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Balancing conditions in global tropical geometry

BALANCING CONDITIONS IN GLOBAL TROPICAL GEOMETRY

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ABSTRACT. — We study tropical geometry in the global setting using Berkovich's deformation retraction. We state and prove the generalized balancing conditions in this setting. Starting with a strictly semi-stable formal scheme, we calculate certain sheaves of vanishing cycles using analytic étale cohomology, then we interpret the tropical weight vectors via these cycles. We obtain the balancing condition for tropical curves on the skeleton associated to the formal scheme in terms of the intersection theory on the special fiber. Our approach works over any complete discrete valuation field.

RÉSUMÉ. — Nous étudions la géométrie tropicale dans le cadre global en utilisant la rétraction par déformation construite par V. Berkovich. Nous montrons les conditions d'équilibre généralisées dans ce cadre. À partir d'un schéma formel strictement semi-stable, nous calculons certains faisceaux de cycles évanescents par la cohomologie étale analytique, puis nous interprétons les vecteurs de poids tropical via ces cycles. Nous obtenons la condition d'équilibre pour les courbes tropicales sur le squelette associé au schéma formel en fonction de la théorie d'intersection sur la fibre spéciale. Notre approche fonctionne pour tout corps complet de valuation discrète.

1. Introduction and statement of result

Tropicalization is a procedure which relates algebraic geometry with tropical geometry. Usually, tropicalization is carried out in the setting of toric varieties [11, 2, 13]. Let us give a quick review in the case of curves.

By definition, a toric variety contains an open dense torus $\mathbb{G}_{\mathrm{m}}^n$. Assume that our torus $\mathbb{G}_{\mathrm{m}}^n$ is defined over a complete discrete valuation field k. Consider the coordinate-wise valuation map

$$(1.1) \quad \mathbb{G}_{\mathrm{m}}^n(k) \simeq (k^\times)^n \to \mathbb{R}^n, \qquad (x_1, \ldots, x_n) \mapsto (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)).$$

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This map can be extended to the Berkovich analytification $(\mathbb{G}_{\mathrm{m}}^n)^{\mathrm{an}}$ and we obtain a continuous surjective map $\tau_0\colon (\mathbb{G}_{\mathrm{m}}^n)^{\mathrm{an}} \to \mathbb{R}^n$.

Now let C be an analytic curve embedded in $(\mathbb{G}_{\mathrm{m}}^n)^{\mathrm{an}}$. Traditionally, the tropicalization of C is by definition the image $\tau_0(C) \subset \mathbb{R}^n$, which we denote by C^{t}. The tropical curve C^{t} has the structure of a metrized graph satisfying the *balancing condition*. One idea of tropical geometry is to study curves in algebraic varieties in terms of such combinatorial gadgets.

Let us recall the classical balancing condition. To each edge e of C^{t} with a chosen orientation, one can associate an element $\widetilde{w}_e \in \mathbb{Z}^n$, parallel to the direction of e inside $\mathbb{R}^n$, called the *tropical weight vector* of e. Then the balancing condition states that for any vertex $v \in C^{\mathrm{t}}$, we have $\sum_{e \ni v} \widetilde{w}_e = 0$, where the sum is taken over all edges that contain v as an endpoint, and the orientation of each edge is chosen to be the one that points away from v. We refer to [17, 18, 20, 2] for the proofs.

In order to go beyond the toric case above, we propose to replace the map $\tau_0\colon (\mathbb{G}_{\mathrm{m}}^n)^{\mathrm{an}} \to \mathbb{R}^n$ by the deformation retraction of a non-archimedean analytic space onto its skeleton constructed by V. Berkovich [7]. Let $\mathfrak{X}$ be a strictly semi-stable formal scheme over the ring of integers k° (see Definition 2.2). Its generic fiber $\mathfrak{X}_\eta$ is a Berkovich space over k and its special fiber $\mathfrak{X}_s$ is a scheme over the residue field $\widetilde{k}$. Let $S_{\mathfrak{X}}$ denote the dual intersection complex of $\mathfrak{X}_s$. Following [7], one can construct an embedding $S_{\mathfrak{X}} \subset \mathfrak{X}_\eta$, and a continuous proper surjective retraction map $\tau\colon \mathfrak{X}_\eta \to S_{\mathfrak{X}}$. So $S_{\mathfrak{X}}$ is called a *skeleton* of $\mathfrak{X}_\eta$. We consider the map $\tau\colon \mathfrak{X}_\eta \to S_{\mathfrak{X}}$ to be the globalization of the map $\tau_0\colon (\mathbb{G}_{\mathrm{m}}^n)^{\mathrm{an}} \to \mathbb{R}^n$ in (1.1).

Now in parallel, let C be a compact quasi-smooth[1] k-analytic curve, and let $f\colon C \to \mathfrak{X}_\eta$ be a k-analytic morphism. We call the image $(\tau \circ f)(C) \subset S_{\mathfrak{X}}$ the associated tropical curve, and denote it again by C^{t}. By working locally, one can show as in the toric case that C^{t} is a graph piecewise linearly embedded in $S_{\mathfrak{X}}$. However, the balancing condition for the tropical curve C^{t} is no longer clear in the global setting, because the vertices of C^{t} may sit on the corners of $S_{\mathfrak{X}}$. So we ask the following question.

Question. Let v be a vertex of the tropical curve C^{t}. What are the constraints on the shape of C^{t} near the vertex v?

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[1] The quasi-smoothness assumption on the curve C is not restrictive because we are considering morphisms from C to $\mathfrak{X}_\eta$ rather than embedded curves, and one can always make desingularizations.

In this paper, we give a necessary condition in terms of the intersection theory on the special fiber.

Let $\{D_i\}_{i \in I_{\mathfrak{X}}}$ denote the set of irreducible components of the special fiber $\mathfrak{X}_s$. We have a natural embedding $S_{\mathfrak{X}} \subset \mathbb{R}^{I_{\mathfrak{X}}}$. Assume that the vertex v sits in the relative interior of the face of $S_{\mathfrak{X}}$ corresponding to a subset $I_v \subset I_{\mathfrak{X}}$. Let D_{I_v} denote the corresponding closed stratum of $\mathfrak{X}_s$, and let $\overline{D}_{I_v}$ denote the base change to the algebraic closure of $\widetilde{k}$.

As in the classical case, to each edge e of C^{t} containing v as an endpoint, we can associate a weight vector $\widetilde{w}_e \in \mathbb{Z}^{I_{\mathfrak{X}}}$ (see Section 5, compare [2, §6]). We denote the sum of weight vectors around v by

$$(1.2) \qquad \sigma_v := \sum_{e \ni v} \widetilde{w}_e \in \mathbb{Z}^{I_{\mathfrak{X}}}.$$

Let L_i be the pullback of the line bundle $\mathcal{O}(D_i)$ to $\overline{D}_{I_v}$, for every $i \in I_{\mathfrak{X}}$. Let α be the map

$$\alpha \colon \mathrm{CH}_1\left(\overline{D}_{I_v}\right) \longrightarrow \mathbb{Z}^{I_{\mathfrak{X}}}$$
$$L \longmapsto \left(L \cdot L_i, \ i \in I_{\mathfrak{X}}\right),$$

that takes a one-dimensional cycle L in $\overline{D}_{I_v}$ to its intersection numbers with the divisors L_i for every $i \in I_{\mathfrak{X}}$.

Our generalized balancing condition is stated in the following theorem. The rest of the paper provides a proof of the theorem.

THEOREM 1.1. — *Assume that the closed stratum D_{I_v} is projective, and that the vertex v does not lie in the image of the boundary $(\tau \circ f)(\partial C)$. Then the sum σ_v of weight vectors lies in the image of the map*

$$\alpha_{\mathbb{Q}} := \alpha \otimes \mathbb{Q} \colon \mathrm{CH}_1\left(D_{I_v}\right)_{\mathbb{Q}} \longrightarrow \mathbb{Q}^{I_{\mathfrak{X}}}.$$

Example 1.2. — Assume that $\mathfrak{X}$ is n-dimensional and that the vertex v sits in the interior of an n-dimensional face of $S_{\mathfrak{X}}$. Then D_{I_v} is a point and the map α is zero. Our balancing condition in this case simply states that the sum σ_v of weight vectors must be zero. So we recover the classical balancing condition in our generalized setting.

