ANNALES DE L'INSTITUT FOURIER

BAHMAN SAFFARI
R. C. VAUGHAN

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Annales de l'institut Fourier, tome 27, nº 2 (1977), p. 31-36
<http://www.numdam.org/item?id=AIF_1977__27_2_31_0>

Ann. Inst. Fourier, Grenoble
27, 2 (1977), 31-36.

ON THE FRACTIONAL PARTS OF x/n AND RELATED SEQUENCES. III

by B. SAFFARI and R.-C. VAUGHAN

1. Introduction.

The object of this paper is to investigate the behaviour of $\Phi_{x,y}(\alpha, h)$ (for notation see [2] and [3]) when

$$h(n) = \frac{1}{\log n}\,(n > 1)$$

and $h(n) = \log n$. In contradistinction to the case $h(n) = 1/n$ it is immediately apparent that the behaviour of $\Phi_{x,y}$ is non-trivial even when y is a large as e^x. For simplicity we only investigate the situation when $\mathcal{A}$ is the Toeplitz transformation formed from the simple Riesz means (R, λ_n) with $\lambda_n = 1$.

Theorems 1 and 2 deal with the case $h(n) = 1/\log n$, whereas Theorem 3 deals with $h(n) = \log n$. While it is well known ([1], Example 2.4, p. 8) that the sequence $\log n$ is not uniformly distributed modulo 1, Theorem 3 shows that it is uniformly distributed in the present context.

2. Theorems and proofs.

2.1. Let

$$(2.1) \qquad \Xi_{x,y}(\alpha) = y^{-1} \sum_{2 \leqslant n \leqslant y} c_\alpha(x/\log n).$$

THEOREM 1. — *Suppose that* $0 < \alpha < 1$ *and* $\log y \leqslant x^{\frac{1}{2}}$. *Then*

$$\Xi_{x,y}(\alpha) = \alpha + \mathrm{O}(xy^{-1}\,(\log x)^{-1}) + \mathrm{O}(x^{-1}\log^2 y).$$

Corollary 1.1. — *Suppose that* $x = o(y \log x)$ *and* $\log y = o\left(x^{\frac{1}{2}}\right)$ *as* $x \to \infty$. *Then*

$$\Xi_{x,y}(\alpha) \to \alpha \quad as \quad x \to \infty.$$

Proof. — Clearly by (2.1),

$$y\Xi_{x,y}(\alpha) = S(0) - S(\alpha) \tag{2.2}$$

where

$$S(\beta) = \sum_{m=1}^{\infty} \sum_{\substack{2 \leqslant n \leqslant y \\ n \leqslant e^{x/(m+\beta)}}} 1 \tag{2.3}$$

and $0 \leqslant \beta < 1$. Let

$$M_\beta = \left[\frac{x}{\log y} - \beta\right] \tag{2.4}$$

and

$$T(\beta) = \sum_{M_\beta < m \leqslant \frac{x}{\log 2} - \beta} \; \sum_{2 \leqslant n \leqslant e^{x/(m+\beta)}} 1. \tag{2.5}$$

Then, by (2.3),

$$S(\beta) = T(\beta) + ([y] - 1)M_\beta. \tag{2.6}$$

By (2.5),

$$T(\beta) = \sum_{M_\beta < m \leqslant H - \beta} e^{x/(m+\beta)} + \sum_{2 \leqslant n \leqslant e^{x/H}} \frac{x}{\log n} - He^{x/H} + O(H) + O(e^{x/H})$$

where H is a real number at our disposal. Hence, by (2.4),

$$T(0) - T(\alpha) = \sum_{M_0 < m \leqslant H} e^{x/m} - \sum_{M_\alpha < m \leqslant H - \alpha} e^{x/(m+\alpha)} + O(H) + O(e^{x/H}) \tag{2.7}$$

whenever $H \geqslant M_0 + 1$. Thus

$$T(0) - T(\alpha) = I(0) - I(\alpha) + O(H) + O(e^{x/H}), \tag{2.8}$$

where

$$I(\beta) = \int_{M_\beta}^{H-\beta} ([u] - M_\beta) e^{x/(u+\beta)} \frac{x\,du}{(u+\beta)^2}. \tag{2.9}$$

