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THE ORBITAL COUNTING PROBLEM FOR HYPERCONVEX REPRESENTATIONS

by Andrés SAMBARINO (*)

Abstract. — We give a precise counting result on the symmetric space of a connected noncompact real-algebraic semisimple Lie group G, for a class of discrete subgroups of G that contains, for example, representations of a surface group on $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, induced by choosing two points on the Teichmüller space of the surface and representations on the Hitchin component of $\mathrm{PSL}(d,\mathbb{R})$. We also prove a mixing property for the Weyl chamber flow in this setting.

Résumé. — Nous trouvons un asymptotique pour le comptage orbitale dans l'espace symétrique d'un groupe de Lie connexe, réel-algébrique, semisimple et non-compact G, pour une classe des sous groupes discrets de G qui contient, par exemple, representations d'un groupe de surface dans $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ induites par la choix de deux éléments de l'espace de Teichmüller de la surface et les representations dans la composante de Hitchin de $\mathrm{PSL}(d,\mathbb{R})$. Nous démontrons aussi, dans ce contexte, une propriété de melange pour le flot des chambres de Weyl.

1. Introduction

The *Orbital Counting Problem* is: given a discrete subgroup Δ of a connected noncompact real-algebraic semisimple Lie group G, find an asymptotic for the growth of

$$\#\{g \in \Delta : d_X(o, g \cdot o) \leqslant t\}$$

as $t \to \infty$, where $o = [K]$ is a basepoint on $X = G/K$, the symmetric space of G, endowed with a G-invariant Riemannian metric.

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When the group Δ is a lattice, this problem has been studied by Eskin-McMullen [9]. They prove that the number of points in $\Delta \cdot o \cap B(o,t)$, is equivalent (modulo a constant) to the volume $\mathrm{vol}(B(o,t))$ of the ball of radius t. Hence, the asymptotic has a polynomial term together with an exponential term. Similar results have been obtained by Duke-Rudnick-Sarnak [7].

We will hence focus on subgroups of infinite covolume. An important tool for such groups, in negative curvature, is the limit set of the group on the visual boundary of the space in consideration. On higher rank, it turns out to be more useful to consider the Furstenberg boundary.

Let P be a minimal parabolic subgroup of G, and denote by $\mathscr{F}_G = \mathscr{F} = G/P$ the *Furstenberg boundary* of X. Benoist [2] has shown that the action of Δ on $\mathscr{F}$ has a smallest closed invariant set, called the *limit set of* Δ *on* $\mathscr{F}$, and denoted by L_Δ.

The limit set is well understood for Schottky groups. These are finitely generated free subgroups of G, for which one has a good control on the relative position of the fixed points on $\mathscr{F}$ of the free generators, together with nice contraction properties.

This precise information allows Quint [23] to build an equivariant continuous map, from the boundary at infinity of the group into $\mathscr{F}$. The limit set is hence identified with a subshift of finite type. Quint [23] uses the Thermodynamic Formalism on this subshift, to obtain an exponential equivalence for the orbital counting problem.

This work consists in studying the orbital counting problem, for a class of subgroups called hyperconvex representations, which we will now define.

The product $\mathscr{F} \times \mathscr{F}$ has a unique open G-orbit, denoted by $\mathscr{F}^{(2)}$. For example, when $G = \mathrm{PGL}(d,\mathbb{R})$, the space $\mathscr{F}$ is the space of complete flags of $\mathbb{R}^d$, i.e. families of subspaces $\{V_i\}_{i=0}^d$ such that $V_i \subset V_{i+1}$ and $\dim V_i = i$; and the set $\mathscr{F}^{(2)}$ is the space of pairs of flags in general position, i.e. pairs $(\{V_i\},\{W_i\})$ such that, for every i, one has

$$V_i \oplus W_{d-i} = \mathbb{R}^d.$$

Let Γ be the fundamental group of a closed connected negatively curved Riemannian manifold (for any basepoint).

Definition 1.0. — *We say that a representation $\rho : \Gamma \to G$ is* hyperconvex *if there exists a Hölder-continuous ρ-equivariant map $\zeta : \partial_\infty \Gamma \to \mathscr{F}$, such that the pair $(\zeta(x),\zeta(y))$ belongs to $\mathscr{F}^{(2)}$ whenever $x, y \in \partial_\infty \Gamma$ are distinct.*

If G is a rank 1 simple group, then its Furstenberg boundary is the visual boundary of the symmetric space, and the open orbit $\mathscr{F}^{(2)}$ is

$$\{(x,y) \in \mathscr{F} \times \mathscr{F} : x \neq y\}.$$

The classical Morse's Lemma implies thus that a quasi-isometric embeding $\Gamma \to G$ is a hyperconvex representation (see Efremovich-Tichonirova [8]).

Hyperconvex representations where introduced by Labourie [16], in his study of the Hitchin component. Consider a closed connected oriented surface Σ of genus $g \geqslant 2$. A representation $\pi_1(\Sigma) \to \mathrm{PSL}(d,\mathbb{R})$ is *Fuchsian* if it factors as

$$\pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(d,\mathbb{R}),$$

where $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(d,\mathbb{R})$ is induced by the irreducible linear action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^d$ (unique modulo conjugation by $\mathrm{SL}(d,\mathbb{R})$) and $\pi_1(\Sigma) \to \mathrm{PSL}(2,\mathbb{R})$ is discrete and faithful. A *Hitchin component* of $\mathrm{PSL}(d,\mathbb{R})$, is a connected component of the space $\hom(\pi_1(\Sigma), \mathrm{PSL}(d,\mathbb{R}))$, containing a Fuchsian representation.

THEOREM (Labourie [16]). — *A representation in a Hitchin component of* $\mathrm{PSL}(d,\mathbb{R})$ *is hyperconvex.*

Finally, recall that if H is also a noncompact real-algebraic semisimple Lie group, then the Furstenberg boundary of $G \times H$ is $\mathscr{F}_G \times \mathscr{F}_H$. Hence, if $\rho : \Gamma \to G$ and $\eta : \Gamma \to H$ are hyperconvex representations, so is the product $\rho \times \eta : \Gamma \to G \times H$.

Denote by $C(Z)$ the Banach space of real continuous functions on a compact space Z (with the uniform topology), and by $C^*(Z)$ its topological dual. Denote by $\overline{X}_F$ the Furstenberg compactification of X (see Section 5).

THEOREM (See Section 5). — *Let $\rho : \Gamma \to G$ be a Zariski-dense hyperconvex representation. Then there exist $h, c > 0$, and a probability measure μ on $\overline{X}_F$, such that*

$$ce^{-ht} \sum_{\gamma \in \Gamma : d_X(o, \rho(\gamma)\cdot o) \leqslant t} \delta_{\rho(\gamma)\cdot o} \otimes \delta_{\rho(\gamma^{-1})\cdot o} \to \mu \otimes \mu,$$

for the weak-star convergence on $C^*(\overline{X}_F^2)$, as $t \to \infty$.

Considering the constant function equal to 1, one obtains the following corollary.

COROLLARY. — *Let $\rho : \Gamma \to G$ be a Zariski-dense hyperconvex representation. Then there exist $h, c > 0$, such that*

$$ce^{-ht} \#\{\gamma \in \Gamma : d_X(o, \rho(\gamma)\cdot o) \leqslant t\} \to 1,$$

as $t \to \infty$.

The exponential growth rate h in Theorem A is explicit: it is the topological entropy of a natural flow we construct, associated to the representation ρ. On the contrary, not much information is known about the constant c.

As first shown by Margulis [19] in negative curvature, in order to obtain a counting theorem, one usually proves a mixing property of a well chosen dynamical system. In closed manifolds with negative curvature, the geodesic flow plays this role. In infinite covolume, for example for convex cocompact groups, one should restrict the geodesic flow to its nonwandering set. When Δ is a lattice in higher rank, Eskin-M cMullen [9] use the mixing property of the Weyl chamber flow, to prove the counting result previously mentioned.

Let τ be the Cartan involution on $\mathfrak{g} = \mathrm{Lie}(G)$, whose fixed point set is the Lie algebra of K. Consider $\mathfrak{p} = \{v \in \mathfrak{g} : \tau v = -v\}$ and $\mathfrak{a}$, a maximal abelian subspace contained in $\mathfrak{p}$. Denote by $\mathfrak{a}^+$ a closed Weyl chamber, and M the centralizer of $\exp(\mathfrak{a})$ on K. The *Weyl chamber flow* is the right action by translations of $\exp(\mathfrak{a})$ on

$$\Delta\backslash G/M.$$

When Δ is a lattice on G, the mixing property of this action is due to Howe-Moore [13].

In this article, we prove a mixing property of the Weyl chamber flow for hyperconvex representations. Before stating the result, let us recall the Patterson-Sullivan theory in higher rank.

Consider a G-invariant Riemannian metric in X, and $\|\ \|$ the induced Euclidean norm on $\mathfrak{a}$, invariant under the Weyl group. Consider the Cartan decomposition $G = K\exp(\mathfrak{a}^+)K$, and $a : G \to \mathfrak{a}^+$ the Cartan projection, then for every $g \in G$, one has $\|a(g)\| = d_X([K], g[K])$. Hence, one is interested in understanding the growth of

$$\#\{g \in \Delta : \|a(g)\| \leqslant t\},$$

as $t \to \infty$. Given an open cone $\mathscr{C}$ in $\mathfrak{a}^+$, consider the exponential growth rate

$$h_{\mathscr{C}} = \limsup_{s\to\infty} \frac{\log \#\{g \in \Delta : a(g) \in \mathscr{C},\ \|a(g)\| \leqslant s\}}{s}.$$

The *growth indicator* of Δ, introduced by Quint [22], is the map $\psi_\Delta : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$, defined by

$$\psi_\Delta(v) = \|v\| \inf h_{\mathscr{C}},$$

where the greatest lower bound is taken over all open cones $\mathscr{C}$ containing v. Remark that $\psi_\Delta(tv) = t\psi_\Delta(v)$ if $t \geqslant 0$.

Benoist [2] has introduced the *limit cone* $\mathscr{L}_\Delta$ of Δ, as the closed cone in $\mathfrak{a}^+$ generated by $\{\lambda(g) : g \in \Delta\}$, where $\lambda : G \to \mathfrak{a}^+$ is the Jordan projection. Quint [22] proves the following theorem.

THEOREM 1.0 (Quint [22]). — *Let Δ be a Zariski-dense discrete subgroup of G. Then ψ_Δ is concave, upper semi-continuous and the space*

$$\{v \in \mathfrak{a} : \psi_\Delta(v) > -\infty\},$$

is the limit cone $\mathscr{L}_\Delta$. Moreover ψ_Δ is nonnegative on $\mathscr{L}_\Delta$, and positive on its interior.

The growth indicator plays the role, in higher rank, of the critical exponent in negative curvature. Denote by P the minimal parabolic group of G, associated to the choice of $\mathfrak{a}^+$. The set $\mathscr{F} = G/P$ is K-homogeneous, the group M is the stabilizer in K of $[P] \in \mathscr{F}$. The *Busemann cocycle* $\sigma : G \times \mathscr{F} \to \mathfrak{a}$ is defined to verify the equation

$$gk = l \exp(\sigma(g, kM))n,$$

for every $g \in G$ and $k \in K$, using Iwasawa's decomposition of $G = K \exp(\mathfrak{a})N$, where N is the unipotent radical of P.

THEOREM 1.0 (Quint [20]). — *Let Δ be a Zariski-dense discrete subgroup of G. Then for each linear form φ, tangent to ψ_Δ in a direction in the interior of $\mathscr{L}_\Delta$, there exists a probability measure ν_φ on $\mathscr{F}$, supported on L_Δ, such that for every $g \in \Delta$ one has,*

$$\frac{dg_*\nu_\varphi}{d\nu_\varphi}(x) = e^{-\varphi(\sigma(g^{-1},x))}.$$

The measure ν_φ is called a φ-Patterson-Sullivan measure of Δ. Denote by u_0 the unique element of the Weyl group that sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The *opposition involution* $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ is defined by $\mathrm{i} = -u_0$. One has $\mathrm{i}(a(g)) = a(g^{-1})$, for every $g \in G$, and thus $\psi_\Delta \circ \mathrm{i} = \psi_\Delta$. Moreover if $\varphi \in \mathfrak{a}^*$ is tangent to ψ_Δ, so is $\varphi \circ \mathrm{i}$. Hence, in higher rank, Patterson-Sullivan's measures come in pairs.

As in negative curvature, one can use these measures to construct invariant measures for the Weyl chamber flow. Consider the action of G on $\mathscr{F}^{(2)} \times \mathfrak{a}$, via Busemann's cocycle, defined by

$$g(x, y, v) = (gx, gy, v - \sigma(g, y)).$$

Denote by $\overline{P}$ the opposite parabolic subgroup of P, associated to the choice of $\mathfrak{a}^+$, the stabilizer in G of the point $([P], [\overline{P}], 0) \in \mathscr{F}^{(2)} \times \mathfrak{a}$ is isomorphic to M, and we get thus an identification $G/M = \mathscr{F}^{(2)} \times \mathfrak{a}$. This is called *Hopf's parametrization* of G.

Using Tits's [31] representations of G, one can define a vector valued Gromov product $\mathscr{G}_\Pi : \mathscr{F}^{(2)} \to \mathfrak{a}$ (see Section 4) such that, for every $g \in G$ and $(x, y) \in \mathscr{F}^{(2)}$,

$$\mathscr{G}_\Pi(gx, gy) - \mathscr{G}_\Pi(x, y) = -(\mathrm{i} \circ \sigma(g, x) + \sigma(g, y)).$$

For a given $\varphi \in \mathfrak{a}^*$ tangent to ψ_Δ, the measure

$$e^{-\varphi(\mathscr{G}_\Pi(\cdot,\cdot))} \nu_{\varphi \circ \mathrm{i}} \otimes \nu_\varphi \otimes \mathrm{Leb}_{\mathfrak{a}}$$

in $\mathscr{F}^{(2)} \times \mathfrak{a}$ is thus Δ-invariant and $\mathfrak{a}$-invariant. Denote by χ_φ the measure induced on the quotient $\Delta \backslash G/M$. We call this measure *the Bowen-Margulis measure* for φ, its support is the set

$$\Delta \backslash (\mathrm{L}^{(2)}_\Delta \times \mathfrak{a}),$$

where $\mathrm{L}^{(2)}_\Delta = (\mathrm{L}_\Delta)^2 \cap \mathscr{F}^{(2)}$. This set is analogous, in higher rank, to the nonwandering set of the geodesic flow in negative curvature. An important contrast though, is that when Δ is not a lattice and G is simple (of higher rank), the measure χ_φ is expected to have infinite total mass. For example, Quint [21] has shown that if $\Delta \backslash (\mathrm{L}^{(2)}_\Delta \times \mathfrak{a})$ is compact, then Δ is a cocompact lattice.

We prove the following mixing property, for hyperconvex representations, inspired by the work of Thirion [30]. He proves an analogous mixing property for ping-pong groups.

Theorem (Theorem 4.0). — *Let $\rho : \Gamma \to G$ be a Zariski-dense hyperconvex representation, and consider $\varphi \in \mathfrak{a}^*$ tangent to ψ_Δ in the direction u_φ. Then there exists $\kappa > 0$ such that, for any two compactly supported continuous functions $f_0, f_1 : \rho(\Gamma) \backslash G/M \to \mathbb{R}$, one has*

$$(2\pi t)^{(\mathrm{rank}(G)-1)/2} \chi_\varphi(f_0 \cdot f_1 \circ \exp(t u_\varphi)) \to \kappa \chi_\varphi(f_0) \chi_\varphi(f_1),$$

as $t \to \infty$.

In Section 2, we recall results on Hölder cocycles from [27], of particular interest is the Reparametrizing Theorem 2.0. This theorem is crucial in understanding the nature of

$$\rho(\Gamma) \backslash (\mathrm{L}^{(2)}_{\rho(\Gamma)} \times \mathfrak{a}),$$

when $\rho : \Gamma \to G$ is hyperconvex (Proposition 3.0). In Section 3, we prove a general mixing property that will imply Theorem B. This is shown in Section 4. In the last section, we prove Theorem A by adapting a method of Roblin [25] and Thirion [29].

2. Hölder cocycles

Reparametrizations. The standard reference for the following is Katok-Hasselblat [15]. Let X be a compact metric space, $\phi = (\phi_t)_{t\in\mathbb{R}}$ a continuous flow on X without fixed points (i.e. no point in X verifies $\phi_t x = x$ for every $t \in \mathbb{R}$), and V a finite dimensional real vector space.

DEFINITION 2.0. — *A* translation cocycle *over ϕ is a map $\kappa : X\times\mathbb{R} \to V$ that verifies the following two conditions:*

- *For every $x \in X$ and $t, s \in \mathbb{R}$, one has*
$$\kappa(x, t + s) = \kappa(\phi_s x, t) + \kappa(x, s).$$
- *For every $t \in \mathbb{R}$, the map $\kappa(\cdot, t)$ is Hölder-continuous with exponent independent of t, and with bounded multiplicative constant when t is bounded.*

Two translation cocycles κ_1 and κ_2 are *Livšic-cohomologous*, if there exists a continuous map $U : X \to V$, such that for all $x \in X$ and $t \in \mathbb{R}$ one has

$$\kappa_1(x, t) - \kappa_2(x, t) = U(\phi_t x) - U(x). \tag{2.1}$$

Denote by $p(\tau)$ the period of a ϕ-periodic orbit τ. If κ is a translation cocycle then the *period* of τ for κ, is defined by

$$L_\kappa(\tau) = \kappa(x, p(\tau)),$$

for any $x \in \tau$. It is clear that $L_\kappa(\tau)$ does not depend on the chosen point $x \in \tau$, and that the set of periods is a cohomological invariant of κ.