Example 1.3. — Let us work out a concrete example for a degeneration of K3 surfaces. Let $k = \mathbb{C}((t))$ be the field of formal Laurent series. Let $\mathfrak{X}^0 \subset \mathbf{P}^3_{\mathbb{C}[\![t]\!]}$ be the formal scheme given by the equation

$$x_0 x_1 x_2 x_3 + t P_4(x_0, x_1, x_2, x_3) = 0,$$

where P_4 is a generic homogeneous polynomial of degree four. We think of $\mathfrak{X}^0$ as a formal family of complex K3 surfaces. The special fiber $\mathfrak{X}_s^0$ consists of the four coordinate hyperplanes in $\mathbf{P}_\mathbb{C}^3$, which we denote by D_0, D_1, D_2, D_3 respectively. The formal scheme $\mathfrak{X}^0$ is not strictly semi-stable. We make a small resolution (cf. [1]) at each of the 24 conical singularities p_α, given by the equations

$$P_4(x_0, x_1, x_2, x_3) = 0, \ x_i = x_j = 0, \quad \text{for } 0 \leqslant i < j \leqslant 3.$$

More precisely, we blow up the divisors D_0, D_1, D_2, D_3 subsequently, and denote by $\mathfrak{X}$ the formal scheme after the blow-ups. Other choices of small resolutions are also possible. We make this particular choice for the simplicity of exposition.

Now the formal scheme $\mathfrak{X}$ is strictly semi-stable. Its special fiber $\mathfrak{X}_s$ has four irreducible components, which are strict transforms of the divisors D_0, D_1, D_2, D_3. We denote them by $\widetilde{D}_0, \widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3$ respectively. We have

$$\widetilde{D}_0 \simeq \mathbf{P}_\mathbb{C}^2,$$
$$\widetilde{D}_1 \simeq \mathrm{Bl}_{\{4 \text{ points}\}} \mathbf{P}_\mathbb{C}^2,$$
$$\widetilde{D}_2 \simeq \mathrm{Bl}_{\{8 \text{ points}\}} \mathbf{P}_\mathbb{C}^2,$$
$$\widetilde{D}_3 \simeq \mathrm{Bl}_{\{12 \text{ points}\}} \mathbf{P}_\mathbb{C}^2,$$

where the symbol Bl means blow-up. The dual intersection complex $S_\mathfrak{X} \subset \mathbb{R}^4$ is a hollow tetrahedron, which is homeomorphic to the sphere S^2. Let v be a point in $S_\mathfrak{X}$. Let us describe explicitly our balancing condition at v. According to Theorem 1.1, it suffices to calculate the map $\alpha\colon \mathrm{CH}_1\left(D_{I_v}\right) \longrightarrow \mathbb{Z}^4$. We distinguish three cases.

First, the point v sits in the relative interior of a 2-dimensional face of $S_\mathfrak{X}$. Then we are in the situation of Example 1.2. So the map α is zero in this case.

Second, the point v sits on a vertex of $S_\mathfrak{X}$. Suppose for example v corresponds to the divisor $\widetilde{D}_0$. We have $\mathrm{CH}_1(\widetilde{D}_0) \simeq \mathrm{CH}_1(\mathbf{P}_\mathbb{C}^2) \simeq \mathbb{Z}$. The map $\alpha\colon \mathrm{CH}_1(\widetilde{D}_0) \to \mathbb{Z}^4$ sends 1 to $(-3, 1, 1, 1)$. We omit the cases where v corresponds to other divisors.

Third, the point v sits in the relative interior of a 1-dimensional face of $S_\mathfrak{X}$. Suppose for example the corresponding closed stratum D_{I_v} is the intersection $\widetilde{D}_0 \cap \widetilde{D}_1$, which we denote by $\widetilde{D}_{01}$. It is isomorphic to the projective line $\mathbf{P}_\mathbb{C}^1$, so $\mathrm{CH}_1(\widetilde{D}_{01}) \simeq \mathbb{Z}$. Let E_1, E_2, E_3, E_4 denote the four exceptional curves in $\widetilde{D}_1$. Let $\pi_1\colon \widetilde{D}_1 \to \mathbf{P}_\mathbb{C}^2$ denote the blow-up. Then

$(\pi_1^* \mathcal{O}(1), [E_1], [E_2], [E_3], [E_4])$ form a basis of $\mathrm{CH}_1(\widetilde{D}_1)$. From the relations

$$\left([\widetilde{D}_0] + [\widetilde{D}_1] + [\widetilde{D}_2] + [\widetilde{D}_3]\right)\Big|_{[\widetilde{D}_0]} = 0 \in \mathrm{CH}_1\left(\widetilde{D}_0\right),$$

$$\left([\widetilde{D}_0] + [\widetilde{D}_1] + [\widetilde{D}_2] + [\widetilde{D}_3]\right)\Big|_{[\widetilde{D}_1]} = 0 \in \mathrm{CH}_1\left(\widetilde{D}_1\right),$$

we have

$$[\widetilde{D}_0]\Big|_{[\widetilde{D}_0]} = -3 \in \mathrm{CH}_1\left(\widetilde{D}_0\right),$$

$$[\widetilde{D}_1]\Big|_{[\widetilde{D}_1]} = (-3, 1, 1, 1, 1) \in \mathrm{CH}_1\left(\widetilde{D}_1\right).$$

Therefore,

$$[\widetilde{D}_0]\Big|_{[\widetilde{D}_{01}]} = -3 \in \mathrm{CH}_1\left(\widetilde{D}_{01}\right),$$

$$[\widetilde{D}_1]\Big|_{[\widetilde{D}_{01}]} = -3 + 1 + 1 + 1 + 1 = 1 \in \mathrm{CH}_1\left(\widetilde{D}_{01}\right).$$

It is obvious that

$$[\widetilde{D}_2]\Big|_{[\widetilde{D}_{01}]} = [\widetilde{D}_3]\Big|_{[\widetilde{D}_{01}]} = 1 \in \mathrm{CH}_1\left(\widetilde{D}_{01}\right).$$

So we conclude that the map $\alpha \colon \mathrm{CH}_1\left(\widetilde{D}_{01}\right) \to \mathbb{Z}^4$ sends 1 to $(-3, 1, 1, 1)$.

Plan. Basic definitions are given in Section 2. In Section 3, we study the geometry of strictly semi-stable formal schemes in terms of vanishing cycles. In Section 5, we define tropical weight vectors. We prove that they are homological in nature. Indeed, they are only related to the "vanishing part" of the first degree cohomology of the generic fiber (Proposition 5.10). In Section 4, we establish an important technical step which allows us to localize our calculation of vanishing cycles to a smaller domain inside the skeleton. In Section 6, we prove a weaker form of our balancing conditions in terms of étale cohomology. The key ingredient is the long exact sequence relating nearby cycles with vanishing cycles. In Section 7, we explain how to use standard arguments in algebraic geometry to obtain the stronger balancing condition (Theorem 1.1) which is stated in terms of algebraic cycles.

essential and useful. I would also like to thank the referees for valuable comments.

2. Strictly semi-stable formal schemes and skeleta

In this article, k always denotes a complete discrete valuation field. Let k° be the ring of integers of k, $k^{\circ\circ}$ the maximal ideal of k°, and $\widetilde{k}$ the residue field. The symbol ℓ always denotes a prime number invertible in the residue field $\widetilde{k}$.

For $n \geqslant 1$, $0 \leqslant d \leqslant n$ and $a \in k^{\circ\circ} \setminus 0$, put

$$(2.1) \quad \mathfrak{S}(n, d, a) = \mathrm{Spf}\left(k^\circ\{T_0, \ldots, T_d, S_{d+1}^\pm, \ldots, S_n^\pm\}/(T_0 \cdots T_d - a)\right).$$

DEFINITION 2.1. — *A formal scheme $\mathfrak{X}$ over k° is said to be* finitely presented *if it is a finite union of open affine subschemes of the form*

$$\mathrm{Spf}\left(k^\circ\{T_0, \ldots, T_n\}/(f_1, \ldots, f_m)\right).$$

DEFINITION 2.2. — *Let $\mathfrak{X}$ be a formal scheme finitely presented over k°. $\mathfrak{X}$ is said to be* strictly semi-stable *if every point x of $\mathfrak{X}$ has an open affine neighbourhood $\mathfrak{U}$ such that the structural morphism $\mathfrak{U} \to \mathrm{Spf}\, k^\circ$ factorizes through an étale morphism $\phi\colon \mathfrak{U} \to \mathfrak{S}(n, d, a)$ for some $0 \leqslant d \leqslant n$ and $a \in k^{\circ\circ} \setminus 0$.*

Recall that for a formal scheme $\mathfrak{X}$ finitely presented over k°, its special fiber $\mathfrak{X}_s$ is a scheme of finite type over $\widetilde{k}$, and its generic fiber $\mathfrak{X}_\eta$ is a compact strictly k-analytic space (cf. [3, 4, 5]). When $\mathfrak{X}$ is polystable in the sense of [7], one can construct a polysimplicial set $\mathbf{C}(\mathfrak{X}_s)$. Its topological realization is denoted by $S_{\mathfrak{X}}$. In [7], Berkovich constructed an embedding $S_{\mathfrak{X}} \subset \mathfrak{X}_\eta$ and a strong deformation retraction from $\mathfrak{X}_\eta$ to $S_{\mathfrak{X}}$. So $S_{\mathfrak{X}}$ is called the *skeleton* of $\mathfrak{X}_\eta$ with respect to $\mathfrak{X}$. In our simplified situation, i.e. when $\mathfrak{X}$ is strictly semi-stable, the skeleton $S_{\mathfrak{X}}$ has a simple description as the dual intersection complex of the special fiber $\mathfrak{X}_s$.