Let $b(u)$ denote the first Bernoulli polynomial modulo one,

$b(u) = \{u\} - 1/2$. Then, by (2.9),

$$(2.10)\quad \mathrm{I}(\beta) = \int_{\mathrm{M}_\beta+\beta}^{\mathrm{H}} (\nu - \mathrm{M}_\beta - \beta - 1/2)e^{x/\nu}x\nu^{-2}\,d\nu - \int_{\mathrm{M}_\beta+\beta}^{\mathrm{H}} b(u-\beta)e^{x/\nu}x\nu^{-2}\,d\nu.$$

The argument now divides into two cases according as $\mathrm{M}_0 = \mathrm{M}_\alpha$ or $\mathrm{M}_0 = \mathrm{M}_\alpha + 1$.

The case $\mathrm{M}_0 = \mathrm{M}_\alpha$. Write M for the common value. Then, by (2.10),

$$\mathrm{I}(0) - \mathrm{I}(\alpha) = \int_{\mathrm{M}}^{\mathrm{M}+\alpha}\left(\nu - \mathrm{M} - \frac{1}{2}\right)e^{x/\nu}x\nu^{-2}\,d\nu + \alpha\int_{\mathrm{M}+\alpha}^{\mathrm{H}} e^{x/\nu}x\nu^{-2}\,d\nu - \int_{\mathrm{M}}^{\mathrm{H}} b(\nu)e^{x/\nu}\,x\nu^{-2}\,d\nu + \int_{\mathrm{M}+\alpha}^{\mathrm{H}} b(\nu-\alpha)e^{x/\nu}\,x\nu^{-2}\,d\nu.$$

The first integral contributes $\ll e^{x/\mathrm{M}}x\mathrm{M}^{-2}$, the second is $\alpha(e^{x/(\mathrm{M}+\alpha)} - e^{x/\mathrm{H}})$ and by partial integration the last two are easily seen to contribute $\ll e^{x/\mathrm{M}}x\mathrm{M}^{-2}$. Hence, by (2.8),

$$(2.11)\quad \mathrm{T}(0) - \mathrm{T}(\alpha) = \alpha e^{x/(\mathrm{M}+\alpha)} + \mathrm{O}(\mathrm{H}) + \mathrm{O}(e^{x/\mathrm{H}}) + \mathrm{O}(e^{x/\mathrm{M}}x\mathrm{M}^{-2}).$$

Recall that $\mathrm{M} = \mathrm{M}_0 = [x/\log y]$ and $\log y \ll x^{1/2}$. Thus

$$e^{x/(\mathrm{M}+\alpha)} = \exp\,(\log y + \mathrm{O}(x^{-1}\log^2 y)) = y(1 + \mathrm{O}(x^{-1}\log^2 y))$$

and $e^{x/\mathrm{M}}x\mathrm{M}^{-2} = \mathrm{O}(yx^{-1}\log^2 y)$. Hence, by (2.2), (2.6) and (2.11)

$$y\Xi_{x,y}(\alpha) = \alpha y + \mathrm{O}(\mathrm{H}) + \mathrm{O}(e^{x/\mathrm{H}}) + \mathrm{O}(yx^{-1}\log^2 y).$$

The choice $\mathrm{H} = \dfrac{x}{\log\,(x/\log x)}$ now gives the desired conclusion.