The standard example of a translation cocycle is obtained by considering a Hölder-continuous map $f : X \to V$, and defining $\kappa_f : X \times \mathbb{R} \to V$ by

$$\kappa_f(x, t) = \int_0^t f(\phi_s x)ds. \tag{2.2}$$

The period of a periodic orbit τ for f is then

$$\int_\tau f = \int_0^{p(\tau)} f(\phi_s x)ds.$$

We say that a map $U : X \to V$ is C^1 *in the direction of the flow* ϕ, if for every $x \in X$, the map $t \mapsto U(\phi_t x)$ is of class C^1, and the map

$$x \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t x)$$

is continuous. Two Hölder-continuous maps $f, g : X \to V$ are *Livšic-cohomologous* if the translation cocycles κ_f and κ_g are. If this is the case,

the map U of equation (2.1) is C^1 in the direction of the flow, and for all $x \in X$ one has

$$f(x) - g(x) = \frac{\partial}{\partial t}\Big|_{t=0} U(\phi_t x).$$

If $f : X \to \mathbb{R}$ is positive, then, since X is compact, f has a positive minimum and for every $x \in X$, the function $\kappa_f(x, \cdot)$ is an increasing homeomorphism of $\mathbb{R}$. We then have a map $\alpha_f : X \times \mathbb{R} \to \mathbb{R}$ that verifies

$$\alpha_f(x, \kappa_f(x,t)) = \kappa_f(x, \alpha_f(x,t)) = t, \tag{2.3}$$

for every $(x,t) \in X \times \mathbb{R}$.

Definition 2.0. — *The* reparametrization *of ϕ by $f : X \to \mathbb{R}_+^*$ is the flow $\psi = \psi^f = (\psi_t)_{t\in\mathbb{R}}$ on X, defined by $\psi_t(x) = \phi_{\alpha_f(x,t)}(x)$, for all $t \in \mathbb{R}$ and $x \in X$. If f is Hölder-continuous, we will say that ψ is a Hölder reparametrization of ϕ.*

Remark 2.0. — If two positive continuous functions $f, g : X \to \mathbb{R}$ are Livšic-cohomologous, then the flows ψ^f and ψ^g are conjugated i.e. there exists a homeomorphism $h : X \to X$ such that, for all $x \in X$ and $t \in \mathbb{R}$, one has[(1)]

$$h(\psi_t^f x) = \psi_t^g(hx).$$

Denote by $\mathcal{M}^\phi$ the set of ϕ-invariant probability measures on X. The *pressure* of a continuous function $f : X \to \mathbb{R}$ is defined by

$$P(\phi, f) = \sup_{m\in\mathcal{M}^\phi} h(\phi, m) + \int_X f dm,$$

where $h(\phi, m)$ is the metric entropy of m for ϕ. A probability measure m, on which the least upper bound is attained, is called an *equilibrium state* of f. An equilibrium state for $f \equiv 0$ is called a *probability measure of maximal entropy*, and its entropy is called the *topological entropy* of ϕ, denoted by $h_{\mathrm{top}}(\phi)$.

If τ is a periodic orbit of ϕ, and $g : X \to \mathbb{R}$ is continuous, then a standard argument shows

$$\int_0^{\kappa_f(x,p(\tau))} g(\psi_s^f x) ds = \int_\tau g f. \tag{2.1}$$

In fact, if m is a ϕ-invariant probability measure on X, then the probability measure $m^\#$, defined by

$$\frac{dm^\#}{dm}(\cdot) = \frac{f(\cdot)}{\int f dm}, \tag{2.2}$$

(1) This is standard, see [26, Remark 2.2.] for a detailed proof.

is invariant under ψ^f.

LEMMA 2.0 ([27, Section 2]). — *If $h = h_{top}(\psi^f) < \infty$, then the map $m \mapsto m^\#$ is a bijection between the set of equilibrium states of $-hf$, and the set of probability measures of maximal entropy of ψ^f.*

Anosov flows and Markov codings. Assume from now on that X is a compact manifold, and that the flow ϕ is C^1. We say that ϕ is *Anosov*, if the tangent bundle of X splits as a sum of three bundles

$$TX = E^s \oplus E^0 \oplus E^u,$$

that are $d\phi_t$-invariant for every $t \in \mathbb{R}$ and, there exist positive constants C and c such that, E^0 is the direction of the flow, and for every $t \geqslant 0$ one has $\|d\phi_t v\| \leqslant Ce^{-ct}\|v\|$ for every $v \in E^s$, and $\|d\phi_{-t} v\| \leqslant Ce^{-ct}\|v\|$ for every $v \in E^u$, for any Riemannian metric on X.

We need the following classical result of Livšic [18]:

THEOREM 2.0 (Livšic [18]). — *Let ϕ be an Anosov flow on X and $\kappa : X \times \mathbb{R} \to V$ a translation cocycle. If $L_\kappa(\tau) = 0$ for every periodic orbit τ, then κ is Livšic-cohomologous to* 0.

As the next lemma proves, one can always chose a translation cocycle of the form κ_f, in the cohomology class of a given translation cocycle κ.

LEMMA 2.0. — *Let ϕ be an Anosov flow on X, and let $\kappa : X \times \mathbb{R} \to V$ be a translation cocycle, then there exists a Hölder-continuous map $f : X \to V$ such that the cocycles κ and κ_f are Livšic-cohomologous.*

Proof. — Fix $C > 0$, and consider the translation cocycle κ^C, defined by

$$\kappa^C(x,t) = \frac{1}{C}\int_0^C \kappa(\phi_s(x),t)ds.$$

The translation cocycles κ^C and κ are Livšic-cohomolgous since they have the same periods. One easily checks that $\kappa^C(\cdot,t)$ is of class C^1 in the direction of the flow and thus, κ^C is the integral of a Hölder-continuous function along the orbits of ϕ. □

The following lemma is useful.

LEMMA 2.0 ([27, Section 3]). — *Consider a Hölder-continuous function $f : X \to \mathbb{R}$, such that*

$$\frac{1}{p(\tau)}\int_\tau f > k,$$

for some positive k and every periodic orbit τ of ϕ. Then f is Livšic-cohomologous to a positive Hölder-continuous function.

In order to study the ergodic theory of Anosov flows, Bowen [5] and Ratner [24] introduced the notion of Markov coding.

DEFINITION 2.0. — *The triple* (Σ, π, r) *is a* Markov coding *for* ϕ, *if* Σ *is an irreducible two-sided subshift of finite type, the maps* $\pi : \Sigma \to X$ *and* $r : \Sigma \to \mathbb{R}_+^*$ *are Hölder-continuous and verify the following conditions: Let* $\sigma : \Sigma \to \Sigma$ *be the shift, and let* $\hat{r} : \Sigma \times \mathbb{R} \to \Sigma \times \mathbb{R}$ *be the homeomorphism defined by*

$$\hat{r}(x,t) = (\sigma x, t - r(x)),$$

then

i) *the map* $\Pi : \Sigma \times \mathbb{R} \to X$ *defined by* $\Pi(x,t) = \phi_t(\pi(x))$ *is surjective and* $\hat{r}$*-invariant,*
ii) *consider the suspension flow* $\sigma^r = (\sigma_t^r)_{t\in\mathbb{R}}$ *on* $(\Sigma \times \mathbb{R})/\hat{r}$, *then the induced map* $\Pi : (\Sigma \times \mathbb{R})/\hat{r} \to X$ *is bounded-to-one and, injective on a residual set which is of full measure for every ergodic invariant measure of total support of* σ^r.

Remark 2.0. — If a flow ϕ admits a Markov coding then every reparametrization ψ of ϕ also admits a Markov coding, simply by considering the new roof function $r'(x) = \int_0^{r(x)} f(\phi_s x)dx$.

A Markov coding is a very accurate measurable model for a flow ϕ. If ϕ admits a Markov coding, then it has a unique probability measure of maximal entropy, and the function $\Pi : (\Sigma \times \mathbb{R})/\hat{r} \to X$ induces an isomorphism between the set probability measures of maximal entropy of σ^r and that of ϕ. In particular the topological entropy of ϕ coincides with that of σ^r.

Recall that a flow ϕ is *transitive* if it has a dense orbit.

THEOREM 2.0 (Bowen [4, 5]). — *A transitive Anosov flow admits a Markov coding.*

The following is standard.

PROPOSITION 2.0 (Bowen-Ruelle [6]). — *Let* ϕ *be a transitive Anosov flow. Then, given a Hölder-continuous function* $f : X \to \mathbb{R}$, *there exists a unique equilibrium state for* f, *moreover, the equilibrium state is ergodic.*

The equilibrium state of the last proposition can be described as follows (see Bowen-Ruelle [6, Proposition 3.1]). If (Σ, π, r) is a Markov coding for the Anosov flow ϕ, then consider the function $F : \Sigma \to \mathbb{R}$ defined by

$$F(x) = \int_0^{r(x)} f(\phi_t(\pi x))dt,$$

and consider the equilibrium state ν, of $F-P(f)$. Then for every measurable function $G : X \to \mathbb{R}$, one has

$$\int_X G dm_f = \frac{1}{\int r d\nu} \int_\Sigma \int_0^{r(x)} G(\phi_t(\pi x)) dt d\nu(x). \tag{2.1}$$

We finish this subsection with the following classical result.

THEOREM 2.0. — *Let M be a closed connected, negatively curved Riemannian manifold. Then the geodesic flow on T^1M is a transitive Anosov flow.*

Hölder cocycles on $\partial_\infty\Gamma$. Let M be a closed connected negatively curved Riemannian manifold M, and denote by $\widetilde{M} \to M$ its universal cover. The group $\Gamma = \pi_1(M)$ is hyperbolic, and the visual boundary of $\widetilde{M}$ is identified with the boundary at infinity $\partial_\infty\Gamma$ of the group, endowed with its usual Hölder structure (see Ghys-de la Harpe [10]). We will now focus on Hölder cocycles on $\partial_\infty\Gamma$.

DEFINITION 2.0. — *A* Hölder cocycle *is a map $c : \Gamma \times \partial_\infty\Gamma \to V$, such that*

$$c(\gamma_0\gamma_1, x) = c(\gamma_0, \gamma_1 x) + c(\gamma_1, x),$$

for any $\gamma_0, \gamma_1 \in \Gamma$ and $x \in \partial_\infty\Gamma$, and such that $c(\gamma, \cdot)$ is Hölder-continuous, for every $\gamma \in \Gamma$ (the same exponent is assumed for every $\gamma \in \Gamma$).

Recall that each $\gamma \in \Gamma - \{e\}$ has two fixed points on $\partial_\infty\Gamma$, γ_+ and γ_-, and that for every $x \in \partial_\infty\Gamma - \{\gamma_-\}$ one has $\gamma^n x \to \gamma_+$, as $n \to \infty$. We will refer to γ_+ as the *attractor* of γ. The *period* of γ for a Hölder cocycle c is defined by

$$\ell_c(\gamma) = c(\gamma, \gamma_+).$$

The cocycle property implies that for all $n \in \mathbb{N}$, one has $\ell_c(\gamma^n) = n\ell_c(\gamma)$, and $\ell_c(\gamma)$ only depends on the conjugacy class $[\gamma]$ of γ.

Two Hölder cocycles c and c' are *cohomologous*, if there exists a Hölder-continuous function $U : \partial_\infty\Gamma \to V$, such that for all $\gamma \in \Gamma$ one has

$$c(\gamma, x) - c'(\gamma, x) = U(\gamma x) - U(x).$$

One easily deduces from the definition that the set of periods of a Hölder cocycle is a cohomological invariant. The following theorem of Ledrappier [17] relates Hölder cocycles with Hölder-continuous maps $T^1M \to V$.

Recall that the set of periodic orbits of the geodesic flow of M is in one-to-one correspondence with the set of conjugacy classes $[\Gamma] - \{e\}$ of $\Gamma - \{e\}$. If $\gamma \in \Gamma$, then $[\gamma]$ will freely represent its conjugacy class in Γ, and its associated periodic orbit on T^1M.

THEOREM 2.0 (Ledrappier [17, page 105]). — *For each Hölder cocycle $c : \Gamma \times \partial_\infty\Gamma \to V$, there exists a Hölder-continuous map $F_c : T^1M \to V$, such that for every $\gamma \in \Gamma - \{e\}$, one has*

$$\ell_c(\gamma) = \int_{[\gamma]} F_c.$$

The map $c \mapsto F_c$ induces a bijection between the set of cohomology classes of V-valued Hölder cocycles and the set of Livšic-cohomology classes of Hölder-continuous maps from $T^1M \to V$.

Two Hölder cocycles c and $\overline{c}$ are *dual cocycles* if for every $\gamma \in \Gamma - \{e\}$, one has $\ell_{\overline{c}}(\gamma) = \ell_c(\gamma^{-1})$. If this is the case we will say that the pair $\{c, \overline{c}\}$ is a pair of dual cocycles.

Denote by $\partial^2_\infty\Gamma$ the set of pairs $(x, y) \in (\partial_\infty\Gamma)^2$ such that $x \neq y$. A function

$$[\cdot, \cdot] : \partial^2_\infty\Gamma \to V$$

is a *Gromov product* for a pair of dual cocycles $\{c, \overline{c}\}$, if for every $\gamma \in \Gamma$ and $(x, y) \in \partial^2_\infty\Gamma$ one has

$$[\gamma x, \gamma y] - [x, y] = -(\overline{c}(\gamma, x) + c(\gamma, y)).$$

Remark 2.0. — The existence of these objects, for a given Hölder cocycle, is a consequence of Ledrappier's Theorem 2.0, see [27, Section 2] for details.

We will now focus on real valued Hölder cocycles with non negative periods, i.e. such that $\ell_c(\gamma) \geqslant 0$ for every $\gamma \in \Gamma - \{e\}$. The *exponential growth rate* of such cocycle is defined by

$$h_c = \limsup_{s\to\infty} \frac{\log \#\{[\gamma] \in [\Gamma] - \{e\} : \ell_c(\gamma) \leqslant s\}}{s} \in (0, \infty],$$

(it is a consequence of Ledrappier's work [17] that a Hölder cocycle c with non negative periods verifies $h_c > 0$).

Remark 2.0. — A simple argument shows that two dual cocycles have the same exponential growth rate, i.e. $h_c = h_{\overline{c}}$.

For $\gamma \in \Gamma - \{e\}$, denote by $|\gamma|$ the period of $[\gamma]$. We will need the following two lemmas.

LEMMA 2.0 (Ledrappier [17, page 106]). — *Let c be a Hölder cocycle with nonnegative periods and finite exponential growth rate, then*

$$\frac{1}{m} < \inf_{\gamma\in\Gamma-\{e\}} \frac{\ell_c(\gamma)}{|\gamma|} \leqslant \sup_{\gamma\in\Gamma-\{e\}} \frac{\ell_c(\gamma)}{|\gamma|} < m,$$

for a positive m.

LEMMA 2.0 ([27, Section 2]). — *Let $c : \Gamma \times \partial_\infty\Gamma \to \mathbb{R}$ be a Hölder cocycle with nonnegative periods and finite exponential growth rate, then the function F_c is Livšic-cohomologous to a positive function.*

If c has finite exponential growth rate then, following Patterson's construction, Ledrappier [17] proves the existence of a *Patterson-Sullivan* probability measure μ on $\partial_\infty\Gamma$ of cocycle $h_c c$, this is to say, μ verifies

$$\frac{d\gamma_*\mu}{d\mu}(x) = e^{-h_c c(\gamma^{-1},x)}$$

for every $\gamma \in \Gamma$ and $x \in \partial_\infty\Gamma$.

THEOREM 2.0 (Ledrappier [17] page 102). — *Let c be a Hölder cocycle with nonnegative periods. If $h_c < \infty$ there exists a unique Patterson-Sullivan probability measure of cocycle $h_c c$. Conversely, if for some positive h, there exists a Patterson-Sullivan measure of cocycle hc, then $h = h_c$.*

Denote by μ and $\overline{\mu}$ the Patterson-Sullivan probability measures associated to c and $\overline{c}$ respectively and consider a Gromov product $[\cdot,\cdot]$, for the pair $\{c, \overline{c}\}$. Remark that the measure

$$e^{-h_c[x,y]} d\overline{\mu}(x) d\mu(y)$$

on $\partial^2_\infty\Gamma$, denoted from now on by $e^{-h_c[\cdot,\cdot]}\overline{\mu}\otimes\mu$, is Γ-invariant. The following theorem is crucial to understand the Weyl chamber flow.