Let $\{\, D_i \mid i \in I_{\mathfrak{X}} = \{0, \ldots, N\} \,\}$ be the set of irreducible components of the special fiber $\mathfrak{X}_s$. For any non-empty subset $I \subset I_{\mathfrak{X}}$, let $D_I = \bigcap_{i \in I} D_i$ and

$$(2.2) \qquad\qquad J_I = \{\, j \mid D_{I \cup \{j\}} \neq \emptyset \,\}.$$

We further assume that the strata D_I are all irreducible. The general constructions in [7] imply the following two lemmas.

LEMMA 2.3. — *The skeleton $S_{\mathfrak{X}}$ is the finite simplicial sub-complex of the simplex $\Delta^{I_{\mathfrak{X}}}$ such that for any $I \subset I_{\mathfrak{X}}$, Δ^I is a face of $S_{\mathfrak{X}}$ if and only if $D_I \neq \emptyset$.*

A face $\Delta^I \subset S_{\mathfrak{X}}$ for $I \subset I_{\mathfrak{X}}$ is called *maximal* if it does not belong to another face of higher dimension. Let Δ^I be a maximal face of $S_{\mathfrak{X}}$ of dimension d. By Definition 2.2, there exists an affine open subscheme $\mathfrak{U}$ in $\mathfrak{X}$ such that the structural morphism $\mathfrak{U} \to \mathrm{Spf}\, k^\circ$ factorizes through an étale morphism $\phi \colon \mathfrak{U} \to \mathfrak{S}(n, d, a)$ for some $n \in \mathbb{N}$, $a \in k^{\circ\circ} \setminus 0$, and that $D_I \cap \mathfrak{U}_s$ is given by the equations $T_0 = \cdots = T_d = 0$. We denote this element $a \in k^{\circ\circ} \setminus 0$ by a_I.

Let

$$S_I = \left\{ \sum_{i \in I} r_i \langle D_i \rangle \;\middle|\; r_i \in \mathbb{R}_{\geqslant 0}, \sum_{i \in I} r_i = \mathrm{val}(a_I) \right\} \subset \mathbb{R}^{I_{\mathfrak{X}}},$$

where $(\langle D_i \rangle)_{i \in I_{\mathfrak{X}}}$ is regarded as the standard basis of $\mathbb{R}^{I_{\mathfrak{X}}}$.

LEMMA 2.4. — *The skeleton $S_{\mathfrak{X}}$ can be identified with the union of the simplexes S_I over all maximal faces $I \subset I_{\mathfrak{X}}$. Thus we obtain an embedding of $S_{\mathfrak{X}}$ into $\mathbb{R}^{I_{\mathfrak{X}}}$.*

We refer to [21, 8, 15, 14] for related constructions.

3. Calculation of vanishing cycles

For any space X, we denote by $X_{\mathrm{\acute{e}t}}^{\sim}$ the category of étale sheaves on X whenever it makes sense. Let $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$ for any positive integer ν. We denote by Λ_X the constant sheaf on X associated to Λ. Let k^s be a separable closure of k, $\widehat{k^s}$ its completion, and $\widetilde{k^s}$ its residue field. For any scheme X defined over $\mathrm{Spec}\, \widetilde{k}$, we denote $\overline{X} = X \times \widetilde{k^s}$. For any k-analytic space X, we denote $\overline{X} = X \times \widehat{k^s}$.

Let $\mathfrak{X}$ be a strictly semi-stable formal scheme over k°. Let $\mathfrak{X}_{\overline{s}}$ (resp. $\mathfrak{X}_{\overline{\eta}}$) denote the special (resp. generic) fiber of the formal scheme $\overline{\mathfrak{X}} := \mathfrak{X} \widehat{\otimes}_{k^\circ} (\widehat{k^s})^\circ$ over $(\widehat{k^s})^\circ$. In [5], Berkovich constructed two functors $\Theta \colon \mathfrak{X}_{\eta\,\mathrm{\acute{e}t}}^{\sim} \to \mathfrak{X}_{s\,\mathrm{\acute{e}t}}^{\sim}$ and $\Psi_\eta \colon \mathfrak{X}_{\eta\,\mathrm{\acute{e}t}}^{\sim} \to \mathfrak{X}_{\overline{s}\,\mathrm{\acute{e}t}}^{\sim}$. We call them the specialization functor and the nearby cycles functor[2] respectively. We denote by $R\Theta$ and $R\Psi$ the corresponding derived functors. Our aim in this section is to compute the sheaf of nearby cycles $R\Psi\Lambda_{\mathfrak{X}_\eta}$ and the sheaf of vanishing cycles $R\Phi\Lambda_{\mathfrak{X}_\eta}$ defined as usual as a cone ([12] XIII 2.1).

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[2] Our terminology differs from [5], where Ψ is called the vanishing cycles functor.

The question being local, we only have to study the affine charts $\phi\colon \mathfrak{U} \to \mathfrak{S}(n, d, a)$ as in Definition 2.2. The formal scheme $\mathfrak{S}(n, d, a)$ is the completion of the scheme

$$\mathrm{Spec}\left(k^{\circ}[T_0, \ldots, T_d, S_{d+1}^{\pm}, \ldots, S_n^{\pm}]/(T_0 \cdots T_d - a)\right)$$

along its special fiber. Therefore, by [5] Corollary 4.5(i) and Corollary 5.3, the calculation is reduced to the case of ordinary schemes.

PROPOSITION 3.1 ([19]). — *We have $R^0 \Psi \Lambda_{\mathfrak{X}_\eta} \simeq \Lambda_{\mathfrak{X}_{\overline{s}}}$, an exact sequence*

$$0 \to \Lambda_{\mathfrak{X}_{\overline{s}}} \xrightarrow{diag} \bigoplus_{i \in I_{\mathfrak{X}}} \Lambda_{\overline{D}_i} \longrightarrow R^1 \Psi \Lambda_{\mathfrak{X}_\eta}(1) \to 0,$$

and isomorphisms

$$\wedge^q R^1 \Psi \Lambda_{\mathfrak{X}_\eta} \simeq R^q \Psi \Lambda_{\mathfrak{X}_\eta}, \quad \text{for } q \geqslant 1.$$

The sheaf of vanishing cycles $R\Phi\Lambda_{\mathfrak{X}_\eta}$ is related to the sheaf of nearby cycles $R\Psi\Lambda_{\mathfrak{X}_\eta}$ through an exact triangle

$$(3.1) \qquad \Lambda_{\mathfrak{X}_{\overline{s}}} \longrightarrow R\Psi\Lambda_{\mathfrak{X}_\eta} \longrightarrow R\Phi\Lambda_{\mathfrak{X}_\eta} \xrightarrow{+1} .$$

Let $\Delta^I, I \subset I_{\mathfrak{X}}$ be a face of $S_{\mathfrak{X}}$ and let $j\colon \overline{D}_I \hookrightarrow \mathfrak{X}_{\overline{s}}$ denote the closed immersion. We assume that $\overline{D}_I$ is a projective variety. Applying j^* to (3.1), we obtain an exact triangle

$$(3.2) \qquad j^* \Lambda_{\mathfrak{X}_{\overline{s}}} \longrightarrow j^* R\Psi\Lambda_{\mathfrak{X}_\eta} \longrightarrow j^* R\Phi\Lambda_{\mathfrak{X}_\eta} \xrightarrow{+1} .$$

Taking global sections $R\Gamma$, we obtain a long exact sequence
$$(3.3)$$
$$\cdots \to R^1\Gamma\left(j^* R\Psi\Lambda_{\mathfrak{X}_\eta}\right) \xrightarrow{\beta^*} R^1\Gamma\left(j^* R\Phi\Lambda_{\mathfrak{X}_\eta}\right) \xrightarrow{\alpha^*} R^2\Gamma\left(j^*\Lambda_{\mathfrak{X}_{\overline{s}}}\right) \to \cdots,$$

where we denote the two arrows above by β^* and α^* respectively.

COROLLARY 3.2. — *We have an isomorphism*

$$R^1\Gamma\left(j^* R\Phi\Lambda_{\mathfrak{X}_\eta}\right) \simeq \mathrm{Coker}\left(\Lambda \xrightarrow{\Delta} \Lambda^J\right)(-1),$$

where Δ denotes the diagonal map that sends an element $\lambda \in \Lambda$ to $(\lambda, \ldots, \lambda) \in \Lambda^J$. Moreover, the map

$$\alpha^*\colon \mathrm{Coker}\left(\Lambda \xrightarrow{\Delta} \Lambda^J\right)(-1) \longrightarrow R^2\Gamma\left(j^*\Lambda_{\mathfrak{X}_{\overline{s}}}\right) \simeq H^2_{\text{ét}}\left(\overline{D}_I, \Lambda\right)$$

is induced by the cycle class map in étale cohomology.

