (J, D) is a special complex structure, i.e. J is integrable and D is flat, torsion-free, such that

$$(d^D J)_{X,Y} := D_X(J)(Y) - D_Y(J)(X) = 0, \quad \forall X, Y \in TM,$$

then J^+ is integrable.

ii) If $D = D^g$ is the Levi-Civita connection of an almost Hermitian structure (g, J), then J^+ is integrable if and only if (g, J) is Kähler and J^- is integrable if and only if J is integrable and the curvature of g satisfies

$$(R^D_{X,Y} - R^D_{JX,JY})(Z) - (R^D_{JX,Y} + R^D_{X,JY})(JZ) = 0, \quad \forall X, Y, Z \in TM.$$

iii) If D is the Chern connection of a Hermitian structure (J, g), then both $J^\pm$ are integrable.

Proof. — The claims follow from Proposition 5.1. For i), we remark that the special complex condition $d^D J = 0$ implies $D_X(J) = JD_{JX}(J)$ for any $X \in TM$. For ii), we use that $D^g_X(J) = -JD^g_{JX}(J)$, for any $X \in TM$, if and only if J is integrable (see [5] or [4, Proposition 1]). This proves the statement for J^-. The statement for J^+ follows as well: if J is integrable and $D^g_X(J) = JD^g_{JX}(J)$, then $D^g J = 0$ and (g, J) is Kähler. For iii), we use that the Chern connection is Hermitian with curvature of type $(1, 1)$. □

The following lemma is a mild improvement of Lemma 6 of [1].

Lemma 5.3. — *Let (M, ω, D) be a manifold with a non-degenerate 2-form ω and a linear connection D. The almost complex structure on T^*M defined by D and the (skew-symmetric) generalized complex structure*

$$\mathcal{J}^\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}$$

is integrable if and only if D is flat and, for any $X, Y, Z \in \mathcal{X}(M)$,

$$(d\omega)(X, Y, Z) - (D_Z\omega)(X, Y) - \omega(T^D_Z X, Y) - \omega(X, T^D_Z Y) = 0.$$

Proof. — The holomorphic bundle of $\mathcal{J}^\omega$ is $L(T^{\mathbb{C}}M, i\omega^{\mathbb{C}})$ and the claim follows from Theorem 3.3 and relation (3.12). □

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