THEOREM 2.0 (The Reparametrizing Theorem [27]). — *Let c be a Hölder cocycle with nonnegative periods such that h_c is finite. Then:*

(1) *the action of Γ in $\partial^2_\infty\Gamma \times \mathbb{R}$ via c, that is,*

$$\gamma(x, y, s) = (\gamma x, \gamma y, s - c(\gamma, y)),$$

is proper and cocompact. Moreover, the flow ψ on $\Gamma\backslash(\partial^2_\infty\Gamma \times \mathbb{R})$, defined by

$$\psi_t\Gamma(x, y, s) = \Gamma(x, y, s - t),$$

is conjugated to a Hölder reparametrization of the geodesic flow on T^1M. The conjugating map is also Hölder-continuous. The topological entropy of ψ is h_c.

(2) *The measure*

$$e^{-h_c[\cdot,\cdot]}\overline{\mu} \otimes \mu \otimes ds$$

on $\partial^2_\infty\Gamma\times\mathbb{R}$ induces on the quotient $\Gamma\backslash(\partial^2_\infty\Gamma\times\mathbb{R})$ a positive multiple of the probability measure of maximal entropy of ψ.

Remark 2.0. — Consider $F_c : T^1M \to \mathbb{R}$ given by Ledrappier's Theorem 2.0 for the cocycle c. Lemma 2.0 implies that F_c is Livšic-cohomologous to a positive function. The reparametrization in Theorem 2.0 is given by this positive function.

3. The action by translations of V on $\Gamma\backslash(\partial^2_\infty\Gamma \times V)$

Recall that M is a closed, connected, negatively curved Riemannian manifold, Γ is its fundamental group (for any base point), and V is a finite dimensional real vector space.

Fix a Hölder cocycle $c : \Gamma \times \partial_\infty\Gamma \to V$, and denote by $\mathscr{L}_c$ the smallest closed, convex cone of V that contains the periods $\{\ell_c(\gamma) : \gamma \in \Gamma - \{e\}\}$. The *dual cone* of $\mathscr{L}_c$ is the set of linear forms that are nonnegative on this cone:

$$\mathscr{L}_c^* = \{\varphi \in V^* : \varphi|_{\mathscr{L}_c} \geqslant 0\}.$$

A direct consequence of the Reparametrizing Theorem 2.0 applied to $\varphi \circ c$ is the following one.

Corollary 3.0. — *If there exists $\varphi \in \mathscr{L}_c^*$ such that $h_{\varphi\circ c}$ is finite, then the action of Γ on $\partial^2_\infty\Gamma \times V$ via c, that is,*

$$\gamma(x, y, v) = (\gamma x, \gamma y, v - c(\gamma, y)),$$

is properly discontinuous.

Denote by $\mathrm{int}(\mathscr{L}_c^*)$ the interior of $\mathscr{L}_c^*$. One has the following lemma.

Lemma 3.0. — *If $\varphi \in \mathscr{L}_c^*$ is such that $h_{\varphi\circ c} < \infty$, then $\varphi \in \mathrm{int}(\mathscr{L}_c^*)$, in particular $\mathrm{int}(\mathscr{L}_c^*)$ is nonempty. Moreover, for every $\theta \in \mathrm{int}(\mathscr{L}_c^*)$, one has $h_{\theta\circ c} < \infty$.*

Proof. — Consider the map $F_c : T^1M \to V$ associated to c by Theorem 2.0. One has

$$\varphi(\int_{[\gamma]} F_c) = \varphi(\ell_c(\gamma)) \geqslant 0.$$

Moreover, since $h_{\varphi\circ c} < \infty$, Ledrappier's Lemma 2.0, applied to $\varphi \circ c$, implies that there exists $k > 0$ such that

$$\varphi(\frac{1}{|\gamma|}\int_{[\gamma]} F_c) = \frac{1}{|\gamma|}\varphi(\ell_c(\gamma)) > k,$$

for every $\gamma \in \Gamma - \{e\}$. Anosov's closing Lemma (c.f. Shub [28]) states that the convex combinations of the Lebesgue measures on periodic orbits are dense in $\mathcal{M}^\phi$, thus

- $\varphi(\int F_c dm) \geqslant k$ for every ϕ-invariant probability measure m,
- the set

$$\{\int F_c dm : m \in \mathcal{M}^\phi\}$$

is compact and generates the cone $\mathscr{L}_c$.

Hence, φ is positive on the cone $\mathscr{L}_c - \{0\}$, i.e. $\varphi \in \mathrm{int}(\mathscr{L}_c^*)$.

If θ belongs to the interior of $\mathscr{L}_c^*$, then $\theta|_{\mathscr{L}_c - \{0\}} > 0$. Hence, there exists a positive a such that $\varphi(v) \leqslant a\theta(v)$, for all $v \in \mathscr{L}_c$. This implies that $h_{\theta \circ c} \leqslant a h_{\varphi \circ c} < \infty$. This finishes the proof. □

Assume from now on the existence of $\varphi \in \mathscr{L}_c^*$ with finite $h_{\varphi \circ c}$. We then have a natural map between $\mathbb{P}(\mathrm{int}(\mathscr{L}_c^*))$ and $\mathbb{P}(\mathscr{L}_c)$ as follows. Fix $F_c : T^1M \to V$ associated to c.

Definition 3.0. — *For $\varphi \in \mathrm{int}(\mathscr{L}_c^*)$, denote by m_φ the equilibrium state, on T^1M, of the function $-h_{\varphi \circ c}\varphi \circ F_c$ (recall Proposition 2.0). The* dual direction *of $\mathbb{R}_+\varphi$, is the direction in $\mathscr{L}_c$ given by the vector*

$$\int F_c dm_\varphi,$$

and is denoted by $\mathsf{u}_\varphi \in \mathbb{P}(\mathscr{L}_c)$.

Remark 3.0. — A change in the Livšic-cohomology class of F_c does not change the value of the integral of F_c over any ϕ-invariant measure. Hence u_φ is well defined, independently of the choice of F_c. Remark also that if $t \in \mathbb{R}_+$, then $h_{t\varphi \circ c} = h_{\varphi \circ c}/t$, hence, the dual direction of $\mathbb{R}_+\varphi$, only depends on the direction given by φ.

Fix also a dual cocycle $\overline{c}$ of c, and a Gromov product $[\cdot,\cdot] : \partial_\infty^2\Gamma \to V$ for the pair $\{c, \overline{c}\}$. Denote by μ_φ and $\overline{\mu}_\varphi$, the Patterson-Sullivan probability measures of cocycles $h_{\varphi \circ c}\varphi \circ c$ and $h_{\varphi \circ c}\varphi \circ \overline{c}$ respectively. The function

$$[\cdot,\cdot]_\varphi = \varphi \circ [\cdot,\cdot]$$

is a Gromov product for the pair $\{\varphi \circ c, \varphi \circ \overline{c}\}$. Denote by Ω_φ the measure on $\Gamma\backslash(\partial_\infty^2\Gamma \times V)$ induced by the measure

$$\widetilde{\Omega_\varphi} = e^{-h_{\varphi \circ c}[\cdot,\cdot]_\varphi}\overline{\mu}_\varphi \otimes \mu_\varphi \otimes \mathrm{Leb}_V,$$

where Leb_V is a fixed Lebesgue measure on V. The measure Ω_φ is called the *Bowen-Margulis measure* of the pair $\{c, \overline{c}\}$ for the linear form φ.

Choose a vector $u_\varphi \in \mathsf{u}_\varphi$ such that $\varphi(u_\varphi) = 1$, and consider the flow $\omega^\varphi = (\omega_t^\varphi)_{t \in \mathbb{R}}$ on $\Gamma\backslash(\partial_\infty^2\Gamma \times V)$ induced on the quotient by

$$(x, y, v) \mapsto (x, y, v - tu_\varphi).$$

Proposition 3.0 (Straightening the action of V). — *For every $\varphi \in \mathscr{L}_c^*$ such that $h_{\varphi\circ c} < \infty$, there exists a Hölder reparametrization of the geodesic flow $\psi = \psi^{c,\varphi}$, a Hölder-continuous map $f : T^1M \to \ker\varphi$, with zero mean for the probability measure of maximal entropy of ψ, denoted by $m_\varphi^\#$, i.e.*

$$\int_{T^1M} f dm_\varphi{}^\# = 0,$$

and a Hölder-continuous homeomorphism

$$\overline{E} : \Gamma\backslash(\partial^2_\infty\Gamma \times V) \to T^1M \times \ker\varphi,$$

that conjugates the flow ω^φ with the flow $\widehat{\psi} = (\widehat{\psi}_t)_{t\in\mathbb{R}}$ on $T^1M \times \ker\varphi$, defined by

$$\widehat{\psi}_t(p, v_0) = (\psi_t(p), v_0 - \int_0^t f(\psi_s p)ds). \tag{3.1}$$

The map $\overline{E}$ also conjugates the actions of $\ker\varphi$, on $\Gamma\backslash(\partial^2_\infty\Gamma \times V)$ and on $T^1M \times \ker\varphi$ (by translation on the fibers), and is an isomorphism, up to a multiplicative constant, between the measures Ω_φ and $m_\varphi{}^\# \otimes \mathrm{Leb}_{\ker\varphi}$.

Proof. — Consider the action of Γ on $\partial^2_\infty\Gamma \times \mathbb{R}$ via $\varphi\circ c$. Then one has a Γ-equivariant fibration $\widehat{\varphi} : \partial^2_\infty\Gamma \times V \to \partial^2_\infty\Gamma \times \mathbb{R}$ with fiber $\ker\varphi$, given by

$$\widehat{\varphi}(x, y, v) = (x, y, \varphi(v)).$$

The measure $\widetilde{\Omega_\varphi}$ disintegrates over the measure

$$e^{-h_{\varphi\circ c}[\cdot,\cdot]_\varphi}\overline{\mu}_\varphi \otimes \mu_\varphi \otimes \mathrm{Leb}_{\mathbb{R}}$$

on $\partial^2_\infty\Gamma \times \mathbb{R}$, with conditional measures the Lebesgue measure on $\ker\varphi$.

Since $h_{\varphi\circ c}$ is finite, the Reparametrizing Theorem 2.0 applies and thus, the action of Γ on $\partial^2_\infty\Gamma \times \mathbb{R}$ via $\varphi \circ c$ is properly discontinuous. Moreover there exists a Hölder-continuous homeomorphism $E : \Gamma\backslash(\partial^2_\infty\Gamma \times \mathbb{R}) \to T^1M$, that conjugates the translation flow with a reparametrization of the geodesic flow. Denote this reparametrization by ψ. The image of the measure induced on the quotient by

$$e^{-h_{\varphi\circ c}[\cdot,\cdot]_\varphi}\overline{\mu}_\varphi \otimes \mu_\varphi \otimes \mathrm{Leb}_{\mathbb{R}},$$

is sent by E to a positive multiple of the (unique) probability measure of maximal entropy of ψ.

The functions $\varphi \circ F_c$ and $F_{\varphi\circ c}$ are Livšic-cohomologous, since they have the same period for every periodic orbit of the geodesic flow. Lemma 2.0 implies then that, $\varphi\circ F_c$ is Livšic-cohomolgous to a positive function, hence we can (and will) assume that $\varphi \circ F_c > 0$. Remark 2.0 states that the flow ψ can be taken as the reparametrization of the geodesic flow ϕ by $\varphi \circ F_c$.

The probability measure of maximal entropy of ψ is $m_\varphi{}^\#$ (recall that m_φ is the equilibrium state of $-h_{\varphi\circ c}\varphi\circ F_c$ and use Lemma 2.0).

Abusing notation, denote again by

$$\widehat{\varphi} : \Gamma\backslash(\partial^2_\infty\Gamma\times V)\to\Gamma\backslash(\partial^2_\infty\Gamma\times\mathbb{R}),$$

the map induced on the quotients by $\widehat{\varphi}:\partial^2_\infty\Gamma\times V\to\partial^2_\infty\Gamma\times\mathbb{R}$. For every $u\in V$, one has

$$E\circ\widehat{\varphi}(x,y,v-u)=\psi_{\varphi(u)}(E(x,y,\varphi(v))),$$

in particular the flow ω^φ is (semi)conjugated to ψ by $E\circ\widehat{\varphi}$, i.e. for every $t\in\mathbb{R}$ one has

$$E\circ\widehat{\varphi}\circ\omega^\varphi_t=\psi_t\circ E\circ\widehat{\varphi}.$$

The action of the abelian group $\ker\varphi$ on $\partial^2_\infty\Gamma\times V$, commutes with the action of Γ and preserves the fibers $\widehat{\varphi}^{-1}(x,y,t)$ of $\widehat{\varphi}$. Hence we have an action of $\ker\varphi$ on the quotient, and one finds that

$$E\circ\widehat{\varphi}:\Gamma\backslash(\partial^2_\infty\Gamma\times V)\to T^1M$$

is a vector bundle with fiber $\ker\varphi$, and the group $\ker\varphi$ acts by Hölder-continuous homeomorphisms on $\Gamma\backslash(\partial^2_\infty\Gamma\times V)$ preserving the fibers, and acting transitively on them. Using the zero section of a vector bundle, and the action of $\ker\varphi$, one can trivialize this bundle. Hence, $\Gamma\backslash(\partial^2_\infty\Gamma\times V)$ is (Hölder) isomorphic to $T^1M\times\ker\varphi$, and this isomorphism is $\ker\varphi$-equivariant.

Denote by $\Psi=(\Psi_t)_{t\in\mathbb{R}}$ the flow on $T^1M\times\ker\varphi$, corresponding to the flow ω^φ via this last identification. Since ω^φ commutes with the action of $\ker\varphi$, the same occurs for Ψ, and thus we can write

$$\Psi_t(p,v_0)=(\psi_t(p),v_0-\kappa(p,t)),$$

where $\kappa : T^1M\times\mathbb{R}\to\ker\varphi$ is a translation cocycle over ψ. Lemma 2.0 implies the existence of a Hölder-continuous map $f : T^1M\to\ker\varphi$, such that the cocycles κ and κ_f are Livšic-cohomologous (for the flow ψ). The flow Ψ is hence conjugated to the flow $\widehat{\psi}=(\widehat{\psi_t})_{t\in\mathbb{R}}$ on $T^1M\times\ker\varphi$, defined by

$$\widehat{\psi_t}(p,v)=(\psi_t(p),v-\int_0^t f(\psi_s(p))ds).$$

Denote by $\overline{E}:\Gamma\backslash(\partial^2_\infty\Gamma\times V)\to T^1M\times\ker\varphi$ the composition of the trivialization of $\Gamma\backslash(\partial^2_\infty\Gamma\times V)$ defined above, with this last conjugacy between Ψ and $\widehat{\psi}$. By definition, $\overline{E}$ conjugates the flows ω^φ and $\widehat{\psi}$, and is $\ker\varphi$-equivariant.

We remark that the image by $\overline{E}$ of the measure Ω_φ on $T^1M \times \ker\varphi$, is a measure that disintegrates as a $\ker\varphi$-invariant measure on the fibers, and a positive constant multiple of $m_\varphi{}^\#$ on T^1M. This measure is then a positive constant multiple of $m_\varphi{}^\# \otimes \mathrm{Leb}_{\ker\varphi}$.

It remains to check that $\int_{T^1M} f dm_\varphi{}^\# = 0$. In order to do this, recall that $\varphi(u_\varphi) = 1$ and that u_φ is collinear to the vector $\int F_c dm_\varphi$, hence

$$\int F_c dm_\varphi = u_\varphi \int \varphi \circ F_c dm_\varphi. \tag{3.2}$$

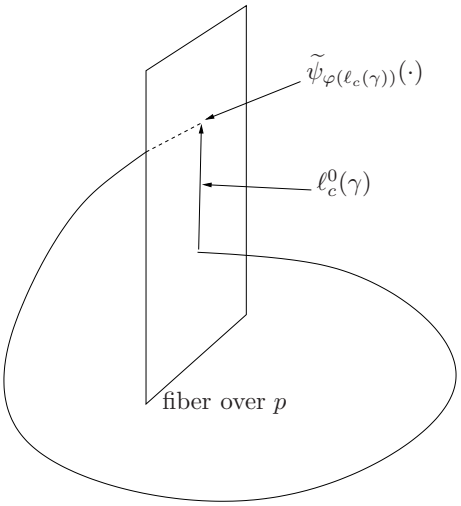

Figure 1. If $p \in T^1M$ belongs to the periodic orbit associated to $[\gamma]$, the translation on the fiber $\ker\varphi$ by the flow $\widehat{\psi}$, at the returning time, is given by $\ell_c^0(\gamma)$.

For every $\gamma \in \Gamma - \{e\}$, let $\ell_c^0(\gamma)$ be the projection of the period $\ell_c(\gamma)$ on $\ker\varphi$, using the decomposition $V = \ker\varphi \oplus \mathsf{u}_\varphi$. Remark that, for any $v \in V$ and $\gamma \in \Gamma - \{e\}$, one has

$$\gamma(\gamma_-, \gamma_+, v + \ell_c^0(\gamma)) = (\gamma_-, \gamma_+, v - \ell_{\varphi\circ c}(\gamma)u_\varphi) = \omega^\varphi_{\ell_{\varphi\circ c}(\gamma)}(\gamma_-, \gamma_+, v).$$

This is to say, $\ell_c^0(\gamma)$ is the displacement on $\ker\varphi$ of the flow ω^φ, over a point of the form (γ_-, γ_+, v), at the return time $\varphi(\ell_c(\gamma)) = \ell_{\varphi\circ c}(\gamma)$.