Proof. — The first statement is an immediate corollary of Proposition 3.1. Let us explain the second statement. Again by comparison theorems between analytic and algebraic vanishing cycles, it suffices to work in the

setting of ordinary schemes. In this proof, let us temporarily assume that $\mathfrak{X}$ is a scheme over $\operatorname{Spec} k^\circ$ instead of a formal scheme. Let

$$i_s \colon \mathfrak{X}_s \to \mathfrak{X}, \quad i_\eta \colon \mathfrak{X}_\eta \to \mathfrak{X}$$

denote the inclusions, and let

$$i_{\bar{s}} \colon \mathfrak{X}_{\bar{s}} \to \mathfrak{X}, \quad i_{\bar{\eta}} \colon \mathfrak{X}_{\bar{\eta}} \to \mathfrak{X}, \quad j_{\overline{D}} \colon \overline{D}_{I_v} \to \mathfrak{X}$$

denote the natural morphisms. We have an exact triangle

(3.4) $$\qquad i_{s!} i_s^! \Lambda_{\mathfrak{X}} \longrightarrow \Lambda_{\mathfrak{X}} \longrightarrow i_{\eta*} i_\eta^* \Lambda_{\mathfrak{X}} \xrightarrow{+1} .$$

By purity, we have an isomorphism

$$R^1 i_{\eta*} i_\eta^* \Lambda_{\mathfrak{X}} \simeq \bigoplus_{i \in I_{\mathfrak{X}}} \Lambda_{D_i}(-1)$$

given by the cohomology classes of the divisors D_i for $i \in I_{\mathfrak{X}}$. Applying $j_{\overline{D}}^*$ to (3.4) and shifting by 1, we obtain an exact triangle

$$j_{\overline{D}}^* \Lambda_{\mathfrak{X}} \longrightarrow j_{\overline{D}}^* i_{\eta*} i_\eta^* \Lambda_{\mathfrak{X}} \longrightarrow j_{\overline{D}}^* i_{s!} i_s^! \Lambda_{\mathfrak{X}}[1] \xrightarrow{+1} .$$

We have a morphism from the exact triangle above to the exact triangle (3.2)

$$
\begin{array}{ccccccc}
j_{\overline{D}}^* \Lambda_{\mathfrak{X}} & \longrightarrow & j_{\overline{D}}^* i_{\eta*} i_\eta^* \Lambda_{\mathfrak{X}} & \longrightarrow & j_{\overline{D}}^* i_{s!} i_s^! \Lambda_{\mathfrak{X}}[1] & \xrightarrow{\ b\ } & j_{\overline{D}}^* \Lambda_{\mathfrak{X}} \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} \\
j^* \Lambda_{\mathfrak{X}_{\bar{s}}} & \longrightarrow & j^* R\Psi\Lambda_{\mathfrak{X}_\eta} & \longrightarrow & j^* R\Phi\Lambda_{\mathfrak{X}_\eta} & \xrightarrow{\ b'\ } & j^* \Lambda_{\mathfrak{X}_{\bar{s}}}[1],
\end{array}
$$

where the maps f_1, f_4 are identities, the map f_2 is obtained from adjunction

$$i_{\eta*} i_\eta^* \Lambda_{\mathfrak{X}} \longrightarrow i_{\bar{\eta}*} i_{\bar{\eta}}^* \Lambda_{\mathfrak{X}},$$

and the map f_3 follows from the properties of triangulated categories. Now the second statement in the corollary follows from the commutativity

$$f_4 \circ b = b' \circ f_3$$

and the definition of the cycle class map in étale cohomology. $\qquad\square$

4. Deformation of analytic tubes

The retraction map $\tau \colon \mathfrak{X}_\eta \to S_{\mathfrak{X}}$ constructed by Berkovich [7] is easy to describe for the standard formal scheme $\mathfrak{B} := \mathfrak{S}(n, d, a)$ defined as in (2.1). By Lemma 2.4, we have

$$S_{\mathfrak{B}} = \left\{ (r_0, \dots, r_d) \in \mathbb{R}_{\geqslant 0}^{d+1} \ \middle|\ \sum_{i=0}^{d} r_i = \operatorname{val} a \right\} \subset \mathbb{R}^{d+1}.$$

LEMMA 4.1. — *The retraction map $\tau_{\mathfrak{B}} \colon \mathfrak{B}_\eta \to S_{\mathfrak{B}}$ takes $x \in \mathfrak{B}_\eta$ to the point*

$$(\operatorname{val} T_0(x), \ldots, \operatorname{val} T_d(x)) \in S_{\mathfrak{B}} \subset \mathbb{R}^{d+1}.$$

For a general strictly semi-stable formal scheme $\mathfrak{X}$, the retraction map $\tau \colon \mathfrak{X}_\eta \to S_{\mathfrak{X}}$ is defined by gluing the construction above in the étale topology. Our aim in this section is to describe the local geometry of this retraction in terms of nearby cycles. More precisely, let v be a point in $S_{\mathfrak{X}}$ sitting in the relative interior of a d_v-dimensional face Δ^{I_v} for $I_v = \{i_0, \ldots, i_{d_v}\} \subset I_{\mathfrak{X}}$. Denote by j the closed immersion $\overline{D}_{I_v} \hookrightarrow \mathfrak{X}_{\overline{s}}$. Let

$$(4.1) \quad V_{I_v}^{s_0 \ldots s_{d_v}} = S_{\mathfrak{X}} \cap \big\{\, (r_0, \ldots, r_N) \in \mathbb{R}^{I_{\mathfrak{X}}} \mid r_{i_j} \geqslant s_j \text{ for } j \in \{0, \ldots, d_v\} \,\big\}$$

where $s_0, \ldots, s_{d_v} \in \operatorname{val} k^{\circ\circ}$. It follow from the definition that

LEMMA 4.2. — $V_{I_v}^{s_0 \ldots s_{d_v}} \neq \emptyset$ *if and only if* $s_0 + \cdots + s_{d_v} \leqslant \operatorname{val} a_{I_v}$.

We suppose from now on that $V_{I_v}^{s_0 \ldots s_{d_v}} \neq \emptyset$, and denote by ι the inclusion map $\tau^{-1} \colon (V_{I_v}^{s_0 \ldots s_{d_v}}) \hookrightarrow \mathfrak{X}_\eta$.

THEOREM 4.3. — *We have a quasi isomorphism*

$$j_* j^* R\Psi\left(\Lambda_{\mathfrak{X}_\eta}\right) \xrightarrow{\sim} R\Psi\left(R\iota_* \iota^* \Lambda_{\mathfrak{X}_\eta}\right).$$

Proof. — By adjunction, we have a morphism $\Lambda_{\mathfrak{X}_\eta} \to R\iota_* \iota^* \Lambda_{\mathfrak{X}_\eta}$, thus a morphism

$$j_* j^* R\Psi\left(\Lambda_{\mathfrak{X}_\eta}\right) \xrightarrow{\gamma} j_* j^* R\Psi\left(R\iota_* \iota^* \Lambda_{\mathfrak{X}_\eta}\right).$$

In order to prove the theorem, we only have to show that

(i) The morphism γ is a quasi-isomorphism.
(ii) The sheaf $R\Psi(R\iota_* \iota^* \Lambda_{\mathfrak{X}_\eta})$ is supported on $\overline{D}_{I_v}$.

The properties being local, we only have to show them for the formal scheme $\mathfrak{B} = \mathfrak{S}(n, d, a)$. We assume that $d_v \leqslant d$ because otherwise it will not have contributions to the stratum D_{I_v}. The special fiber $\mathfrak{B}_s$ is given by the equation $T_0 \cdots T_d = 0$, and we assume that $D_{I_v} \subset \mathfrak{B}_s$ is further cut out by the equations $T_{d_v} = \cdots = T_d = 0$.