The case $\mathrm{M}_0 = \mathrm{M}_\alpha + 1$. Write M for M_α. Then, by (2.10),

$$\mathrm{I}(0) - \mathrm{I}(\alpha) = (\alpha - 1)\int_{\mathrm{M}+1}^{\mathrm{H}} e^{x/\nu}x\nu^{-2}\,d\nu - \int_{\mathrm{M}+\alpha}^{\mathrm{M}+1}\left(\nu - \mathrm{M} - \alpha - \frac{1}{2}\right)e^{x/\nu}\,x\nu^{-2}\,d\nu + \mathrm{O}(e^{x/(\mathrm{M}+\alpha)}x(\mathrm{M}+\alpha)^{-2}).$$

Now proceeding as in the previous case we obtain

$$\mathrm{T}(0) - \mathrm{T}(\alpha) = (\alpha - 1)y + \mathrm{O}(\mathrm{H}) + \mathrm{O}(e^{x/\mathrm{H}}) + \mathrm{O}(yx^{-1}\log^2 y).$$

Since $M_0 = M_\alpha + 1$, this with (2.6) and (2.2) and the choice $H = \dfrac{x}{\log(x/\log x)}$ gives the required result once more.

2.2. One might expect that the theorem holds even when y is close to e^x, but this is false. In fact the next theorem indicates that Theorem 1 is essentially best possible, at least as for as the upper bound on y is concerned.

THEOREM 2. — *Suppose that* $0 < \alpha < 1$, $\frac{1}{2} < \theta < 1$ *and* $y = \exp(x^\theta)$. *Then* $\limsup_{x \to \infty} \Xi_{x,y}(\alpha) = 1$ *and*

$$\liminf_{x \to \infty} \Xi_{x,y}(\alpha) = 0.$$

Proof. — We begin by following the proof of Theorem 1 as far as (2.7). Suppose that $0 < \beta < 1$,

$$y = \exp(x^\theta) \tag{2.12}$$

and

$$H = x. \tag{2.13}$$

Then, by (2.4),

$$\frac{x}{(M_\beta + 2)(M_\beta + 3)} \gg x^{-1}(\log y)^2 = x^{2\theta-1}.$$

Thus, by (2.13),

$$\sum_{M_\beta+1<m\leqslant H-\beta} e^{x/(m+\beta)} \ll x e^{x/(M_\beta+1+\beta)} \exp(-C_1 x^{2\theta-1}).$$

Hence, by (2.7) and (2.13),

$$T(0) - T(\alpha) = \left(e^{x/(M_0+1)} - e^{x/(M_\alpha+1+\alpha)}\right)(1 + O(x^{-1})) + O(x). \tag{2.14}$$

To obtain the inferior limit, let N be a large natural number and let

$$x = x_N = (N + \alpha)^{\frac{1}{1-\theta}}. \tag{2.15}$$

Then, by (2.4) and (2.12), $M_0 = M_\alpha = N$. Hence, by (2.2), (2.6), (2.12), (2.14) and (2.15),

$$y\Xi_{x,y}(\alpha) = e^{x/(N+1)} = o(y)$$

as $N \to \infty$.

For the superior limit, take instead

$$x = x_{\mathrm{N}} = \mathrm{N}^{\frac{1}{1-\theta}}. \tag{2.16}$$

Then, by (2.4) and (2.12), $\mathrm{M}_\alpha = \mathrm{M}_0 - 1 = \mathrm{N} - 1$, so that, by (2.2), (2.6), (2.12), (2.14) and (2.16),

$$y\Xi_{x,y}(\alpha) \sim -e^{x/(\mathrm{N}+\alpha)} + y \sim y$$

as $\mathrm{N} \to \infty$.

2.3. The latter part of the paper is devoted to $h(n) = \log n$. It is well known that the sequence $\log n$ is not uniformly distributed modulo 1, and in view of this the next theorem is perhaps rather surprising. However, one can take the view that x being permitted to go to infinity, however slowly by comparison with y, crushes any unruly behaviour of the logarithmic function.