Consider also $F_c = F_c^0 + (\varphi \circ F_c)u_\varphi$, using this same decomposition. Equation (3.2) implies that

$$\int_{T^1M} F_c^0 dm_\varphi = 0,$$

moreover one has

$$\ell_c^0(\gamma) = \int_{[\gamma]} F_c^0.$$

Since $\widehat{\psi}$ and ω^φ are conjugated one has, for $p \in T^1M$ of the form (γ_-, γ_+, t),

$$\widehat{\psi}_{\ell_{\varphi\circ c}(\gamma)}(p, v) = (p, v - \int_0^{\ell_{\varphi\circ c}(\gamma)} f(\psi_s p)ds) = (p, v + \ell_c^0(\gamma)).$$

Hence,

$$\ell_c^0(\gamma) = -\int_0^{\ell_{\varphi\circ c}(\gamma)} f(\psi_s p)ds = -\int_{[\gamma]} f\varphi \circ F_c,$$

by equation (2.1) with f therein equal to $\varphi \circ F_c$. Livšic's Theorem 2.0 implies that the functions F_c^0 and $-f\varphi \circ F_c$ are Livšic-cohomologous for the flow ϕ, thus

$$0 = \int F_c^0 dm_\varphi = -\int f\varphi \circ F_c dm_\varphi = -\int f dm_\varphi{}^{\#} \int \varphi \circ F_c dm_\varphi.$$

This finishes the proof. □

Mixing properties of the action of V on $\Gamma\backslash(\partial^2_\infty\Gamma \times V)$. Now that we have a good description of $\Gamma\backslash(\partial^2_\infty\Gamma \times V)$, together with the action of V, we can use Markov codings and a theorem of Thirion [30], to prove a mixing property.

Consider $\varphi \in \mathscr{L}_c^*$, with $h_{\varphi\circ c} < \infty$, and $u_\varphi \in \mathfrak{u}_\varphi$ such that $\varphi(u_\varphi) = 1$. For $v_0 \in \ker\varphi$ and $t \in \mathbb{R}^+$, denote by $\omega_t^{\varphi,v_0} : \Gamma\backslash(\partial^2_\infty\Gamma \times V) \to \Gamma\backslash(\partial^2_\infty\Gamma \times V)$ the map induced on the quotient by

$$(x, y, v) \mapsto (x, y, v - tu_\varphi - \sqrt{t}\, v_0).$$

If $|\cdot|$ is a Euclidean norm on V, denote by $I = I^{|\cdot|} : \ker\varphi \to \mathbb{R}$ the function defined by

$$I(v) = \frac{|v|^2|u_\varphi|^2 - \langle v, u_\varphi\rangle^2}{|u_\varphi|^2}. \tag{3.3}$$

THEOREM 3.0. — *Let $c : \Gamma \times \partial_\infty\Gamma \to V$ be a Hölder cocycle, such that the group generated by its periods is dense in V. Fix a linear form $\varphi \in \mathscr{L}_c^*$ such that $h_{\varphi\circ c} < \infty$. Then there exists $c > 0$ and a Euclidean norm*

$|\cdot|$ *on* V *such that given two compactly supported continuous functions* $f_0, f_1 : \Gamma\backslash(\partial^2_\infty\Gamma \times V) \to \mathbb{R}$, *one has, for every* $v_0 \in \ker\varphi$,

$$(2\pi t)^{(\dim V-1)/2}\Omega_\varphi(f_0 \cdot f_1 \circ \omega_t^{\varphi,v_0}) \to ce^{-I(v_0)/2}\Omega_\varphi(f_0)\Omega_\varphi(f_1),$$

as $t \to \infty$.

The remainder of the section is devoted to the proof of Theorem 3.0.

Applying Proposition 3.0, we get a Hölder reparametrization of the geodesic flow ψ, together with a Hölder-continuous map $f : T^1M \to \ker\varphi$ and $\overline{E} : \Gamma\backslash(\partial^2_\infty\Gamma \times V) \to T^1M \times \ker\varphi$ that conjugates:

- the actions of $\ker\varphi$ on $\Gamma\backslash(\partial^2_\infty\Gamma \times V)$ and on $T^1M \times \ker\varphi$,
- the flow ω^φ on $\Gamma\backslash(\partial^2_\infty\Gamma \times V)$ with the flow $\widehat{\psi} = (\widehat{\psi}_t)_{t\in\mathbb{R}}$ on $T^1M \times \ker\varphi$, defined by equation (3.1).

We will thus study mixing properties of

$$t \cdot (x, v) \mapsto (\psi_t(p), v - \int_0^t f(\psi_s p)ds - \sqrt{t}\, v_0).$$

Consider a Markov coding (Σ, π, r) for ψ (Remark 2.0). According to equation (2.1), there exists an equilibrium state of the shift $\sigma : \Sigma \to \Sigma$, denoted by ν_φ, corresponding to the measure $m_\varphi{}^\#$ via the Markov coding, i.e. for every measurable function $G : T^1M \to \mathbb{R}$ one has

$$\int_{T^1M} G dm_\varphi{}^\# = \frac{1}{\int r d\nu_\varphi} \int_\Sigma \int_0^{r(x)} G(\psi_s(\pi x))ds d\nu_\varphi(x). \tag{3.1}$$

Define $K : \Sigma \to V$ by

$$K(x) = r(x)u_\varphi + \int_0^{r(x)} f(\psi_s(\pi x))ds,$$

and $\hat{K} : \Sigma \times V \to \Sigma \times V$ by $\hat{K}(x, v) = (\sigma x, v - K(x))$.

Lemma 3.0. — *The map* $\underline{\pi} : \Sigma \times V \to T^1M \times \ker\varphi$, *defined by*

$$\underline{\pi}(x, v) = (\psi_{\varphi(v)}(\pi x), v - \varphi(v)u_\varphi - \int_0^{\varphi(v)} f(\psi_s(\pi x))ds)$$

$$= \widehat{\psi}_{\varphi(v)}(\pi x, v - \varphi(v)u_\varphi),$$

is $\hat{K}$*-invariant, and induces a measurable isomorphism between the measure induced on* $(\Sigma\times V)/\hat{K}$ *by* $\nu_\varphi \otimes \mathrm{Leb}_V$ *and a positive multiple of the measure* $m_\varphi{}^\# \otimes \mathrm{Leb}_{\ker\varphi}$ *on* $T^1M \times \ker\varphi$.

Proof. — Let's show that $\underline{\pi}$ is $\hat{K}$-invariant, the proof is an explicit computation. Remark that Property i) in the definition of Markov coding states that, for every $x \in \Sigma$ and $t \in \mathbb{R}$, one has $\psi_{t-r(x)}(\pi(\sigma x))) = \psi_t(\pi(x))$. Now,

$$\underline{\pi}(\hat{K}(x,v)) = \widehat{\psi}_{\varphi(v-K(x))}(\pi(\sigma x), v - K(x) - \varphi(v - K(x))u_\varphi).$$

Observe that

$$-\int_0^{-r(x)} f(\psi_s(\pi(\sigma x)))ds = -\int_{r(x)}^0 f(\psi_{s-r(x)}(\pi(\sigma x)))ds =$$

$$-\int_{r(x)}^0 f(\psi_s(\pi x))ds = \int_0^{r(x)} f(\psi_s(\pi(\sigma x)))ds.$$

Recall that $K(x) = r(x)u_\varphi + \int_0^{r(x)} f(\psi_s(\pi x))ds$, hence

$$\underline{\pi}(\hat{K}(x,v)) = \widehat{\psi}_{\varphi(v)-r(x)}(\pi(\sigma x), v - \int_0^{r(x)} f(\psi_s(\pi x))ds - \varphi(v)u_\varphi) =$$

$$\widehat{\psi}_{\varphi(v)}(\pi(x), v - \varphi(v)u_\varphi - \int_0^{r(x)} f(\psi_s(\pi x))ds - \int_0^{-r(x)} f(\psi_s(\pi(\sigma x)))ds.$$

This proves the $\hat{K}$-invariance. The remaining statements follow from equation (3.1) and Property ii) of Markov codings. □

Hence, the flow $\widehat{\psi}$ is measurably conjugated to the translation flow on $(\Sigma \times V)/\hat{K}$, in the direction given by u_φ. Remark that, since Proposition 3.0 states that $\int f dm_\varphi^\# = 0$, equation (3.1) applied to $G = f$ yields

$$\int_\Sigma K d\nu_\varphi = (u_\varphi + \int f dm_\varphi^\#) \int_\Sigma r d\nu_\varphi = u_\varphi \int_\Sigma r d\nu_\varphi.$$

Moreover, this conjugation also conjugates the actions of $\ker \varphi$ on $T^1M \times \ker \varphi$ and on $(\Sigma \times V)/\hat{K}$.

Observe that the periods of K are the periods of the Hölder cocycle c, and remark that $\varphi \circ K = r > 0$. Theorem 3.0 is thus a consequence of Proposition 3.0, and the following theorem due to Thirion [30], applied to $\nu = \nu_\varphi$.

THEOREM 3.0 (Thirion [30]). — *Let Σ be a subshift of finite type and $K : \Sigma \to V$ a Hölder-continuous map such that the group generated by its periods is dense in V. Assume there exists $\varphi \in V^*$ such that $\varphi \circ K$ is Livšic-cohomologous to a positive function. Consider an equilibrium state ν of σ and denote by*

$$\tau = \int_\Sigma K d\nu \in V.$$

Define $\hat{K} : \Sigma \times V \to \Sigma \times V$ *by* $\hat{K}(x,v) = (\sigma(x), v - K(x))$. *Then there exists a Euclidean norm* $|\cdot|$ *on* V *such that given two compactly supported continuous functions* $f_0, f_1 : (\Sigma \times V)/\hat{K} \to \mathbb{R}$, *and* $v_0 \in \ker \varphi$, *one has*

$$(2\pi t)^{(\dim V - 1)/2} \int_{(\Sigma\times V)/\hat{K}} f_0(x,v) f_1(x, v - t\tau - \sqrt{t}\, v_0) d(\nu \otimes \mathrm{Leb}_V)$$

converges, as $t \to \infty$, *to*

$$c e^{-I(v_0)/2} \int_{(\Sigma\times V)/\hat{K}} f_0 d(\nu \otimes \mathrm{Leb}_V) \int_{(\Sigma\times V)/\hat{K}} f_1 d(\nu \otimes \mathrm{Leb}_V),$$

where $c > 0$ *is a constant and* $I(v_0) = (|v_0|^2|\tau|^2 - \langle v_0, \tau\rangle^2)/|\tau|^2$.

Proof. — Let us give some hints on the proof for completeness, the basic method is that of Guivarc'h-Hardy [11]. Consider a Hölder-continuous function $g : \Sigma \to \mathbb{R}$, and the associated Ruelle operator, defined by

$$L_g(T)(x) = \sum_{y \in \Sigma : \sigma(y) = x} e^{-g(y)} T(y),$$

where $T : \Sigma \to \mathbb{R}$ is Hölder-continuous. It is a standard fact that g can be assumed to be normalized such that the equilibrium state ν, is the unique probability measure on Σ such that $L_g^* \nu = \nu$. One then considers the semi-Markovian chain on $\Sigma \times V$ defined by

$$\mathbb{P}_{(x,v)} = \sum_{y \in \Sigma : \sigma(y) = x} e^{-g(y)} \delta_{(y, v + K(y))}.$$

The proof then consists on explicitly verifying the hypothesis of Babillot [1, Theorem 2.9], see Thirion [30] for details. □

4. Convex representations and the Weyl chamber flow

We are now interested in studying representations $\Gamma \to G$, of the fundamental group Γ of a closed connected negatively curved Riemannian manifold, admitting equivariant maps from $\partial_\infty \Gamma$ to some flag space of a connected, noncompact real-algebraic semisimple Lie group G.

Let K be a maximal compact subgroup of G, and consider τ, the Cartan involution on $\mathfrak{g} = \mathrm{Lie}(G)$ whose fixed point set is the Lie algebra of K. Consider $\mathfrak{p} = \{v \in \mathfrak{g} : \tau v = -v\}$ and $\mathfrak{a}$ a maximal abelian subspace contained in $\mathfrak{p}$.

Let Σ be the set of roots of $\mathfrak{a}$ on $\mathfrak{g}$. Consider a closed Weyl chamber $\mathfrak{a}^+$, Σ^+ the set of positive roots associated to $\mathfrak{a}^+$, and Π the set of simple roots determined by Σ^+. Let W be the Weyl group of Σ, and denote by

$u_0 : \mathfrak{a} \to \mathfrak{a}$ the longest element in W, which is the unique element in W that sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The *opposition involution* $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ is defined by $\mathrm{i} = -u_0$.

To each subset θ of Π, one associates two opposite parabolic subgroups of G, P_θ and $\check{P}_\theta$, whose Lie algebras are, by definition,

$$\mathfrak{p}_\theta = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \langle \Pi - \theta \rangle} \mathfrak{g}_{-\alpha},$$

and

$$\check{\mathfrak{p}}_\theta = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\alpha \in \langle \Pi - \theta \rangle} \mathfrak{g}_\alpha,$$

where $\langle \theta \rangle$ is the set of positive roots generated by θ, and

$$\mathfrak{g}_\alpha = \{w \in \mathfrak{g} : [v, w] = \alpha(v) w \ \forall v \in \mathfrak{a}\}.$$

Every pair of opposite parabolic subgroups of G is conjugated to $(P_\theta, \check{P}_\theta)$ for a unique θ, and every opposite parabolic subgroup of P_θ is conjugated to $P_{\mathrm{i}\,\theta}$: the parabolic group associated to

$$\mathrm{i}\,\theta = \{\alpha \circ \mathrm{i} : \alpha \in \theta\}.$$

Fix from now on a nonempty subset of simple roots $\theta \subset \Pi$ and let $\mathscr{F}_\theta = G/P_\theta$. The space $\mathscr{F}_{\mathrm{i}\,\theta} \times \mathscr{F}_\theta$ has a unique open G-orbit, denoted by $\mathscr{F}_\theta^{(2)}$.

DEFINITION 4.0. — *A representation $\rho : \Gamma \to G$ is* θ-convex *if it admits two Hölder-continuous ρ-equivariant maps, $\xi = \xi_\rho : \partial_\infty \Gamma \to \mathscr{F}_\theta$ and $\eta = \eta_\rho : \partial_\infty \Gamma \to \mathscr{F}_{\mathrm{i}\,\theta}$, such that whenever $x \neq y$ in $\partial_\infty \Gamma$, the pair $(\eta(x), \xi(y))$ belongs to $\mathscr{F}_\theta^{(2)}$.*

The space $\mathscr{F}_\Pi = \mathscr{F}$ is the Furstenberg boundary of the symmetric space of G, hence, a Π-convex representation is called *hyperconvex*.

We recall some definitions from Benoist [2]. An element $g \in G$ is *proximal in $\mathscr{F}_\theta$* if it has an attracting fixed point on $\mathscr{F}_\theta$. This attractor is unique and is denoted by g_+^θ. The element g also has a fixed point g_-^θ on $\mathscr{F}_{\mathrm{i}\,\theta}$, which is the attractor for g^{-1} on $\mathscr{F}_{\mathrm{i}\,\theta}$. For every $x \in \mathscr{F}_\theta$ such that $(g_-^\theta, x) \in \mathscr{F}_\theta^{(2)}$, one has $g^n x \to g_+^\theta$. The point g_-^θ is called the repelling hyperplane of g.

LEMMA 4.0 ([26, Section 3]). — *Let $\rho : \Gamma \to G$ be a Zariski-dense θ-convex representation. Then for every $\gamma \in \Gamma - \{e\}$, $\rho(\gamma)$ is proximal in $\mathscr{F}_\theta$, $\xi(\gamma_+)$ is its attracting fixed point and $\eta(\gamma_-)$ is the repelling hyperplane.*

The equivariant functions ξ and η of the definition are hence unique, since attracting points γ_+ are dense in $\partial_\infty \Gamma$.

Busemann cocycle of ρ. To a θ-convex representation $\rho : \Gamma \to G$, one associates a Hölder cocycle on $\partial_\infty\Gamma$. In order to do so, we need *Busemann's cocycle* of G, introduced by Quint [20].

The set $\mathscr{F}$ is K-homogeneous, denote by M the stabilizer of $[P]$ in K. One defines $\sigma_\Pi : G \times \mathscr{F} \to \mathfrak{a}$ to verify the following equation

$$gk = l \exp(\sigma_\Pi(g, kM))n,$$

for every $g \in G$ and $k \in K$, using Iwasawa's decomposition of $G = K\exp(\mathfrak{a})N$, where N is the unipotent radical of P.

In order to obtain a cocycle only depending on the set $\mathscr{F}_\theta$ (and G), one considers

$$\mathfrak{a}_\theta = \bigcap_{\alpha\in\Pi-\theta} \ker\alpha,$$

the Lie algebra of the center of the reductive group $P_\theta \cap \check{P}_\theta$. Consider also $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$, the only projection invariant under the group $W_\theta = \{w \in W : w(v) = v\, \forall v \in \mathfrak{a}_\theta\}$.

Remark 4.0. — One easily verifies the following relation: $p_{\mathrm{i}\,\theta} = \mathrm{i} \circ p_\theta \circ \mathrm{i}$.