Now pick any elements $a_0, \ldots, a_{d_v} \in k^{\circ\circ}$ such that $\operatorname{val} a_j = s_j$. Let $a' = a \cdot a_0^{-1} \cdots a_{d_v}^{-1}$, $\widetilde{\mathfrak{B}} = \mathfrak{S}(n, d, a')$. Let $f \colon \widetilde{\mathfrak{B}} \to \mathfrak{B}$ be the morphism given by the algebra homomorphism

$$k^{\circ}\{T_0, \ldots, T_d, S_{d+1}^{\pm}, \ldots, S_n^{\pm}\}/(T_0 \cdots T_d - a) \longrightarrow$$
$$k^{\circ}\{T_0, \ldots, T_d, S_{d+1}^{\pm}, \ldots, S_n^{\pm}\}/(T_0 \cdots T_d - a'),$$

which takes

$$T_j \longmapsto a_j T_j \qquad \text{for } j = 0, \ldots, d_v,$$
$$T_j \longmapsto T_j \qquad \text{for } j = d_v + 1, \ldots, d,$$
$$S_j \longmapsto S_j \qquad \text{for } j = d + 1, \ldots, n.$$

We have

(4.2) $$Rf_{\eta *} \Lambda_{\widetilde{\mathfrak{B}}_\eta} \simeq R\iota_* \iota^* \Lambda_{\mathfrak{B}_\eta}.$$

By [5] Corollary 4.5(ii), we have

(4.3) $$R\Psi\left(Rf_{\eta *} \Lambda_{\widetilde{\mathfrak{B}}_\eta}\right) \xrightarrow{\sim} Rf_{\bar{s}*}\left(R\Psi\left(\Lambda_{\widetilde{\mathfrak{B}}_\eta}\right)\right).$$

Combining (4.2) and (4.3), we obtain

$$R\Psi\left(R\iota_* \iota^* \Lambda_{\mathfrak{B}_\eta}\right) \simeq Rf_{\bar{s}*}\left(R\Psi\left(\Lambda_{\widetilde{\mathfrak{B}}_\eta}\right)\right).$$

This shows (ii). Then (i) follows from the calculation of nearby cycles in the last section and the fact that the fibers of $f_{\bar{s}}$ are all cohomologically trivial. $\qquad\square$

Remark 4.4. — The geometry behind Theorem 4.3 is that the étale cohomology does not change when we deform certain tubular neighborhoods. A scheme theoretic analog can be found in [9]. Our approach may be adapted to give a better understanding of the result loc. cit.

5. Cohomological interpretation of tropical weight vectors

Let C be a compact quasi-smooth k-analytic curve, and $f \colon C \to \mathfrak{X}_\eta$ a k-analytic morphism. The image of C under $\tau \circ f$ is a one-dimensional[3] polyhedral complex embedded in the skeleton $S_{\mathfrak{X}}$. We denote it by C^{t}, and call it the associated *tropical curve*. We call the 0-dimensional faces of C^{t} vertices and the 1-dimensional faces of C^{t} edges. We denote by e° the interior of an edge e.

PROPOSITION 5.1. — *There exists a subdivision of the edges of the tropical curve C^{t} such that, if we denote by $\overline{C^{\mathrm{t}}}$ the polyhedral complex after the subdivision of edges, then*

(1) *For any edge e of $\overline{C^{\mathrm{t}}}$, each connected component of $(\tau \circ f)^{-1}(e^\circ)$ is isomorphic to an open annulus.*

---

[3] In degenerate situations, the tropical curve C^{t} can be zero-dimensional. We are not interested in such cases.

(2) *For each annulus A as above, there exists an open subscheme $\mathfrak{U}$ in $\mathfrak{X}$,
equipped with an étale morphism $\phi\colon \mathfrak{U} \to \mathfrak{S}(n,d,a)$ as in Definition
2.2, such that $f(A)$ is contained in $\mathfrak{U}_\eta$.*

Proof. — [4] Let us choose affine charts $\mathfrak{U}_1,\dots,\mathfrak{U}_m$ as in Definition 2.2
such that $\bigcup_{i=1}^m \mathfrak{U}_i = \mathfrak{X}$. Assume that for each $i \in \{1,\dots,m\}$, the structural
morphism $\mathfrak{U}_i \to \mathrm{Spf}\, k^\circ$ factorizes through an étale morphism

$$\phi_i\colon \mathfrak{U}_i \to \mathrm{Spf}\left(k^\circ\{T_0^{(i)},\dots,T_{d_i}^{(i)},S_{d_i+1}^{(i)\pm},\dots,S_{n_i}^{(i)\pm}\}/(T_0^{(i)}\cdots T_{d_i}^{(i)} - a_i)\right)$$

for some $0 \leqslant d_i \leqslant n_i$, $a_i \in k^{\circ\circ}\backslash 0$. Let $C_i = f^{-1}(\mathfrak{U}_{i,\eta})$, $t_{ij} = |T_j^{(i)} \circ \phi_{i,\eta} \circ f_{|C_i}|$,
for $i = 1,\dots,m$, $j = 0,\dots,d_i$, where $|\cdot|$ denotes the absolute value. We
have the following fact concerning the variation of holomorphic functions.

LEMMA 5.2 ([10](4.4.35)). — *For each $i = 1,\dots,m$, there exists a finite
graph $\Gamma_i \subset C_i$ such that the functions t_{ij} are locally constant over $C_i \setminus \Gamma_i$,
and piecewise linear over Γ_i.*

By extending Γ_i, we can assume that it contains an analytic skeleton of
C_i (cf. [10](5.1.8)). Let Γ be the union of the analytic skeleton of C and
$\bigcup \Gamma_i$. Let $\overline{\Gamma}$ be the convex hull of Γ, and let $\overline{\Gamma}_i = \overline{\Gamma} \cap C_i$. By [10](5.1), we
have a strong deformation retraction $r\colon C \to \overline{\Gamma}$, and $r_{|C_i}$ gives a retraction
of C_i onto $\overline{\Gamma}_i$. Let K_i be the set of knot points[5] for the subgraph $\overline{\Gamma}_i \subset C_i$
for $i = 1,\dots,m$, and let K be the set of knot points for the subgraph
$\overline{\Gamma} \subset C$. Let $P_0 = (\tau \circ f)\big(K \cup \bigcup_{i=1}^m K_i\big)$, and let P_1 be the union of the
points $p \in C^{\mathrm{t}}$ such that $((\tau \circ f)_{|\overline{\Gamma}})^{-1}(p)$ contains an infinite number of
points. The set of points P_0 is finite by [10](5.1.12.2). The set of points P_1
is also finite by Lemma 5.2. Therefore, the union $P := P_1 \cup P_2$ is a finite set.
Now it suffices to make the subdivision of the edges of our tropical curve
C^{t} by adding the points in P as new vertices. Indeed, for each edge e of
the subdivided tropical curve $\overline{C^{\mathrm{t}}}$, $\overline{\Gamma} \cap (\tau \circ f)^{-1}(e^\circ)$ is a finite disjoint union
of open segments in $\overline{\Gamma}$. Any such segment s is by construction contained
in a certain $\overline{\Gamma}_i$. By [10](5.1.12.3), $r^{-1}(s)$ is isomorphic to an open annulus,
which we denote by A. By Lemma 5.2, we have $f_{|\overline{\Gamma}} \circ r_{|A} = \tau \circ f_{|A}$. So we
have proved the first assertion of our proposition. Moreover, $s \subset \overline{\Gamma}_i$ implies
that $A_i := (r_{|C_i})^{-1}(s)$ is an open annulus inside C_i. We have an inclusion
of two open annuli $A_i \subset A$, and both of them retract to the same segment
s. Therefore we have $A_i = A$, and $f(A) = f(A_i) \subset \mathfrak{U}_{i,\eta}$. So we have proved
the second assertion as well. $\square$

---

[4] Many thanks to Antoine Ducros for his help with this proof.
[5] The notion of knot point is defined in [10](5.1.12) for an analytically admissible and
locally finite subgraph inside a generically quasi-smooth k-analytic curve.

Remark 5.3. — It is pointed out by the referee that Proposition 5.1 is the analog of [2, Proposition 6.4(2)] in our global setting.

From now on, we replace our tropical curve C^{t} by the subdivided curve $\overline{C^{\mathrm{t}}}$ produced by Proposition 5.1.

We explain in the following how to equip every edge e of C^{t} a *tropical weight vector* $\widetilde{w}_e \in \mathbb{Z}^{I_{\mathfrak{x}}}$ with respect to a choice of orientation of the edge e. An orientation of an edge e is a choice of a direction parallel to e among the two possible choices. We will see that the weight vector $\widetilde{w}_e$ is multiplied by -1 if we reverse the orientation of e.