Let

$$\Omega_{x,y}(\alpha) = y^{-1} \sum_{n \leqslant y} c_\alpha(x \log n). \tag{2.17}$$

Theorem 3. — *Suppose that* $0 < \alpha < 1$, $x \geqslant 2$ *and* $y \geqslant 2$. *Then*

$$\Omega_{x,y}(\alpha) = \alpha + \mathrm{O}(x^{-1} \log x + x^{1/3} y^{-2/3} (\log xy)^{2/3}).$$

Corollary 3.1. — *Suppose that* $x^{1/2} \log x = o(y)$ *as* $x \to \infty$. *Then*

$$\Omega_{x,y}(\alpha) \to \alpha \quad \textit{as} \quad x \to \infty.$$

Proof. — Let

$$\mathrm{M} = [y^{2/3} x^{-1/3} (\log xy)^{-2/3}] + 1. \tag{2.18}$$

Then, by Theorem 1 of [2] and (2.17),

$$\Omega_{x,y}(\alpha) - \alpha \ll y^{-1} + \mathrm{M}^{-1} + y^{-1} \sum_{k=1}^{\mathrm{M}} k^{-1} \Big| \sum_{n \leqslant y} e(kx \log n) \Big|. \tag{2.19}$$

Let

$$\mathrm{Y} = [y] + \frac{1}{2} \tag{2.20}$$

and

$$\mathrm{T} = 4\pi k x. \tag{2.21}$$

Then, by Lemma 3.12 of Titchmarsh [4],

$$\sum_{n\leq y} e(kx\log n) = \frac{1}{2\pi i}\int_{1+\frac{1}{\log y}-i\mathrm{T}}^{1+\frac{1}{\log y}+i\mathrm{T}} \zeta(s-2\pi ikx)\frac{\mathrm{Y}^s}{s}\,ds + \mathrm{O}\left(\left(\frac{\mathrm{Y}}{\mathrm{T}}+1\right)\log xy\right)$$

where ζ is the Riemann zeta function. By moving the path of integration to the line $\sigma = 1/\log y$, one obtains

$$\sum_{n\leq y} e(kx\log n) = \frac{y^{1+2\pi ikx}}{1+2\pi ikx} + \frac{1}{2\pi i}\int_{\frac{1}{\log y}-i\mathrm{T}}^{\frac{1}{\log y}+i\mathrm{T}} \zeta(s-2\pi ikx)\frac{\mathrm{Y}^s}{s}\,ds + \mathrm{O}(((kx)^{1/2}+y\log kx)\mathrm{T}^{-1}).$$

Hence, by (2.21),

$$\sum_{n\leq y} e(kx\log n) \ll (kx)^{1/2}\int_0^{\mathrm{T}} \frac{dt}{t+\dfrac{1}{\log y}} + (kx)^{-1/2} + \frac{y\log kx}{kx}$$
$$\ll (kx)^{1/2}(\log\log y + \log kx) + y(\log kx)(kx)^{-1}.$$

Thus

$$\sum_{k=1}^{\mathrm{M}} k^{-1}\left|\sum_{n\leq y} e(kx\log n)\right| \ll (\mathrm{M}x)^{1/2}(\log\log y + \log \mathrm{M}x) + yx^{-1}\log x.$$

Therefore, by (2.18) and (2.19), we have the theorem.

BIBLIOGRAPHY

[1] L. Kuipers and H. Niederreiter, Uniform distribution of sequences, Wiley, New York, 1974.

[2] B. Saffari and R.-C. Vaughan, On the fractional parts of x/n and related sequences. I, *Annales de l'Institut Fourier*, 26, 4 (1976), 115-131.

[3] B. Saffari and R.-C. Vaughan, On the fractional parts of x/n and related sequences, II, *Annales de l'Institut Fourier*, 27, 2 (1977), 1-30.

[4] E. C. Titchmarsh, Theory of the Riemann zeta function, Clarendon Press, Oxford, 1967.

Manuscrit reçu le 23 janvier 1976
Proposé par C. Chabauty.

R.-C. Vaughan,
Imperial College
Dept of Mathematics
Huxley Building
180 Queen's Gate
London Sw7 2Bz (G. B.).

B. Saffari,
1, place Corneille
92100 Boulogne-Billancourt.