Quint [20] proves the following lemma.

Lemma 4.0 (Quint [20, Lemmas 6.1 and 6.2]). — *The map $p_\theta \circ \sigma_\Pi$ factors trough a map $\sigma_\theta : G \times \mathscr{F}_\theta \to \mathfrak{a}_\theta$. The map σ_θ verifies the cocycle relation: for every $g, h \in G$ and $x \in \mathscr{F}_\theta$, one has*

$$\sigma_\theta(gh, x) = \sigma_\theta(g, hx) + \sigma_\theta(h, x).$$

The cocycle associated to a θ-convex representation $\beta^\rho_\theta = \beta_\theta : \Gamma\times\partial_\infty\Gamma \to \mathfrak{a}_\theta$ is defined by

$$\beta_\theta(\gamma, x) = \sigma_\theta(\rho(\gamma), \xi(x)).$$

Denote by $\lambda : G \to \mathfrak{a}^+$ the *Jordan projection*, and define $\lambda_\theta : G \to \mathfrak{a}_\theta$ by $\lambda_\theta(g) = p_\theta(\lambda(g))$.

Lemma 4.0. — *Let $\rho : \Gamma \to G$ be a Zariski-dense θ-convex representation. Then the period of β_θ for $\gamma \in \Gamma - \{e\}$, is*

$$\beta_\theta(\gamma, \gamma_+) = \lambda_\theta(\rho(\gamma)).$$

Proof. — The proof follows from Lemma 4.0. See [27, Lemma 7.5] for details. □

Remark that a θ-convex representation is also (by definition), $\mathrm{i}\,\theta$-convex. Define then $\overline{\beta_\theta} : \Gamma \times \partial_\infty\Gamma \to \mathfrak{a}_\theta$ by $\overline{\beta_\theta} = \mathrm{i}\,\beta_{\mathrm{i}\,\theta}$. One has the following.

Lemma 4.0. — *The pair $\{\beta_\theta, \overline{\beta_\theta}\}$ is a pair of dual cocycles.*

Proof. — The proof follows from Remark 4.0, together with Lemma 4.0, and the fact that $\mathrm{i}(\lambda(g)) = \lambda(g^{-1})$, for every $g \in G$. □

Consider $\mathscr{L}_{\beta_\theta}$, the closed cone associated to β_θ. Since $\mathscr{L}_{\beta_\theta}$ is contained in $p_\theta(\mathfrak{a}^+)$, it does not contain any line, and thus the dual cone $\mathscr{L}^*_{\beta_\theta}$ has non empty interior.

LEMMA 4.0. — *Let $\rho : \Gamma \to G$ be a Zariski-dense θ-convex representation, and consider φ in the interior of the dual cone $\mathscr{L}^*_{\beta_\theta}$, then the cocycle $\varphi \circ \beta_\theta : \Gamma \times \partial_\infty\Gamma \to \mathbb{R}$ has finite and positive exponential growth rate h_φ.*

Proof. — The proof follows exactly as [27, Lemma 7.7]. □

Applying Corollary 3.0 to the cocycle β_θ, one directly obtains:

COROLLARY 4.0. — *Let $\rho : \Gamma \to G$ be Zariski-dense θ-convex representation, then the action of Γ on $\partial^2_\infty\Gamma \times \mathfrak{a}_\theta$ via β_θ is properly discontinuous.*

Even though we will not use it on this work, we remark that Lemma 4.0, together with [27, Corollary 4.1], imply the following counting result:

COROLLARY 4.0. — *Let $\rho : \Gamma \to G$ be a Zariski-dense θ-convex representation, and consider φ in the interior of $\mathscr{L}^*_{\beta_\theta}$. Then there exists $h_\varphi > 0$, such that*

$$h_\varphi t e^{-h_\varphi t} \#\{[\gamma] \in [\Gamma] \text{ primitive} : \varphi(\lambda_\theta(\rho\gamma)) \leqslant t\} \to 1,$$

as $t \to \infty$.

Gromov product. The purpose of this section is to define a Gromov product for the pair $\{\beta_\theta, \overline{\beta_\theta}\}$. We begin with the following result of Tits [31] (see also Humphreys [14, Chapter XI]). Recall that a representation $\Lambda : G \to \mathrm{PGL}(d, \mathbb{R})$ is *proximal* if there exists $g \in G$ such that $\Lambda(g)$ is proximal in $\mathbb{P}(\mathbb{R}^d)$.

PROPOSITION 4.0 (Tits [31]). — *For each $\alpha \in \Pi$ there exists a finite dimensional proximal irreducible representation $\Lambda_\alpha : G \to \mathrm{PGL}(V_\alpha)$, such that the highest weight χ_α of Λ_α is an integer multiple of the fundamental weight ω_α. Moreover, any other weight of Λ_α is of the form*

$$\chi_\alpha - \alpha - \sum_{\beta \in \Pi} n_\beta \beta,$$

with $n_\beta \in \mathbb{N}$.

Fix a nonempty subset θ of Π and consider $\Lambda_\alpha : G \to \mathrm{PGL}(V_\alpha)$, a representation given by Tits's proposition for $\alpha \in \theta$. Since Λ_α is proximal

and $\alpha \in \theta$, the parabolic group P_θ is contained in the stabilizer of a line in $\mathbb{P}(V_\alpha)$. Thus one obtains a continuous equivariant map $\xi_\alpha : \mathscr{F}_\theta \to \mathbb{P}(V_\alpha)$.

The dual representation $\Lambda_\alpha^* : G \to \mathrm{PGL}(V_\alpha^*)$ is also proximal, and its highest weight is $\chi_\alpha \mathrm{i}$. Hence, one obtains another equivariant map $\eta_\alpha = \xi_{\mathrm{i}\,\alpha} : \mathscr{F}_{\mathrm{i}\,\theta} \to \mathbb{P}(V_\alpha^*)$. Moreover, if $(x, y) \in \mathscr{F}_\theta^{(2)}$ then

$$\eta_\alpha(x)(\xi_\alpha(y)) \neq 0.$$

Consider a scalar product on V_α invariant under $\Lambda_\alpha(K)$ such that $\Lambda_\alpha(\exp \mathfrak{a})$ is symmetric. The Euclidean norm $\| \ \|_\alpha$ induced by this scalar product verifies

$$\log \|\Lambda_\alpha(g)\|_\alpha = \chi_\alpha(a(g)),$$

for every $g \in G$, where $a : G \to \mathfrak{a}^+$ is the Cartan projection (observe that the operator norm only depends on $\mathbb{R}_+ \| \ \|_\alpha$).

Lemma 4.0 (Quint [20, Lemma 6.4]). — *For every $\alpha \in \theta$ and $v \in \xi_\alpha(x)$ one has*

$$\chi_\alpha(\sigma_\theta(g, x)) = \log \frac{\|\Lambda_\alpha(g)v\|_\alpha}{\|v\|_\alpha}.$$

The set $\{\omega_\alpha|_{\mathfrak{a}_\theta} : \alpha \in \theta\}$ is a basis of $\mathfrak{a}_\theta^*$ and hence so is $\{\chi_\alpha|_{\mathfrak{a}_\theta}\}_{\alpha\in\theta}$. Thus, defining

$$\chi_\alpha(\mathscr{G}_\theta(x, y)) = \log \frac{|\varphi(v)|}{\|\varphi\|_\alpha \|v\|_\alpha}$$

for any $\varphi \in \eta_\alpha(x)$ and $v \in \xi_\alpha(y)$, provides a definition of $\mathscr{G}_\theta$. Moreover, notice that if $(x, y) \in \mathscr{F}_\theta^{(2)}$ are such that $\xi_\alpha(x) \perp \ker \eta_\alpha(y)$ for the Euclidean norm $\| \ \|_\alpha$ and all $\alpha \in \theta$, then

$$\mathscr{G}_\theta(x, y) = 0. \tag{4.1}$$

Lemma 4.0. — *For every $g \in G$ and $(x, y) \in \mathscr{F}_\theta^{(2)}$, one has*

$$\mathscr{G}_\theta(gx, gy) - \mathscr{G}_\theta(x, y) = -(\mathrm{i}\,\sigma_{\mathrm{i}\,\theta}(g, x) + \sigma_\theta(g, y)).$$

Proof. — For any norm $\| \ \|$ on a vector space V, every $g \in \mathrm{PGL}(V)$ and every $(\varphi, v) \in \mathbb{P}(V^*) \times \mathbb{P}(V) - \{(\varphi, v) \in \mathbb{P}(V^*) \times \mathbb{P}(V) : \varphi(v) = 0\}$, one has

$$\log \frac{|\varphi \circ g^{-1}(gv)|}{\|\varphi \circ g^{-1}\|\|gv\|} - \log \frac{|\varphi(v)|}{\|\varphi\|\|v\|} = -\log \frac{\|g\varphi\|}{\|\varphi\|} - \log \frac{\|gv\|}{\|v\|}.$$

The lemma follows from this formula together with the definition of $\mathscr{G}_\theta$ and Quint's Lemma 4.0. □

The following corollary is immediate.

COROLLARY 4.0. — *Let $\rho : \Gamma \to G$ be a Zariski-dense θ-convex representation. The function $[\cdot,\cdot] : \partial_\infty \Gamma^{(2)} \to \mathfrak{a}_\theta$ defined by*

$$[x, y] = \mathscr{G}_\theta(\eta(x), \xi(y)),$$

is a Gromov product for the pair $\{\beta_\theta, \overline{\beta_\theta}\}$.

Mixing. We need the following theorem of Benoist [3]:

THEOREM 4.0 (Benoist [3, Main Proposition]). — *Consider a Zariski-dense subgroup Δ of G. Then the group generated by $\{\lambda(g) : g \in \Delta\}$ is dense in $\mathfrak{a}$.*

Recall that the Bowen-Margulis measure of the pair $\{\beta_\theta, \overline{\beta_\theta}\}$ for $\varphi \in \mathrm{int}(\mathscr{L}^*_{\beta_\theta})$ is the measure Ω_φ on $\Gamma\backslash(\partial^2_\infty\Gamma \times \mathfrak{a}_\theta)$, induced on the quotient by

$$e^{-h_\varphi[\cdot,\cdot]_\varphi}\overline{\mu_\varphi} \otimes \mu_\varphi \otimes \mathrm{Leb}_{\mathfrak{a}_\theta},$$

where μ_φ and $\overline{\mu_\varphi}$ are the Patterson-Sullivan probability measures with cocycles $h_\varphi\varphi \circ \beta_\theta$ and $h_\varphi\varphi \circ \overline{\beta_\theta}$, respectively. Benoist's theorem (and the continuity of p_θ) guarantees the missing hypothesis of Theorem 3.0, applied using $c = \beta_\theta$, and we obtain the following result.

THEOREM 4.0. — *Let $\rho : \Gamma \to G$ be a Zariski-dense θ-convex representation, and consider $\varphi \in \mathrm{int}(\mathscr{L}^*_{\beta_\theta})$. Then there exists a Euclidean norm $|\cdot|$ on $\mathfrak{a}$ such that, for any two compactly supported continuous functions $f_0, f_1 : \Gamma\backslash(\partial^2_\infty\Gamma \times \mathfrak{a}_\theta) \to \mathbb{R}$ and any $v_0 \in \ker\varphi$, one has*

$$(2\pi t)^{(\dim \mathfrak{a}_\theta - 1)/2}\Omega_\varphi(f_0 \cdot f_1 \circ \omega_t^{\varphi, v_0}) \to e^{-I(v_0)/2}\Omega_\varphi(f_0)\Omega_\varphi(f_1),$$

as $t \to \infty$.

The growth indicator function. Consider a G-invariant Riemannian metric on X, and $\|\ \|$ the induced Euclidean norm on $\mathfrak{a}$, invariant under the Weyl group. Recall that if $g \in G$, then $\|a(g)\| = d_X([K], g[K])$. Consider a Zariski-dense discrete subgroup Δ of G, and define

$$h_\Delta = \limsup_{s\to\infty} \frac{\log \#\{g \in \Delta : \|a(g)\| \leqslant s\}}{s}.$$

Recall that in the introduction we have defined ψ_Δ, the growth indicator of Δ.

LEMMA 4.0 (Quint [22, Corollaire 3.1.4]). — *Let Δ be a Zariski-dense subgroup of G, then one has*

$$\sup_{v\in\mathfrak{a}-\{0\}} \frac{\psi_\Delta(v)}{\|v\|} = h_\Delta.$$

If $\varphi \in \mathfrak{a}^*$ is such that $\varphi(v) \geqslant \psi_\Delta(v)$ for all $v \in \mathfrak{a}^+$ then $\|\varphi\| \geqslant h_\Delta$. One is thus interested in the convex set

$$D_\Delta = \{\varphi \in \mathfrak{a}^* : \varphi \geqslant \psi_\Delta\}.$$

This set is nonempty (Quint [22]) and the linear form $\Theta_\Delta \in D_\Delta$ closest to the origin is called the *the growth form* of Δ. One has

$$\|\Theta_\Delta\| = h_\Delta. \tag{4.1}$$

Since ψ_Δ is concave (recall Theorem 1.0), and the balls of $\| \ \|$ are strictly convex, one obtains a unique direction $\mathbb{R}_+ u_\Delta$ in $\mathscr{L}_\Delta$, which realizes the upper bound

$$\sup_{v \in \mathfrak{a} - \{0\}} \frac{\psi_\Delta(v)}{\|v\|},$$

this is called *the growth direction* of Δ. Choose u_Δ in the growth direction such that $\Theta_\Delta(u_\Delta) = 1$.

A linear form $\varphi \in \mathfrak{a}^*$ is *tangent* to ψ_Δ at x if $\varphi \in D_\Delta$ and $\varphi(x) = \psi_\Delta(x)$. We say that ψ_Δ has *vertical tangent* at x, if for every $\varphi \in D_\Delta$, one has $\varphi(x) > \psi_\Delta(x)$.

The following remarks are direct consequences of the definitions:

Remark 4.0. — For every $v \in \mathbb{R}_+ u_\Delta$, one has $|\Theta_\Delta(v)| = \|\Theta_\Delta\|\|v\| = \psi_\Delta(v)\|v\|$, consequently $\ker \Theta_\Delta$ and $\mathbb{R}_+ u_\Delta$ are orthogonal for $\| \ \|$, and Θ_Δ is tangent to ψ_Δ at every point of the growth direction $\mathbb{R}_+ u_\Delta$.

Remark 4.0. — The number of elements of $a(\Delta)$ that lie outside a given open cone containing u_Δ has exponential growth rate strictly smaller than h_Δ.

Fix from now on a Zariski-dense hyperconvex representation $\rho : \Gamma \to G$, and denote by $\zeta : \partial_\infty \Gamma \to \mathscr{F}$ its ρ-equivariant map. The image $\zeta(\partial_\infty \Gamma)$ is the limit set $\mathrm{L}_{\rho(\Gamma)}$, and thus

$$\zeta \times \zeta : \partial^2_\infty \Gamma \to \mathrm{L}^{(2)}_{\rho(\Gamma)}$$

is a ρ-equivariant Hölder-continuous homeomorphism. Also, the cone $\mathscr{L}_{\beta_\Pi}$ is the limit cone $\mathscr{L}_{\rho(\Gamma)}$ of $\rho(\Gamma)$. One has the following results.

Proposition 4.0 ([26, Corollary 3.13]). — *The limit cone of a Zariski-dense hyperconvex representation is contained in the interior of the Weyl chamber.*

Theorem 4.0 ([26, Theorem A + Corollary 4.9]). — *The growth indicator of a Zariski-dense hyperconvex representation ρ is strictly concave,*

analytic on the interior of $\mathscr{L}_{\rho(\Gamma)}$, and with vertical tangent on the boundary. If $\varphi \in \mathrm{int}(\mathscr{L}^*_{\rho(\Gamma)})$ then $h_\varphi \varphi$ is tangent to $\psi_{\rho(\Gamma)}$ at every point of the dual direction u_φ.

Remark 4.0. — Hence, Remark 4.0 and Theorem 4.0 imply that for a Zariski-dense hyperconvex representation ρ of Γ, the growth direction $\mathbb{R}_+ u_{\rho(\Gamma)}$ is the dual direction (see Definition 3.0) of the growth form $\Theta_{\rho(\Gamma)}$. Moreover, since $\psi_{\rho(\Gamma)}$ has vertical tangent on the boundary of $\mathscr{L}_{\rho(\Gamma)}$, the growth direction $\mathbb{R}_+ u_{\rho(\Gamma)}$ is contained in the interior of the limit cone.

Recall that if φ is in the interior of $\mathscr{L}^*_{\rho(\Gamma)}$, then the Hölder cocycle $\varphi \circ \beta_\Pi$ has finite and positive exponential growth rate h_φ. Ledrappier's Theorem 2.0 guarantees the existence of a Patterson-Sullivan probability measure μ_φ on $\partial_\infty \Gamma$, with cocycle $h_\varphi \varphi \circ \beta_\Pi$. The following corollary of Theorem 1.0 and Theorem 4.0 is hence direct.