Now fix an edge e of our tropical curve C^{t} and an orientation of e. Assume that the interior e° of the edge e is contained in the relative interior of a d_e-dimensional face Δ^{I_e} for $I_e \subset I_{\mathfrak{x}}$. Let A be a connected component of $(\tau \circ f)^{-1}(e^{\circ})$. Fix $i \in I_{\mathfrak{x}}$ and denote by $p_i \colon \mathbb{R}^{I_{\mathfrak{x}}} \to \mathbb{R}$ the projection to the ith coordinate. Let r_i be the composition of $p_i \circ \tau \circ f_{|A}$ using the embedding $S_{\mathfrak{x}} \subset \mathbb{R}^{I_{\mathfrak{x}}}$. By Proposition 5.1(2) and the explicit description of the retraction map in Lemma 4.1, the map r_i equals $\mathrm{val}(f_i)$ for some invertible function f_i on the open annulus A. Choose a coordinate z on A such that A is given by $c_1 < |z| < c_2$ for two positive real numbers $c_1 < c_2$ and that the image $(\tau \circ f)(z)$ moves along the orientation of the edge e as $\mathrm{val}(z)$ increases. Write

$$f_i = \sum_{m \in \mathbb{Z}} f_{i,m} z^m$$

with $f_{i,m} \in k$. By [4] Lemma 6.2.2, there exists $m_i \in \mathbb{Z}$ with $|f_{i,m_i}| r^{m_i} > |f_{i,m}| r^m$ for all $m \neq m_i$, $c_1 < r < c_2$. Therefore, for $c_1 < |z| < c_2$, we have

(5.1) $$\mathrm{val}(f_i(z)) = \mathrm{val}(f_{i,m_i}) + m_i \cdot \mathrm{val}(z).$$

We define the ith component $\widetilde{w}_A^i$ of the weight vector $\widetilde{w}_A$ to be m_i for every $i \in I_{\mathfrak{x}}$.

LEMMA 5.4. — *The weight vector $\widetilde{w}_A \in \mathbb{Z}^{I_{\mathfrak{x}}}$ defined as above does not depend on the choice of the invertible function f_i or the coordiante z. It is multiplied by -1 if we reverse the orientation of the edge e. It is parallel to the direction of the edge e sitting inside $\mathbb{R}^{I_{\mathfrak{x}}}$. In particular, it is an element of $\mathrm{Ker}\left(\mathbb{Z}^{I_e} \xrightarrow{\Sigma} \mathbb{Z}\right) \subset \mathbb{Z}^{I_{\mathfrak{x}}}$, where $\Sigma \colon \mathbb{Z}^{I_e} \to \mathbb{Z}$ sends $(x^1, \ldots, x^q) \in \mathbb{Z}^{I_e}$ to $x^1 + \cdots + x^q \in \mathbb{Z}$.*

Proof. — All the assertions follow from Eq. (5.1). □

DEFINITION 5.5. — *Let $\widetilde{w}_e$ be the sum of $\widetilde{w}_A$ over every connected component A of $(\tau \circ f)^{-1}(e)$. The element $\widetilde{w}_e \in \mathrm{Ker}\left(\mathbb{Z}^{I_e} \xrightarrow{\Sigma} \mathbb{Z}\right) \subset \mathbb{Z}^{I_{\mathfrak{x}}}$ is called*

the tropical weight vector *associated to the edge e with the chosen orientation.*

Remark 5.6. — Definition 5.5 is a globalization of the classical notion of tropical weight vector. We refer to [2, §6] for similar considerations in the toric case.

Let us explain the homological nature of the tropical weight vectors. Let A be an open annulus as above and let $\phi\colon \mathfrak{U} \to \mathfrak{S}(n,d,a)$ be as in Proposition 5.1(2). We assume for simplicity that the divisors $D_0, \ldots, D_d$ restricted to $\mathfrak{U}$ are given by the equations $T_0 \circ \phi = 0, \ldots, T_d \circ \phi = 0$ respectively, where $T_0, \ldots, T_d$ are coordinates on $\mathfrak{S}(n,d,a)$ (see Eq. (2.1)). We further assume that $I_e = \{0, \ldots, d_e\}$. Let $\mathfrak{B} = \mathfrak{S}(n,d,a)$, and let $\pi_{\mathfrak{U}}\colon \mathfrak{U}_\eta \to \mathfrak{U}_s$, $\pi_{\mathfrak{B}}\colon \mathfrak{B}_\eta \to \mathfrak{B}_s$ be the reduction maps (cf. [5, §1]).

LEMMA 5.7. — *There exists a closed point $p \in \mathfrak{U}_s$ such that $f(A) \subset \pi_{\mathfrak{U}}^{-1}(p)$.*

Proof. — Consider the composition $\phi_\eta \circ f_{|A}\colon A \to \mathfrak{B}_\eta$. In terms of affinoid algebras, it is given by n power series:

$$T_i \mapsto \sum_{m \in \mathbb{Z}} f_{i,m} z^m \qquad \text{for } i = 0, \ldots, d,$$

$$S_i \mapsto \sum_{m \in \mathbb{Z}} f_{i,m} z^m \qquad \text{for } i = d+1, \ldots, n.$$

From the definition of the weight vector $\widetilde{w}_A$, for any $i = 0, \ldots, n$, $c_1 < r < c_2$, $m \neq \widetilde{w}_A^i$, we have

$$\left| f_{i, \widetilde{w}_A^i} \right| r^{\widetilde{w}_A^i} > |f_{i,m}| r^m.$$

Therefore, if $\widetilde{w}_A^i = 0$, then the ith-coordinate of all the points in $(\pi_{\mathfrak{B}} \circ \phi_\eta \circ f)(A)$ is $\overline{f}_{i,0}$, where $\overline{f}_{i,0}$ denotes the image of $f_{i,0}$ in the residue field $\widetilde{k}$. If $\widetilde{w}_A^i \neq 0$, then the ith-coordinate of all the points in $(\pi_{\mathfrak{B}} \circ \phi_\eta \circ f)(A)$ is zero. This shows that the image $(\pi_{\mathfrak{B}} \circ \phi_\eta \circ f)(A)$ is a $\widetilde{k}$-rational point in $\mathfrak{B}_s$, which we denote by $p_{\mathfrak{B}}$. By the commutativity

$$\pi_{\mathfrak{B}} \circ \phi_\eta = \phi_s \circ \pi_{\mathfrak{U}}\colon \mathfrak{U}_\eta \to \mathfrak{B}_s,$$

we have

$$(\pi_{\mathfrak{U}} \circ f)(A) \subset \phi_s^{-1}(p_{\mathfrak{B}}).$$

Since ϕ_s is étale, $\phi_s^{-1}(p_{\mathfrak{B}})$ is discrete. Then by the connectedness of the annulus A, there exists a point $p \in \phi_s^{-1}(p_{\mathfrak{B}})$ such that $(\pi_{\mathfrak{U}} \circ f)(A) = p$. $\quad\square$

Since $\phi\colon \mathfrak{U} \to \mathfrak{B}$ is étale, and ϕ_s induces an isomorphism between the point p and the point $p_{\mathfrak{B}} = \phi_s(p)$, ϕ_η induces an isomorphism between $\pi_{\mathfrak{U}}^{-1}(p)$ and $\pi_{\mathfrak{B}}^{-1}(p_{\mathfrak{B}})$ ([7] Lemma 4.4). Now $\pi_{\mathfrak{B}}^{-1}(p_{\mathfrak{B}})$ is very easy to describe. It is isomorphic to the generic fiber of the special formal scheme

$$(5.2) \qquad \mathrm{Spf}\left(k^\circ[[T_0, \ldots, T_{d_e}, S_{d_e+1}, \ldots, S_n]]/(T_0 \cdots T_{d_e} - a')\right).$$

For $i \in I_e$, let A_i be the generic fiber of the special formal scheme

$$\mathrm{Spf}\left(k^\circ[[T_i, T']]/(T_i \cdot T' - a')\right),$$

and let c_i denote the morphism of k-analytic spaces

$$\pi_{\mathfrak{B}}^{-1}(p_{\mathfrak{B}}) \to A_i$$

induced by the homomorphism of algebras

$$k^\circ[[T_i, T']]/(T_i \cdot T' - a) \longrightarrow k[[T_0, \ldots, T_{d_e}, S_{d_e+1}, \ldots, S_n]]/(T_0 \cdots T_{d_e} - a'),$$

which takes

$$T_i \longmapsto T_i$$
$$T' \longmapsto T_0 \cdots \widehat{T_i} \cdots T_{d_e}.$$

LEMMA 5.8. — *Let $g_i = c_i \circ \phi_\eta \circ f_{|A}\colon A \to A_i$, and let $g_i^*\colon H_{\text{ét}}^1\left(\overline{A_i}, \mathbb{Q}_\ell\right) \to H_{\text{ét}}^1\left(\overline{A}, \mathbb{Q}_\ell\right)$ be the induced homomorphism of étale cohomology groups. Then $\widetilde{w}_A^i = g_i^*(1) \in \mathbb{Q}_\ell$ for all $i \in I_e$.*

Proof. — It follows from [4] Lemma 6.2.5 that the winding numbers in terms of étale cohomology equals exactly the numbers m_i in Eq. (5.1). $\square$

Our next step is to relate the tropical weight vectors to vanishing cycles. Since our tropical weight vectors are defined using étale cohomology with $\mathbb{Q}_\ell$-coefficients, we shall take the inverse limit over ν for all our reasonings in Sections 3 and 4. By abuse of notation, we will just replace Λ by $\mathbb{Q}_\ell$, although it should be understood that the inverse limit is taken in the last step rather than from the beginning.