COROLLARY 4.0. — *Let $\rho : \Gamma \to G$ be a Zariski-dense hyperconvex representation. For each φ tangent to $\psi_{\rho(\Gamma)}$, there exists a unique φ-Patterson-Sullivan measure of $\rho(\Gamma)$, denoted by ν_φ. Moreover, ζ induces an isomorphism between μ_φ and ν_φ.*

Consequently, the map

$$\zeta \times \zeta \times \mathrm{id} : \partial^2_\infty \Gamma \times \mathfrak{a} \to \mathrm{L}^{(2)}_{\rho(\Gamma)} \times \mathfrak{a}$$

is a ρ-equivariant homeomorphism, and induces on the quotients a map still denoted by $\zeta \times \zeta \times \mathrm{id} : \Gamma \backslash (\partial^2_\infty \Gamma \times \mathfrak{a}) \to \rho(\Gamma) \backslash (\mathrm{L}^{(2)}_{\rho(\Gamma)} \times \mathfrak{a})$, which is a measurable isomorphism between the φ-Bowen-Margulis measures of $\rho(\Gamma)$ on each side:

$$(\zeta \times \zeta \times \mathrm{id})_* \Omega_\varphi = \chi_\varphi,$$

where χ_φ is the φ-Bowen-Margulis measure of $\rho(\Gamma)$, defined in the introduction.

Theorem 4.0 together with Remark 4.0 imply the following mixing property of the Weyl chamber flow. Recall that the rank of G is the dimension of $\mathfrak{a}$, and that the Weyl chamber flow is the right action by translations of $\exp(\mathfrak{a})$ on $\rho(\Gamma) \backslash G/M$. If $f : \rho(\Gamma) \backslash G/M \to \mathbb{R}$ and $v \in \mathfrak{a}$ we denote the composition of f with the right action of $\exp(v)$ on $\rho(\Gamma) \backslash G/M$ by

$$f \circ \exp(v).$$

THEOREM 4.0. — *Let $\rho : \Gamma \to G$ be a Zariski-dense hyperconvex representation. Consider $\varphi \in \mathrm{int}(\mathscr{L}^*_\rho)$. Then there exists a Euclidean norm $|\cdot|$*

on $\mathfrak{a}$ such that, for all compactly supported continuous functions $f_0, f_1 : \rho(\Gamma)\backslash G/M \to \mathbb{R}$ and for all $v_0 \in \ker\varphi$, one has

$$(2\pi t)^{(\mathrm{rank}(G)-1)/2}\chi_\varphi(f_0 \cdot f_1 \circ \exp(tu_\varphi + \sqrt{t}\, v_0))$$

converges to

$$ce^{-I(v_0)/2}\chi_\varphi(f_0)\chi_\varphi(f_1)$$

as $t \to \infty$, for a constant $c > 0$.

The following corollary will be most useful to us.

COROLLARY 4.0. — *Let $\rho : \Gamma \to G$ be a Zariski-dense hyperconvex representation. Then there exists $C > 0$ such that given two compactly supported continuous functions $f_0, f_1 : \rho(\Gamma)\backslash G/M$ one has*

$$e^{-\|\Theta_{\rho(\Gamma)}\|T}\int_{B(0,T)\cap\mathfrak{a}^+} e^{\Theta_{\rho(\Gamma)}(u)}\chi_{\Theta_{\rho(\Gamma)}}(f_0 \cdot f_1 \circ \exp(u))d\,\mathrm{Leb}_{\mathfrak{a}}(u) \\ \to C\chi_{\Theta_{\rho(\Gamma)}}(f_0)\chi_{\Theta_{\rho(\Gamma)}}(f_1),$$

as $T \to \infty$.

The proof of the corollary follows the exact same lines as Thirion [29, §12.k] for Ping-Pong groups. We give a sketch of this proof for completeness.

Proof. — In order to simplify notation, denote by $\Theta = \Theta_{\rho(\Gamma)}$, $H = \ker\Theta$ and $u_\rho = u_{\rho(\Gamma)}$. Consider the change of variables $G : \mathbb{R} \times H \to \mathfrak{a}$ given by

$$G(t, v) = t\frac{u_\rho}{\|u_\rho\|} + \sqrt{t}\, v.$$

Since u_ρ is orthogonal to H (see Remark 4.0), its Jacobian is $(\sqrt{t})^{\dim H} = t^{(\mathrm{rank}(G)-1)/2}$. The integral we are interested in becomes

$$\int_H e^{-\|\Theta\|T}\int_0^\infty e^{\|\Theta\|t}(\sqrt{t})^{\dim H}\chi_\Theta(f_0 \cdot f_1 \circ \exp(G(t, v)))\mathbf{1}_{B(T)}(t, v)dt\,d\,\mathrm{Leb}_H(v),$$

where $B(T) = \{(t, v) \in \mathbb{R} \times H : G(t, v) \in \mathfrak{a}^+,\ \|G(t, v)\| \leqslant T\}$, and $\mathbf{1}_A$ is the characteristic function of a subset A.

The conditions $t > 0$ and $\|G(t, v)\| \leqslant T$ imply that

$$0 < t < \frac{1}{2}(\sqrt{\|v\|^4 + 4T^2} - \|v\|^2) = R(T, v).$$

Note that $R(T, v) - T \to -\|v\|^2/2$ as $T \to \infty$, and observe that for every $v \in H$ there exists t_0 such that, for all $t \geqslant t_0$, one has $G(t, v) \in \mathfrak{a}^+$. This, together with Theorem 4.0 applied to

$$G(t, v) = t\|\Theta\|u_\rho + \sqrt{t\|\Theta\|}\frac{v}{\sqrt{\|\Theta\|}},$$

implies the existence of a Euclidean norm $|\cdot|$ and $c>0$ such that

$$e^{-\|\Theta\|T}\int_0^{R(T,v)} e^{\|\Theta\|t}(\sqrt{t})^{\dim H}\chi_\Theta(f_0\cdot f_1\circ\exp(G(t,v)))\mathbf{1}_{B(T)}(t,v)dt$$

converges to

$$ce^{-(\|\Theta\|\|v\|^2+I^{|\cdot|}(\|\Theta\|-1/2v))/2}\chi_\Theta(f_0)\chi_\Theta(f_1)$$

as $T\to\infty$.

We must now integrate both sides of this limit with respect to Leb_H, in order to so we will apply the dominated convergence theorem. Hence, we need to find an integrable function $F:H\to\mathbb{R}$, such that for every $v\in H$ one has

$$e^{\|\Theta\|T}\int_0^{R(T,v)} e^{\|\Theta\|t}(\sqrt{t})^{\dim H}\chi_\Theta(f_0\cdot f_1\circ\exp(G(t,v)))\mathbf{1}_{B(T)}(t,v)dt\leqslant F(v).$$

Remark that, since $I(v)>0$ for all $v\in H$, Theorem 4.0 implies that for all large enough t, one has

$$(\sqrt{t})^{\dim H}\chi_\Theta(f_0\cdot f_1\circ\exp(G(t,v)))\leqslant K,$$

for a constant K independent of v.

Lemma 4.0 below states that there exists a constant $\kappa>0$ such that, for all $(t,v)\in H\times\mathbb{R}$, with $G(t,v)\in B(T)$ one has $R(T,v)-T\leqslant-\kappa\|v\|^2/2$. Hence

$$e^{-\|\Theta\|T}\int_0^{R(T,v)} e^{\|\Theta\|t}(\sqrt{t})^{\dim H}\chi_\Theta(f_0\cdot f_1\circ\exp(G(t,v)))\mathbf{1}_{B(T)}(t,v)dt\leqslant$$

$$Ke^{-\|\Theta\|T}\int_0^{R(T,v)} e^{\|\Theta\|t}\mathbf{1}_{B(T)}(t,v)dt\leqslant Ke^{\|\Theta\|(R(T,v)-T)}\leqslant Ke^{-\kappa\|\Theta\|\|v\|^2/2},$$

for a constant $K>0$. This last function is clearly integrable on H. This finishes the proof.

□

Lemma 4.0. — *There exists $\kappa>0$ such that for every $T\geqslant 0$, if $(t,v)\in B(T)$ then $R(T,v)-T\leqslant-\kappa\|v\|^2/2$.*

Proof. — Recall that the angle between two walls of $\mathfrak{a}^+$ is at most $\pi/2$, hence, since $u_\rho\in\operatorname{int}\mathfrak{a}^+$, there exists $\theta_0\in(0,\pi/2)$ such that if $G(t,v)\in\mathfrak{a}^+$, then the angle between $G(t,v)$ and $tu_\rho/\|u_\rho\|$ is at most θ_0, i.e.

$$\frac{\|\sqrt{t}v\|}{t}\leqslant\tan(\theta_0).$$

From now on, standard computations imply the lemma, see Thirion [29, page 184] for details.

□

5. The orbital counting problem

General aspects. The standard reference for this subsection is the book by Guivarc'h-Ji-Taylor [12]. Recall that G is a connected, noncompact real-algebraic semisimple Lie group, X its symmetric space and Γ is the fundamental group of a closed connected negatively curved Riemannian manifold.

Recall that we have denoted by $a : G \to \mathfrak{a}^+$ the Cartan projection of G. We will define a new projection $\underline{a} : X \times X \to \mathfrak{a}^+$ by $\underline{a}(g \cdot o, h \cdot o) = a(g^{-1}h)$. Notice that $\underline{a}$ is G-invariant for the diagonal action of G on $X \times X$, that $\|\underline{a}(p,q)\| = d_X(p,q)$ and that

$$\mathrm{i}(\underline{a}(p,q)) = \underline{a}(q,p). \tag{5.1}$$

By definition, one has $q \in K_p \exp(\underline{a}(p,q)) \cdot p$, where K_p is the stabilizer in G of p.

Remark 5.0. — Observe that there exists $\kappa_0 > 0$ such that for every $g \in G$ one has $\|\underline{a}(p, gq) - a(g)\| \leqslant \kappa_0$.

Similarly (and abusing notation), we will define the *Busemann cocycle* $\sigma : \mathscr{F} \times X \times X \to \mathfrak{a}$ by

$$(x, g \cdot o, h \cdot o) \mapsto \sigma_x(g \cdot o, h \cdot o) = \sigma(g^{-1}, x) - \sigma(h^{-1}, x).$$

A *parametrized flat* is a map $\mathsf{f} : \mathfrak{a} \to X$, defined by $\mathsf{f}(v) = g \exp(v) \cdot o$, for some $g \in G$. Observe that G acts transitively on the set of parametrized flats and that the stabilizer of $\mathsf{f}_0 : v \mapsto \exp(v) \cdot o$ is the group M of elements in K commuting with $\exp(\mathfrak{a})$. We will hence identify the space of parametrized flats with G/M.

A *maximal flat* is the image on X of a parametrized flat i.e. the maximal flat associated to f is defined by $[\mathsf{f}] = \mathsf{f}(\mathfrak{a}) = \{g \exp(v) \cdot o : v \in \mathfrak{a}\}$. The space of maximal flats is naturally identified with $G/MA = \mathscr{F}^{(2)}$ (recall Hopf's parametrization of G on the Introduction). Denote by $(\check{\mathsf{Z}}, \mathsf{Z}) : G/M \to \mathscr{F}^{(2)} = G/MA$ the canonical projection.

The following proposition is standard.

Proposition 5.0 (see [12, Chapter III]). —

(1) *Let f, g be two parametrized flats, then the function $\mathfrak{a} \to \mathbb{R}$, defined by*

$$v \mapsto d_X(\mathsf{f}(v), \mathsf{g}(v)),$$

is bounded on the Weyl chamber $\mathfrak{a}^+$ if and only if $\mathsf{Z}(\mathsf{f}) = \mathsf{Z}(\mathsf{g})$.

(2) *A pair $(p, x) \in X \times \mathscr{F}$ determines a unique parametrized flat f such that $\mathsf{f}(0) = p$ and $\mathsf{Z}(\mathsf{f}) = x$.*

(3) *A point* $(x,y) \in \mathscr{F}^{(2)}$ *determines a unique maximal flat* $[\mathsf{f}_{xy}]$ *such that* $\check{\mathsf{Z}}(\mathsf{f}_{xy}) = x$ *and* $\mathsf{Z}(\mathsf{f}_{xy}) = y$.

The usual relation between the Cartan projection and Busemann's cocycle is given by the following lemma of Quint [20]. Observe that if $p, q \in X$ are such that $\underline{a}(p,q) \in \mathrm{int}(\mathfrak{a}^+)$ then, there is a unique parametrized flat f_{pq} such that $\mathsf{f}_{pq}(0) = p$, and $\mathsf{f}_{pq}(\underline{a}(p,q)) = q$. Denote by $x_{pq} = \mathsf{Z}(\mathsf{f}_{pq})$ and recall that Π is the set of simple roots of G.

LEMMA 5.0 (Quint [20, Lemma 6.6]). — *Fix* $p, q \in X$, *then*

$$\underline{a}(p,z) - \underline{a}(q,z) - \sigma_{x_{pz}}(p,q) \to 0,$$

as $\min_{\alpha\in\Pi} \alpha(\underline{a}(p,z)) \to \infty$.

Given $r > 0$, define the *shadow (on* $\mathscr{F}$*) of* q *seen from* p *of size* r, by

$$\mathcal{O}_r(p,q) = \{\mathsf{Z}(\mathsf{f}) : \mathsf{f} \in G/M,\ \mathsf{f}(0) = p,\ \exists v \in \mathrm{int}(\mathfrak{a}^+),\ d_X(\mathsf{f}(v), q) < r\}.$$

Denote by $B_X(p,r)$ the ball in X of radius r centered at p, and define by

$$\mathcal{O}_r^+(p,q) = \bigcup_{p_0 \in B_X(p,r)} \mathcal{O}_r(p_0,q),$$

and

$$\mathcal{O}_r^-(p,q) = \bigcap_{p_0 \in B_X(p,r)} \mathcal{O}_r(p_0,q).$$

Finally, for $x \in \mathscr{F}$ define the *shadow of* q *seen from* x *of size* r, by

$$\mathcal{O}_r(x,q) = \{\mathsf{Z}(\mathsf{f}) : \mathsf{f} \in G/M,\ d_X(\mathsf{f}(0), q) < r,\ \check{\mathsf{Z}}(\mathsf{f}) = x\}.$$

LEMMA 5.0 (Thirion [29, Proposition 8.66]). — *There exists* $\kappa > 0$ *such that, if* $p, p_0 \in X$ *and* $r > 0$, *then for all* $x \in \mathcal{O}_r^+(p,p_0)$ *one has*

$$\|\sigma_x(p,p_0) - \underline{a}(p,p_0)\| \leqslant \kappa r.$$

Let Δ be a Zariski-dense discrete subgroup of G, and consider a linear form $\varphi \in \mathfrak{a}^*$ tangent to ψ_Δ on a direction in the interior of $\mathscr{L}_\Delta$. Denote by ν_φ the φ-Patterson-Sullivan measure of Δ (recall Quint's Theorem 1.0). Define the *φ-Patterson-Sullivan density* $(\mu_p)_{p\in X}$ by $\mu_o = \nu_\varphi$ and

$$\frac{d\mu_p}{d\mu_o}(x) = e^{-\varphi(\sigma_x(p,o))}.$$

Since $\mathscr{F}$ is K_p-homogeneous and K_p is compact, there is a unique K_p-invariant probability measure on $\mathscr{F}$. This gives an embedding of X on the space $\mathcal{M}(\mathscr{F})$ of probability measures on $\mathscr{F}$. The closure of this embedding,

denoted by $\overline{X}_F$, is called *the Furstenberg compactification of* X. Observe that if $v \in \mathrm{int}(\mathfrak{a}^+)$ and $k \in K$, then

$$ke^{tv} \cdot o \to \delta_{kM}$$

as $t \to \infty$, for the weak-star convergence on $\mathcal{M}(\mathscr{F})$, where δ_{kM} is the unit Dirac mass at kM.

A pair $(p, x) \in X \times \mathscr{F}$ is *in good position* (w.r.t. Δ) if the parametrized flat f determined by p and x verifies $\check{\mathsf{Z}}(\mathsf{f}) \in \mathrm{L}_\Delta$. Given $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we will say that $a \overset{\varepsilon}{\sim} b$ if $e^{-\varepsilon} a \leqslant b \leqslant e^{\varepsilon} a$.

Lemma 5.0 (Thirion [29, Lemma 10.7]). — *Fix a pair in good position* $(p, x) \in X \times \mathscr{F}$. *Then for all but countably many* $r \in \mathbb{R}_+$ *one has* $\mu_p(\partial \mathcal{O}_r(x, p)) = 0$. *Moreover, given* $\varepsilon > 0$ *there exists a neighborhood* V_x *of* x *in* $\overline{X}_F$ *such that for all* $z \in V_x$ *and all (but countably many) small enough* r *one has*

$$\mu_p(\mathcal{O}_r(z, p)) \overset{\varepsilon}{\sim} \mu_p(\mathcal{O}_r(x, p))$$

and if $z \in V_x \cap X$ *then* $\mu_p(\mathcal{O}_r^{\pm}(z, p)) \overset{\varepsilon}{\sim} \mu_p(\mathcal{O}_r(x, p))$.

Proof. — Indeed, the function $\mathbb{R}_+^* \to [0, 1]$ defined by $r \mapsto \mu_p(\mathcal{O}_r(x, p))$ is the distribution function of a probability measure in $\mathbb{R}_+^*$, and has only a countable number of discontinuity points. See Thirion [29, Lemma 10.7] for details.

□

If $p = g \cdot o \in X$, define the *φ-Gromov product* (or simply *Gromov product*) *based at* p as the map $[\cdot, \cdot]_p = [\cdot, \cdot]_p^{\varphi} : \mathscr{F}^{(2)} \to \mathbb{R}$ with

$$[x, y]_{g \cdot o} = \varphi(\mathscr{G}_\Pi(g^{-1}x, g^{-1}y)).$$

Remark 5.0. — Observe that $[\cdot, \cdot]_p$ is continuous, and that if p belongs to the maximal flat determined by $(x, y) \in \mathscr{F}^{(2)}$ then $[x, y]_p = 0$ (recall equation (4.1)).

Denote by $(\overline{\mu}_p)_{p \in X}$ the $\varphi \circ \mathrm{i}$-Patterson-Sullivan density of Δ. The following remark follows from the definitions of $(\mu_p)_{p \in X}$, $(\overline{\mu}_p)_{p \in X}$ and Lemma 4.0.

Remark 5.0. — The measure $e^{-[\cdot,\cdot]_p} \overline{\mu}_p \otimes \mu_p \otimes \mathrm{Leb}_{\mathfrak{a}}$ is independent of p. As said in the introduction, this measure is called the *φ-Bowen-Margulis measure* of Δ, and is denoted by $\widetilde{\chi}_\varphi$.

The main theorem. This section is devoted to the proof of the following theorem. The method is that of Roblin [25]. Indeed, his method adapts to our situation with minor arrangements, provided Corollary 4.0. This was noticed by Thirion [29], who extended Roblin's method to some higher rank situations. We will explain here how to overpass the main difficulties and refer the reader to Roblin [25, Chapitre 4] or Thirion [29, Chapitre 10] for the minor details.

If $\rho : \Gamma \to G$ is a Zariski-dense hyperconvex representation, denote by $\Theta = \Theta_{\rho(\Gamma)}$ its growth form. Recall that $\|\Theta\| = h_{\rho(\Gamma)}$ and that Θ is i-invariant and tangent to the growth indicator of $\rho(\Gamma)$. Denote by $(\mu_p)_{p\in X}$ the Θ-Patterson-Sullivan density of $\rho(\Gamma)$. Note that, with the above notation, $\overline{\mu}_p = \mu_p$ for all $p \in X$.

THEOREM 5.0. — *Let $\rho : \Gamma \to G$ be a Zariski-dense hyperconvex representation, and consider $p, q \in X$, then there exists $c = c(p, q) > 0$ such that*

$$e^{-\|\Theta\|T} \sum_{\gamma\in\Gamma : d_X(p,\rho(\gamma)q)\leqslant T} \delta_{\rho(\gamma)q} \otimes \delta_{\rho(\gamma^{-1})p} \to c\mu_p \otimes \mu_q$$

as $T \to \infty$, for the weak-star convergence on $C^*(\overline{X}_F \times \overline{X}_F)$.

A Zariski-dense hyperconvex representation $\rho : \Gamma \to G$ is fixed from now on. In order to simplify notation, we will identify Γ with $\rho(\Gamma)$, i.e. if $p \in X$ then γp means $\rho(\gamma)p$.

For $T \in \mathbb{R}_+$, let $\lambda^T(p, q)$ be the measure on $\overline{X}_F \times \overline{X}_F$ defined by

$$\lambda^T(p, q) = e^{-\|\Theta\|T} \sum_{\gamma\in\Gamma : d_X(p,\gamma q)\leqslant T} \delta_{\gamma q} \otimes \delta_{\gamma^{-1}p}.$$

If $A \subset \mathscr{F}$ and $r > 0$, consider the subset $C_r^+(p, A)$ of X, defined as the r-neighborhood of

$$\{\mathsf{f}(\mathfrak{a}^+) : \mathsf{f} \in G/M,\ d_X(\mathsf{f}(0), p) \leqslant r,\ \mathsf{Z}(\mathsf{f}) \in A\},$$

and consider the set $C_r^-(p, A)$ defined by

$$\{y \in X : B_X(y, r) \subset \bigcap_{\{q\in X : d_X(q,p)\leqslant r\}} \bigcup_{\{\mathsf{f}\in G/M : \mathsf{f}(0)=q,\ \mathsf{Z}(\mathsf{f})\in A\}} \mathsf{f}(\mathfrak{a}^+)\}.$$

The following proposition is the main step of the proof of Theorem 5.0.

PROPOSITION 5.0. — *Consider $p, q \in X$ and $x, y \in \mathscr{F}$ such that (p, x) and (q, y) are in good position. Then there exists $c > 0$ that verifies the following: for every $\varepsilon > 0$ there exists a neighborhood W of (x, y) on*

$\overline{X}_F \times \overline{X}_F$, *such that for every Borel sets* $A, B \subset \mathscr{F}$ *with* $A \times B \subset W$, *one has*

$$\limsup_{T\to\infty} \lambda^T(p,q)(C_1^-(p,A) \times C_1^-(q,B)) \leqslant e^{\varepsilon} c\mu_p(A)\mu_q(B)$$

and

$$\liminf_{T\to\infty} \lambda^T(p,q)(C_1^+(p,A) \times C_1^+(q,B)) \geqslant e^{-\varepsilon} c\mu_p(A)\mu_q(B).$$

Proof. — If $[\mathsf{f}]$ is a maximal flat and $p \in X$, denote by f^p the parametrized flat such that $[\mathsf{f}^p] = [\mathsf{f}]$, and such that $\mathsf{f}^p(0)$ is the orthogonal projection of p on $[\mathsf{f}]$. If $v \in \mathfrak{a}$, denote by $\mathrm{tr}_v : \mathfrak{a} \to \mathfrak{a}$ the translation by v, i.e. $\mathrm{tr}_v(u) = u+v$. For $A \subset \mathscr{F}$ and $r > 0$, consider the subsets of G/M defined by

$$K_r^+(p,A) = \{\mathsf{f}^p \circ \mathrm{tr}_v : v \in B_{\mathfrak{a}}(0,r),\ d_X(\mathsf{f}^p(0),p) \leqslant r,\ \mathsf{Z}(\mathsf{f}) \in A\}$$

and

$$K_r^-(p,A) = \{\mathsf{f}^p \circ \mathrm{tr}_v : v \in B_{\mathfrak{a}}(0,r),\ d_X(\mathsf{f}^p(0),p) \leqslant r,\ \check{\mathsf{Z}}(\mathsf{f}) \in A\},$$

where $B_{\mathfrak{a}}(0,r)$ is the ball on $\mathfrak{a}$ of radius r centered at 0, for the Euclidean norm $\|\ \|$. Denote by $K_r(p) = K_r^+(p,\mathscr{F}) = K_r^-(p,\mathscr{F})$.

Fix $\varepsilon > 0$. Lemma 5.0 applied to (p,x) and (q,y) provides neighborhoods V_x of x and V_y of y, such that for all (but countably many) small enough $r > 0$ one has $\mu_p(\partial\mathcal{O}_r(x,p)) = \mu_p(\partial\mathcal{O}_r(y,q)) = 0$ and if $z \in V_x$ then

$$\mu_p(\mathcal{O}_r(z,p)) \overset{\varepsilon}{\sim} \mu_p(\mathcal{O}_r(x,p))$$

and if $w \in V_y$ then

$$\mu_p(\mathcal{O}_r(w,q)) \overset{\varepsilon}{\sim} \mu_p(\mathcal{O}_r(y,q)).$$

Moreover, for all $z \in V_x \cap X$ and $w \in V_y \cap X$, Lemma 5.0 states that

$$\mu_p(\mathcal{O}_r^{\pm}(z,p)) \overset{\varepsilon}{\sim} \mu_p(\mathcal{O}_r(x,p))$$

and

$$\mu_p(\mathcal{O}_r^{\pm}(w,q)) \overset{\varepsilon}{\sim} \mu_p(\mathcal{O}_r(y,q)).$$

Consider $r < \min\{1, \varepsilon/\|\Theta\|, \varepsilon/\kappa, \varepsilon/\|\Theta\|\kappa\}$ such that the last paragraph holds, where κ is the constant given by Lemma 5.0, and such that $|\Theta(u)| < \varepsilon$ for all $u \in B_{\mathfrak{a}}(0,r)$.

We can assume also that r is small enough such that if $z \in V_x$ and $w \in \mathcal{O}_r(z,p)$ then, $e^{-[w,z]_p} \overset{\varepsilon}{\sim} 1$ (Remark 5.0) and similarly for V_y and q.

We will show that $V_x \times V_y$ is the desired neighborhood. Consider then A, B Borel subsets of $\mathscr{F}$ such that $A \times B \subset V_x \times V_y$. Let us simplify notation and write $K^+ = K_r^+(p,A)$ and $K^- = K_r^-(q,B)$.

Given $\gamma \in \Gamma$ and $T > 0$, define $\Xi(\gamma, T)$ by

$$\Xi(\gamma, T) = \int_{B_{\mathfrak{a}}(0,T)\cap \mathfrak{a}^+} e^{\Theta(v)} \widetilde{\chi}_\Theta(K^+ \cdot \exp(v) \cap \gamma \cdot K^-) d\,\mathrm{Leb}_{\mathfrak{a}}(v).$$

Following Roblin's [25] method (see also Thirion [29]), we will compute

$$e^{-\|\Theta\|T} \sum_{\gamma\in\Gamma} \Xi(\gamma, T)$$

in two different ways. Observe first that Corollary $4.0^{(2)}$ gives

$$e^{-\|\Theta\|T} \sum_{\gamma\in\Gamma} \Xi(\gamma, T) \overset{\varepsilon}{\sim} c\widetilde{\chi}_\Theta(K^+)\widetilde{\chi}_\Theta(K^-)$$

for all big enough T and a constant $c > 0$. Let's compute then $\widetilde{\chi}_\Theta(K^+)$ and $\widetilde{\chi}_\Theta(K^-)$. Remark 5.0 states that

$$\widetilde{\chi}_\Theta = e^{-[\cdot,\cdot]_p} \mu_p \otimes \mu_p \otimes \mathrm{Leb}_{\mathfrak{a}},$$

hence

$$\widetilde{\chi}_\Theta(K^+) = \int_{v\in B_{\mathfrak{a}}(0,r)} \int_{z\in A} \int_{w\in\mathcal{O}_r(z,p)} e^{-[w,z]_p} \mathbf{1}_{K_r(p)}(\mathsf{f}^p_{wz} \circ \mathrm{tr}_v) d\mu_p(w) d\mu_p(z) d\,\mathrm{Leb}_{\mathfrak{a}}(v).$$

Since $w \in \mathcal{O}_r(z, p)$ (and by our choice of r) one has $e^{-[w,z]_p} \overset{\varepsilon}{\sim} 1$, thus

$$\widetilde{\chi}_\Theta(K^+) \overset{\varepsilon}{\sim} \mathrm{vol}(B_{\mathfrak{a}}(0, r)) \int_A \mu_p(\mathcal{O}_r(z, p)) d\mu_p(z).$$

Since $z \in A \subset V_x$ and by the definition of V_x, we have $\mu_p(\mathcal{O}_r(z, p)) \overset{\varepsilon}{\sim} \mu_p(\mathcal{O}_r(x, p))$, hence

$$\widetilde{\chi}_\Theta(K^+) \overset{2\varepsilon}{\sim} \mathrm{vol}(B_{\mathfrak{a}}(0, r))\mu_p(\mathcal{O}_r(x, p))\mu_p(A).$$

Analogous reasoning, using the equality $\widetilde{\chi}_\Theta = e^{-[\cdot,\cdot]_q} \mu_q \otimes \mu_q \otimes \mathrm{Leb}_{\mathfrak{a}}$, gives $\widetilde{\chi}_\Theta(K^-) \overset{2\varepsilon}{\sim} \mathrm{vol}(B_{\mathfrak{a}}(0, r))\mu_q(\mathcal{O}_r(y, q))\mu_q(B)$. Hence, if we denote by

$$\mathsf{H} = \mathrm{vol}(B_{\mathfrak{a}}(0, r))^2 \mu_q(\mathcal{O}_r(y, q))\mu_p(\mathcal{O}_r(x, p)),$$

one has

$$e^{-\|\Theta\|T} \sum_{\gamma\in\Gamma} \Xi(\gamma, T) \overset{4\varepsilon}{\sim} c\mu_p(A)\mu_q(B)\mathsf{H}, \tag{5.1}$$

for all big enough T. Notice that, since (p, x) and (q, y) are in good position, one has $\mathsf{H} \neq 0$. This will allow us later to divide by H.

(2) Even though Corollary 4.0 is stated for continuous functions with compact support, a standard measure-theoretic argument permits to extend it to characteristic functions of compact sets.

We will now explicitly compute $\sum_{\gamma\in\Gamma}\Xi(\gamma,T)$.

Remark 5.0. — Denote by $V_A^+ = C_1^+(p,A)\cap V_x$ and $V_B^+ = C_1^+(q,B)\cap V_y$, and denote by $V_A^- = C_1^-(p,A)\cap V_x$ and $V_B^- = C_1^-(q,B)\cap V_y$. Then there exist constants $L>0$, independent of ε and T, and $C>0$ independent of T, such that for all big enough T one has

$$\sum_{\gamma\in\Gamma}\Xi(\gamma,T-2r)\leqslant C+e^{L\varepsilon}\mathsf{H}\sum \mathbf{1}_{V_A^+}(\gamma q)\mathbf{1}_{V_B^+}(\gamma^{-1}p),$$

where the sum is over all $\gamma\in\Gamma$ such that $d_X(p,\gamma q)\leqslant T$ and $(\gamma q,\gamma^{-1}p)\in C_1^+(p,A)\times C_1^+(q,B)$, and moreover

$$\sum_{\gamma\in\Gamma}\Xi(\gamma,T+2r)\geqslant -C+e^{-L\varepsilon}\mathsf{H}\sum \mathbf{1}_{V_A^-}(\gamma q)\mathbf{1}_{V_B^-}(\gamma^{-1}p),$$

where the last sum is over all $\gamma\in\Gamma$ such that $d_X(p,\gamma q)\leqslant T$ and $(\gamma q,\gamma^{-1}p)\in C_1^-(p,A)\times C_1^-(q,B)$.

Proof. — We will only show the upper bound (the lower bound being analogous). Observe that, if for some $\gamma\in\Gamma$ one has $\Xi(\gamma,T-2r)\neq 0$, then

$$K^+\cdot\exp(v)\cap\gamma\cdot K^-\neq\emptyset,$$

for some $v\in B_{\mathfrak{a}}(0,T-2r)\cap\mathfrak{a}^+$. This intersection is contained in

$$(\mathcal{O}_r^+(\gamma q,p)\cap\gamma B)\times(\mathcal{O}_r^+(p,\gamma q)\cap A)\times\mathfrak{a},$$

and necessarily one has:

i) $v\in B_{\mathfrak{a}}(\underline{a}(p,\gamma q),2r)$, in particular $d_X(p,\gamma q)\leqslant T$, and
ii) $(\gamma q,\gamma^{-1}p)\in C_1^+(p,A)\times C_1^+(q,B)$.