Let v, I_v, $V_{I_v}^{s_0 \ldots s_{d_v}}$, j be as in Section 4. Let A, $\phi\colon \mathfrak{U} \to \mathfrak{B}$, $p \in \mathfrak{U}_s$ be as before. Denote $V \coloneqq V_{I_v}^{s_0 \ldots s_{d_v}}$ for simplicity. We assume that $f(A) \subset \tau^{-1}(V)$. By Theorem 4.3, we have

$$H_{\text{ét}}^1\left(\tau^{-1}(V) \times \widehat{k^s}, \mathbb{Q}_\ell\right) \simeq R^1\Gamma\left(j^* R\Psi\mathbb{Q}_{\ell, \mathfrak{X}_\eta}\right).$$

LEMMA 5.9. — *We have an isomorphism*

$$(5.3) \qquad R^1\Gamma\left(j_p^* R\Psi\mathbb{Q}_{\ell, \mathfrak{U}_\eta}\right) \xrightarrow{\sim} H_{\text{ét}}^1\left(\left(\tau^{-1}(V) \cap \pi_{\mathfrak{U}}^{-1}(p)\right) \times \widehat{k^s}, \mathbb{Q}_\ell\right).$$

Proof. — By [6], we have

$$(5.4) \qquad R^1\Gamma\left(j_p^* R\Psi\mathbb{Q}_{\ell,\mathfrak{U}_\eta}\right) \xrightarrow{\sim} H^1_{\text{ét}}\left(\pi_{\mathfrak{U}}^{-1}(p) \times \widehat{k}^s, \mathbb{Q}_\ell\right).$$

Using the isomorphism $\pi_{\mathfrak{U}}^{-1}(p) \simeq \pi_{\mathfrak{B}}^{-1}(p_\mathfrak{B})$ and the explicit description of $\pi_{\mathfrak{B}}^{-1}(p_\mathfrak{B})$ in Eq. (5.2), the same arguments as in the proof of Theorem 4.3 give an isomorphism

$$(5.5) \qquad H^1_{\text{ét}}\left(\left(\tau^{-1}(V) \cap \pi_{\mathfrak{U}}^{-1}(p)\right) \times \widehat{k}^s, \mathbb{Q}_\ell\right) \xrightarrow{\sim} H^1_{\text{ét}}\left(\pi_{\mathfrak{U}}^{-1}(p) \times \widehat{k}^s, \mathbb{Q}_\ell\right).$$

Now our lemma follows from (5.4) and (5.5). □

Now assume that v is an endpoint of the edge e. Let $J := J_{I_v}$ (see Eq. (2.2)). Since $I_e \subset J$, by Lemma 5.4, the weight vector $\widetilde{w}_A$ is an element in $\text{Ker}\left(\mathbb{Z}^J \xrightarrow{\Sigma} \mathbb{Z}\right)$. Using the natural inclusion $\text{Ker}(\mathbb{Z}^J \xrightarrow{\Sigma} \mathbb{Z}) \hookrightarrow \text{Ker}(\mathbb{Q}_\ell^J \xrightarrow{\Sigma} \mathbb{Q}_\ell)$, the weight vector $\widetilde{w}_A$ induces a map

$$\widetilde{w}_A^*\colon \text{Coker}\left(\mathbb{Q} \xrightarrow{\Delta} \mathbb{Q}^J\right) \longrightarrow \mathbb{Q}_\ell.$$

PROPOSITION 5.10. — *We have the following commutative diagram*

$$\begin{array}{ccccc}
R^1\Gamma\left(j^* R\Psi\mathbb{Q}_{\ell,\mathfrak{X}_\eta}\right) & \xrightarrow{\beta^*} & R^1\Gamma\left(j^* R\Phi\mathbb{Q}_{\ell,\mathfrak{X}_\eta}\right) & \xrightarrow{\sim} & \text{Coker}\left(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^J\right)(-1) \\
\downarrow\sim & & & & \downarrow\widetilde{w}_A^* \\
H^1_{\text{ét}}\left(\tau^{-1}(V) \times \widehat{k}^s, \mathbb{Q}_\ell\right) & \xrightarrow{f_{|A}^*} & H^1_{\text{ét}}\left(\overline{A}, \mathbb{Q}_\ell\right) & \xrightarrow{\quad\sim\quad} & \mathbb{Q}_\ell(-1).
\end{array}$$

Proof. — In order to simplify notation, let us temporarily denote

$$\widetilde{V} = \tau^{-1}(V) \times \widehat{k}^s,$$
$$\widetilde{V}_p = \left(\tau^{-1}(V) \cap \pi_{\mathfrak{U}}^{-1}(p)\right) \times \widehat{k}^s,$$
$$\mathbb{Q}_\ell^{[I_e]} = \text{Coker}\left(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^{I_e}\right),$$
$$\mathbb{Q}_\ell^{[J]} = \text{Coker}\left(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^J\right).$$

Let j_p denote the inclusion $\overline{\{p\}} \hookrightarrow \mathfrak{U}_{\overline{s}}$. By the description of the point p in the proof of Lemma 5.7, we have

$$(5.6) \quad R^1\Gamma\left(j_p^* R\Psi\mathbb{Q}_{\ell,\mathfrak{U}_\eta}\right) \xrightarrow{\sim} R^1\Gamma\left(j_p^* R\Phi\mathbb{Q}_{\ell,\mathfrak{U}_\eta}\right) \xrightarrow{\sim} \text{Coker}\left(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^{I_e}\right)(-1).$$

Combining (5.6) and (5.3), we have isomorphisms

$$H^1_{\text{ét}}\left(\widetilde{V}_p, \mathbb{Q}_\ell\right) \xleftarrow{\sim} R^1\Gamma\left(j_p^* R\Psi\mathbb{Q}_{\ell,\mathfrak{U}_\eta}\right) \xrightarrow{\sim} R^1\Gamma\left(j_p^* R\Phi\mathbb{Q}_{\ell,\mathfrak{U}_\eta}\right) \xrightarrow{\sim} \mathbb{Q}_\ell^{[I_e]}(-1).$$

Then the following commutative diagram follows from the cohomological interpretations of tropical weight vectors (Lemma 5.8).

(5.7)
$$
\begin{array}{ccc}
R^1\Gamma\big(j_p^* R\Psi\mathbb{Q}_{\ell,\mathfrak{U}_\eta}\big) \xrightarrow{\ \sim\ } R^1\Gamma\big(j_p^* R\Phi\mathbb{Q}_{\ell,\mathfrak{U}_\eta}\big) \xrightarrow{\ \sim\ } \mathbb{Q}_\ell^{[I_e]}(-1) \\
\Big\downarrow{\scriptstyle\sim} \qquad\qquad\qquad\qquad\qquad \Big\downarrow{\scriptstyle\widetilde{w}_A^*} \\
H^1_{\text{ét}}\big(\widetilde{V}_p,\mathbb{Q}_\ell\big) \xrightarrow{\ f_{|A}^*\ } H^1_{\text{ét}}\big(\overline{A},\mathbb{Q}_\ell\big) \xrightarrow{\ \sim\ } \mathbb{Q}_\ell(-1).
\end{array}
$$

We conclude by the functoriality of the formation of vanishing cycles, i.e. the following commutative diagram:

$$
\begin{array}{ccccc}
H^1_{\text{ét}}\big(\widetilde{V},\mathbb{Q}_\ell\big) & \xleftarrow{\ \sim\ } & R^1\Gamma\big(j^* R\Psi\mathbb{Q}_{\ell,\mathfrak{X}_\eta}\big) & \xrightarrow{\ \beta^*\ } R^1\Gamma\big(j^* R\Phi\mathbb{Q}_{\ell,\mathfrak{X}_\eta}\big) & \xrightarrow{\ \sim\ } \mathbb{Q}_\ell^{[J]}(-1) \\
\downarrow & & \downarrow & \downarrow & \downarrow \\
H^1_{\text{ét}}\big(\widetilde{V}_p,\mathbb{Q}_\ell\big) & \xleftarrow{\ \sim\ } & R^1\Gamma\big(j_p^* R\Psi\mathbb{Q}_{\ell,\mathfrak{U}_\eta}\big) & \xrightarrow{\ \sim\ } R^1\Gamma\big(j_p^* R\Phi\mathbb{Q}_{\ell,\mathfrak{U}_\eta}\big) & \xrightarrow{\ \sim\ } \mathbb{Q}_\ell^{[I_e]}(-1).
\end{array}
$$

$\square$

Remark 5.11. — Intuitively, Proposition 5.10 says that the image of the annulus A under the morphism f can only go around vanishing cycles in $\tau^{-1}(V)$ rather than arbitrary homology cycles.

6. Balancing condition in terms of étale cohomology

We use the settings in Section 1. The aim of this section is to prove the following theorem.

THEOREM 6.1. — *Let σ_v denote the sum of weight vectors around the vertex v as in (1.2). Let $J := J_{I_v}$ as in Section 5. Then σ_v lies in the image of the following map:*

$$
\alpha_\ell\colon H^{2(\dim D_{I_v}-1)}_{\text{ét}}\left(D_{I_v}\times_{\widetilde{k}}\operatorname{Spec}\widetilde{k}^s,\mathbb{Q}_\ell\right)(\dim D_{I_v}-1) \longrightarrow \operatorname{Ker}\big(\mathbb{Q}_\ell^J \xrightarrow{\ \Sigma\ } \mathbb{Q}_\ell\big)
$$
$$
L \longmapsto \big(L\cdot L_j,\ j\in J\big)
$$

where α_ℓ is dual to the map α^* in Corollary 3.2 for $\mathbb{Q}_\ell$ coefficients.

We begin with a simple observation.

LEMMA 6.2. — *For any extension $k\subset k'$ of non-archimedean fields, if we apply extension of ground fields to $\mathfrak{X}_\eta$ and to the map $f\colon C\to\mathfrak{X}_\eta$, the skeleton $S_{\mathfrak{X}}$ and the tropical curve C^{t} remains unchanged.*

We assume that our tropical curve C^{t} is already subdivided according to Proposition 5.1. We choose a sufficiently small convex neighborhood V of v inside $S_{\mathfrak{X}}$ which does not contain any other vertex of C^{t}. We further require that V is of the form (4.1). This can be achieved by making a finite extension of our field k.

LEMMA 6.3. — *Let k' be a separably closed non-archimedean field, C° a connected compact quasi-smooth k'-analytic curve, and let $b_1, \ldots, b_m$ be the boundary points of C°. We assume that there are neighborhoods $A_1, \ldots, A_m$ of $b_1, \ldots, b_m$ which are pairwise disjoint and are isomorphic to annuli. Let $A = \coprod A_i$ and let ι_b denote the inclusion $A \hookrightarrow C^\circ$. Then the composition $\Sigma \circ \iota_b^*$*

$$H_{\text{ét}}^1(C^\circ, \mathbb{Q}_\ell)(1) \xrightarrow{\iota_b^*} H_{\text{ét}}^1(A, \mathbb{Q}_\ell)(1) \simeq \mathbb{Q}_\ell^m \xrightarrow{\Sigma} \mathbb{Q}_\ell$$

is zero, where Σ denotes the sum.

Proof. — By gluing discs onto $b_1, \ldots, b_m$, we embed C° into a proper smooth k'-analytic curve, which we denote by $\widehat{C^\circ}$. Let $z_1, \ldots, z_m$ be the centers of the discs, and let $Z = \coprod_{i=1}^m z_i$. Now the lemma follows from the Gysin exact sequence

$$0 \to H_{\text{ét}}^1(\widehat{C^\circ}, \mathbb{Q}_\ell)(1) \to H_{\text{ét}}^1(C^\circ, \mathbb{Q}_\ell)(1) \to H_{\text{ét}}^0(Z, \mathbb{Q}_\ell)$$
$$\to H_{\text{ét}}^2(\widehat{C^\circ}, \mathbb{Q}_\ell)(1) \simeq \mathbb{Q}_\ell \to \cdots .$$

$\square$

Let r be a positive real number. Let

$$V^r = \left\{ x \in V \subset S_{\mathfrak{X}} \subset \mathbb{R}^{I_{\mathfrak{X}}} \mid \operatorname{dist}(x, V^c) > r \right\},$$

where dist denotes the standard Euclidean metric in $\mathbb{R}^{I_{\mathfrak{X}}}$ and V^c denotes the complement of V in $S_{\mathfrak{X}}$. Put $V^b = V \setminus V^r$, $C^\circ = (\tau \circ f)^{-1}(V)$, $C^b = (\tau \circ f)^{-1}(V^b)$, $\iota_b\colon C^b \hookrightarrow C^\circ$. We choose r sufficiently small, such that C^b does not contain any vertices of C^{t}. For each segment e in $C^{\mathrm{t}} \cap V^b$, we choose the orientation of e to be the one that points away from v. The definition of tropical weight vectors in Section 5 gives an element $\widetilde{w}_e \in \operatorname{Ker}\left(Z^J \xrightarrow{\Sigma} Z\right)$, which induces a map

$$\widetilde{w}_e^*\colon \operatorname{Coker}\left(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^J\right) \longrightarrow \mathbb{Q}_\ell.$$

Then σ_v is the sum of $\widetilde{w}_e$ over all segments e in $C^{\mathrm{t}} \cap V^b$. It induces a map

$$\sigma_v^*\colon \operatorname{Coker}\left(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^J\right) \longrightarrow \mathbb{Q}_\ell.$$

By Proposition 5.10, we have a commutative diagram
(6.1)

$$
\begin{array}{ccc}
R^1\Gamma\left(j^*R\Psi\mathbb{Q}_{\ell,\mathfrak{x}_\eta}\right) & \xrightarrow{\beta^*} & R^1\Gamma\left(j^*R\Phi\mathbb{Q}_{\ell,\mathfrak{x}_\eta}\right) \xrightarrow{\sim} \mathrm{Coker}\left(\mathbb{Q}_\ell \xrightarrow{\Delta} \mathbb{Q}_\ell^J\right)(-1) \\
\downarrow{\scriptstyle\sim} & & \qquad\qquad\qquad\qquad\qquad\downarrow{\scriptstyle\sigma_v^*} \\
H^1_{\text{ét}}(\tau^{-1}(V)\times\widehat{k^s},\mathbb{Q}_\ell) & \xrightarrow{f^*_{|C^b}} & H^1_{\text{ét}}(\overline{C^b},\mathbb{Q}_\ell) \xrightarrow{\qquad\Sigma\qquad} \mathbb{Q}_\ell(-1),
\end{array}
$$

where the bottom row factorizes as

$$
H^1_{\text{ét}}(\tau^{-1}(V)\times\widehat{k^s},\mathbb{Q}_\ell) \xrightarrow{f^*_{|C^\circ}} H^1_{\text{ét}}(\overline{C^\circ},\mathbb{Q}_\ell) \xrightarrow{\iota_b^*} H^1_{\text{ét}}(\overline{C^b},\mathbb{Q}_\ell) \xrightarrow{\Sigma} \mathbb{Q}_\ell(-1).
$$

By Lemma 6.3, we have $\Sigma\circ\iota_b^* = 0$. By the commutativity of (6.1), we have

(6.2) $$\sigma_v^*\circ\beta^* = 0.$$

By (3.3), we have a long exact sequence
(6.3)

$$
\cdots \to R^1\Gamma(j^*R\Psi\mathbb{Q}_{\ell,\mathfrak{x}_\eta}) \xrightarrow{\beta^*} R^1\Gamma\left(j^*R\Phi\mathbb{Q}_{\ell,\mathfrak{x}_\eta}\right) \xrightarrow{\alpha^*} R^2\Gamma(j^*\mathbb{Q}_{\ell,\mathfrak{x}_{\overline{s}}}) \to \cdots.
$$

Combining (6.2) and the exactness of (6.3), we have proved that the sum σ_v of weight vectors lies in the image of the map α_ℓ.

7. From cohomological classes to algebraic cycles

The passage from Theorem 6.1 to Theorem 1.1 is a simple application of the standard conjectures on algebraic cycles, which is easy to prove in codimension one. More precisely, for divisors with rational coefficients in a projective variety, Matsusaka [16] proved that numerical equivalence implies algebraic equivalence. So in particular, numerical equivalence implies homological equivalence.

Let us suppose that $\mathrm{Im}(\alpha_\mathbb{Q}) \subset \mathbb{Q}^{I_\mathfrak{x}}$ is contained in a hyperplane given by $f = 0$. Let x_i, $i \in I_\mathfrak{x}$ be the coordinates on $\mathbb{Q}^{I_\mathfrak{x}}$ and write f as $\sum_{i\in I_\mathfrak{x}} a_i x_i$, for some $a_i \in \mathbb{Q}$. The fact that $\mathrm{Im}(\alpha_\mathbb{Q})$ is contained in the hyperplane $f = 0$ implies that the $\mathbb{Q}$-divisor $\sum_{i\in I_\mathfrak{x}} a_i L_i$ is numerically equivalent to 0, it is thus homologically equivalent to 0. Therefore, the image $\mathrm{Im}(\alpha_\ell)$ in Theorem 6.1 is also contained in the hyperplane in $\mathbb{Q}_\ell^{I_\mathfrak{x}}$ defined by the same equation $f = 0$. To conclude, we have shown that $\mathrm{Im}\,\alpha_\mathbb{Q}\otimes\mathbb{Q}_\ell \simeq \mathrm{Im}\,\alpha_\ell$. So we have deduced Theorem 1.1 from Theorem 6.1.

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