Observe that $\widetilde{\chi}_\Theta(K^+\cdot\exp(v)\cap\gamma K^-)=$

$$\int_{u\in B_{\mathfrak{a}}(0,r)}\int_{z\in\mathcal{O}_r^+(p,\gamma q)\cap A}\int_{w\in\mathcal{O}_r^+(\gamma q,p)\cap\gamma B} e^{-[w,z]_p}\times\mathbf{1}_{K_r(\gamma q)}(\mathsf{f}_{zw}^p\circ\mathrm{tr}_{u+v})d\mu_p(z)d\mu_p(w)d\,\mathrm{Leb}_{\mathfrak{a}}(u).$$

For $(z,w)\in(\mathcal{O}_r^+(p,\gamma q)\cap A)\times(\mathcal{O}_r^+(\gamma q,p)\cap\gamma B)$, one has $e^{-[w,z]_p}\overset{\varepsilon}{\sim}1$ and

$$\int_{B_{\mathfrak{a}}(0,T-2r)} e^{\Theta(v)}e^{-[w,z]_p}\mathbf{1}_{K_r(\gamma q)}(\mathsf{f}_{zw}^p\circ\mathrm{tr}_{u+v})d\,\mathrm{Leb}_{\mathfrak{a}}(v)\leqslant e^{3\varepsilon}e^{\Theta(\underline{a}(p,\gamma q))}\,\mathrm{vol}(B_{\mathfrak{a}}(0,r)).$$

One concludes that $\sum_{\gamma\in\Gamma}\Xi(\gamma,T-2r)\leqslant$

$$C+e^{L_0\varepsilon}\,\mathrm{vol}(B_{\mathfrak{a}}(0,r))^2\sum e^{\Theta(\underline{a}(p,\gamma q))}\mu_p(\mathcal{O}_r^+(\gamma q,p))\mu_p(\mathcal{O}_r^+(p,\gamma q))\mathbf{1}_{V_A^+}(\gamma q)\mathbf{1}_{V_B^+}(\gamma^{-1}p),$$

for some $L_0 > 0$ independent of T, where the sum is over all $\gamma \in \Gamma$ that verify i) and ii) above, and C is a constant independent of T, determined by the (finitely many) $\gamma \in \Gamma$ such that

$$(\gamma q, \gamma^{-1} p) \in C_r^+(p, A) \times C_r^+(q, B) - V_x \times V_y,$$

(which is bounded in $X \times X$). Since $\gamma q \in V_x$ one has $\mu_p(\mathcal{O}_r^+(\gamma q, p)) \overset{\varepsilon}{\sim} \mu_p(\mathcal{O}_r(x, p))$ and the right hand side of the last equation becomes

$$C + e^{L_0 \varepsilon} \operatorname{vol}(B_{\mathfrak{a}}(0, r))^2 \mu_p(\mathcal{O}_r(x, p)) \times \sum e^{\Theta(\underline{a}(p, \gamma q))} \mu_p(\mathcal{O}_r^+(p, \gamma q)) \mathbf{1}_{V_A^+}(\gamma q) \mathbf{1}_{V_B^+}(\gamma^{-1} p).$$

Using Lemma 5.0 and the fact that γq belongs to $V_A^+ \subset V_x$, one obtains

$$e^{\Theta(\underline{a}(p, \gamma q))} \overset{\varepsilon}{\sim} e^{\Theta(\sigma_z(p, \gamma q))},$$

for any $z \in \mathcal{O}_r^+(p, \gamma q)$. Applying the definition of $(\mu_m)_{m \in X}$, one has that

$$e^{\Theta(\underline{a}(p, \gamma q))} \mu_p(\mathcal{O}_r^+(p, \gamma q)) \overset{\varepsilon}{\sim} \int_{\mathcal{O}_r^+(p, \gamma q)} e^{\Theta(\sigma_z(p, \gamma q))} d\mu_p(z) = \mu_{\gamma q}(\mathcal{O}_r^+(p, \gamma q))$$

$$= \mu_q(\mathcal{O}_r^+(\gamma^{-1} p, q)) \overset{\varepsilon}{\sim} \mu_q(\mathcal{O}_r(y, q)),$$

since $\gamma^{-1} p \in V_B^+ \subset V_y$. Hence, for some constant $L > 0$

$$\sum_{\gamma \in \Gamma} \Xi(\gamma, T - 2r) \leqslant C + e^{L\varepsilon} \mathsf{H} \sum \mathbf{1}_{V_A^+}(\gamma q) \mathbf{1}_{V_B^+}(\gamma^{-1} p),$$

where the sum is over all $\gamma \in \Gamma$ that verify i) and ii) above. This finishes the proof of the remark. □

The proof of the proposition will be completed when we compute

$$e^{-\|\Theta\|(T-2r)} \sum_{\gamma \in \Gamma} \Xi(\gamma, T - 2r),$$

assembling equation (5.1) and Remark 5.0. For all big enough T, one has

$$e^{-4\varepsilon} c \mu_p(A) \mu_q(B) \mathsf{H} \leqslant e^{-\|\Theta\|(T-2r)} \sum_{\gamma \in \Gamma} \Xi(\gamma, T - 2r) \leqslant$$

$$e^{-\|\Theta\| T} (C_0 + e^{L_0 \varepsilon} \mathsf{H} \sum \mathbf{1}_{V_A^+}(\gamma q) \mathbf{1}_{V_B^+}(\gamma^{-1} p)),$$

for some L_0 independent of ε and T (recall that $r < \varepsilon / \|\Theta\|$) and C_0 independent of T, where the sum is over all $\gamma \in \Gamma$ that verify i) and ii) above. Since C_0 is independent of T and since $\mathsf{H} \neq 0$, one obtains

$$\liminf_{T \to \infty} \lambda^T(p, q) (C_1^+(p, A) \times C_1^+(q, B)) \geqslant e^{-L_0 \varepsilon} c \mu_p(A) \mu_q(B).$$

The other inequality follows similarly. □

We continue with the proof of Theorem 5.0. For $\mathscr{A} \subset \mathfrak{a}^+$, define the measure $\lambda^T(p,q,\mathscr{A})$ on $\overline{X}_F \times \overline{X}_F$ by

$$\lambda^T(p,q,\mathscr{A}) = e^{-\|\Theta\|T} \sum_{\gamma\in\Gamma:\underline{a}(p,\gamma q)\in\mathscr{A},\ d_X(p,\gamma q)\leqslant T} \delta_{\gamma q}\otimes\delta_{\gamma^{-1}p}.$$

Observe that $\lambda^T(p,q) = \lambda^T(p,q,\mathfrak{a}^+)$.

We will need the following lemma.

LEMMA 5.0. — *Let Δ be a Zariski-dense subgroup of G. Consider a continuous function $f : \overline{X}_F \times \overline{X}_F \to \mathbb{R}$ and an open cone $\mathscr{C}$ with $u_\Delta \in \mathscr{C}$. Then*

$$e^{-h_\Delta t}\sum f(gq, g^{-1}p) \to 0$$

as $t \to \infty$, where the sum is over all $g \in \Delta$ such that $d_X(p,gq) \leqslant t$ and $\underline{a}(p,gq) \notin \mathscr{C}$.

Proof. — The lemma follows directly from Remark 4.0 together with Remark 5.0. □

In our current notation, the last lemma reads as follows.

LEMMA 5.0. — *Let $\mathscr{C} \subset \mathfrak{a}^+$ be an open cone with $u_\Gamma \in \mathscr{C}$, then*

$$\lambda^T(p,q,\mathscr{C}) - \lambda^T(p,q) \to 0,$$

for the weak-star convergence on $C(\overline{X}_F \times \overline{X}_F)$, as $T \to \infty$.

Proof of Theorem 5.0. — It remains to overpass the good position hypothesis on Proposition 5.0.

Notice that if $x \in \mathscr{F}$ then one can choose $z \in \partial_\infty\Gamma$ such that $(x,\zeta(z)) \in \mathscr{F}^{(2)}$, where $\zeta : \partial_\infty\Gamma \to \mathscr{F}$ is the equivariant map. Fix then (x,y) in $\mathscr{F}\times\mathscr{F}$ and consider $(z,w) \in \zeta(\partial_\infty\Gamma)^2$ such that (x,z) and (y,w) belong to $\mathscr{F}^{(2)}$. Choosing p_0 on the maximal flat determined by (x,z), and q_0 on the maximal flat determined by (y,w), one gets that (p_0,x) and (q_0,y) are both in good position.

Applying Proposition 5.0 to the pairs (p_0,x) and (q_0,y) and a given $\varepsilon > 0$, one obtains a neighborhood W of $(x,y) \in \overline{X}_F^2$ such that if $A \times B$ is a Borel set contained in $\mathscr{F}^2 \cap W$, then

$$\liminf_{T\to\infty} \lambda^T(p_0,q_0)(C_1^+(p_0,A)\times C_1^+(q_0,B)) \geqslant e^{-\varepsilon} c\mu_{p_0}(A)\mu_{q_0}(B). \tag{5.1}$$

Discarding finitely many $\gamma \in \Gamma$, we can assume that if $\gamma q_0 \in C_1^+(p_0,A)$ and $\gamma^{-1}p \in C_1^+(q,B)$, then $(\gamma q_0, \gamma^{-1}p) \in W$. Moreover, if W is small enough, Quint's Lemma 5.0 together with Proposition 4.0 imply that for all such $\gamma \in \Gamma$ one has

$$\|\underline{a}(p_0,\gamma q_0) - \underline{a}(p,\gamma q_0) - \sigma_x(p_0,p)\| \leqslant \varepsilon$$

and

$$\|\underline{a}(q_0,\gamma^{-1}p)-\underline{a}(q,\gamma^{-1}p)-\sigma_y(q_0,q)\|\leqslant\varepsilon.$$

Equation (5.1) and G-invariance of $\underline{a}$ imply that $\underline{a}(q_0,\gamma^{-1}p)=\mathrm{i}(\underline{a}(p,\gamma q_0))$, and since $\mathrm{i}^2=\mathrm{id}$ one has

$$\|\underline{a}(p,\gamma q_0)-\underline{a}(p,\gamma q)-\mathrm{i}\,\sigma_y(q_0,q)\|<\varepsilon.$$

Consequently,

$$\|\underline{a}(p_0,\gamma q_0)-\underline{a}(p,\gamma q)-(\sigma_x(p_0,p)+\mathrm{i}\,\sigma_y(q_0,q))\|\leqslant 2\varepsilon.$$

Hence,

$$\Theta(\underline{a}(p_0,\gamma q_0))\leqslant\Theta(\underline{a}(p,\gamma q))+\Theta(\sigma_x(p_0,p)+\mathrm{i}\,\sigma_y(q_0,q))+\delta,$$

for some δ (Θ is continuous at 0).

Recall that if $v\in\mathbb{R}\cdot u_{\rho(\Gamma)}$ then $|\Theta(v)|=\|\Theta\|\|v\|$ (Remark 4.0). Consider then a closed cone $\mathscr{C}$, with $u_{\rho(\Gamma)}\in\operatorname{int}\mathscr{C}$, such that for all $v\in\mathscr{C}$ one has

$$\Theta(v)=|\Theta(v)|\overset{\varepsilon}{\sim}\|\Theta\|\|v\|.$$

Notice that, since $\underline{a}(p_0,\gamma q_0)$ is at bounded distance from $\underline{a}(p,\gamma q)$ (independently of $\gamma\in\Gamma$, see Remark 5.0), we can consider an open cone $\mathscr{C}'$ with $u_{\rho(\Gamma)}\in\mathscr{C}'$, such that if γ is big enough, and $\underline{a}(p,\gamma q)\in\mathscr{C}'$, then $\underline{a}(p_0,\gamma q_0)\in\mathscr{C}$. Hence, for all big enough $\gamma\in\Gamma$ such that $\underline{a}(p,\gamma q)\in\mathscr{C}'$ one has

$$d_X(p_0,\gamma q_0)\leqslant d_X(p,\gamma q)+\frac{1}{\|\Theta\|}(\Theta(\sigma_x(p_0,p)+\mathrm{i}(\sigma_y(q_0,q)))+\delta).$$

Lemma 5.0 together with equation (5.1), imply that

$$\liminf_{T\to\infty}\lambda^T(p_0,q_0,\mathscr{C})(C_1^+(p_0,A)\times C_1^+(q_0,B)\geqslant e^{-\varepsilon}c\mu_{p_0}(A)\mu_{q_0}(B).$$

Denoting by

$$T'=T+\frac{1}{\|\Theta\|}\Theta(\sigma_x(p_0,p)+\mathrm{i}(\sigma_y(q_0,q)))+\delta,$$

one concludes that (using again Lemma 5.0) for all big enough T one has $\lambda^T(p,q)(C_1^+(p,A)\times C_1^+(q,B))\geqslant\lambda^T(p,q,\mathscr{C}')(C_1^+(p,A)\times C_1^+(q,B))-\varepsilon\geqslant$

$$e^{-\varepsilon}e^{\Theta(\sigma_x(p_0,p)+\mathrm{i}(\sigma_y(q_0,q)))+\delta}\lambda^{T'}(p_0,q_0,\mathscr{C})(C_1^+(p_0,A)\times C_1^+(q_0,B))-\varepsilon.$$

Thus, $\liminf_T\lambda^T(p,q)(C_1^+(p,A)\times C_1^+(q,B))\geqslant$

$$e^{-2\varepsilon}ce^{\Theta(\sigma_x(p_0,p)+\mathrm{i}\,\sigma_y(q_0,q)))}\mu_{p_0}(A)\mu_{q_0}(B)-\varepsilon.$$

Finally, by definition of $(\mu_m)_{m\in X}$, one has

$$e^{\Theta(\sigma_x(p_0,p))}\mu_{p_0}(A)\overset{\varepsilon}{\sim}\mu_p(A),$$

and

$$e^{\Theta(\mathrm{i}\,\sigma_y(q_0,q))}\mu_{q_0}(B) \overset{\varepsilon}{\sim} \mu_q(B).$$

One concludes that

$$\liminf_{T\to\infty} \lambda^T(p,q)(C_1^+(p,A)\times C_1^+(q,B)) \geqslant e^{-4\varepsilon}c\mu_p(A)\mu_q(B) - \varepsilon,$$

as desired. The other inequality is analogous, and a standard partition of unity argument finishes the proof of the theorem (see Roblin [25, pages 62-63] for more details). □

BIBLIOGRAPHY

[1] M. Babillot, "Théorie du renouvellement pour des chaînes semi-markoviennes transientes", *Ann. Inst. H. Poincaré Probab. Statist.* **24** (1988), no. 4, p. 507-569.

[2] Y. Benoist, "Propriétés asymptotiques des groupes linéaires", *Geom. Funct. Anal.* **7** (1997), no. 1, p. 1-47.

[3] Y. Benoist, "Propriétés asymptotiques des groupes linéaires. II", in *Analysis on homogeneous spaces and representation theory of Lie groups, Okayama–Kyoto (1997)*, Adv. Stud. Pure Math., vol. 26, Math. Soc. Japan, Tokyo, 2000, p. 33-48.

[4] R. Bowen, "Periodic orbits for hyperbolic flows", *Amer. J. Math.* **94** (1972), p. 1-30.

[5] ———, "Symbolic dynamics for hyperbolic flows", *Amer. J. Math.* **95** (1973), p. 429-460.

[6] R. Bowen & D. Ruelle, "The ergodic theory of Axiom A flows", *Invent. Math.* **29** (1975), no. 3, p. 181-202.

[7] W. Duke, Z. Rudnick & P. Sarnak, "Density of integer points on affine homogeneous varieties", *Duke Math. J.* **71** (1993), no. 1, p. 143-179.

[8] V. A. Efremovič & E. S. Tihomirova, "Equimorphisms of hyperbolic spaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), p. 1139-1144.

[9] A. Eskin & C. McMullen, "Mixing, counting, and equidistribution in Lie groups", *Duke Math. J.* **71** (1993), no. 1, p. 181-209.

[10] É. Ghys & P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1990, Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988, xii+285 pages.

[11] Y. Guivarc'h & J. Hardy, "Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov", *Ann. Inst. H. Poincaré Probab. Statist.* **24** (1988), no. 1, p. 73-98.

[12] Y. Guivarc'h, L. Ji & J. C. Taylor, *Compactifications of symmetric spaces*, Progress in Mathematics, vol. 156, Birkhäuser Boston, Inc., Boston, MA, 1998, xiv+284 pages.

[13] R. E. Howe & C. C. Moore, "Asymptotic properties of unitary representations", *J. Funct. Anal.* **32** (1979), no. 1, p. 72-96.

[14] J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York-Heidelberg, 1975, Graduate Texts in Mathematics, No. 21, xiv+247 pages.

[15] A. Katok & B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza, xviii+802 pages.

[16] F. Labourie, "Anosov flows, surface groups and curves in projective space", *Invent. Math.* **165** (2006), no. 1, p. 51-114.

[17] F. Ledrappier, "Structure au bord des variétés à courbure négative", in *Séminaire de Théorie Spectrale et Géométrie, No. 13, Année 1994–1995*, Sémin. Théor. Spectr. Géom., vol. 13, Univ. Grenoble I, Saint-Martin-d'Hères, 1995, p. 97-122.

[18] A. N. Livšic, "Cohomology of dynamical systems", *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), p. 1296-1320.

[19] G. A. Margulis, "Certain applications of ergodic theory to the investigation of manifolds of negative curvature", *Funkcional. Anal. i Priložen.* **3** (1969), no. 4, p. 89-90.

[20] J.-F. Quint, "Mesures de Patterson-Sullivan en rang supérieur", *Geom. Funct. Anal.* **12** (2002), no. 4, p. 776-809.

[21] ———, "Groupes convexes cocompacts en rang supérieur", *Geom. Dedicata* **113** (2005), p. 1-19.

[22] J.-F. Quint, "Divergence exponentielle des sous-groupes discrets en rang supérieur", *Comment. Math. Helv.* **77** (2002), no. 3, p. 563-608.

[23] ———, "Groupes de Schottky et comptage", *Ann. Inst. Fourier (Grenoble)* **55** (2005), no. 2, p. 373-429.

[24] M. Ratner, "Markov partitions for Anosov flows on n-dimensional manifolds", *Israel J. Math.* **15** (1973), p. 92-114.

[25] T. Roblin, "Ergodicité et équidistribution en courbure négative", *Mém. Soc. Math. Fr. (N.S.)* (2003), no. 95, p. vi+96.

[26] A. Sambarino, "Hyperconvex representations and exponential growth", *Ergodic Theory Dynam. Systems* **34** (2014), no. 3, p. 986-1010.

[27] A. Sambarino, "Quantitative properties of convex representations", *Comment. Math. Helv.* **89** (2014), no. 2, p. 443-488.

[28] M. Shub, *Global stability of dynamical systems*, Springer-Verlag, New York, 1987, With the collaboration of Albert Fathi and Rémi Langevin, Translated from the French by Joseph Christy, xii+150 pages.

[29] X. Thirion, "Sous-groupes discrets de $SL(d, \mathbb{R})$ et equidistribution dans les espaces symétriques", PhD Thesis, Université de Tours (France), 2007.

[30] ———, "Propriétés de mélange du flot des chambres de Weyl des groupes de ping-pong", *Bull. Soc. Math. France* **137** (2009), no. 3, p. 387-421.

[31] J. Tits, "Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque", *J. Reine Angew. Math.* **247** (1971), p. 196-220.

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